Final, Math 672, SP 04; Yuval FLICKER; do questions 1-5 only; YOUR NAME:

1. Suppose that $f = x^3 + px + q \in \mathbb{Z}[x]$ is irreducible. Show that $D = -4p^3 - 27q^2$ is not 0, 1 or -1.

2. Let F be a field of characteristic zero such that every cubic over F has a root (in F). What is the Galois group of an irreducible $f \in F[x]$ of degree 4 whose discriminant is in F^2?

3. Let p, q, r be primes in $\mathbb{Z}$ with $q \neq r$. Show that $\mathbb{Q}(\sqrt[p]{q}) \neq \mathbb{Q}(\sqrt[p]{r})$ where $\sqrt[p]{q}$ is any root of $x^p - q$ and $\sqrt[p]{r}$ is any root of $x^p - r$.

4. If $a \in F$ where F is a field in $\mathbb{R}$, and $K = F(\sqrt[n]{a})$ where $\sqrt[n]{a}$ is an nth root of a in $\mathbb{R}$, then any Galois extension L of F in K has degree 1 or 2 over F.

5. Let E be a cyclic extension of F of degree n. Suppose r divides n and $c \in F^\times$ has $c^r = N_{E/F}(u)$ for some u in E. Show that there is a v in the unique subfield K of E of dimension $m = n/r$ over F with $N_{K/F}(v) = c$. Please denote by σ a generator of $\mathrm{Gal}(E/F)$. (This is the hardest question in the exam. K is the subfield of E fixed by σ^m. Put $\beta = \prod_{i=0}^{m-1} \sigma^i u \in E$. Compute $N_{E/K}\beta$. Is there an x with $c/\beta = x/\sigma^m x$? Take $v = \frac{ux}{\sigma(x)}$.)

6. Show that the norm map N from the field $\mathbb{F}_{q^n}^\times$ of q^n elements to the field $\mathbb{F}_q^\times$ of q elements is onto.

7. Show that $\mathbb{Q}(\sqrt[3]{5})$ is not a subfield of any cyclotomic field over $\mathbb{Q}$.

8. Determine all rational solutions of $x^2 + Dy^2 = 1$ where D is a positive integer which is not a square in $\mathbb{Z}$.