Mathematics 581
Midterm I
Friday, February 8, 2008

There are four problems worth 25 points each and two extra credit problems worth something or other.

1. Let $\mathbf{A} = \mathbf{Z}_{54}$.

   (a) Show that $\mathbf{A}$ is not an integral domain.

   (b) Let $I = 2\mathbf{A}$. Show that I is a maximal ideal of $\mathbf{A}$ and describe $\mathbf{A}/I$. (How big is this factor ring? Is it a field? Is it an integral domain? What does it look like? Be as explicit as possible.)

   (c) Let $J = 9\mathbf{A}$. Show that J is an ideal of $\mathbf{A}$ which is not maximal. Describe $\mathbf{A}/J$. (How big is this factor ring? Is it a field? Is it an integral domain? What does it look like? Be as explicit as possible.) Find an ideal $K \neq \mathbf{A}$ which strictly contains J.

2. (a) State the well-ordering principle for the ring $\mathbf{Z}$ of integers.

   (b) Use the well-ordering principle (and long division) to prove that every ideal of $\mathbf{Z}$ is a principal ideal.

3. Let $a = 10358$ and $b = 2035$.

   (a) Find the greatest common divisor d of a and b.

   (b) Find integers x and y such that $ax + by = d$.

4. Suppose that p is a prime integer. Prove that if $p|ab$ then either $p|a$ or $p|b$ (or both). Give an example to show that this statement is false for composite integers. You may assume (without proof) the $ax + by = d$ theorem for greatest common divisors.

5. (Extra Credit. Do Problem 4 first)

   Factor the numbers a and b of Problem 3 into prime factors. Be sure to verify that your factors are really prime.

6. (A hard extra credit problem. Do this one last of all.) Let $\mathbf{A} = \mathbf{Q}[x]$ be the ring of polynomials in one variable with rational coefficients. Let $\mathbf{R}$ be the field of real numbers and let $f : \mathbf{A} \to \mathbf{R}$ be the mapping that sends x to $\sqrt{2}$. Explain why f is a ring homomorphism. Find its kernel. (Note: f is *not* onto. For even more extra credit, describe its range.)