COUNTING LOCAL SYSTEMS WITH PRINCIPAL UNIPOTENT LOCAL MONODROMY

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Abstract. Let X_1 be a curve of genus g, projective and smooth over $\mathbb{F}_q$. Let $S_1 \subset X_1$ be a reduced divisor consisting of N_1 closed points of X_1. Let (X, S) be obtained from (X_1, S_1) by extension of scalars to an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. Fix a prime l not dividing q. The pullback by the Frobenius endomorphism Fr of X induces a permutation Fr^* of the set of isomorphism classes of rank n irreducible $\overline{\mathbb{Q}}_l$-local systems on $X - S$. It maps to itself the subset of those classes for which the local monodromy at each $s \in S$ is unipotent, with a single Jordan block. Let $T(X_1, S_1, n, m)$ be the number of fixed points of Fr^{*m} acting on this subset.

Under the assumption that $N_1 \geq 2$, we show that $T(X_1, S_1, n, m)$ is given by a formula reminiscent of a Lefschetz fixed point formula: the function $m \mapsto T(X_1, S_1, n, m)$ is of the form $\sum n_i \gamma_i^m$ for suitable integers n_i and "eigenvalues" γ_i.

We use Lafforgue [L] to reduce the computation of $T(X_1, S_1, n, m)$ to counting automorphic representations of $\mathrm{GL}(n)$, and the assumption $N_1 \geq 2$ to move the counting to the multiplicative group of a division algebra, where the trace formula is easier to use.

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Date: July 30, 2012.

2000 *Mathematics Subject Classification.* Primary 11F70, 11F72; Secondary 22E35, 22E55, 11G20, 11R39, 11R52, 11R58, 14H30, 11S37.

Key words and phrases. Local systems, principal unipotent local monodromy, trace formula, automorphic representations, $\mathrm{GL}(n)$, function fields.

Introduction

We keep the notations of the abstract. They are elaborated upon in 0.1. What are "$\overline{\mathbb{Q}}_l$-local systems" is explained in 1.1. A better terminology, which we will use from now on, is "$\overline{\mathbb{Q}}_l$-smooth sheaf". Actions of Frobenius are explained in 1.2.

The present work is motivated by Drinfeld's 1981 paper [D]. In it, Drinfeld considers irreducible $\overline{\mathbb{Q}}_l$-smooth sheaves of rank 2 on X, and computes the number of fixed points of Fr^{*m} on the set of their isomorphism classes. He uses the trace formula for GL(2) and the then not yet wholly available correspondence between irreducible $\overline{\mathbb{Q}}_l$-smooth sheaves of rank 2 over X_1 and everywhere unramified cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A})$, where $\mathbb{A}$ is the adèle ring of the function field F_1 of X_1. He shows in particular that, as a function of m, the number of fixed points is of the form $\sum n_i \gamma_i^m$. Our result is a simple yet higher dimensional analogue of Drinfeld's deep and beautiful analysis. We hope similar counting formulas of Lefschetz type hold for any number of ramification points and any imposed local monodromy at those points. See 6.29, 6.30. The case of tame local monodromy, given at each point by n characters of the residue field, has been considered by Arinkin (unpublished), under a "generic position" assumption for the characters involved.

We now describe what is done in each section. In §1, we review relations between $\overline{\mathbb{Q}}_l$-smooth sheaves on $X_1 - S_1$, l-adic representations of $\mathrm{Gal}(\overline{F}_1/F_1)$, and automorphic representations of $\mathrm{GL}(n)$ and of the multiplicative groups of some division algebras.

Our proof relies on the relation between automorphic representations for $\mathrm{GL}(n)$ and for multiplicative groups of division algebras. More precisely, our results use the statement 1.13. How to extract this statement from the literature is explained in the Appendix. As computations of numbers of various kinds of automorphic representations for multiplicative groups of some division algebras, our results are independent of 1.13. It is the interpretation of our results as computations of numbers of fixed points which makes our results interesting.

In §2, we state a first form (2.3) of our result. This form is the one to which an application of the trace formula (compact quotient case) leads. It does not make clear that, as a function of m, the number $T(X_1, S_1, n, m)$ of fixed points of Fr^{*m} has the form $m \mapsto \sum n_i \gamma_i^m$.

After a discussion of division algebras and Tamagawa numbers in §3, the proof of 2.3 is given in §4 and §5. It applies the trace formula in the compact quotient case of multiplicative groups of division algebras. A categorical approach to the proof of the trace formula in the compact quotient case we need is presented in §4, and the building of $\mathrm{SL}(n)$ is used in the computations of §5.

The reader is invited, at first reading, to jump from §2 to §6. In §6, 2.3 is massaged into a form which cries for a geometric explanation, and makes clear that, as a function of m, $T(X_1, S_1, n, m)$ has the form

$$m \longmapsto \sum n_i \gamma_i^m, \tag{1}$$

where the n_i are nonzero integers and the γ_i are distinct q-Weil numbers: for each γ_i there exists an integer w, its weight, such that all complex conjugates of γ_i have absolute value $q^{w/2}$. For $g = 0$ and $\deg(S_1) = 2$, as well as for $g = 0$, $n = 2$ and $\deg(S_1) = 3$, the sum (1) is empty: $T(X_1, S_1, n, m)$ is identically zero.

After excluding these cases, we find the γ_i with the largest complex absolute value. It occurs with multiplicity one and is an integral power of q. We also show that each γ_i is q times an algebraic integer. Next, we consider how $T(X_1, S_1, n, m)$ varies with (X_1, S_1), and in 6.3.1 we ask for a topological understanding.

We do not understand the meaning of the change of summations from $\Sigma_a \Sigma_b$ in the expression (6.5.2) for $T(X_1, S_1, n, 1)$ to $\Sigma_b \Sigma_a$ in the expression (6.6.4). It leads to a decomposition of $T(X_1, S_1, n, m)$ as a sum over the divisors b of n, each term of which has the form (1).

In §7, we look at one of the simplest examples: the case where X is of genus 0 and where S consists of four points. An amusing trick allows us in that case to remove the assumption $N_1 \geq 2$. The same trick implies a symmetry property of some automorphic representations, of which we do not know of a proof not using [L].

0. Notations

0.1 We fix a finite field $\mathbb{F}_q$ with q elements and its algebraic closure $\mathbb{F}$. The characteristic is p, and $\mathbb{F}_{q^m}$ is the degree m extension of $\mathbb{F}_q$ in $\mathbb{F}$.

0.2. We fix a projective and smooth curve X_1 over $\mathbb{F}_q$, absolutely irreducible of genus g, and a reduced divisor $S_1 \subset X_1$ consisting of N_1 closed points. As $\mathbb{F}_q$ is a perfect field, it is automatically étale over $\mathrm{Spec}(\mathbb{F}_q)$. The field of rational functions on X_1 will be denoted by F_1. It is a global field of characteristic p, with field of constants $\mathbb{F}_q$. Closed points of X_1 correspond one to one to places of F_1; we identify the two. If s is a closed point, we denote the local ring of X_1 at s by $\mathcal{O}_{(s)}$. It is a discrete valuation ring. We denote its *completion* by $\mathcal{O}_s$. This is contrary to the usual usage. The residue field at s we denote by $k(s)$. The completion of F_1 at the place s we denote by F_{1s}. They are, respectively, the quotient of the complete valuation ring $\mathcal{O}_s$ by its maximal ideal m_s, and the fraction field of $\mathcal{O}_s$.

0.3. Suppressing the index 1 indicates an extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}$, and replacing it by m an extension of scalars to $\mathbb{F}_{q^m}$. For instance, N is the number of points of $X = X_1 \otimes_{\mathbb{F}_q} \mathbb{F}$ in the inverse image S of S_1. We write $\mathbb{A}$ for the ring of F_1-adèles.

Exception: from §3 on, we do not use the function field $F = F_1 \otimes_{\mathbb{F}_q} \mathbb{F}$ of X. To simplify notations we will write simply F for F_1, and D for a division algebra with center F_1 (renamed F).

0.4. We fix a prime $l \neq p$ and an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$.

0.5. The element $\sigma\colon \mathbb{F} \to \mathbb{F}$, $x \mapsto x^q$, of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ is called the *Frobenius substitution*. Its inverse, denoted Frob, is the *geometric Frobenius*.

0.6. For any scheme Y over $\mathbb{F}_q$, let Φ_Y be the endomorphism of Y which is the identity on the underlying set, and for which $\Phi_Y^*(f) = f^q$. For any étale sheaf $\mathcal{F}$ on Y, the pullback $\Phi_Y^*\mathcal{F}$ is canonically isomorphic to $\mathcal{F}$. If one pictures $\mathcal{F}$ as an (algebraic) space over Y, this expresses the functoriality of Φ, which gives rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Phi_{\mathcal{F}}} & \mathcal{F} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Phi_Y} & Y. \end{array} \tag{0.6.1}$$

Special case: Φ_{X_1} is an endomorphism of the $\mathbb{F}_q$-scheme X_1. Extending scalars to $\mathbb{F}$, one obtains the *Frobenius endomorphism* Fr of X.

0.7. The multiplicative group of an algebra A is denoted by A^*.

1. Dictionaries

The reader familiar with $\overline{\mathbb{Q}}_l$-smooth sheaves and actions of Frobenius is invited to jump to 1.4.

1.1. Warning: $\overline{\mathbb{Q}}_l$-smooth sheaves, called "$\overline{\mathbb{Q}}_l$-local systems" in the abstract, are not defined to be locally constant for the étale topology. Rather, the category of $\overline{\mathbb{Q}}_l$-smooth sheaves is defined by limiting processes from categories of local systems with finite fibers, which are locally constant for the étale topology. One proceeds in three steps:

a) Let E_λ be a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$, $\mathcal{O}_\lambda$ its valuation ring, and m_λ the maximal ideal of $\mathcal{O}_\lambda$. The category of $\mathcal{O}_\lambda$-smooth sheaves on a scheme Y is the category of projective systems $(\mathcal{F}_k)_{k\,\geq 1}$ of locally constant sheaves of $\mathcal{O}_\lambda/m_\lambda^k$-modules of finite type, with

$$\mathcal{F}_k/m_\lambda^{k'}\mathcal{F}_k \xrightarrow{\sim} \mathcal{F}_{k'} \text{ for } k' \leq k.$$

b) [Assuming Y to be normal] The category of E_λ-smooth sheaves on Y is obtained from it by tensoring over $\mathcal{O}_\lambda$ by E_λ: same objects, and $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_\lambda}(\mathcal{F},\mathcal{G}) \otimes_{\mathcal{O}_\lambda} E_\lambda$.

When Y is not assumed to be normal, a more complicated definition is required if one wants categories of E_λ-smooth sheaves to form a stack for the étale topology. We will not need this more general setting.

c) The set of finite extensions E_λ of $\mathbb{Q}_l$ contained in $\overline{\mathbb{Q}}_l$, ordered by inclusion, is filtering. The category of $\overline{\mathbb{Q}}_l$-smooth sheaves is the 2-inductive limit, along this set, of the categories of E_λ-smooth sheaves. Those categories do not form an inductive system in the category of categories, only a 2-inductive system in the 2-category of categories, hence the appearance of 2-inductive limits.

Suppose the normal scheme Y is connected. If o is a geometric point of Y, the functor "fiber at o" is an equivalence of categories from the category of $\overline{\mathbb{Q}}_l$-smooth sheaves to the category $\mathrm{Rep}(\pi_1(Y,o),\overline{\mathbb{Q}}_l)$ of continuous linear representations of the profinite fundamental group of Y:

$$\pi_1(Y,o) \to \mathrm{GL}(V),$$

for V a finite dimensional vector space over $\overline{\mathbb{Q}}_l$. In this statement, the topology used on $\mathrm{GL}(n,\overline{\mathbb{Q}}_l)$ is any such that the $\mathrm{GL}(n,E_\lambda)$ are closed subgroups, and the induced topology on $\mathrm{GL}(n,E_\lambda)$ is the l-adic topology. A Baire category argument shows that a continuous homomorphism from $\pi_1(Y,o)$ to $\mathrm{GL}(n,\overline{\mathbb{Q}}_l)$ factors through a $\mathrm{GL}(n,E_\lambda)$.

1.2. The Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts on $\mathbb{F}$, hence on the scheme $X-S = (X_1-S_1)\otimes_{\mathbb{F}_q}\mathbb{F}$, and, by transport of structures, on the set of isomorphism classes of $\overline{\mathbb{Q}}_l$-smooth sheaves on $X-S$. Transport of structures (Bourbaki Ens Ch. IV) is the principle that any isomorphism $Y_1 \to Y_2$ extends to objects constructed from Y_1 and Y_2. When $Y_1 = Y_2$: symmetries extend. The construction has to be canonical: not involving choices.

Example: If $\tau\colon\ K_1 \to K_2$ is an isomorphism between fields, the corresponding isomorphism $[\tau]\colon\ \mathrm{Spec}(K_1) \to \mathrm{Spec}(K_2)$ is such that $[\tau]^*(a) = \tau^{-1}(a)$ for a in K_2. If τ is an automorphism of K, the action by transport of structures of τ on $\mathrm{Spec}(K)$ is $[\tau] : \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ such that $[\tau]^*(a) = \tau^{-1}(a)$.

Lemma 1.3. *The action, by transport of structures, of the geometric Frobenius* Frob *in* $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ (0.5) *on the set of isomorphism classes of* $\overline{\mathbb{Q}}_l$*-smooth sheaves on* $X-S$ *coincides with the pullback by the Frobenius endomorphism* $\mathrm{Fr}\colon X-S\to X-S$.

More precisely, if $\mathcal{F}$ is $\overline{\mathbb{Q}}_l$-smooth sheaf on $X-S$, one has a canonical isomorphism between $\mathrm{Fr}^*\mathcal{F}$, and the $\overline{\mathbb{Q}}_l$-smooth sheaf obtained from $\mathcal{F}$ by transport of structures using Frob. This is deduced from a similar result for étale sheaves, which we now explain, following SGA 5 XIV, §1,2.

Proof. The endomorphism Φ_{X-S} (0.6) of $X-S=(X_1-S_1)\times_{\mathrm{Spec}(\mathbb{F}_q)}\mathrm{Spec}(\mathbb{F})$ is the composite of the endomorphisms induced by $\Phi_{X_1-S_1}$ and by $\Phi_{\mathrm{Spec}(\mathbb{F})}$. The first is $\mathrm{Fr}\colon X-S\to X-S$. By the example in 1.2, $\Phi_{\mathrm{Spec}(\mathbb{F})}=[\mathrm{Frob}]$. By (0.6.1), the pullback functor Φ^*_{X-S} is isomorphic to the identity. It is the composite of Fr^*, and of $(X_1-S_1)\times_{\mathrm{Spec}(\mathbb{F}_1)}[\mathrm{Frob}]^*$. The latter is the inverse of the action of Frob by transport of structures, and the claim follows. □

1.4. We now explain how the action 1.3 is expressed in the languages of representations of Galois groups or of fundamental groups.

The function field F_1 of X_1 is a global field of characteristic p, with field of constants $\mathbb{F}_q$. The function field F of X is $F_1\otimes_{\mathbb{F}_q}\mathbb{F}$. Fix a separable closure $\overline{F}$ of F. The Galois group $\mathrm{Gal}(\overline{F}/F_1)$ is an extension

$$1\to\mathrm{Gal}(\overline{F}/F)\to\mathrm{Gal}(\overline{F}/F_1)\to\mathrm{Gal}(F/F_1)\to 1 \tag{1.4.1}$$

and

$$\mathrm{Gal}(F/F_1)\xrightarrow{\sim}\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q). \tag{1.4.2}$$

We recall the identification 0.2 of closed points and places. If s is a closed point of X, with image s_1 in X_1, the choice of a place $\bar{s}$ of $\overline{F}$ above s defines a decomposition (= inertia) group $I_s\subset\mathrm{Gal}(\overline{F}/F)$ as well as a decomposition group $D_{s_1}\subset\mathrm{Gal}(\overline{F}/F_1)$, and I_s is the inertia subgroup of D_{s_1}. If the closed point s_1 is of degree d over $\mathbb{F}_q$, the residue field $k(s_1)$, naturally embedded in $k(s)\xleftarrow{\sim}\mathbb{F}$, is $\mathbb{F}_{q^d}$. We have a comparison morphism relating (1.4.1) (with (1.4.2) used to replace $\mathrm{Gal}(F/F_1)$ by $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$) and its local analogue:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_s & \longrightarrow & D_{s_1} & \longrightarrow & \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{q^d}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Gal}(\overline{F}/F) & \longrightarrow & \mathrm{Gal}(\overline{F}/F_1) & \longrightarrow & \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) & \longrightarrow & 1. \end{array} \tag{1.4.3}$$

On the right, $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ is $\widehat{\mathbb{Z}}$, generated by Frob, $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{q^d})$ is $\widehat{\mathbb{Z}}$, generated by its d^{th} power, and the vertical map is $d\colon\widehat{\mathbb{Z}}\to\widehat{\mathbb{Z}}$.

The geometric point $\mathrm{Spec}(\overline{F}) \to X - S$ of $X - S$, as well as its image $\mathrm{Spec}(\overline{F}) \to X_1 - S_1$ in $X_1 - S_1$, will be denoted by o. Let $\overline{F}^S$ be the maximal extension of F in $\overline{F}$ which is unramified outside of S. The fundamental group $\pi_1(X - S, o)$ (resp. $\pi_1(X_1 - S_1, o)$) is the Galois group $\mathrm{Gal}(\overline{F}^S/F)$ (resp. $\mathrm{Gal}(\overline{F}^S/F_1)$), and the homotopy sequence of fundamental groups for the fibration $X_1 - S_1 \to \mathrm{Spec}(\mathbb{F}_q)$ is the quotient of the second line of (1.4.3) obtained by replacing $\overline{F}$ by $\overline{F}^S$:

$$1 \to \pi_1(X - S, o) \to \pi_1(X_1 - S_1, o) \to \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \to 1. \tag{1.4.4}$$

As explained in 1.1, the functor "fiber at o" is an equivalence of categories from the category of $\overline{\mathbb{Q}}_l$-smooth sheaves on $X - S$ to the category of continuous representations of $\pi_1(X - S, o)$ on finite dimensional $\overline{\mathbb{Q}}_l$-vector spaces. To compute in this language the action by transport of structures of τ in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, one should first lift τ to an automorphism of the geometric point o, i.e. of $\overline{F}$. The induced automorphism $\tilde{\tau}$ of $\overline{F}^S$ is an element of $\pi_1(X_1 - S_1, o)$. It acts on $\pi_1(X_1 - S_1, o)$ by the corresponding inner automorphism. The induced action on $\pi_1(X - S, o)$ gives the action on representations: $\rho \mapsto (g \mapsto \rho(\tilde{\tau}^{-1} g \tilde{\tau}))$. As was clear a priori, the action obtained on isomorphism classes of representations (i.e. $\overline{\mathbb{Q}}_l$-smooth sheaves) does not depend on the lifting chosen. This will be used for $\tau = \mathrm{Frob}$.

1.5. To state the relation between Frobenius fixed points and automorphic representations, it is convenient to replace Galois groups by Weil groups. Let $W(X_1 - S_1, o)$ be the inverse image in $\pi_1(X_1 - S_1, o)$ of the subgroup $\mathbb{Z}$ of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) = \widehat{\mathbb{Z}}$. One gives it the topology for which $\pi_1(X - S, o)$ is an open subgroup.

Locally, with the notations $s, s_1, \bar{s}, I_s, D_{s_1}$ and d of 1.4, let W_{s_1} be the inverse image in D_{s_1} of the subgroup $\mathbb{Z}$ of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{q^d}) = \widehat{\mathbb{Z}}$, with the topology for which I_s is an open subgroup. From (1.4.3) we get

$$\begin{tikzcd} 1 \arrow[r] & I_s \arrow[r] \arrow[d] & W_{s_1} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \pi_1(X - S, o) \arrow[r] & W(X_1 - S_1, o) \arrow[r] & \mathbb{Z} \arrow[r] & 1. \end{tikzcd} \tag{1.5.1}$$

The left vertical map is trivial when $s \notin S$, in which case we get from (1.5.1) a map $\mathbb{Z} \to W(X_1 - S_1, o)$. The image of 1 is called a Frobenius at s_1.

From the computation 1.4 of the Frobenius action, we get

Lemma 1.6. *Let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_l$-smooth sheaf on $X - S$. Its isomorphism class is fixed by* Frob $\in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ *(cf.* 1.3*) if and only if the corresponding representation of $\pi_1(X - S, o)$ can be extended to $W(X_1 - S_1, o)$.*

1.7. In the case of function fields, the notion of cuspidal automorphic representation is purely algebraic. One can use as field of coefficients any algebraically closed field k of characteristic zero, for instance $\overline{\mathbb{Q}}_l$. Let $\mathbb{A}$ be the adèle ring of F_1. In the case of $\mathrm{GL}(n)$, the multiplicity one theorem allows us not to make a distinction between cuspidal automorphic representations π of $\mathrm{GL}(n,\mathbb{A})$, viewed as subrepresentations of the space of k-valued locally constant cuspidal functions on $\mathrm{GL}(n,F_1)\backslash\mathrm{GL}(n,\mathbb{A})$, on which $\mathrm{GL}(n,\mathbb{A})$ acts by right translations, or viewed as isomorphism classes of irreducible representations of $\mathrm{GL}(n,\mathbb{A})$ occuring in that space. As representations, they are restricted tensor products of representations π_s of the local groups $\mathrm{GL}(n,F_{1s})$, s a place of F_1. We will say that π is *unramified* at s if π_s admits a nonzero vector fixed by the maximal compact subgroup $\mathrm{GL}(n,\mathcal{O}_s)$ of $\mathrm{GL}(n,F_{1s})$.

By Lafforgue [L], the isomorphism classes of n-dimensional irreducible $\overline{\mathbb{Q}}_l$-linear continuous representations of $W(X_1-S_1,o)$ are in a natural bijective correspondence with the $\overline{\mathbb{Q}}_l$-cuspidal automorphic representations of $\mathrm{GL}(n,\mathbb{A})$ which are unramified outside of S_1 (global Langlands correspondence). If ρ corresponds to π, the restriction (1.5.1) of ρ to W_{s_1} corresponds to π_{s_1} by the local Langlands correspondence.

A surprising consequence: an algebraic isomorphism $\overline{\mathbb{Q}}_l \xrightarrow{\sim} \overline{\mathbb{Q}}_{l'}$ induces a bijection between isomorphism classes of irreducible $\overline{\mathbb{Q}}_l$-linear continuous representations of $W(X_1-S_1,o)$, and the same for $\overline{\mathbb{Q}}_{l'}$, in spite of the fact that we are considering continuous representations for the l- or l'-adic topologies. Representations correspond if the characteristic polynomials of Frobeniuses do. Caveat: if one wants to use the so-called unitary correspondence between representations of $W(X_1-S_1,o)$ and automorphic representations, one needs to choose a square root of q in $\overline{\mathbb{Q}}_l$.

1.8. Let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_l$-smooth sheaf on $X-S$. Let $V:=\mathcal{F}_o$ be the corresponding representation of $\pi_1(X-S,o)$. Suppose that the isomorphism class of $\mathcal{F}$ is fixed by the Frobenius, and choose an extension of the representation V to a representations V_1 of $W(X_1-S_1,o)$ (1.6). In general, irreducibility of V_1 does not imply irreducibility of V (equivalently: of $\mathcal{F}$). The 1981 work of Drinfeld [D], in which $S=\emptyset$, suggests that it is the irreducibility of $\mathcal{F}$ we should be concerned with. We are interested in the case when $S\neq\emptyset$ and when the local monodromy at each s in S is principal unipotent. This case is simpler.

As in 1.4, let I_s be an inertia group at s. The largest pro-l quotient of I_s is isomorphic to $\mathbb{Z}_l$. "*Principal unipotent local monodromy at s*" means that the action of I_s factors through this $\mathbb{Z}_l$, with an element of I_s with image a in $\mathbb{Z}_l$ acting as $\exp(aN)$, where N is nilpotent with one Jordan block.

Lemma 1.9. (i) *If at one $s \in S$ the local monodromy is principal unipotent, then V is irreducible as soon as V_1 is.*
(ii) *If V is irreducible, any extension V_1' of V to a representation of $W(X_1 - S_1, o)$ is of the form $V_1 \otimes \chi$, for $\chi \colon \mathbb{Z} \to \overline{\mathbb{Q}}_l^*$ a character of the quotient $\mathbb{Z}$ of $W(X_1 - S_1, o)$, and the $V_1 \otimes \chi$ are all nonisomorphic.*

Proof of (i). Any subrepresentation V' of V, being stable by the $\exp(aN)$ ($a \in \mathbb{Z}_l$), and hence by N, will be of the form $N^k V$. It will hence be the only subrepresentation of V of its dimension. As $\pi_1(X - S, a)$ is an invariant subgroup of $W(X_1 - S_1, 0)$, any element of $W(X_1 - S_1, o)$ will map V' to a subrepresentation of V, of the same dimension, hence to itself: V' is a subrepresentation of V_1.

Proof of (ii). By assumption, V_1 and V_1' are identical as representations of $\pi_1(X, o)$. by Schur's lemma,

$$\mathrm{Hom}_{\pi_1(X-S,o)}(V_1, V_1')$$

is reduced to the line $\overline{\mathbb{Q}}_l$ of multiplication by scalars. It is a representation of the quotient $\mathbb{Z}$ of $W(X_1 - S_1, o)$, and is given by a character χ of $\mathbb{Z}$. That

$$V_1 \otimes \mathrm{Hom}_{\pi_1(X-S,o)}(V_1, V_1') \xrightarrow{\sim} V_1'$$

gives that V_1' is $V_1 \otimes \chi$. Conversely, if we take $V_1' = V_1 \otimes \eta$, the character χ we obtain is η, so that twists by distinct characters are non isomorphic.

1.10. The *divisor* of an idèle $a = (a_s)$ is $\sum$ valuation$(a_s) \cdot s$, where the sum is over all closed points of X_1. The *degree* of a is the degree $\sum$ valuation$(a_s) \cdot \deg(s)$ of its divisor. Equivalently, the degree map, from the idèle class group $F_1^* \backslash \mathbb{A}^*$ to $\mathbb{Z}$, is characterized by

$$\|a\| = q^{-\deg(a)}. \tag{1.10.1}$$

Let $\chi \colon \mathbb{Z} \to \overline{\mathbb{Q}}_l^*$ be a character of $\mathbb{Z}$, where $\mathbb{Z}$ is viewed as the quotient $\deg \circ \det \colon \mathrm{GL}(n, \mathbb{A}) \to \mathbb{Z}$ of $\mathrm{GL}(n, \mathbb{A})$ (resp. as the quotient $\mathbb{Z}$ of $W(X_1 - S_1, o)$). The twist by χ, denoted $\pi\,\chi$ (resp, $V_1\chi$), of a representation π of $\mathrm{GL}(n, \mathbb{A})$ (resp. V_1 of $W(X_1 - S_1, o)$) is $\pi \otimes \chi$ (resp. $V_1 \otimes \chi$). Such twists we name $\mathbb{F}_q$-*twists*. If $\mathcal{F}_1$ is a $\overline{\mathbb{Q}}_l$-smooth sheaf on $X_1 - S_1$, if V_1 is the restriction of the corresponding representation of $\pi_1(X_1 - S_1, o)$ to $W(X_1 - S_1, o)$, and if χ is with values in the units of $\overline{\mathbb{Q}}_l^*$, hence extends to $\widehat{\mathbb{Z}}$, then $V_1\chi$ is obtained from the tensor product of $\mathcal{F}_1$ with the pullback to $X_1 - S_1$ of a rank one $\overline{\mathbb{Q}}_l$-sheaf on $\mathrm{Spec}(\mathbb{F}_q)$. This is what motivates the terminology.

If π is automorphic, hence is a space of functions on $\mathrm{GL}(n,F_1)\backslash\mathrm{GL}(n,\mathbb{A})$, then so is $\pi\,\chi$: it is the space of functions $f\cdot\chi(\deg\circ\det)$ for f in π. With the notations of 1.7, if an irreducible representation V_1 of $W(X_1-S_1,o)$ corresponds by the global Langlands correspondence to the cuspidal automorphic representation π of $\mathrm{GL}(n,\mathbb{A})$, then $V_1\chi$ corresponds to $\pi\,\chi$.

By the local Langlands correspondence, the condition that $\mathcal{F}$ has principal unipotent local monodromy at $s\in S$, with image s_1 in S_1, corresponds to the condition on the automorphic representation π that the local component π_{s_1} be of the form

$$\text{Steinberg representation} \otimes \chi(\det), \tag{1.10.2}$$

for χ an unramified character $F^*_{1s_1}\to\overline{\mathbb{Q}}_l^*$.

Applying 1.9 we conclude:

Scholium 1.11. *Assume that S is not empty.*

(i) *There is a bijective correspondence between*

(A) *isomorphism classes of rank n irreducible $\overline{\mathbb{Q}}_l$-smooth sheaves on $X-S$, fixed by the Frobenius, and with principal unipotent monodromy at each $s\in S$, and*

(B) *classes modulo $\mathbb{F}_q$-twisting of cuspidal automorphic $\overline{\mathbb{Q}}_l$-representations π of* $\mathrm{GL}(n,\mathbb{A})$, *unramified outside of S_1, such that for each $s\in S_1$ the representation π_s is of the form* (1.10.2).

(ii) *For π as in* (B), *the $\mathbb{F}_q$-twists are all distinct.*

1.12. Suppose that D is a rank n division algebra over F_1, unramified outside a subset ${}_0S_1$ of the set of places S_1, and for which at each $s\in{}_0S_1$ the completion D_s is a division algebra over F_{1s}. Such a division algebra exists if and only if $|S_1|\geq 2$.

By abuse of notations, we denote by D^* the algebraic group over F such that for any commutative F-algebra R, the group $D^*(R)$ of R-points of D^* is the multiplicative group $(D\otimes_F R)^*$ of $D\otimes_F R$. The group of F-points of D^* is simply the multiplicative group of D.

The reduced norm defines an homomorphism det of algebraic groups from D^* to the multiplicative group $\mathbb{G}_m$, and $\mathbb{F}_q$-twists of automorphic representations of $D^*(\mathbb{A})$ are defined as for $\mathrm{GL}(n)$ (1.10).

Our results depend on the following statement. How to extract this statement from the literature is explained in the Appendix.

Statement 1.13. *Suppose that* $n \geq 2$. *There is then a bijective correspondence, compatible with* $\mathbb{F}_q$*-twists, between*

(A) *cuspidal automorphic representations* π *of* $\mathrm{GL}(n, \mathbb{A})$, *whose local components at each* $s \in {}_0S_1$ *is of the form* (1.10.2), *and*

(B) *automorphic representations* π' *of* $D^*(\mathbb{A})$, *other than one-dimensional, whose local components at each* $s \in {}_0S_1$ *is one-dimensional and of the form* $\chi(\det)$ *for* χ *an unramified character of* F_{1s}^*.

The representation π (*resp.* π') *occurs with multiplicity one in the cuspidal spectrum of* $\mathrm{GL}(n, \mathbb{A})$ (*resp.* $D^*(\mathbb{A})$). *If* π *corresponds to* π', *at each* $v \notin {}_0S_1$ (*so that* $D_s^* \simeq \mathrm{GL}(n, F_{1s})$), π_s *is isomorphic to* π_s'.

The number of classes 1.11(A), or (B), is hence also the number of classes modulo $\mathbb{F}_q$-twisting of automorphic representations π' of $D^*(\mathbb{A})$ other than one-dimensional, with local components at each $s \in {}_0S_1$ as in 1.13 (B), with local components at each $s \in S_1 - {}_0S_1$ of the form (1.10.2), and unramified outside of S_1. Further, for π' as in (B), the $\mathbb{F}_q$-twists $\pi'\chi$ are all distinct.

So far, we have considered $\overline{\mathbb{Q}}_l$-automorphic representations. The theory being algebraic, nothing changes if one considers instead the usual $\mathbb{C}$-automorphic representations.

2. Statement of the theorem: first form

2.1. Let $\mathcal{T}^{(n)}(X, S)$, or simply $\mathcal{T}^{(n)}$, be the set of isomorphism classes of rank n irreducible $\overline{\mathbb{Q}}_l$-smooth sheaves on $X - S$, with principal unipotent local monodromy at each $s \in S$ (see 1.8).

The geometric Frobenius $\mathrm{Frob} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts on $\mathcal{T}^{(n)}$ by transport of structures. This action coincides with the pullback by the Frobenius endomorphism $\mathrm{Fr}\colon X - S \to X - S$ (1.3). Our aim is to compute the number $T(X_1, S_1, n)$ of its fixed points. For each $m \geq 1$, the Frobenius endomorphism for $(X_m - S_m)/\mathbb{F}_{q^m}$ is Fr^m. The number $T(X_1, S_1, n, m)$ of fixed points of the m^{th} iterate of Frobenius is hence given by

$$T(X_1, S_1, n, m) = T(X_m, S_m, n). \tag{2.1.1}$$

Before stating the result, we introduce some notations.

2.2. The zeta function of X_1 is a function of a complex variable usually called s. We will call it z, to avoid confusion with points of X_1. Rather than this variable, we will systematically use the variable $t = q^{-z}$. The function $Z(X_1, t)$ is defined by

$$Z(X_1, t) = \zeta(X_1, z) \tag{2.2.1}$$

when $t = q^{-z}$. It depends not only on the scheme X_1, but also on q such that X_1 is a $\mathbb{F}_q$-scheme. It has the cohomological description

$$\begin{aligned} Z(X_1, t) &= \prod \det(1 - \mathrm{Frob} \cdot t, H^i(X))^{(-1)^{i+1}} \qquad (2.2.2) \\ &= \frac{\det(1 - \mathrm{Frob} \cdot t, H^1(X))}{(1-t)(1-qt)} \end{aligned}$$

where Frob stands for the geometric Frobenius $\mathrm{Frob} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, acting on the l-adic cohomology groups $H^*(X)$ by transport of structures (1.2). Put

$$f(t) := \det(1 - \mathrm{Frob} \cdot t, H^1(X)).$$

It is a polynomial of degree $2g$ with integral coefficients. We will write α for a quantity running over the inverse roots of $f(t)$, counted with their multiplicities. One has

$$f(t) = \prod (1 - \alpha\, t). \tag{2.2.3}$$

When $X_1/\mathbb{F}_q$ is replaced by $X_m/\mathbb{F}_{q^m}$, the corresponding polynomial f_m is

$$f_m(t) = \prod (1 - \alpha^m t). \tag{2.2.4}$$

In (2.2.4), the relation between the variables t and z is $t = q^{-mz}$. The value of f_m at $t = 1$ is the number h_m of $\mathbb{F}_{q^m}$-points of the jacobian $\mathrm{Pic}^0(X_1)$ of X_1:

$$f_m(1) = \prod (1 - \alpha^m) = h_m := |\mathrm{Pic}^0(X_1)(\mathbb{F}_{q^m})|. \tag{2.2.5}$$

We define

(2.2.6) $(n/S_1) :=$ the largest divisor of n which is prime to all $\deg(s)$, for s in S_1.

$${}_nT_1 := f(1) \cdot \frac{1}{q^n - 1} \cdot \prod_{j=1}^{n-1} \left\{ (1 - q^j)^{-2} \cdot f(q^j) \cdot \prod_{s \in S_1} (1 - q^{j \deg s}) \right\}. \tag{2.2.7}$$

(2.2.8) ${}_nT_m := {}_nT_1$ for (X_m, S_m) over $\mathbb{F}_{q^m}$.

$$c_m := \#\{x \in \mathbb{F}_{q^m}^* \mid x \text{ generates } \mathbb{F}_{q^m} \text{ over } \mathbb{F}_q\}. \tag{2.2.9}$$

If m is prime to the degree of a closed point s_1 of X_1, then $\mathbb{F}_{q^m} \otimes_{\mathbb{F}_q} k(s_1)$ is a field and

$$[\mathbb{F}_{q^m} \otimes_{\mathbb{F}_q} k(s_1) \colon \mathbb{F}_{q^m}] = [k(s_1) \colon \mathbb{F}_q].$$

In geometric terms: there is a unique point s_m of X_m above s_1, and its degree, as a closed point of $X_m/\mathbb{F}_{q^m}$, coincides with $\deg(s_1)$. It follows that, when $m \mid (n/S_1)$, the étale divisor S_m maps bijectively to S_1 and if $s_m \in S_m$ has image $s_1 \in S_1$, the degree $[k(s_m)\colon \mathbb{F}_{q^m}]$ of the closed point s_m of X_m is equal to the degree $[k(s_1)\colon \mathbb{F}_q]$ of the closed point s_1 of X_1: $N_m = N_1$, and ${}_nT_m$ is given by the formula (2.2.7) with f replaced by f_m and q by q^m.

Theorem 2.3. *Suppose that n and N_1 are ≥ 2. One then has*

$$f(1) + (-1)^{N_1(n-1)} T(X_1, S_1, n) = \sum_{m|(n/S_1)} c_m \cdot m^{N_1-2} \cdot {}_{n/m}T_m. \tag{2.3.1}$$

In section 4 (resp. 5), we will prove (2.3.1) under the assumption that n is odd or N_1 even (resp. that $N_1 \geq 3$ and that n or N_1 is odd).

3. Division algebras and Tamagawa numbers

We will no more use the field $F_1 \otimes_{\mathbb{F}_q} \mathbb{F}$ previously noted F. To lighten the notations we will simply write F for F_1. We fix a finite set S of places of F.

3.1. The central simple algebras D over F of dimension n^2 are classified by their local invariants $\mathrm{inv}(D_v) \in (\mathbb{Q}/\mathbb{Z})_n$, where D_v is the completion $D \otimes_F F_v$ of D at the place v of F. The only constraints on the local invariants are that almost all are zero and that their sum is zero. The completion D_v is a matrix algebra (resp. a division algebra) if and only if $\mathrm{inv}(D_v) = 0$ (resp. $\mathrm{inv}(D_v)$ is of exact order n).

Proposition 3.2. *Assume that $|S| \geq 2$. Then, except in the case where n is even and $|S|$ is odd, there exists a D as in* 3.1*, such that D_v is a matrix algebra for $v \notin S$, and a division algebra for $v \in S$.*

Proof. For D as in 3.1, define $a_v = n \,\mathrm{inv}(D_v) \in \mathbb{Z}/n$. The problem is to find a family of a_v, for $v \in S$, of exact order n and sum zero. If $|S|$ is even, take half of the a_v to be 1, and the other half to be -1. If $|S| \geq 3$ is odd and n is also odd, take two a_v's to be $(n-1)/2$, one to be 1, and the others in $(1, -1)$ pairs. ☐

3.3. If γ in D^* is of finite order, it generates over $\mathbb{F}_q \subset F$ a finite extension, isomorphic to $\mathbb{F}_{q^m}$ for some m: γ is in the image of a morphism of $\mathbb{F}_q$-algebras: $\mathbb{F}_{q^m} \to D$. Such morphisms correspond one-to-one to morphisms of F-algebras: $F_m = \mathbb{F}_{q^m} \otimes F \to D$. By a special case of the Skolem-Noether theorem, a proof of which is recalled at the end of the proof of 3.5, any two such morphisms are conjugate by some d in the multiplicative group D^* of D. With the notation (n/S) of (2.2.6), one has:

Proposition 3.4. *Let D be as in 3.2. There exists a morphism of F-algebras: $F_m \to D$ if and only if m divides (n/S).*

Before giving the proof of 3.4, we observe that 3.3 and 3.4 imply that there exists a morphism of F-algebras $\varphi\colon F_{(n/S)} \to D$, inducing $\varphi_0\colon \mathbb{F}_{q^{(n/S)}} \to D$, that any conjugacy class of elements of finite order of D^* meets $\varphi_0(\mathbb{F}^*_{q^{(n/S)}})$, and that two elements of $\mathbb{F}^*_{q^{(n/S)}}$ have images in the same conjugacy class if and only if they are $\mathrm{Gal}(\mathbb{F}_{q^{(n/S)}}/\mathbb{F}_q)$-conjugate.

We will deduce 3.4 from the following well-known lemma:

Lemma 3.5. *Let F be a field, D a division algebra with center F of dimension n^2 over F, and F_m a field extension of degree m of F. The central simple algebra $D_m := F_m \otimes_F D$ over F_m is isomorphic to a matrix algebra $M_k(C)$ over a division algebra C with center F_m. One has $k \mid m$ and the following conditions are equivalent:*

- (i) *F_m can be embedded, as an F-algebra, in D;*
- (ii) *$k = m$.*

The proof will repeat that of the Skolem-Noether theorem.

Proof. Conjugacy classes of F-algebra embeddings $F_m \to D$ correspond one-to-one to isomorphism classes of (F_m, D)-bimodules, of dimension one over D. By "bimodule" we mean a bimodule for which the two induced F-module structures are equal; this is the same as a right D_m-module. Indeed, if M is a (F_m, D)-bimodule, of dimension one over D, each $e \neq 0$ in M is a basis of M over D, the map $x \mapsto e^{-1}xe\colon F_m \to D$ such that $xe = e(e^{-1}xe)$ is an embedding, and the $x \mapsto e^{-1}xe$ for $e \neq 0$ in M form a conjugacy class of embeddings $F_m \to D$. If φ belong to that conjugacy class, corresponding to $e \in M$, M is isomorphic to D, with D acting by right multiplication and $x \in F_m$ acting by left multiplication by $\varphi(x)$. The isomorphism is given by $1 \mapsto e$. The same construction shows that any embedding φ is obtained from some (M, e): take $M = D$, $e = 1$ and actions of F_m and D as above.

If C is of dimension c^2 over F_m, one has $n = kc$, and simple right D_m-modules are of dimension kc^2 over F_m, hence mkc^2 over F, and $mkc^2/n^2 = m/k$ over D. It follows that k divides m, and that the dimensions over D of (F_m, D)-bimodules are the multiples of m/k. The dimension one is possible if and only if $k = m$. If $k = m$, the (F_m, D)-bimodules of dimension one over D are the simple D_m-modules, they are all isomorphic, and this proves that the embeddings $F_m \to D$ are all conjugate (Skolem-Noether theorem). $\square$

Proof of 3.4. In our case, 3.5 shows that F_m embeds in D if and only if m divides n and the local invariants of D_m are of order dividing n/m. If the place v of F_m is above the place s of F, by extension of scalars the local invariant gets multiplied by $[F_{m,v}\colon F_s]$. This degree is divisible by m if and only if s is inert in F_m. This is the case if and only if $\mathbb{F}_{q^m} \otimes_{\mathbb{F}_q} F_s$, or equivalently $\mathbb{F}_{q^m} \otimes_{\mathbb{F}_q} k(s)$, is a field, i.e. when m is prime to the degree of $k(s)$ over $\mathbb{F}_q$. We need this to be the case at each s in S. $\square$

3.6. We fix D as in 3.2. By an "order of D" we will mean an order containing the maximal order of F. Orders cannot be defined, as in the number field case, as suitable subalgebras. Such a description is available only on each affine chart of the projective curve X_1. The constant sheaf D on X_1 is a quasi-coherent sheaf of algebras. An *order* of D (over X_1) is a coherent subsheaf of algebras, whose fiber at the generic point is D. Being contained in the constant sheaf D, an order $\mathcal{O}_D$ is torsion free, hence locally free as a sheaf of $\mathcal{O}$-modules. It is of rank n^2, and over some open subset of X_1 is a sheaf of Azumaya algebras.

Similarly, *modules* over an order $\mathcal{O}_D$ are sheaves of $\mathcal{O}_D$-modules, quasi-coherent as sheaves of $\mathcal{O}$-modules. We will denote by $\mathrm{Mod}(-\mathcal{O}_D)$ the category of right $\mathcal{O}_D$-modules which are coherent as sheaves of $\mathcal{O}$-modules, and $\mathrm{Mod}^*(-\mathcal{O}_D)$ the subcategory of those which are *invertible*, that is, locally free of rank one over $\mathcal{O}_D$. In parallel to the notations $\mathcal{O}_{(s)}$, $\mathcal{O}^\wedge_{(s)}$, F_s of (0.2), we denote by $\mathcal{O}_{D,(s)}$ the stalk of $\mathcal{O}_D$ at the closed point s of X_1, $\mathcal{O}_{D,s} = \mathcal{O}_{D,(s)} \otimes_{\mathcal{O}_{(s)}} \mathcal{O}_s$ its completion, and $D_s = \mathcal{O}_{D,s} \otimes_{\mathcal{O}_s} F_s$ the completion of D at s. Any other order $\mathcal{O}'_D$ coincides with $\mathcal{O}_D$ outside of a finite set of closed points, call it T, and is determined by the $\mathcal{O}'_{D,t} \subset D_t$ for t in T. The $\mathcal{O}'_{D,t}$ are arbitrary orders of the D_t: $\mathcal{O}_t$-subalgebras, generated as $\mathcal{O}_t$-modules by a basis of D_t over F_t.

If $v \notin S$, D_v is isomorphic to $\mathrm{End}(V)$ for V a vector space of dimension n over F_v. The maximal orders of $\mathrm{End}(V)$ are the $\mathrm{End}_{\mathcal{O}_v}(V^0)$ for V^0 a *lattice* in V: an $\mathcal{O}_v$-submodule generated over $\mathcal{O}_v$ by a basis of V over F_v. If $\mathcal{O}_{D,v}$ is a maximal order of D_v, $\mathcal{O}_D$ is Azumaya at v.

If $v \in S$, the division algebra D_v admits a valuation and its unique maximal order is its valuation ring. The order $\mathcal{O}_D$ is maximal if and only if, at each closed point v of X_1, $\mathcal{O}_{D,v}$ is a maximal order in D_v.

3.7. As in 1.12, we denote by D^* the obvious affine algebraic group over F with group of rational points the multiplicative group of D. For any order $\mathcal{O}_D$ of D, the adelic group $D^*(\mathbb{A})$ ($\mathbb{A}$ the ring of adèles of F) is the restricted product of the D_v^*, relative to the compact open subgroups $\mathcal{O}_{D,v}^*$. Its center is the group $\mathbb{A}^*$ of idèles of F.

3.8. We will use that the Tamagawa number $\tau(D^*)$ is 1. As D^* is not semisimple, we use Ono's definition of Tamagawa numbers for reductive groups [O]. We need to explain how Ono's definition extends to the function field case.

Let G be a unimodular connected smooth linear algebraic group over F. A translation invariant differential form of maximal degree on G, or equivalently, $\omega \in \overset{\max}{\wedge} \mathrm{Lie}(G)^\vee$, defines at each place v of F a measure $\|\omega_v\|$ on $G(F_v)$. One would like to define the Tamagawa measure on $G(\mathbb{A})$ to be the product measure $\|\omega\| = \prod_v \|\omega_v\|$, independent of $\omega \neq 0$ by the product formula, times $q^{(1-g)\dim G}$. When G is semisimple, this makes sense: the product is absolutely convergent. When G is reductive and does not admit the multiplicative group $\mathbb{G}_m$ as a quotient, the product is conditionally convergent in the following sense: if $K = \prod K_v$ is a compact open subgroup of $G(\mathbb{A})$, the product of the $\int_{K_v} \|\omega_v\|$ converges if one first groups together the factors for which v has a given degree: the product over j

$$\prod_j \left(\prod_{\deg(v)=j} \int_{K_v} \|\omega_v\| \right)$$

is absolutely convergent.

When G admits $\mathbb{G}_m$ as a quotient, the same product vanishes, essentially because $\prod(1 - q^{-\deg v})$ does. As it is the only case we will need, we will explain what is to be done under the additional assumption that G is given with $d\colon G \to \mathbb{G}_m$, an epimorphism whose kernel is the derived group of G. This applies to $\mathrm{GL}(n)$ and to D^*, with $d = \det$.

For $|t| < 1/q$, the rational function $Z(X_1, t)$ (see 2.2) is the product over all places of the local factors $Z_v(t) = (1 - t^{\deg v})^{-1}$. At the simple poles $t = 1$, $1/q$, a regularized value $Z^\star$ is defined to be the negative of the residue of $Z(X_1, t)\frac{dt}{t}$.

Under our assumptions Ono's *Tamagawa measure* on $G(\mathbb{A})$ is

$$\mu := Z^\star(X_1, 1/q)^{-1} \cdot q^{(1-g)\dim G} \cdot \prod_v Z_v(1/q) \cdot \|\omega_v\|. \tag{3.8.1}$$

Let $G(\mathbb{A})^0$ be the kernel of the homomorphism

$$\deg d\colon\ G(\mathbb{A}) \to \mathbb{Z},$$

and $k\mathbb{Z}$ be its image. If $G(\mathbb{A})^{(i)}$ is the inverse image of i, the $G(F)\backslash G(\mathbb{A})^{(i)}$ for $i \in k\mathbb{Z}$ have all the same volume. It follows that $\mu(G(F)\backslash G(\mathbb{A}))$ is infinite. This is a counterpart to the divergence of $Z(X_1, t)$ at $t = 1/q$, which forced us to put $Z^\star(X_1, 1/q)$ rather than $Z(X_1, 1/q)$ in (3.8.1). Under our assumptions Ono's *Tamagawa number* $\tau(G)$ of G is

$$\tau(G) := \mu(G(F)\backslash G(\mathbb{A})^0)/k. \tag{3.8.2}$$

For $G = \mathrm{GL}(n)$ or D^*, deg d is onto and we have simply $\tau(G) := \mu(G(F)\backslash G(\mathbb{A})^0)$. Ono's general results imply that

$$\tau(\mathrm{GL}(n)) = 1 \text{ and } \tau(D^*) = 1. \tag{3.8.3}$$

3.9. Remark. Ono considers the number field case. He uses the complex variable s (which we call z, see 2.2), and the morphism $\log \|d\|\colon G(\mathbb{A}) \to \mathbb{R}$ to define $\tau(G)$. His proofs work as well in the function field case. One should, however, be aware of two differences between his definition of $\tau(G)$ and the one we explained. They cancel each other.

(a) Ono uses as regularized value $\zeta^\star(X_1, 1)$ the value at $z = 1$ of $(z-1)\zeta(X_1, z)$, equal to the residue at 1 of $\zeta(X_1, z)dz$. As $dt/t = -\log q\, dz$, one has

$$Z^\star(X_1, 1/q) = \log q \cdot \zeta^\star(X_1, 1).$$

In the case we are considering, the measure μ_{O} used by Ono in [O] is hence $\log q$ times the measure μ defined by (3.8.1).

(b) The right side of (3.8.2) should be viewed as the volume modulo $G(F)$ of the "complex" $G(\mathbb{A}) \to \mathbb{Z}$, for the measure μ on $G(\mathbb{A})$ and the counting measure on $\mathbb{Z}$. Ono uses the "complex" $\log \|d\|\colon G(\mathbb{A}) \to \mathbb{R}$, μ_{O} on $G(\mathbb{A})$ and dx on $\mathbb{R}$. In the number field case, this amounts to using the measure μ_{O}/dx on $G(\mathbb{A})^0$.

In the function field case, the two constructions are possible. They are related by the commutative diagram

$$\begin{array}{ccc} G(\mathbb{A}) & \longrightarrow & \mathbb{Z} \\ \| & & \big\downarrow{\scriptstyle -\log q} \\ G(\mathbb{A}) & \longrightarrow & \mathbb{R}. \end{array}$$

When one uses the measure μ on $G(\mathbb{A})$, the counting measure on $\mathbb{Z}$, and the measure dx on $\mathbb{R}$, the volume modulo $G(F)$ of the second line is equal to the volume modulo $G(F)$ of the first line, divided by $\log q = \int_{\mathbb{R}/\log q\cdot\mathbb{Z}} dx$.

The following is well known.

Proposition 3.10. *For $G = \mathrm{GL}(n)$, the Tamagawa volume of* $\prod \mathrm{GL}(n, \mathcal{O}_v)$ *is*

$$(-Z^\star(1)Z(q)\cdots Z(q^{n-1}))^{-1}.$$

Sketch of Proof. For the additive group $\mathbb{G}_a$, and $\omega = dx$, the volume of $\mathcal{O}_v$ for $\|\omega_v\|$, and the volume of $\prod \mathcal{O}_v$ for $\|\omega\|$, are 1. The intersection $F \cap \prod \mathcal{O}_v$ is $\mathbb{F}_q$, $F\backslash\mathbb{A}/\prod \mathcal{O}_v$ is $H^1(\mathcal{O})$ and the volume of $F\backslash\mathbb{A} = \mathbb{G}_a(F)\backslash\mathbb{G}_a(\mathbb{A})$ for $\|\omega\|$ is hence q^{g-1}. This, and the wished for multiplicativity in G, are the reasons for the factor $q^{(1-g)\dim G}$ in the definition of the Tamagawa measure: it ensures that $\tau(\mathbb{G}_a) = 1$.

Let us temporarily forget that $\prod(1 - q^{-\deg v})$ diverges. The Tamagawa measure of $\prod \mathrm{GL}(n, \mathcal{O}_v)$ would then be the measure of $\prod \mathrm{GL}(n, \mathcal{O}_v)$, viewed as a subspace of $\prod M_n(\mathcal{O}_v)$. The case of the additive group tells us that the Tamagawa volume of $\prod M_n(\mathcal{O}_v)$ is $q^{(1-g)n^2}$. For $\prod \mathrm{GL}(n, \mathcal{O}_v)$, the Tamagawa volume is then

$$\begin{aligned} q^{(1-g)n^2} \cdot \prod \frac{|\mathrm{GL}(n, k(v))|}{|M_n(k(v))|} &= q^{(1-g)n^2} \cdot \prod_v ((1 - q_v^{-1}) \cdots (1 - q_v^{-n})) \\ &= q^{(1-g)n^2} \cdot (Z(q^{-1}) \cdots Z(q^{-n}))^{-1}. \end{aligned}$$

The required regularization replaces $Z(q^{-1})$ by $Z^\star(q^{-1})$.

One then uses the functional equation

$$Z(t) = (qt^2)^{g-1} Z(1/qt).$$

As $t \mapsto 1/qt$ maps $\frac{dt}{t}$ to $\frac{-dt}{t}$, the same holds up to sign for the regularized values at $t = 1$, $1/q$. We get

$$\begin{aligned} Z^\star(q^{-1})Z(q^{-2}) \cdots Z(q^{-n}) &= -(q^{-1} \cdots q^{-(2n-1)})^{g-1} Z^\star(1)Z(q) \cdots Z(q^{n-1}) \\ &= -q^{(1-g)n^2} Z^\star(1)Z(q) \cdots Z(q^{n-1}) \end{aligned}$$

and the proposition. □

One has $Z(t) = f(t)/(1-t)(1-qt)$ and $Z^\star(1)$ is the value at 1 of $f(t)/(1-qt)$. One can hence rewrite 3.10 as

$$\mu\left(\prod \mathrm{GL}(n, \mathcal{O}_v)\right) = \left[f(1) \cdot \frac{1}{q^n - 1} \cdot \prod_{j=1}^{n-1} f(q^j)/(1-q^j)^2 \right]^{-1}. \tag{3.10.1}$$

The analogous question for D^* is reduced to 3.10 by a standard argument:

Proposition 3.11. *Let $\mathcal{O}_D$ be a maximal order of D as in 3.2. Define $q_v = |k(v)| = q^{\deg(v)}$. Then, for $G = D^*$, the Tamagawa volume of $\prod \mathcal{O}^*_{D,v}$ is*

$$\mu\left(\prod \mathcal{O}^*_{D,v}\right) = \prod_{v\in S} \delta_v \cdot \mu\left(\prod_v \mathrm{GL}(n, \mathcal{O}_v)\right) \tag{3.11.1}$$

with

$$\delta_v = \left[(q_v - 1)\cdots(q_v^{n-1} - 1)\right]^{-1}. \tag{3.11.2}$$

Plugging (3.10.1) in (3.11.1) and using that $(n-1)|S|$ is even, one can rewrite (3.11.1) as

$$\mu\left(\prod \mathcal{O}^*_{D,v}\right) = \left[f(1)\cdot\frac{1}{q^2-1}\cdot\prod_{j=1}^{n-1}\left\{(1-q^j)^{-2}\cdot f(q^j)\prod_{s\in S}(1-q^{j\deg(s)})\right\}\right]^{-1}. \tag{3.11.3}$$

For $S = S_1$, (3.11.3) is the inverse of ${}_nT_1$ (2.2.7).

Sketch of Proof. The algebra D over F is obtained from the $n \times n$ matrix algebra M_n by twisting by a $\mathrm{PGL}(n)$-torsor. The algebraic group D^* as well as $\overset{\max}{\wedge}\mathrm{Lie}(D^*)^\vee$ are similarly obtained from $\mathrm{GL}(n)$ and $\overset{\max}{\wedge}\mathrm{Lie}(\mathrm{GL}(n))^\vee$. As $\mathrm{PGL}(n)$ acts trivially on $\overset{\max}{\wedge}\mathrm{Lie}\,\mathrm{GL}(n)^\vee$, $\overset{\max}{\wedge}\mathrm{Lie}(D^*)^\vee$ is canonically isomorphic to $\overset{\max}{\wedge}\mathrm{Lie}(\mathrm{GL}(n))^\vee$. This isomorphism is compatible with extensions of scalars. It induces a correspondence between the Haar measures on $\mathrm{GL}(n, F_v)$ and D^*_v. Taking products, it induces also a correspondence between the Haar measures of $\mathrm{GL}(n, \mathbb{A})$ and $D^*(\mathbb{A})$. The Tamagawa measures μ correspond to each other. If the μ_v are corresponding Haar measures for the local groups, this gives

$$\mu\left(\prod \mathcal{O}^*_{D,v}\right)/\mu\left(\prod \mathrm{GL}(n, \mathcal{O}_v)\right) = \prod \mu_v(\mathcal{O}^*_{D,v})/\mu_v(\mathrm{GL}(n, \mathcal{O}_v)). \tag{3.11.4}$$

It suffices to extend the product on the right only over the set S of places where D_v is not isomorphic to $M_n(F_v)$.

For $v \in S$, $\mathcal{O}^*_{D,v}$ is an Iwahori subgroup of D^*_v. Let T_v be the subalgebra of $M_n(\mathcal{O}_v)$ consisting of the matrices whose reduction in $M_n(k(v))$ is upper triangular. The multiplicative group $I_v := T_v^*$ is an Iwahori subgroup of $\mathrm{GL}(n, F_v)$. As multiplicative groups of $\mathcal{O}_v$-algebras which as $\mathcal{O}_v$-modules are free of finite type, $\mathcal{O}^*_{D,v}$ and I_v are the groups of $\mathcal{O}_v$-points of group schemes smooth over $\mathrm{Spec}(\mathcal{O}_v)$.

Suppose that G is a reductive group over F_v, F'_v an unramified extension of F_v and $\mathcal{O}'_v$ its valuation ring. By Bruhat-Tits, an Iwahori subgroup of G, that is of $G(F_v)$, is in a natural way the group of $\mathcal{O}_v$-points of a smooth group scheme $\mathbf{I}$ over $\mathrm{Spec}(\mathcal{O}_v)$, with generic fiber G. Further, $\mathbf{I}(\mathcal{O}'_v)$ is an Iwahori subgroup of $G' := G \otimes_{F_v} F'_v$ and $\mathbf{I} \otimes_{\mathcal{O}_v} \mathcal{O}'_v$ is the corresponding smooth group scheme over $\mathrm{Spec}(\mathcal{O}'_v)$. We use here that, as the residue fields are finite, we are in the residually split case. This is in fact how Iwahori subgroups are constructed: one finds an Iwahori subgroup I' of G' fixed by $\mathrm{Gal}(F'_v/F_v)$, and one constructs $\mathbf{I}$ from $\mathbf{I}'$ by étale descent.

In our case, choose F'_v such that D_v and M_n become isomorphic after extension of scalars to F'_v. For $\mathrm{GL}(n)$, an Iwahori subgroup I' determines the corresponding order T' of M_n, of which it is the multiplicative group. Let I' be an Iwahori subgroup of $(D_v \otimes F'_v)^* \simeq \mathrm{GL}(n, F'_v)$ and T' be the order of $D_v \otimes F'_v$ of which I' is the multiplicative group. If I' is stable by $\mathrm{Gal}(F'_v/F_v)$, T' descends to an order T of D_v. As T^* is the Iwahori subgroup of D_v^*, T must be $\mathcal{O}_{D,v}$.

We conclude that $\mathcal{O}_{D,v} \otimes F'_v \subset D_v \otimes F'_v \simeq M_n(F'_v)$ is conjugate to T_v (we will also check this directly in 3.12), and that the Haar measures μ_v on D_v^* and $\mathrm{GL}(n, F_v)$ defined by generators of $\overset{\max}{\wedge} \mathrm{Lie}(\mathcal{O}_{D,v}^*)$ and $\overset{\max}{\wedge} \mathrm{Lie}(T_v^*)$ correspond to each other.

On $\mathcal{O}_{D,v}^*$, μ_v is induced by the Haar measure on D_v for which $\mathcal{O}_{D,v}$ has volume one. On $\mathrm{GL}(n, \mathcal{O}_v)$, μ_v is determined by the Haar measure on $M_n(F_v)$ for which T_v has volume one. Using that $\mathcal{O}_{D,v}$ is a valuation ring with residue field $\mathbb{F}_{q_v^n}$, we get

$$\mu_v(\mathcal{O}_{D,v}^*) = 1 - \frac{1}{q_v^n}.$$

The μ_v-volume of $\mathrm{GL}(n, \mathcal{O}_v)$ is

$$\begin{aligned} &|\text{image of } \mathrm{GL}(n, \mathcal{O}_v) \text{ in } M_n(k(v))| / |\text{image of } T_v \text{ in } M_n(k(v))| \\ &= q_v^{n^2}\Big(1 - \frac{1}{q_v}\Big) \cdots \Big(1 - \frac{1}{q_v^n}\Big) / q_v^{n(n+1)/2} \\ &= (q_v - 1) \cdots (q_v^{n-1} - 1)\Big(1 - \frac{1}{q_v^n}\Big). \end{aligned}$$

This gives

$$\mu_v(\mathcal{O}_{D,v}^*)/\mu_v(\mathrm{GL}(n, \mathcal{O}_v)) = \big[(q_v - 1) \cdots (q_v^{n-1} - 1)\big]^{-1}$$

and (3.11.1) follows from (3.11.4). □

3.12. Let k be the residue field of F_v, and choose an isomorphism of F_v with $k((t))$. Let k' be a degree n extension of k, and $F'_v := k'((t))$. For some generator τ of $\mathrm{Gal}(F'_v/F_v) \simeq \mathrm{Gal}(k'/k) \simeq \mathbb{Z}/n$, the division algebra D_v admits the following model as a crossed product: one adds to F'_v an element $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}^n = t$ and $\boldsymbol{\pi} f = \tau(f)\boldsymbol{\pi}$ for f in F'_v.

The elements of D_v can be written as Laurent series $\sum a_n \boldsymbol{\pi}^n$, with a_n in k', and the product is such that $\boldsymbol{\pi}\, a = \tau(a)\boldsymbol{\pi}$ for a in F'_v: D_v is a twisted Laurent formal power series field $k'(\!(\boldsymbol{\pi})\!)_\tau$. It admits the valuation

$$\sum a_n \boldsymbol{\pi}^n \longmapsto \inf\{n \mid a_n \neq 0\}.$$

The valuation ring is the twisted formal power series ring $k'[\![\boldsymbol{\pi}]\!]_\tau$.

The tensor product $F'_v \otimes_{F_v} F'_v$ is a product of copies of F'_v, indexed by $\mathrm{Gal}(F'_v/F_v)$. The coordinates of the isomorphism are the

$$\mathrm{pr}_\alpha\colon\ F'_v \otimes_{F_v} F'_v \to F'_v\colon\ x \otimes y \longmapsto \alpha(x)y.$$

They are permuted by τ, acting on the first factor F'_v: $\mathrm{pr}_\alpha(\tau(x) \otimes y) = \alpha\tau(x) \otimes y = \mathrm{pr}_{\alpha\tau}(x \otimes y)$. The algebra $D'_v := D_v \otimes_{F_v} F'$ is obtained by adding to this product of copies of F'_v an element $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}^n = t$ and $\boldsymbol{\pi}(x_\alpha) = (x_{\alpha\tau})\boldsymbol{\pi}$.

Let us identify $\mathrm{Gal}(k'_v/k_v)$ with $\mathbb{Z}/n$ by using τ as a generator. One then defines an isomorphism of D'_v with $M_n(F'_v)$ by mapping $F'_v \otimes F'_v = F'^{\mathbb{Z}/n}_v$ to the diagonal matrices, and $\boldsymbol{\pi}$ to the matrix 1 just above the diagonal, t on the lower left corner, and 0 elsewhere. With T_v as in 3.11, by this isomorphism, $\mathcal{O}_{D,v} \otimes k'[\![t]\!]$ maps to $T_v \otimes k'[\![t]\!]$.

4. Proof of 2.3: Masses of categories

In this section, we prove 2.3 under the assumption that there exists a division algebra D with center F of dimension n^2 over F, such that D_v is a division algebra for $v \in S_1$, and a matrix algebra over F_v for $v \notin S_1$. As explained in 3.2, this amounts to assuming that n is odd, or that N_1 (assumed to be ≥ 2) is even. In both cases, $N_1(n-1)$ is even and the sign $(-1)^{N_1(n-1)}$ in (2.3.1) is $+1$.

4.1. As we recalled in 1.11 and 1.13, the number $T(X_1, S_1, n)$ we want to compute is also the number of classes modulo $\mathbb{F}_q$-twists of automorphic representations π of $D^*(\mathbb{A})$ such that

(i) π is unramified outside of S_1;

(ii) for v in S_1, the local component π_v is of the form $\chi(\det)$ for χ an unramified character of F_v^*;

(iii) π is not one-dimensional.

To count these representations up to $\mathbb{F}_q$-twists, we will proceed in two steps. Let $\mathbf{a}$ be an idèle of F of positive degree: $\deg(\mathbf{a}) > 0$. Identifying $\mathbb{A}^*$ with the center of $D^*(\mathbb{A})$, one has

$$\deg\, \det(\mathbf{a}) = \deg(\mathbf{a}^n) = n\, \deg(\mathbf{a}).$$

It follows that for χ a character of the quotient $\mathbb{Z}$ of $D^*(\mathbb{A})$,

$$\omega_{\pi\chi}(\mathbf{a}) = \omega_\pi(\mathbf{a})\chi(n \deg \mathbf{a})$$

and any π has an $\mathbb{F}_q$-twist π' such that $\omega_{\pi'}(\mathbf{a}) = 1$. This π' is not unique: one remains free to twist it by a character of $\mathbb{Z}/n\deg(\mathbf{a})$. We conclude that $T(X_1, S_1, n)$ is the number of automorphic representations of $D^*(\mathbb{A})$ obeying (i), (ii), (iii) and

(iv) $\omega_\pi(\mathbf{a}) = 1$,

taken modulo $\mathbb{F}_q$-twists by a character of $\mathbb{Z}/n\deg \mathbf{a}$. These twists being distinct, we have

Lemma 4.2. *The number $T(X_1, S_1, n)$ is $1/n\deg(\mathbf{a})$ times the number of automorphic representations of $D^*(\mathbb{A})$ obeying* (i), (ii), (iii), (iv) *of* 4.1.

In the rest of this section, we take $\mathbf{a}$ to be of degree one.

4.3. Let us fix a maximal order $\mathcal{O}_D$ of D (3.6). The space of locally constant functions on $D^*\backslash D^*(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}$ is the direct sum of the automorphic representations π for which $\omega_\pi(\mathbf{a}) = 1$. This direct sum decomposition, being $D^*(\mathbb{A})$-equivariant, is compatible with taking the invariants by $\prod \mathcal{O}_{D,v}^*$. The π for which (i), (ii) hold are those for which the subspace of vectors fixed by $\prod \mathcal{O}_{D,v}^*$ is nontrivial. This subspace is then of dimension one. It follows that the number of automorphic representations of $D^*(\mathbb{A})$ for which (i), (ii) holds is the dimension of the space of functions on $D^*\backslash D^*_{\mathbb{A}}/\prod \mathcal{O}_{D,v}^* \cdot \mathbf{a}^{\mathbb{Z}}$:

Lemma 4.4. *The number of automorphic representations of $D^*(\mathbb{A})$ for which* (i), (ii), (iv) *hold is the number of elements of the finite set*

$$D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}_{D,v}^* \cdot \mathbf{a}^{\mathbb{Z}}. \tag{4.4.1}$$

Lemma 4.5. *There are $n\,h_1$ one-dimensional automorphic representations of $D^*(\mathbb{A})$ for which* (i), (ii), (iv) *hold.*

Proof. The map det maps $D^*(\mathbb{A})$ (resp. D^*, resp. $\mathcal{O}_{D,v}^*$) onto $\mathbb{A}^*$ (resp. F^*, resp. $\mathcal{O}_v^*$). The representations considered may hence be identified with the unramified characters χ of the idèle class group such that $\chi(\mathbf{a}^n) = 1$. As $F^*\backslash\mathbb{A}^*/\prod \mathcal{O}_v^* \cdot \mathbf{a}^{n\mathbb{Z}}$ is an extension of $\mathbb{Z}/n$ by $\mathrm{Pic}^0(X_1)(\mathbb{F}_q)$, the lemma follows. □

From 4.4 and 4.5, we get

Lemma 4.6. *We have* $T(X_1, S_1, n) + h_1 = \frac{1}{n}|D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{vD} \cdot \mathbf{a}^{\mathbb{Z}}|$.

Let $D^*(\mathbb{A})^{(i)}$ be the coset of $D^*(\mathbb{A})^0$ (notation of 3.8) on which deg det $= i$. One has

$$\coprod_{0\leq i<n} D^*\backslash D^*(\mathbb{A})^{(i)}/\prod \mathcal{O}^*_{D,v} \xrightarrow{\sim} D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}}.$$

We will not need the following variant of 4.6.

Proposition 4.7. *The* $D^*\backslash D^*(\mathbb{A})^{(i)}/\prod \mathcal{O}^*_{D,v}$ *have all the same number of elements. As a consequence,*

$$T(X_1, S_1, n) + h_1 = |D^*\backslash D^*(\mathbb{A})^0/\prod \mathcal{O}^*_{D,v}|.$$

Proof. The group $(\mathbb{Z}/n)^\vee$ of characters χ of $\mathbb{Z}/n$ acts on the space L of functions on $D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}}$ by multiplication by $\chi(\deg\det)$. The space L is a direct sum of lines, indexed by the automorphic representations obeying (i), (ii), (iv), and $(\mathbb{Z}/n)^\vee$ acts freely on this set of lines: L is a multiple of the regular representation of $(\mathbb{Z}/n)^\vee$ and each character of $(\mathbb{Z}/n)^\vee$ occurs in it with the same multiplicity. One concludes by observing that the multiplicity of the character $\chi \mapsto \chi(i)$ is $|D^*\backslash D^*(\mathbb{A})^{(i)}/\prod \mathcal{O}^*_{D,v}|$. $\square$

4.8. Let φ_0 be the Haar measure with mass one of $\prod \mathcal{O}^*_{D,v}$, extended by zero to a measure on $D^*(\mathbb{A})$. Convolution with φ_0 is an idempotent projection from locally constant functions on $D^*\backslash D^*(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}$ to functions on $D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}}$. Its trace is $|D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}}|$. The trace formula (compact quotient case) expresses it as a sum over the conjugacy classes of γ in D^* contained in a $D^*(\mathbb{A})$-conjugate of $\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}}$.

Lemma 4.9. *Any* γ *in* D^* *contained in a* $D^*(\mathbb{A})$*-conjugate of* $\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}}$ *is of finite order.*

Proof. As $\deg\det(\gamma) = 0$, γ will be in a $D^*(\mathbb{A})$-conjugate of $\prod \mathcal{O}^*_{D,v}$. For any d in $D^*(\mathbb{A})$, the intersection $D^* \cap d\prod \mathcal{O}^*_{D,v}d^{-1}$ is discrete and compact, hence finite. $\square$

With essentially no change in content, the trace formula computation can be expressed using masses of suitable categories (cf. 4.18). The direct use of the trace formula has the advantages of being more directly applicable in the number field case, and of clearly separating local and global questions. The use of masses is more geometric. As the equivalence between the two approaches seemed to us interesting, we will use masses. We first review their formalism.

4.10. The *mass* of a category $\mathcal{C}$ is the sum, over the isomorphism classes of objects,

$$\text{mass}(\mathcal{C}) := \sum 1/|\text{Aut}(X)|. \tag{4.10.1}$$

Let $\mathcal{C}_{\text{is}}$ be the subcategory of $\mathcal{C}$ with the same objects, and for which the morphisms are the isomorphisms in $\mathcal{C}$. The category $\mathcal{C}_{\text{is}}$ is a groupoid. It has the same mass as $\mathcal{C}$. A category and its opposite have the same mass. Two equivalent categories have the same mass. If $\mathcal{C}$ is a finite sum (resp. product) of categories $\mathcal{C}_i$, one has

$$\text{mass}\left(\coprod \mathcal{C}_i\right) = \sum \text{mass}(\mathcal{C}_i) \tag{4.10.2}$$

$$\text{mass}\left(\prod \mathcal{C}_i\right) = \prod \text{mass}(\mathcal{C}_i). \tag{4.10.3}$$

More generally, suppose each object X of $\mathcal{C}$ is given a weight $w(X)$, and that isomorphic objects have the same weight. The *weighted mass* of $(\mathcal{C}, w)$ is the sum, over isomorphism classes of objects,

$$\text{mass}(\mathcal{C}, w) = \sum w(X)/|\text{Aut}(X)|. \tag{4.10.4}$$

In our applications, the groups of automorphisms $\text{Aut}(X)$ will all be finite.

Suppose $T\colon \mathcal{C} \to \mathcal{D}$ is a functor. The *fiber* $\mathcal{C}_Y$ of $\mathcal{C}/\mathcal{D}$ at the object Y of $\mathcal{D}$ is the category of objects X of $\mathcal{C}$, given with an isomorphism $\alpha\colon T(X) \to Y$. A morphism from (X', α') to (X'', α'') is $u\colon X' \to X''$ such that $\alpha'' T(u) = \alpha'$. Maybe this should be called "homotopy fiber". The naive definition of fiber: $(T^{-1}(Y), T^{-1}(\text{Id}_Y))$ is no good, as it is not compatible with equivalences. If, however, $\mathcal{C}$ is fibered over $\mathcal{D}$, the two definitions give equivalent categories. If Y and Y' are isomorphic, the fibers $\mathcal{C}_Y$ and $\mathcal{C}_{Y'}$ are equivalent, hence have the same mass.

Lemma 4.11. *Let $T\colon \mathcal{C} \to \mathcal{D}$ be a functor. The mass of $\mathcal{C}$ is the weighted mass of $\mathcal{D}$, weighted by the mass of the fibers. Special case: if all fibers have the same mass, one has*

$$\text{mass}(\mathcal{C}) = \text{mass}(\mathcal{D}) \cdot \text{mass(any fiber)}. \tag{4.11.1}$$

Proof. One reduces to the case where all morphisms in $\mathcal{C}$ and $\mathcal{D}$ are isomorphisms. Using (4.10.2) and its analogue for weighted masses, one may assume that $\mathcal{C}$ and $\mathcal{D}$ have only one isomorphism class of objects. Replacing $\mathcal{C}$, $\mathcal{D}$ by equivalent categories, one may assume that $\mathcal{C}$ (resp. $\mathcal{D}$) has only one object X (resp. Y). The functor T is then given by a morphism of groups T from $G = \text{Aut}(X)$ to $H = \text{Aut}(Y)$.

An object of the fiber $\mathcal{C}_Y$ is the data of $h \in H$: $T(X) = Y \to Y$. A morphism from (X, h') to (X, h'') is g in G such that $h'' T(g) = h'$. This identifies the set of isomorphism classes in $\mathcal{C}_Y$ with $H/T(G)$, and the group of automorphisms of any object with $\mathrm{Ker}(T)$. Then (4.11.1) reduces to

$$1/|G| = 1/|H| \cdot (|H/T(G)|/|\mathrm{Ker}(T)|). \qquad \square$$

The same proof shows that if $\mathcal{C}$ is given with a weight, its weighted mass is the weighted mass of $\mathcal{D}$, for the weighted masses of the fibers.

4.12. Suppose a group Γ acts on a set E. The *category* $[\Gamma\backslash E]$ is defined as follows: the set of objects is E, a morphism from x to y is γ in Γ such that $y = \gamma x$, and the composition of morphisms is the product in Γ. Variant: suppose given a (left) action of Γ_1, and a right action of Γ_2, that is an action of the opposite group Γ_2^0. If the actions commute, $\Gamma_1 \times \Gamma_2^0$ acts and one defines $[\Gamma_1\backslash E/\Gamma_2] = [\Gamma_1 \times \Gamma_2^0\backslash E]$. If Γ_2 acts freely, the natural functor $[\Gamma_1\backslash E/\Gamma_2] \to [\Gamma_1\backslash(E/\Gamma_2)]$ is an equivalence. More generally, if the normal subgroup Γ_0 of Γ acts freely on E, $[\Gamma\backslash E] \to [(\Gamma/\Gamma_0)\backslash(\Gamma_0\backslash E)]$ is an equivalence.

The mass of $[\Gamma\backslash E]$ is the sum, over representatives of the orbits of Γ,

$$\mathrm{mass}([\Gamma\backslash E]) = \sum_{\text{orbits}} 1/|\,\mathrm{Stabilizer}(e)|. \tag{4.12.1}$$

Lemma 4.13. *If Γ' is a subgroup of finite index of Γ, then*

$$\mathrm{mass}([\Gamma'\backslash E]) = |\Gamma/\Gamma'| \cdot \mathrm{mass}([\Gamma\backslash E]). \tag{4.13.1}$$

Proof. We apply (4.11.1) to the natural functor $[\Gamma'\backslash E] \to [\Gamma\backslash E]$. An object of the fiber at e is (e', γ) with $\gamma\, e' = e$. It is determined by γ, has no nontrivial automorphism, and $(\gamma^{-1}e, \gamma)$ is isomorphic to $(\delta^{-1}e, \delta)$ if and only if γ and δ have the same image in $\Gamma'\backslash\Gamma$. All fibers have hence mass $|\Gamma/\Gamma'|$. $\square$

Example 4.14. If E and Γ are finite,

$$\mathrm{mass}([\Gamma\backslash E]) = |E|/|\Gamma|. \tag{4.14.1}$$

Indeed, take Γ' in 4.13 to be trivial.

Example 4.15. Let a finite group Γ act on itself by conjugation. For this action, the stabilizer of γ in Γ is the centralizer $Z(\gamma)$ of γ. Applying (4.12.1) and (4.14.1), we get that the sum over representatives of conjugacy classes

$$\sum 1/|Z(\gamma)| = 1 \qquad \text{(sum over conjugacy classes)}. \tag{4.15.1}$$

The same holds for twisted conjugacy and twisted centralizer: if Γ_1 is an extension of $\mathbb{Z}$ by the finite group Γ, one applies (4.14.1) to the conjugation action of Γ on the inverse image of 1.

For $\mathcal{C}$ a category, let $\mathcal{C}^\star$ be the category of objects X of $\mathcal{C}$, given with an automorphism α. A morphism from (X', α') to (X'', α'') is a morphism $f\colon X' \to X''$ such that $\alpha'' f = f\alpha'$.

Proposition 4.16. *If the groups of automorphisms of objects of $\mathcal{C}$ are finite, the mass of the category $\mathcal{C}^\star$ is the number of isomorphism classes of $\mathcal{C}$.*

Proof. As $(\mathcal{C}^\star)_{\mathrm{is}} = (\mathcal{C}_{\mathrm{is}})^\star$, we may and shall assume that $\mathcal{C} = \mathcal{C}_{\mathrm{is}}$. We apply 4.11 to the forgetful functor $(X, \alpha) \mapsto X$ from $\mathcal{C}^\star$ to $\mathcal{C}$. The fiber at X is equivalent to the discrete category with set of objects $\mathrm{Aut}(X)$. The mass of $\mathcal{C}^\star$ is the weighted mass of $\mathcal{C}$, for the weight $|\mathrm{Aut}\, X|$: the sum over isomorphism classes in $\mathcal{C}$

$$\sum |\mathrm{Aut}(X)|/|\mathrm{Aut}(X))| = \sum 1. \qquad \square$$

If we take for $\mathcal{C}$ a category with one object whose group of automorphisms is Γ, and compute the mass of $\mathcal{C}^\star$ by (4.10.1), we recover (4.15.1).

Example 4.17. For E and Γ as in 4.12, the objects of $[\Gamma\backslash E]^\star$ are the pairs (e, γ) with e fixed by γ: $e \in E^\gamma$. The objects isomorphic to (e, γ) are the $(\delta e, \delta\gamma\delta^{-1})$. This shows that the conjugacy class of γ depends only on the isomorphism class of (e, γ), and that if we fix a set R of representatives of the conjugacy classes, any object of $[\Gamma\backslash E]^\star$ is isomorphic to a (e, γ) with γ in R. A morphism from (e, γ) to (e', γ) is δ in Γ such that $\delta e = e'$ and $\gamma\delta = \delta\gamma$. The category $[\Gamma\backslash E]^\star$ is hence equivalent to the disjoint sum, over representatives γ of the conjugacy classes, of the categories $[Z(\gamma)\backslash E^\gamma]$. We get

$$|\Gamma\backslash E| =_{4.16} \mathrm{mass}([\Gamma\backslash E]^\star) = \sum \mathrm{mass}([Z(\gamma)\backslash E^\gamma]), \tag{4.17.1}$$

the sum being over a set of representatives of the conjugacy classes of Γ. It suffices to consider those for which E^γ is not empty.

Example 4.18. Let G be a totally disconnected locally compact group, K an open compact subgroup, Γ a cocompact discrete subgroup and consider the action of Γ on $E = G/K$. The formula (4.17.1) expresses the number of double cosets of Γ, K in G as a sum over conjugacy classes in Γ. For γ in Γ, E^γ is the set of $g^{-1}K$ such that $\gamma g^{-1}K = g^{-1}K$, that is $\gamma \in g^{-1}Kg$. Let $Z_\Gamma(\gamma)$ (resp. $Z_G(\gamma)$) be the centralizer of γ in Γ (resp. G). Let us check that the term $\mathrm{mass}([Z_\Gamma(\gamma)\backslash E^\gamma])$ in (4.17.1) is an orbital integral. Recall that orbital integrals associate a function on the set of conjugacy classes to a density. The density one takes here is the

Haar measure of K giving it volume 1, extended by 0, that is $1_K dg$, for 1_K the characteristic function of K and dg the Haar measure of G giving K the volume 1:

$$\text{mass}([Z_\Gamma(\gamma)\backslash E^\gamma]) = \int_{G/Z_\Gamma(\gamma)} 1_K(g\gamma g^{-1})dg. \tag{4.18.1}$$

Indeed, the double coset $Z_\Gamma(\gamma)g^{-1}K$ (resp. $KgZ_\Gamma(\gamma)$) contributes to the mass (resp. integral) if and only if $g\gamma g^{-1} \in K$, in which case the contribution is $1/|gZ_\Gamma(\gamma)g^{-1} \cap K|$.

The formula (4.17.1) hence gives

$$|\Gamma\backslash G/K| = \sum \int_{G/Z_\Gamma(\gamma)} 1_K(g\gamma g^{-1})dg \tag{4.18.2}$$

(sum over conjugacy classes in Γ). The trace formula (compact quotient case), after telling the same, introduces a Haar measure dz on the centralizer $Z_G(\gamma)$ of γ in G and rewrites each term as

$$\int_{Z_G(\gamma)/Z_\Gamma(\gamma)} dz \cdot \int_{G/Z_G(\gamma)} 1_K(g\gamma g^{-1})dg/dz.$$

4.19. The number of double cosets $|D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}}|$ appearing in 4.6 will now be computed using these methods. By 4.9, only the conjugacy classes of elements of finite order of D^* need to be considered. We recall that (n/S_1) is the largest divisor of n prime to the degrees $\deg(v)$ for v in S_1. It will be convenient to choose an $\mathbb{F}_q$-algebra embedding of $\mathbb{F}_{q^{(n/S_1)}}$ in D (3.4), and to choose the maximal order $\mathcal{O}_D$ to contain this $\mathbb{F}_{q^{(n/S_1)}}$. This is possible: as $\mathbb{F}_{q^{(n/S_1)}}$ is finite, it is contained in a maximal compact subgroup of $D^*(\mathbb{A})$, and these are the $\prod \mathcal{O}^*_{D,v}$ for $\mathcal{O}_D$ a maximal order.

Fix γ in $\mathbb{F}_{q^{(n/S_1)}}$. Let m be the degree $[\mathbb{F}_q(\gamma) \colon \mathbb{F}_q]$: the element γ is in the image of $\mathbb{F}_{q^m}$ and generates it over $\mathbb{F}_q$. The contribution of γ to the number of double cosets is

$$\text{mass}([Z(\gamma)\backslash(D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}})^\gamma], \tag{4.19.1}$$

where $Z(\gamma)$ is the centralizer of γ in D^* and $(\dots)^\gamma$ is the set of fixed points of γ. The centralizer $Z(\gamma)$ is the multiplicative group of D^γ, the subalgebra of D consisting of the elements which commute with γ or, what amounts to the same, with each element of $\mathbb{F}_q(\gamma) \subset \mathbb{F}_{q^{(n/S_1)}} \subset D$. As we used in the proof of 4.9, $(D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}})^\gamma = (D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v})^\gamma/\mathbf{a}^{\mathbb{Z}}$, and $(D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v})^\gamma$ consists of the cosets of d in $D^*(\mathbb{A})$ such that γ, or, what amounts to the same, $\mathbb{F}^*_{q^m}$, is contained in $d \cdot \prod \mathcal{O}^*_{D,v} \cdot d^{-1}$. Such a coset is fixed, not only by γ, but by all elements of $\mathbb{F}^*_{q^m}$. It follows that (4.19.1) depends only on m. As the orbits of $\text{Gal}(\mathbb{F}_{q^{(n/S_1)}}/\mathbb{F}_q)$ acting on $\mathbb{F}^*_{q^{(n/S_1)}}$ map bijectively onto the conjugacy classes of elements of finite order of D^*

(3.4), the number of double cosets is

$$|D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v}\cdot \mathbf{a}^{\mathbb{Z}}| = \sum_{m|(n/S_1)} \frac{c_m}{m}(\text{mass}(4.19.1) \text{ for } \mathbb{F}_q(\gamma)=\mathbb{F}_{q^m}). \tag{4.19.2}$$

The category $\mathcal{C} = [D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v}]$, the disjoint summand $[D^{\gamma *}\backslash(D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v})^\gamma]$ of $\mathcal{C}^\star$, and their analogues when one divides also by $\mathbf{a}^{\mathbb{Z}}$, have a concrete interpretation, which we now explain (4.20–4.23). Our proof of 2.3 will not rely on this interpretation.

Construction 4.20. *The category* $[D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v}]$ *is naturally equivalent to the category* $\mathrm{Mod}^*_{\mathrm{is}}(-\mathcal{O}_D)$.

For the notation $\mathrm{Mod}^*_{\mathrm{is}}$, see 3.6 and 4.10.

Construction. To $x=(x_v)$ in $D^*(\mathbb{A})$ one attaches the coherent sub-$\mathcal{O}_D$-module of D whose completion at v is $x_v\mathcal{O}_{v,D}\subset D_v$. Call it $x\mathcal{O}_D$. It depends only on the coset $x\cdot\prod \mathcal{O}^*_{D,v}$. For any invertible right $\mathcal{O}_D$-module $\mathcal{E}$, a trivialization of $\mathcal{E}$ at the generic point defines an isomorphism of it with some $x\mathcal{O}_D$. An $\mathcal{O}_D$-module isomorphism from $x\mathcal{O}_D$ to $y\mathcal{O}_D$ has the form $f\mapsto df$, where d in D^* is such that $dx\prod \mathcal{O}^*_{D,v} = y\prod \mathcal{O}^*_{D,v}$. This defines the announced equivalence to $\mathrm{Mod}^*_{\mathrm{is}}(-\mathcal{O}_D)$. □

Define $\mathrm{Mod}(\mathbb{F}_{q^m}-\mathcal{O}_D)$ to be the category of $(\mathbb{F}_{q^m},\mathcal{O}_D)$-bimodules, where the two implied $\mathbb{F}_q$-module structures are assumed to agree. A decoration * means that we consider only those which are invertible as $\mathcal{O}_D$-modules.

Construction 4.21. *The category* $[D^{\gamma *}\backslash(D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v})^\gamma]$ *is naturally equivalent to the category* $\mathrm{Mod}^*_{\mathrm{is}}(\mathbb{F}_{q^m}-\mathcal{O}_D)$.

Construction. The coset $x\prod \mathcal{O}^*_{D,v}$ is fixed by γ if and only if $x\mathcal{O}_D\subset D$ is a left $\mathbb{F}_{q^m}$-module. If the cosets of x and y are fixed by γ, an $\mathcal{O}_D$-module isomorphism, defined as in 4.20 by d in D^*, respects the $\mathbb{F}_{q^m}$-module structure if and only if d is in $D^{\gamma *}$. This defines the functor, and proves it is fully faithful. To see that it is essentially surjective, one uses that an $\mathbb{F}_{q^m}$-module structure on $x\mathcal{O}_D$ is given by an embedding $\mathbb{F}_{q^m}\to D$, and that two such embeddings are D^*-conjugate. □

Variants 4.22. (i) Define A to be the divisor of the idèle $\mathbf{a}$. The $\mathcal{O}_D$-module $x\mathbf{a}\mathcal{O}_D$ is naturally isomorphic to $x\mathcal{O}_D(-A) = x\mathcal{O}_D\otimes_{\mathcal{O}}\mathcal{O}(-A)$, and the $x\mathcal{O}_D(kA)$ for $k\in\mathbb{Z}$ are nonisomorphic: their degrees are distinct. It follows that $[D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v}\cdot\mathbf{a}^{\mathbb{Z}}]$ is naturally equivalent to the category $\mathrm{Mod}^*_{\mathrm{is}}(-\mathcal{O}_D)/\mathbf{a}$ of invertible right $\mathcal{O}_D$-modules $\mathcal{E}$, taken up to

$\mathcal{E} \mapsto \mathcal{E}(kA)$. In this category, $\mathcal{E}'$ and $\mathcal{E}''$ are isomorphic if and only if for some k', k'' the modules $\mathcal{E}'(k'A)$ and $\mathcal{E}'(k''A)$ are isomorphic. In this case, (k', k'') is unique up to $(k', k'') \mapsto (k'+c, k''+c)$, and $\mathrm{Hom}_{/\mathbf{a}}(\mathcal{E}', \mathcal{E}'')$ is any of the $\mathrm{Isom}(\mathcal{E}'((k'+c)A), \mathcal{E}''((k''+c)A))$, between which one has a transitive system of bijections.

(ii) Similarly, $[Z(\gamma)\backslash(D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v} \cdot \mathbf{a}^{\mathbb{Z}})^\gamma]$ is naturally equivalent to the category $\mathrm{Mod}^*_{\mathrm{is}}(\mathbb{F}_{q^m} - \mathcal{O}_D)/\mathbf{a}$ of bimodules $\mathcal{E}$ in $\mathrm{Mod}^*_{\mathrm{is}}(\mathbb{F}_{q^m} - \mathcal{O}_D)$, taken up to $\mathcal{E} \mapsto \mathcal{E}(kA)$.

Remark 4.23. The decomposition 4.17

$$[D^*\backslash D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v}]^\star = \coprod [Z(\gamma)\backslash(D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v})^\gamma]$$

becomes – via 4.20 and 4.21 – the fact that the algebra of endomorphisms of an invertible left $\mathcal{O}_D$-module $\mathcal{E}$ is a finite field, and that the category of the $(\mathcal{E}, \alpha), \alpha$ an automorphism of $\mathcal{E}$ which generates an extension of degree m of $\mathbb{F}_q$ and has a specified minimal polynomial over $\mathbb{F}_q$, is equivalent to the category $\mathrm{Mod}^*_{\mathrm{is}}(\mathbb{F}_{q^m} - \mathcal{O}_D)$ of $(\mathbb{F}_{q^m}, \mathcal{O}_D)$-bimodules.

4.24. The tensor product $\mathbb{F}_{q^m} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ is the product of copies of $\mathbb{F}_{q^m}$, indexed by $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$. The projections are the

$$x \otimes y \longmapsto \tau(x)y\colon\ \mathbb{F}_{q^m} \otimes \mathbb{F}_{q^m} \to \mathbb{F}_{q^m}.$$

An $\mathbb{F}_{q^m} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$-module structure on M hence gives a decomposition of M in submodules M_τ, which we will view as a grading, with group of degrees $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \simeq \mathbb{Z}/m$.

We can apply this to D, as well as to $\mathcal{O}_D$, on which $\mathbb{F}_{q^m}$ acts by left and right multiplications. On the τ-component, $\lambda d = d\tau(\lambda)$, for λ in $\mathbb{F}_{q^m}$. The degree zero component of D (resp. $\mathcal{O}_D$) is the commutant D^γ (resp. $\mathcal{O}^\gamma_D$) of $\mathbb{F}_{q^m}$. The embedding of $\mathbb{F}_{q^m}$ in D (resp. $\mathcal{O}_D$) extends to an embedding of $F_m = F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ (resp. $\mathcal{O}_m = \mathcal{O} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$), and D^γ (resp. $\mathcal{O}^\gamma_D$) is also the commutant of F_m (resp. $\mathcal{O}_m$). The division algebra D^γ is of degree $(n/m)^2$ over its center F_m.

So far, we have viewed $\mathcal{O}_m$ and $\mathcal{O}^\gamma_D$ as sheaves over X_1. The sheaf $\mathcal{O}_m$ is the direct image, from X_m to X_1, of the structural sheaf of X_m. As X_m is finite over X_1, the direct image functor is an equivalence from coherent sheaves on X_m to coherent sheaves of $\mathcal{O}_m$-modules on X_1. We will tacitly use this equivalence to view $\mathcal{O}^\gamma_D$ as an order, over X_m, of the central simple algebra D^γ over F_m. The completion $\mathcal{O}^\gamma_{D,v}$ is the product, over the places $v(i)$ of X_m above v, of the completions $\mathcal{O}^\gamma_{D,v(i)}$ of $\mathcal{O}^\gamma_D$, viewed as an order over X_m. The proofs of 4.25 and 4.26 will show that the $\mathcal{O}^\gamma_{D,v(i)}$ are maximal orders of the $D^\gamma_{v(i)}$: $\mathcal{O}^\gamma_D$ is a maximal order of D^γ over X_m.

To compute the category (4.19.1), we will first consider the fixed point set of γ, acting on $D^*(\mathbb{A})/\prod \mathcal{O}^*_{D,v}$. It is the restricted product of the fixed points of γ acting on the $D^*_v/\mathcal{O}^*_{D,v}$.

Proposition 4.25. *If $v \notin S_1$, $\mathcal{O}^{\gamma *}_{D,v}$ is a maximal compact subgroup in $D^{\gamma *}_v$ and*

$$D^{\gamma *}_v/\mathcal{O}^{\gamma *}_{D,v} \xrightarrow{\sim} (D^*_v/\mathcal{O}^*_{D,v})^\gamma.$$

Proof. Let V be a vector space of dimension n over F_v. As $v \notin S_1$, there exists an isomorphism $D_v \xrightarrow{\sim} \mathrm{End}(V)$. We choose one, and identify D_v with $\mathrm{End}(V)$. The maximal order $\mathcal{O}_{D,v} \subset D_v$ is $\mathrm{End}(\mathcal{E}_0)$, for $\mathcal{E}_0$ a lattice in V: a free $\mathcal{O}_v$-module such that $F_v \otimes_{\mathcal{O}_v} \mathcal{E}_0 \xrightarrow{\sim} V$. The completion $F_{m,v} = F_v \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ is a product of local fields $F^{(i)}$: the product of the completions of F_m at the points of X_m above v. The completion $\mathcal{O}_{m,v} = \mathcal{O}_v \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ is the product of their valuation rings $\mathcal{O}^{(i)}$. By assumption, $F_{m,v}$ embeds in $\mathrm{End}(V)$, turning V into an $F_{m,v}$-module, i.e. into a product of vector spaces $V^{(i)}$ over the $F^{(i)}$, while $\mathcal{O}_{m,v}$ embeds into $\mathrm{End}(\mathcal{E}_0)$, turning $\mathcal{E}_0$ into a product of lattices $\mathcal{E}^{(i)}_0$ in $V^{(i)}$.

The map $d \mapsto d\mathcal{E}_0$ induces a bijection from $D^*_v/\mathcal{O}^*_{D,v}$ to the set of lattices in V. Indeed, any lattice $\mathcal{E}_1$, being a free $\mathcal{O}_v$-module, is the image of $\mathcal{E}_0$ by some element of $\mathrm{GL}(V)$. The fixed points by γ are the lattices which are $\mathcal{O}_{m,v}$-modules, that is product of lattices $\mathcal{E}^{(i)}_1$ in $V^{(i)}$. They are the images of $\mathcal{E}_0$ by some d in $\prod \mathrm{GL}_{F^{(i)}}(V^{(i)}) = D^{\gamma *}_v$, with d unique up to $\prod \mathrm{GL}_{\mathcal{O}^{(i)}}(\mathcal{E}^{(i)}_0) = \mathcal{O}^{\gamma *}_{D,v}$. The claim follows. □

4.26. For v in S_1, D_v is a division algebra with center F_v and $\mathcal{O}_{D,v}$ is its valuation ring. The map $\det\colon D^*_v \to F^*_v$ is onto, and induces a bijection from $D^*_v/\mathcal{O}^*_{D,v}$, the group of the valuation of D_v, to the group $\mathbb{Z}$ of the valuation of F_v. The action of d in D^* is by "adding the valuation of d at v". Special case: γ, being of finite order, acts trivially.

Recall that m is prime to $\deg(v)$, and that the field $(F_m)_v = F_v \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ is contained in D_v, with commutant D^γ_v. The reduced norms for D^γ_v, with center $F_{m,v}$, and for D_v, with center F_v, are related by the commutative diagram

(4.26.1)
$$\begin{tikzcd} D^{\gamma *}_v \arrow[r] \arrow[d, hook] & F^*_{m,v} \arrow[d, "\mathrm{Norm}"] \\ D_v \arrow[r] & F_v. \end{tikzcd}$$

Identifying the groups of the valuations of D_v^γ and D_v with $\mathbb{Z}$, a quotient of (4.26.1) is

$$\begin{array}{ccc} D_v^{\gamma *}/\mathcal{O}_{D,v}^{\gamma *} & \xrightarrow{\sim} & \mathbb{Z} \\ \big\downarrow\rlap{\ \supset} & & \big\downarrow m \\ D_v^{*}/\mathcal{O}_{D,v}^{*} & \xrightarrow{\sim} & \mathbb{Z}. \end{array}$$

We conclude that

$$(D_v^*/\mathcal{O}_{D,v}^*)^\gamma = D_v^*/\mathcal{O}_{D,v}^* = \frac{1}{m}(D_v^{\gamma *}/\mathcal{O}_{D,v}^{\gamma *}). \tag{4.26.2}$$

4.27. From (4.26.2) we get a decomposition of the category

$$[D^{\gamma *}\backslash(D^*(\mathbb{A})/\prod_v \mathcal{O}_{D,v}^*)^\gamma]$$

as the disjoint sum of m^{N_1} subcategories, indexed by $(\mathbb{Z}/m)^{S_1}$. By 4.25, the subcategory with index 0 is

$$[D^{\gamma *}\backslash D^{\gamma *}(\mathbb{A})/\prod \mathcal{O}_{D,v}^{\gamma *}].$$

The others are equivalent to it: if $\boldsymbol{\pi}_v$ is of valuation one in D_v ($v \in S_1$), the equivalences are given by right multiplication by the $\prod \boldsymbol{\pi}_v^{a_v}$, $0 \leq a_v < m$.

Similarly,

$$[D^{\gamma *}\backslash(D^*(\mathbb{A})/\prod \mathcal{O}_{D,v}^*)^\gamma/\mathbf{a}^{\mathbb{Z}}]$$

is the disjoint sum of m^{N_1} subcategories equivalent to

$$[D^{\gamma *}\backslash D^{\gamma *}(\mathbb{A})/\prod \mathcal{O}_{D,v}^{\gamma *} \cdot \mathbf{a}^{\mathbb{Z}}]. \tag{4.27.1}$$

In (4.27.1), D^γ is a central simple division algebra of dimension $(n/m)^2$ over F_m. The adelization $D^\gamma(\mathbb{A}) = D^\gamma \otimes_F \mathbb{A}$ ($\mathbb{A}$ adèles of F) can be viewed also as the adelization of D^γ, viewed as an F_m-algebra, that is $D^\gamma \otimes_{F_m} \mathbb{A}_m$ for $\mathbb{A}_m$ the adèles of F_m, while $\prod_v \mathcal{O}_{D,v}^\gamma$ can be viewed as the product of the completions of a maximal order of D^γ over X_m. As an adèle of F_m, $\mathbf{a}$ is still of degree one, and the corresponding divisor on $X_m/\mathbb{F}_{q^m}$ is of degree one. One has $S_m \xrightarrow{\sim} S_1$, D^γ is a division algebra at each v in S_m, and a matrix algebra elsewhere. The same assumptions as those we started with, but over X_m, and a dimension $(n/m)^2$ for D^γ.

4.28. We now work over X_m. The mass of the category (4.27.1) is the volume of the double coset $D^{\gamma *}\backslash D^{\gamma *}(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}$ for the Haar measure on $D^{\gamma *}(\mathbb{A})$ which gives $\prod \mathcal{O}_{D,v}^{\gamma *}$ the volume one. The product is over the places of X_m, and by 4.25 and 4.26 $\mathcal{O}_D^\gamma$ is a maximal order in D^γ. Let $D^{\gamma *}(\mathbb{A})^0$ be the subgroup of d in $D^{\gamma *}(\mathbb{A})$ on which the degree of $\det(d)$ is zero, and $D^{\gamma *}(\mathbb{A})^{(i)}$ the coset on which it is i. The $D^{\gamma *}\backslash D^{\gamma *}(\mathbb{A})^{(i)}$ have all the same volume, and their disjoint sum, for $0 \le i < n/m$, maps bijectively onto $D^{\gamma *}\backslash D^{\gamma *}(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}$. If μ is the Tamagawa measure on $D^{\gamma *}(\mathbb{A})$, the Tamagawa number $\tau(D^{\gamma *}) = \mu(D^{\gamma *}\backslash D^{\gamma *}(\mathbb{A})^0)$ is 1 and we conclude that

$$\text{mass}(4.27.1) = \frac{n}{m} \cdot \mu\Big(\prod \mathcal{O}^{\gamma *}_{D,v}\Big)^{-1}. \tag{4.28.1}$$

Lemma 4.6 and 4.19.2 give

Lemma 4.29. *We have* $T(X_1, S_1, n) + h_1 = \sum_{m|(n/S_1)} \frac{c_m}{m} \cdot m^{N_1} \cdot \frac{1}{m} \cdot \mu(\prod_v \mathcal{O}^{\gamma *}_{D,v})^{-1}$ *where* γ *is such that* $\mathbb{F}_q(\gamma) = \mathbb{F}_{q^m}$, μ *is the Tamagawa measure for* $D^{\gamma *}$, *viewed as a reductive group over* F_m, *and the product is over the places* v *of* F_m. □

Proof of 2.3 *when* $N_1(n-1)$ *is even and* $N_1 \ge 2$. As $\mathcal{O}^{\gamma}_D$ is a maximal order of D^{γ} over X_m, the Tamagawa volumes in 4.29 have been computed in (3.11.1). Applying (3.11.3) to the $(X_m, S_m, D^{\gamma}, \mathcal{O}^{\gamma}_D)$, one obtains 2.3. □

5. Proof of 2.3: Using the building

In this section, we prove 2.3 under the assumption that for some w in S_1, there exists a division algebra D with center F of dimension n^2 over F such that D_v is a division algebra for $v \in S_1^w := S_1 - \{w\}$, and a matrix algebra over F_v for $v \notin S_1^w$. As explained in 3.2, this amounts to assuming that $N_1 \ge 3$, and that n or N_1 is odd; the place w can then be chosen freely in S_1. The integers $N_1(n-1)$ and $n-1$ have the same parity, and the sign $(-1)^{N_1(n-1)}$ in (2.3.1) is $(-1)^{n-1}$.

The constructions made in section 4 apply to D and S_1^w. With the notation of 2.2, one has that (n/S_1) divides (n/S_1^w) which divides n. As in 4.19, we choose an $\mathbb{F}_q$-algebra embedding of $\mathbb{F}_{q^{(n/S_1^w)}}$ in D, and a maximal order $\mathcal{O}_D$ of D containing this $\mathbb{F}_{q^{(n/S_1^w)}}$.

5.1. As recalled in 1.11 and 1.13, the number $T(X_1, S_1, n)$ we want to compute is equal to the number of classes modulo $\mathbb{F}_q$-twists of automorphic representations π of $D^*(\mathbb{A})$ such that

(i) π is unramified outside of S_1;

(ii) for v in S_1^w, the local component π_v is of the form $\chi(\det)$ for χ an unramified character of F_v^*;

(iii) the local component of π at w is of the form

$$\text{Steinberg representation } \otimes \chi(\det)$$

for χ an unramified character of F_v^*.

Let $\mathbf{a}$ be an idèle of degree > 0. Let (iv) be the condition

(iv) The central character ω_π of π is trivial on $\mathbf{a}$.

The same proof as in 4.2 gives

Lemma 5.2. *The number* $T(X_1, S_1, n)$ *is* $1/n\ \deg(\mathbf{a})$ *times the number of automorphic representations of* $D^*(\mathbb{A})$ *for which* (i), (ii), (iii), (iv) *hold.*

5.3. Let (iii)$'$ be the condition

(iii)$'$ the local component of π at w is of the form $\chi(\det)$, for χ an unramified character of F_w^*.

The algebraic subgroup $SD^* := \mathrm{Ker}(\det\colon\ D^* \to \mathbb{G}_m)$ of D^*, being a form of $\mathrm{SL}(n)$, is simply connected. By the strong approximation theorem applied to SD^*, an automorphic representation π of $D^*(\mathbb{A})$ for which (iii)$'$ holds factors through $\det\colon\ D^*(\mathbb{A}) \to \mathbb{A}^*$. Indeed, for any automorphic function f in π, and any d in $D^*(\mathbb{A})$, f is constant on $SD^*(F)\cdot d \cdot SD^*(F_w) = SD^*(F)\cdot SD^*(F_w)\cdot d$, which is dense in $SD^*(\mathbb{A})d$. As in 4.5, if π satisfies (i), (ii), (iii)$'$, (iv), it is of the form $\chi(\det)$, for χ an unramified character of the idèle class group of F, trivial on $\mathbf{a}^n$. There are $n\deg(\mathbf{a})h_1$ such characters.

5.4. The space of locally constant functions on $D^*\backslash D^*(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}$ is the direct sum of the automorphic representations π of $D^*(\mathbb{A})$ for which (iv) holds. Those for which (i), (ii) hold are those whose local component π_v have, for $v \neq w$, a nontrivial subspace of vectors fixed by $\mathcal{O}_{D,v}^*$, and this subspace is then of dimension 1. The space L of locally constant functions on $D^*\backslash D^*(\mathbb{A})/\prod_{v\neq w}\mathcal{O}_{D,v}^*\cdot\mathbf{a}^{\mathbb{Z}}$ is hence isomorphic, as a representation of D_w^*, to the direct sum of the components π_w of the automorphic representations of $D^*(\mathbb{A})$ for which (i), (ii), (iv) hold. We get:

Lemma 5.5. *The number of automorphic representations of* $D^*(\mathbb{A})$ *for which* (i), (ii), (iii), (iv) *hold is the sum, over the unramified characters* χ *of* F_w^* *for which* $\chi^n(\mathbf{a}) = 1$,

$$T(X_1, S_1, n) = \frac{1}{n\ \deg(\mathbf{a})}\sum_{\chi}[L\colon\ \text{Steinberg}\otimes\chi(\det)]. \tag{5.5.1}$$

5.6. Let E be a nonarchimedian local field, $\mathrm{val}\colon\ E^* \to \mathbb{Z}$ its valuation, $\mathcal{O}$ its valuation ring and t a uniformizing parameter. The case we need is $E = F_w$. Define $\mathrm{GL}(n, E)^0$ by the short exact sequence

$$1 \to \mathrm{GL}(n,E)^0 \to \mathrm{GL}(n,E) \xrightarrow{\mathrm{val}\det} \mathbb{Z} \to 1. \tag{5.6.1}$$

Define $\mathrm{GL}(n,E)^{(i)}$ to be the coset of $\mathrm{GL}(n,E)^0$ on which val(det) is i. Dividing by the center E^* of $\mathrm{GL}(n,E)$, its intersection $\mathcal{O}^*$ with $\mathrm{GL}(n,E)^0$ and its image $n\mathbb{Z}$ in $\mathbb{Z}$, we obtain from (5.6.1) an exact sequence

$$1 \to \mathrm{PGL}(n,E)^0 \to \mathrm{PGL}(n,E) \xrightarrow{\mathrm{val}\det} \mathbb{Z}/n \to 1. \tag{5.6.2}$$

The group $\mathrm{GL}(n,E)$ acts by conjugation on $\mathrm{SL}(n,E)$, and hence on its building. This action factors through $\mathrm{PGL}(n,E)$. The vertices of the building $\mathcal{B}$ of $\mathrm{SL}(n,E)$ are the lattices (= $\mathcal{O}$-submodules, free of rank n) Λ in E^n, taken up to dilations $\Lambda \mapsto t^i\Lambda$. If $e_1,\dots,e_n$ is a basis of Λ, the valuation of $e_1\wedge\cdots\wedge e_n \in \overset{n}{\wedge} E^n \simeq E$ depends only on Λ. Its class in $\mathbb{Z}/n$ is the same for Λ and $t^i\Lambda$. We call it the *type* of the corresponding vertex. The group $\mathrm{SL}(n,E)$ acts transitively on the vertices of each type. The group $\mathrm{GL}(n,E)$ permutes the types: g in $\mathrm{GL}(n,E)$ maps the vertex $\{t^i\Lambda\}$ to the vertex $\{t^ig\Lambda\}$, of type that of $\{t^i\Lambda\}$ plus val(det g).

The types of vertices (of the building) are the vertices of the affine Dynkin diagram of $\mathrm{SL}(n,E)$. The subgroup $\mathrm{PGL}(n,E)^0$ of $\mathrm{PGL}(n,E)$ is the subgroup acting trivially on this Dynkin diagram. By [BTI, 1.2.13–1.2.17], applied to the natural homomorphism $\varphi\colon \mathrm{SL}(n,E) \to \mathrm{PGL}(n,E)^0$ or to $\mathrm{SL}(n,E) \to \mathrm{GL}(n,E)^0$, the building of $\mathrm{SL}(n,E)$ is also the building of a Tits system of $\mathrm{PGL}(n,E)^0$, as well as of $\mathrm{GL}(n,E)^0$.

A *chamber* of the building is spanned by vertices represented by lattices Λ_i $(0 \le i \le n-1)$ forming a cyclic chain of distinct lattices

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{n-1} \subset t^{-1}\Lambda_0.$$

If $(e_i)_{1\le i\le n}$ is the standard basis of E^n, the *fundamental chamber* C is the chamber obtained when Λ_i is spanned by $t^{-1}e_1,\dots,t^{-1}e_i,e_{i+1},\dots,e_n$.

The *type* of a face (= simplex) of $\mathcal{B}$ is the set of types of its vertices. By the properties of Tits systems, any face of $\mathcal{B}$ is $\mathrm{GL}(n,E)^0$-conjugate to the unique face of C of the same type, and the stabilizer in $\mathrm{GL}(n,E)^0$ of a face (a *parahoric* subgroup) fixes it. We will orient each face by the ordering of its vertices induced by the ordering $0<1<\cdots<n-1$ of the types of vertices. If σ is the face of type $S\subset\mathbb{Z}/n$ of C and $K_\sigma \subset \mathrm{GL}(n,E)^0$ its stabilizer, the set of (oriented) faces of $\mathcal{B}$ of type S is the orbit $\mathrm{GL}(n,E)^0/K_\sigma$.

For any i, one still has a bijection

$$\coprod_\sigma \mathrm{GL}(n,E)^{(i)}/K_\sigma \sim \text{ set of faces of } \mathcal{B}, \tag{5.6.3}$$

compatible with the action of $\mathrm{GL}(n,E)^0$. This times, if σ is the face of C of type $S \subset \mathbb{Z}/n$, $\mathrm{GL}(n,E)^{(i)}/K_\sigma$ is the set of faces of type $S+i$.

The component $C_d(\mathcal{B})$ of the chain complex $C_*(\mathcal{B})$ of oriented chains, computing the homology of $\mathcal{B}$, is

$$C_d(\mathcal{B}) = \bigoplus_{\dim \sigma = d} \mathbb{C}^{(\mathrm{GL}(n,E)^0/K_\sigma)}. \tag{5.6.4}$$

The group $\mathrm{GL}(n,E)^0$ acts on this complex. As $\mathcal{B}$ is contractible, it is a resolution of the trivial representation 1. We will not use the fact that $\mathrm{GL}(n,E)$ acts too. Its action introduces signs, because $\mathrm{GL}(n,E)$ permutes types and does not respect the orientations of faces which we used.

5.7. By *representation* we will always mean C^∞ representation: the stabilizers of vectors are open. The cohomology used below is the analogue, in that setting, of continuous cohomology, as in Casselman [C]. To apply 5.5, we need to detect, in a unitary representation of $\mathrm{GL}(n,E)$, the occurrences of the representations $\mathrm{Steinberg} \otimes \chi(\det)$, for χ a unitary unramified character of E^*. The following lemma is a corollary of [C]. (We write unitary for unitarizable.)

Lemma 5.8. *Put $H^*(\pi)$ for $H^*(\mathrm{GL}(n,E)^0,\pi)$. The irreducible unitary representations π of $\mathrm{GL}(n,E)$ with $H^*(\pi) \neq 0$ are the representations $\mathrm{Steinberg} \otimes \chi(\det)$ and $\chi(\det)$, for χ a unitary unramified character of E^*. The nonzero H^i are a one-dimensional H^{n-1} in the Steinberg case, a one-dimensional H^0 in the other.*

Proof. If the cohomology is not zero, the center $\mathcal{O}^*$ of $\mathrm{GL}(n,E)^0$ must act trivially. Twisting by a $\chi(\det)$, which is trivial on $\mathrm{GL}(n,E)^0$, has no effect on the cohomology. Replacing π by a twist, we are reduced to considering only representations with trivial central character, i.e. of $\mathrm{PGL}(n,E)$. As $\mathcal{O}^*$ is compact,

$$H^*(\mathrm{PGL}(n,E)^0,\pi) \xrightarrow{\sim} H^*(\mathrm{GL}(n,E)^0,\pi).$$

On the left, we have the cohomology of π, restricted to $\mathrm{PGL}(n,E)^0$. As $\mathrm{PGL}(n,E)^0$ is of finite index in $\mathrm{PGL}(n,E)$, it is the same as the cohomology of $\mathrm{PGL}(n,E)$, with coefficient in the induced (= coinduced) of this restriction to $\mathrm{PGL}(n,E)$. This is the direct sum of the twists $\pi \otimes \chi(\mathrm{val}(\det))$, for χ a character of $\mathbb{Z}/n$. By Casselman, $H^* \neq 0$ is nonzero if and only if one of these twists is Steinberg or trivial, and such a twist contributes respectively to a one-dimensional H^{n-1} or H^0. As the twists are nonisomorphic, the lemma follows. □

5.9. For L a representation of $\mathrm{GL}(n,E)$, to compute

$$H^*(\mathrm{GL}(n,E)^0, L) = \mathrm{Ext}^*_{\mathrm{GL}(n,E)^0}(1, L)$$

one can use the resolution (5.6.4) of the trivial representation 1 of $\mathrm{GL}(n,K)^0$. It gives us a complex $C^*(L)$ computing the cohomology, with components the

$$C^d(L) = \bigoplus_{\dim\,\sigma = d} L^{K_\sigma} \tag{5.9.1}$$

(sum over faces σ of the fundamental chamber C).

In particular, if the L^{K_σ} are finite dimensional, we have

$$\sum(-1)^i \dim\, H^i(\mathrm{GL}(n,E)^0, L) = \sum(-1)^{\dim\,\sigma} \dim\, L^{K_\sigma}.$$

Moreover, if L is the direct sum of irreducible unitary representations of $\mathrm{GL}(n,E)$, one has

$$\sum_\sigma(-1)^{\dim\,\sigma} \dim\, L^{K_\sigma} = \sum_\chi [L\colon\ \chi(\det)] + (-1)^{n-1}\sum_\chi [L\colon\ \text{Steinberg }\otimes\chi] \tag{5.9.2}$$

with χ running over the unitary unramified characters of E^*.

5.10. We now take $E = F_w$, choose an isomorphism of D_w with the matrix algebra $M_n(F_w)$ and identify D_w^* with $\mathrm{GL}(n,F_w)$. For L the representation 5.4 of D_w^*, L^{K_σ} can be identified with the space of functions on the finite set $D^*\backslash D^*(\mathbb{A})/\prod_{v\neq w} \mathcal{O}^*_{D,v}\cdot K_\sigma\cdot \mathbf{a}^{\mathbb{Z}}$.

By 5.3 and 5.4, the sum of the one-dimensional subrepresentations of L which are isomorphic to a $\chi(\det)$ with χ unramified, is of dimension $n\deg(\mathbf{a})h_1$. By (5.5.1) and (5.9.2), one has

Lemma 5.11. *With the notations of* 5.10*, we have*

$$h_1 + (-1)^{n-1}T(X_1, S_1, n) = \frac{1}{n\ \deg(\mathbf{a})}\sum_\sigma (-1)^{\dim\,\sigma}|D^*\backslash D^*(\mathbb{A})/\prod_{v\neq w}\mathcal{O}^*_{D,v}\cdot K_\sigma \mathbf{a}^{\mathbb{Z}}|, \tag{5.11.1}$$

the sum being over the faces of the fundamental chamber of the building of $\mathrm{SL}(n,F_v)$. □

The trace formula (compact quotient case) expresses the right side of (5.11.1) in terms of alternating sums ($\sum(-1)^{\dim\,\sigma}\dots$) of orbital integrals (cf. 4.18). Similar sums appear in Kottwitz [K], and we might have quoted the computations of [K], except that they are written for the characteristic zero case, and mainly for semisimple simply connected groups. We prefer to use the methods of [K], restricted to the case we need, couched in the language of masses.

5.12. We use the arguments of 4.16–4.18 to express the numbers of double cosets appearing in (5.11.1) as sums of masses of categories. Each sum is over conjugacy classes in D^*. As in 4.19, only the conjugacy classes of elements of finite order need to be considered. They have representatives in our chosen $\mathbb{F}_{q^{(n/S_1^w)}} \subset D$, and this representative is unique up to the $\mathrm{Gal}(\mathbb{F}_{q^{(n/S_1^w)}}/\mathbb{F}_q)$ action. Fix γ in $\mathbb{F}^*_{q^{(n/S_1^w)}}$. Define $m := [\mathbb{F}_q(\gamma)\colon \mathbb{F}_q]$. As in 4.19 and 4.24, the commutant D^γ of γ in D is also the commutant of $\mathbb{F}_{q^m} \subset \mathbb{F}_{q^{(n/S_1^w)}}$, as well as of F_m. The contribution of the conjugation class of γ to the right side of (5.11.1) is

$$\frac{1}{n\ \deg(\mathbf{a})}\sum_\sigma (-1)^{\dim\sigma}\mathrm{mass}([D^{\gamma *}\backslash(D^*(\mathbb{A})/\prod_{v\neq w}\mathcal{O}^*_{D,v}\cdot K_\sigma\cdot \mathbf{a}^{\mathbb{Z}})^\gamma]). \tag{5.12.1}$$

It depends only on m, not on the chosen γ. Indeed, K_σ is the multiplicative group of an order in $\mathcal{O}_{D,w}$, and one argues as in 4.19. The right side of (5.11.1) becomes

$$\sum_{m|(n/S_1^w)}\frac{c_m}{m}\{(5.12.1) \text{ for } \gamma \text{ such that } \mathbb{F}_q(\gamma)=\mathbb{F}_{q^m}\}. \tag{5.12.2}$$

It will be convenient to choose $\mathbf{a} = (\mathbf{a}_v)$ such that $\mathbf{a}_v = 1$ for $v \neq w$, and that $\mathbf{a}_w$ is a uniformizing parameter of F_w. The degree of $\mathbf{a}$ is $\deg(w)$. By abuse of notations, $\mathbf{a}_w$ will also be denoted $\mathbf{a}$. The fixed point sets occuring in (5.12.1) are the restricted products over v of

- for v in S_1^w: $\frac{1}{m}(D_v^\gamma/\mathcal{O}^{\gamma *}_{D,v})$, as in (4.26.2);
- for v not in S_1: $D_v^{\gamma *}/\mathcal{O}^{\gamma *}_{D,v}$, as in 4.25;
- for $v = w$: $(D^*_w/K_\sigma\cdot\mathbf{a}^{\mathbb{Z}})^\gamma$.

The corresponding categories decompose into the disjoint sum of $m^{|S_1^w|}$ subcategories, each equivalent to the one obtained by replacing $\frac{1}{m}D_v^{\gamma *}/\mathcal{O}^{\gamma *}_{D,v}$ by $D_v^{\gamma *}/\mathcal{O}^{\gamma *}_{D,v}$. The proof is as in 4.27. Denoting restricted direct product by $\prod$, we conclude:

Lemma 5.13. *The contribution* (5.12.1) *of the conjugacy class of* γ *to the right side of* (5.11.1) *is*

$$\frac{1}{n\ \deg(\mathbf{a})}m^{N_1-1}\sum_\sigma(-1)^{\dim\sigma}\mathrm{mass}[D^{\gamma *}\backslash\prod_{v\neq w}D_v^{\gamma *}/\mathcal{O}^{\gamma *}_{D,v}\times(D^*_w/K_\sigma\cdot\mathbf{a}^{\mathbb{Z}})^\gamma], \tag{5.13.1}$$

where the sum is over the faces σ *of the fundamental chamber of* $\mathrm{SL}(n, F_w)$.

5.14. Let $D^*(\mathbb{A})^0$ (resp. D_v^{*0}) be the kernel of $\deg\circ\det\colon D^*(\mathbb{A})\to\mathbb{Z}$ (resp. $\mathrm{val}\circ\det\colon D^*_v\to\mathbb{Z}$), and $D^{\gamma *}(\mathbb{A})^0$ (resp. $D_v^{\gamma *0}$) be its intersection with $D^{\gamma *}(\mathbb{A})$ (resp. $D_v^{\gamma *}$). As in 4.27 and 4.28, D^γ is a division algebra with center F_m, and $D^{\gamma *}(\mathbb{A})$ can be identified with its adelic multiplicative group, over F_m. The diagram

$$\begin{tikzcd} D^{\gamma *}(\mathbb{A}) \arrow[r] \arrow[d, hook] & \text{idèles of } F_m \arrow[r, "\deg"] \arrow[d, "\text{Norm}"] & \mathbb{Z} \arrow[d, "m"] \\ D^*(\mathbb{A}) \arrow[r] & \text{idèles of } F \arrow[r] & \mathbb{Z} \end{tikzcd}$$

is commutative. The notation $D^{\gamma *}(\mathbb{A})^0$ above hence agrees with that ("over F_m") of 4.28.

If δ in $D^{*\gamma}$ fixes x_v in $D_v^{\gamma *}/\mathcal{O}_{D,v}^{\gamma *}$, its image in $D_v^{\gamma *}$ is in the conjugate $x_v \mathcal{O}_{D,v}^{\gamma *} x_v^{-1}$ of $\mathcal{O}_{D,v}^{\gamma *}$. Hence it is in D_v^{*0}. By the product formula, $\deg(\det(\delta)) = 0$. It follows that if δ fixes x in $\prod_{v \neq w} D_v^{\gamma *}/\mathcal{O}_{D,v}^{\gamma *}$, then its image in $D_w^{\gamma *}$ is in $D_w^{\gamma *0}$. As D^γ is a division algebra, $D^{\gamma *}$ is cocompact in $D^{\gamma *}(\mathbb{A})^0$: it acts with finitely many orbits in $\prod_{v \neq w} D_v^{\gamma *}/\mathcal{O}_v^{\gamma *}$, and for each x in $\prod_{v \neq w} D_v^{\gamma *}/\mathcal{O}_v^{\gamma *}$, the stabilizer Γ_x of x in $D^{\gamma *}$ is cocompact in $D_w^{\gamma *0}$. The group Γ_x admits torsion free subgroups of finite index (Serre [S, Th. 4(b)]). For any place $v \neq w$, the kernel of the reduction $\bmod\, v$, from Γ_x to $(x_v \mathcal{O}_{D,v}^{\gamma} x_v^{-1} \otimes \mathcal{O}_v/m_v)^*$, is such a subgroup.

5.15. Recall that we have chosen an isomorphism of D_w with $M_n(F_w)$, that $D_w^{*(i)}$ is the coset of $D_w^{*0} \subset D_w^*$ on which the valuation of det is i, and that for each i the $D_w^{*(i)}/K_\sigma = \mathrm{GL}(n, F_w)^{(i)}/K_\sigma$ are the facets of one copy $\mathcal{B}^{(i)}$ of the building of $\mathrm{SL}(n, F_v)$ (5.6.4). Because γ is in $\mathrm{GL}(n, F_w)^0$, the fixed locus $\mathcal{B}^{(i)\gamma}$ of γ on $\mathcal{B}^{(i)}$ is a union of facets. Because γ is of finite order, this fixed locus is not empty, hence is contractible. As in [K], p. 635, all the conditions of [S], 3.3, are satisfied by the action of $D_w^{\gamma *0}$ on $\mathcal{B}^{(i)\gamma}$.

If Γ'_x is a torsion free subgroup of finite index of Γ_x, its action on $\mathcal{B}^{(i)}$ is free, its cohomology is that of $\Gamma'_x \backslash \mathcal{B}^{(i)}$, and its Euler-Poincaré characteristic is

$$\chi(\Gamma'_x) = \sum (-1)^{\dim \Gamma} \cdot |\Gamma'_x \backslash (D_w^{*(i)}/K_\sigma)^\gamma|.$$

Dividing both sides by $[\Gamma_x \colon \Gamma'_x]$, one obtains that the virtual Euler-Poincaré characteristic of Γ_x is

$$\chi(\Gamma_x) = \sum_\sigma (-1)^{\dim \sigma} \cdot \mathrm{mass}[\Gamma_x \backslash (D_w^{*(i)}/K_\sigma)^\gamma]. \tag{5.15.1}$$

Let $\mu_{w,EP}$ be the invariant measure on $D_w^{\gamma *}$ whose restriction to $D_w^{\gamma *0}$ is the Euler-Poincaré measure of $D_w^{\gamma *0}$. By the definition of $\mu_{w,EP}$,

$$\sum (-1)^{\dim \sigma} \mathrm{mass}[\Gamma_x \backslash (D_w^{*(i)}/K_\sigma)^\gamma] = \mu_{w,EP}(\Gamma_x \backslash D_w^{\gamma 0}).$$

As $(D_w^*/K_\sigma \mathbf{a}^{\mathbb{Z}})^\gamma$ is the disjoint sum of the $(D_w^{*(i)}/K_\sigma)^\gamma$ for $0 \leq i < n$, we also have

$$\sum (-1)^{\dim \sigma} \mathrm{mass}[\Gamma_x \backslash (D_w^*/K_\sigma \mathbf{a}^{\mathbb{Z}})^\gamma] = n \mu_{w,EP}(\Gamma_x \backslash D_w^{\gamma 0}).$$

Hence

$$\sum_\sigma (-1)^{\dim\sigma}\mathrm{mass}([D^{\gamma*}\backslash \prod_{v\neq w}(D_v^{\gamma*}/\mathcal{O}_{D,v}^{\gamma*})\times(D_w^*/K_\sigma \mathbf{a}^{\mathbb{Z}})^\gamma]) \tag{5.15.2}$$
$$= n\sum_x \mu_{w,EP}(\Gamma_x\backslash D_w^{\gamma 0}),$$

the sum being over representatives of the orbits of $D^{\gamma*}$ acting on $\prod_{v\neq w} D_v^{\gamma*}/\mathcal{O}_{D,v}^{\gamma*}$.

5.16. Let c be the greatest common divisor of m and $\deg(w)$ and suppose that $c>1$. There are c places of X_m above w, and $(F_m)_w = F_w\otimes_{\mathbb{F}_q}\mathbb{F}_{q^m}$ is the product of the completions of F_m at these places. The completion $(F_m)_w$ is hence the product of $c>1$ fields, and D_w^γ is a product of matrix algebras over these fields. The multiplicative group $D_w^{\gamma*}$ admits a quotient $\mathbb{Z}^c$, and $D_w^{\gamma*0}$ a quotient $\mathbb{Z}^{c-1}$. As $c>1$, its Euler-Poincaré measure is 0. If $m=[\mathbb{F}_q(\gamma)\colon \mathbb{F}_q]$ is not prime to $\deg(w)$, (5.15.2) implies that

$$\sum_\sigma (-1)^{\dim\sigma}\mathrm{mass}([D^{\gamma*}\backslash \prod_{v\neq w}(D_v^{\gamma*}/\mathcal{O}_v^*)\times(D_w^*/K_\sigma\mathbf{a}^{\mathbb{Z}})^\gamma])=0$$

and γ does not contribute to the right side of (5.11.1).

5.16. Suppose now that m is prime to $\deg(w)$, that is that $m\mid(n/S_1)$. In this case, $(F_m)_w$ is the local field of X_m at the unique place above w, and $D_w^{\gamma*}$ is a matrix algebra $M_{n/m}(F_{m,w})$. Define μ_{EP} to be the measure on $D^{\gamma*}(\mathbb{A})$ which is the product of the Haar measures on the $D_v^{\gamma*}$ giving volume one to $\mathcal{O}_{D,v}^{\gamma*}$, for $v\neq w$, and of $\mu_{w,EP}$ on $D_w^{\gamma*}$. As $D_w^{\gamma*0}$ is of index $\frac{n}{m}$ in $D_w^{\gamma*}/\mathbf{a}^{\mathbb{Z}}$, (5.15.2) is equal to

$$m\sum \mu_{w,EP}(\Gamma_x\backslash D_w^\gamma/\mathbf{a}^{\mathbb{Z}}) = m\mu_{w,EP}(D^{\gamma*}\backslash D^{\gamma*}(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}); \tag{5.16.1}$$

the sum is over representatives of orbits, as in (5.15.2). As D^γ is central simple of dimension $\frac{n}{m}$ over F_m, and $\deg(\mathbf{a})$ is the same, whether $\mathbf{a}$ is viewed as an idèle of $F/\mathbb{F}_q$, or of $F_m/\mathbb{F}_{q^m}$, when we plug (5.15.2) and (5.16.1) in 5.13, we get that (5.13.1) equals

$$\frac{1}{n\deg(\mathbf{a})}m^{N_1}\mu_{EP}(D^{\gamma*}\backslash D^{\gamma*}(\mathbb{A})/\mathbf{a}^{\mathbb{Z}}) = \frac{1}{n\deg(\mathbf{a})}m^{N_1}\frac{n}{m}\deg(\mathbf{a})\mu_{EP}(D^{\gamma*}\backslash D^{\gamma*}(\mathbb{A})^0). \tag{5.16.2}$$

By (5.12.2), (5.11.1) becomes

$$h_1+(-1)^{n-1}T(X_1,S_1,n) = \sum_{m\mid(n/S_1)}\frac{c_m}{m}\cdot m^{N_1}\cdot\frac{1}{m}\mu_{EP}(D^{\gamma*}\backslash D^{\gamma*}(\mathbb{A})^0). \tag{5.16.3}$$

If μ is the Tamagawa measure on $D^{\gamma*}$, and μ_w the measure on $D_w^{\gamma*}$ giving $\mathcal{O}_{D,w}^{\gamma*}$ the volume one, the Tamagawa number $\tau(D^{\gamma*})=\mu(D^{\gamma*}\backslash D^{\gamma*}(\mathbb{A})^0)$ is 1 and the term of index m in the sum (5.16.3) can be rewritten as

$$\left\{\frac{c_m}{m}\cdot m^{N_1}\cdot\frac{1}{m}\cdot\mu\Big(\prod\mathcal{O}^*_{D,v}\Big)^{-1}\right\}\cdot\mu_{w,EP}/\mu_w. \tag{5.16.4}$$

The factor in curly brackets appeared in 4.29. As there, it is ${}_{n/m}T_n$ for (X_1,S_w). The subgroups $\mathrm{SL}(n/m,F_{m,w})$ of $\mathrm{GL}(n/m,F_{m,w})^0$, and $\mathrm{SL}(n/m,\mathcal{O}_{m,w})$ of $\mathrm{GL}(n/m,\mathcal{O}_{m,w})$ are both of the same finite index q_w^*-1. It follows that the restriction of $\mu_{w,EP}$ to $\mathrm{SL}(n/m,F_{m,w})$ is (q_w^m-1) times its Euler-Poincaré measure, and that the ratio of measures in (5.16.4) is the Euler-Poincaré volume of the subgroup $\mathrm{SL}(n/m,\mathcal{O}_{m,w})$ of $\mathrm{SL}(n/m,F_{m,w})$. By [S, Th. 7], it is

$$\prod_{i=1}^{n/m-1}(1-q_w^{mi})$$

and 2.3 follows. □

6. Lefschetz type form of the theorem

We keep the notation of section 2.

6.1. Formula (2.3.1), applied to (X_m,S_m), gives the number $T(X_1,S_1,n,m)=T(X_m,S_m,n)$ of fixed points of Fr^{*m} acting on the set $\mathcal{T}^{(n)}$ of 2.1. In the form given, it is not helpful to understand how this number varies with m. One of the difficulties is that when (X_1,S_1) over $\mathbb{F}_q$ is replaced by (X_m,S_m) over $\mathbb{F}_{q^m}$, the divisor (n/S_1) of n can change. The cardinality N_1 of S_1 changes too (it becomes N_m), as well as the degrees of the elements of S_1. Our first aim in this section is to give the right side of (2.3.1) a more convenient form. In this rewriting, until 6.16 we do not assume that $n\ge 2$. *We do assume that* $n\ge 1$ *and that* $N_1\ge 2$.

6.2. Let B' be the sum of the multisets (a) and (b_s) for s in S_1 described below.

(a) The multiset of the eigenvalues of Frob acting on $H^1(X)$, counted with their multiplicity. The polynomial $f(t):=\det(1-\mathrm{Frob}\cdot t,H^1(X))$ has integral coefficients. It is the product, over the multiset (a), of the $(1-\alpha t)$. The complex absolute value of each α is $q^{1/2}$.

(b_s) The set of $\deg(s)^{\mathrm{th}}$ roots of unity.

The multiset (a) and the sets (b_s) are viewed as multisets in a fixed algebraically closed extension $\mathbb{Q}$. It does not matter which.

The number 1 appears once in each (b_s). It hence appears N_1 times in B'. We define

$$B:=B'\text{ minus twice }\{1\}. \tag{6.2.1}$$

The sum of the sets (b_s) is the multiset of the eigenvalues of the Frob acting on $\mathbb{Q}^S$. Indeed, S is the disjoint union of the fibers of the projection $S \to S_1$. The fiber at $s \in S_1$ has $\deg(s)$ elements, permuted cyclically by the Frobenius. The exact sequence

$$0 \to \mathbb{Q}_l \to \mathbb{Q}_l^S \to H_c^1(X - S) \to H^1(X) \to 0 \tag{6.2.2}$$

shows that B is the multiset of eigenvalues of Frob acting on $H_c^1(X-S)$, minus $\{1\}$. It has $2g + N - 2$ elements.

Fix a divisor m of n. We will write $\prod_{\zeta^m=1}$ (resp. $\prod'_{\zeta^m=1}$) or simply $\prod$ and $\prod'$, for the product over the m^{th} roots of unity (resp. m^{th} roots of unity other than 1). We define

$$T'_m = \prod_{\beta\in B}\left[\prod{}'(1-\zeta\beta)\cdot\prod_{j=1}^{n/m-1}\prod(1-\zeta\beta q^j)\right]. \tag{6.2.3}$$

Proposition 6.3. (i) *If the divisor m of n does not divide (n/S_1), then $T'_m = 0$.*
(ii) *If m divides (n/S_1), one has*

$$m^{N_1-2}\cdot{}_{n/m}\,T_m = f(1)\cdot\frac{1}{q^n-1}\cdot T'_m. \tag{6.3.1}$$

Proof of (i). Suppose $m \nmid (n/S_1)$. For some s in S_1, m is not prime to $\deg(s)$: there is a $\deg(s)^{\text{th}}$-root of unity $\beta \neq 1$ which is an m^{th} root of unity, and the m^{th}-root of unity $\zeta := \beta^{-1}$ contributes a factor $1 - \zeta\beta = 0$ to T'_m.

Proof of (ii). Suppose that m divides (n/S_1). As observed at the end of 2.2, ${}_{n/m}T_m$ is then the product of

$$f_m(1) = \prod_\alpha(1-\alpha^m) = \prod_\alpha\prod(1-\zeta\alpha) = f(1)\cdot\prod_\alpha\prod{}'(1-\zeta\alpha),$$
$$\frac{1}{(q^m)^{n/m}-1} = \frac{1}{q^n-1}$$

and for each j, $1 \leq j < n/m$,

$$\begin{aligned}(1-q^{mj})^{-2}f_m(q^{mj})\prod_{s\in S_1}(1-q^{mj\deg(s)}) &= \prod_{\beta\in B}(1-\beta^m q^{mj})\\ &= \prod_{\beta\in B}\prod(1-\zeta\beta q^j).\end{aligned}$$

The right side of (6.3.1) is the product of these same factors, and of

$$\prod_\beta\prod{}'(1-\zeta\beta) \text{ for } \beta \text{ in the multiset sum of the sets } (b_s)\text{, minus twice } \{1\}. \tag{6.3.2}$$

For each s in S_1, $\zeta \mapsto \zeta^{\deg(s)}$ is a permutation of the m^{th} roots of unity other than 1. This allows the rewriting of (6.3.2) as

$$\begin{aligned}&\prod{}'(1-\zeta)^{-2} \cdot \prod_{s\in S_1} \prod_{\eta^{\deg(s)}=1} \prod{}'(1-\zeta\eta)\\ &= \prod{}'(1-\zeta)^{-2} \cdot \prod_{s\in S_1} \prod{}'(1-\zeta^{\deg(s)}) = \prod{}'(1-\zeta)^{N_1-2}.\end{aligned}$$

One has

$$\prod{}'(1-\zeta) = \frac{1-t^m}{1-t}\Big|_{t=1} = m,$$

and (6.3.2) is equal to the factor m^{N_1-2} on the left of (6.3.1). □

Corollary 6.4. *The right side of* (2.3.1) *is equal to*

$$f(1) \cdot \frac{1}{q^n-1} \cdot \sum_{m|n} c_m T'_m. \tag{6.4.1}$$

We recall that the Möbius function $\mu(a)$ is $(-1)^d$ when a is the product of d distinct primes, and is 0 otherwise.

Lemma 6.5. *One has*

$$c_m = \sum_{a|m} \mu(a)(q^{m/a}-1). \tag{6.5.1}$$

Proof. Any element of $\mathbb{F}^*_{q^m}$ is, for some divisor a of m, a generator of $\mathbb{F}_{q^a}$ over $\mathbb{F}_q$: for each m,

$$q^m - 1 = \sum_{a|m} c_a.$$

The formula (6.5.1) follows by Möbius inversion. □

Substituting (6.5.1) in the last factor of (6.4.1), we get

$$\sum_{m|n} c_m T'_m = \sum_{ab|n} \mu(a)(q^b-1)T'_{ab}, \tag{6.5.2}$$

(the sum on the right is over all integers $a, b \geq 1$ with $ab|n$), which, plugged in (6.4.1), gives

Proposition 6.6. *The right side of* (2.3.1) *is equal to*

$$f(1) \sum_{ab|n} \mu(a)\, \frac{q^b-1}{q^n-1}\, T'_{ab} \tag{6.6.1}$$

(*sum over* (a,b) *such that* ab *divides* n). □

Let us, in the second factor of (6.6.1), "consider q and the β in B as indeterminates". We introduce an indeterminate Q and $2g+N-2$ indeterminates X_i, and define for $m|n$ (resp. $b|n$) polynomials with integral coefficients, symmetric in the X_i

$$T'_m(Q;(X_i)) := \prod_i \Big\{ \prod_{\zeta^m=1}{}' (1-\zeta X_i) \prod_{1\le j<n/m} \prod_{\zeta^m=1} (1-\zeta X_i Q^j) \Big\}, \tag{6.6.2}$$

$$S^n_b(Q;(X_i)) := \sum_{ab|n} \mu(a) T'_{ab}(Q;(X_i)) \quad \text{(sum over } a\text{)}. \tag{6.6.3}$$

The second factor of (6.6.1) is obtained by evaluating Q at q, and the X_i at the β in B, in the sum over a, b given by

$$\sum_{ab|n} \mu(a) \frac{Q^b-1}{Q^n-1} T'_{ab}(Q;(X_i)) = \sum_{b|n} \frac{Q^b-1}{Q^n-1} S^n_b(Q;(X_i)). \tag{6.6.4}$$

Proposition 6.7. *For each divisor b of n, the rational function*

$$\frac{Q^b-1}{Q^n-1}\, S^n_b(Q;(X_i))$$

lies in $\mathbb{Z}[Q,(X_i)]$.

Proof for $b=1$. We have to show that the polynomial with integral coefficients $S^n_1(Q;(X_i))$ is divisible by the polynomial $(Q^n-1)/(Q-1)$. This is equivalent to the vanishing of the polynomial in the X_i

$$S^n_1(u;(X_i)) = \sum_{a|n} \mu(a) \prod_i \prod_{\zeta^a=1}{}' (1-\zeta X_i) \prod_{j=1}^{n/a-1} \prod_{\zeta^a=1} (1-\zeta X_i u^j)$$

whenever u is an n^{th} root of unity other than 1. It suffices to prove this vanishing for the product of $S^n_1(u;(X_i))$ with $\prod(1-X_i)$. This product is

$$\sum_{a|n} \mu(a) \prod_i \prod_{\zeta^a=1} \prod_{0\le j<n/a} (1-\zeta X_i u^j). \tag{6.7.1}$$

We will show that the terms in the sum over a cancel two by two. More precisely, let r be a prime which divides the order m of u. We will show that the coefficients of $\mu(a)$, for $a=a_0$ prime to r, and for $a=a_0 r$, are equal (recall that $\mu(a)$ is zero unless a is a product of distinct primes). This means the equality, for these two values of a, of the multisets

$$\{\text{the } \zeta u^j \text{ for } \zeta^a=1 \text{ and } 0\le j<n/a\}. \tag{6.7.2}$$

The multiset (6.7.2) is the inverse image, by $z\mapsto z^a$, of the multiset of the u^{aj} $(0\le j<n/a)$. The root of unity u^a has order $m/(m,a)$. As the order of u^a divides n/a, the multiset of the

u^{aj} is a multiple of the set of roots of unity of order dividing $m/(m,a)$. Its inverse image by $z \mapsto z^a$ is the same multiple of the set of roots of unity of order dividing $am/(a,m)$, the lowest common multiple of a and m. As $r \mid m$, this l.c.m is the same for a_0 and for $a_0 r$. The multiplicity will be the same as well, as both multisets have the same number n of elements.

Proof of 6.7 (*general case*). For each divisor b of n, we have

$$\begin{aligned}\prod_{\zeta^{ab}=1}(1-\zeta X_i Q^j) &= 1 - X_i^{ab}Q^{abj} = \prod_{\zeta^a=1}(1-\zeta X_i^b Q^{bj}), \text{ and}\\ \prod_{\zeta^{ab}=1}{}'\,(1-\zeta X_i) &= (1-X_i^{ab})/(1-X_i) = \frac{1-X_i^b}{1-X_i}\cdot\frac{1-X_i^{ab}}{1-X_i^b}\\ &= \frac{1-X_i^b}{1-X_i}\prod_{\zeta^a=1}{}'(1-\zeta X_i^b).\end{aligned}$$

It follows that

$$\frac{Q^b-1}{Q^n-1}S_b^n(Q;(X_i)) = \prod\nolimits_i \frac{(1-X_i^b)}{1-X_i}\cdot\frac{Q^b-1}{(Q^b)^{n/b}-1}\cdot S_1^{n/b}(Q^b;(X_i^b)).$$

This identity reduces 6.7 to the case $b=1$, n being replaced by n/b. ☐

Corollary 6.8. *When* $(X_1,S_1)/\mathbb{F}_q$ *is replaced by* $(X_m,S_m)/\mathbb{F}_{q^m}$*, the second factor of* (6.6.1)*, as a function of* m*, has the form*

$$\sum_j n_j\gamma_j^m. \tag{6.8.1}$$

In (6.8.1)*, the* n_j *are integers, and each* γ_j *is the product of a root of unity and of a monomial in* q *and the eigenvalues of* Frob *acting on* $H^1(X)$.

The expression (6.8.1) is not unique. It becomes unique if one imposes that the γ_j be distinct and the $n_j \neq 0$. We then call γ_j the *eigenvalues occuring with nonzero multiplicity*; we call n_j the *multiplicity* of γ_j. The motivation for this terminology is 6.25.

Proof. The polynomial (6.6.4) depends only on n and the number of X_i. One obtains a description (6.8.1) of the right side of (6.6.1), for (X_m,S_m), by decomposing it as a linear combination of monomials. Indeed, when passing from $(X_1,S_1)/\mathbb{F}_q$ to $(X_m,S_m)/\mathbb{F}_{q^m}$, q is replaced by q^m, and the β in B by their m^{th} powers: B is the multiset of eigenvalues of Frob acting on $H^1_c(X-S)$, minus $\{1\}$, and Frob gets replaced by Frob^m. By definition, the β in B are roots of unity or eigenvalues of Frobenius acting on $H^1(X)$. ☐

Corollary 6.9. *The same property holds for* (6.6.1)*, the right side of* (2.3.1)*.*

Proof. The product of two functions of m of the form (6.8.1) is again of this form, and the first factor $f(1)$ of (6.6.1) is

$$\prod(1-\alpha)$$

for α the eigenvalues of Frob acting on $H^1(X)$. Again, when one goes from $X_1/\mathbb{F}_q$ to $X_m/\mathbb{F}_{q^m}$, Frob is replaced by Frob^m. $\square$

6.10. We define polynomials R_k in the variables X_i by

$$\sum\nolimits_k Q^k R_k(X_i) := \sum_{b|n} \frac{Q^b-1}{Q^n-1} S_b^n(Q;(X_i)) \tag{6.10.1}$$
$$= \sum_{ab|n} \mu(a) \frac{Q^b-1}{Q^n-1} T'_{ab}(Q;(X_i)).$$

Each R_k is a symmetric polynomial with integer coefficients.

Example 6.11. If $n=2$, one has

$$R_0 = 1, \tag{6.11.1}$$

$$R_k = (-1)^{k+1} \sum_{j>k} \sigma_j\,, \qquad \text{(for } k \geq 1\text{)}, \tag{6.11.2}$$

where the σ_j are the elementary symmetric polynomials.

Proof. When n is prime, the sum (6.10.1) over (a,b) has only the three terms for $(a,b) = (1,1), (n,1)$ or $(1,n)$. When $ab=1$, the factor $\prod'$ in $T'_{ab}(Q;(X_i))$ is one. When $ab=n$, the product over j in $T'_{ab}(Q;(X_i))$ is one. For n prime, (6.10.1) hence reduces to

$$\frac{Q-1}{Q^n-1}\Big[\prod_i \prod_{1\leq j<n} (1-X_iQ^j) - \prod_i \prod_{\zeta^n=1}{}' (1-\zeta X_i)\Big] + \prod_i \prod_{\zeta^n=1}{}' (1-\zeta X_i). \tag{6.11.1}$$

For $n=2$, this reduces to

$$\frac{1}{Q+1}\Big[\prod_i (1-X_iQ) - \prod_i (1+X_i)\Big] + \prod_i (1+X_i), \tag{6.11.2}$$

which we rewrite as

$$\begin{aligned}\frac{-1}{1-(-Q)}\sum_j(1-(-Q)^j)\sigma_j+\sum\sigma_j\\ &= -\sum_j\sigma_j\sum_{k<j}(-Q)^k+\sum\sigma_j\\ &= \sum_{k\geq 0}(-1)^{k+1}Q^k\sum_{j>k}\sigma_j+\sum\sigma_j\\ &= 1+\sum_{k\geq 1}(-1)^{k+1}Q^k\sum_{j>k}\sigma_j.\end{aligned}\tag{6.11.3}$$

□

Example 6.12. If $n=1$, or if B is empty (which occurs only for $g=0$ and $N=2$), (6.10.1) reduces to 1.

Proof. If $n=1$, only the term $(a,b)=(1,1)$ occurs in (6.10.1), and all its factors are 1.

If B is empty, the polynomials T'_m are 1, and (6.10.1) reduces to

$$\sum_{ab|n}\mu(a)\frac{Q^b-1}{Q^n-1}=\sum_{b|n}\frac{Q^b-1}{Q^n-1}\sum_{a|(n/b)}\mu(a).$$

The sum over a vanishes, except when $n=b$ and the claim follows. □

Proposition 6.13. $R_0=1$.

Proof. The polynomial R_0 is obtained by taking $Q=0$ in (6.10.1). At $Q=0$, the quotient $(Q^b-1)/(Q^n-1)$ is one, and (6.10.1) reduces to

$$\sum_{m|n}T'_m(Q;(X_i))\cdot\sum_{a|m}\mu(a).$$

Except for $m=1$, the sum over a vanishes, and at $Q=0$, $T'_1(Q,(X_i))$ is one. □

Proposition 6.14. (i) *In the following two cases,* (6.10.1) *is reduced to* $1:|B|=0$ (*that is* $g=0$, $N=2$); $|B|=1$ (*that is* $g=0$, $N=3$) *and* $n=2$.

(ii) *Let d be the largest of the integers k such that $R_k\neq 0$. Excluding the cases* (i), *one has*

$$d=|B|\,\frac{n(n-1)}{2}+1-n,\tag{6.14.1}$$

$$R_d=(-1)^{N(n-1)}\Big(\prod X_i\Big)^{n-1}.\tag{6.14.2}$$

(iii) *Each R_k is a linear combination of monomials dividing* $\Big(\prod X_i\Big)^{n-1}$.

Proof. The cases (i) are covered by 6.11 and 6.12. To compute d and R_d, we will expand (6.10.1) around $Q=\infty$. This means embedding the ring of polynomials in Q and the X_i in the ring of Laurent formal power series in Q^{-1}, with cofficients polynomials in the X_i, that is in $\mathbb{Z}[(X_i)][\![Q^{-1}]\!][Q]$, and computing there, using that

$$\frac{Q^b-1}{Q^n-1} = Q^{b-n}\frac{1-Q^{-b}}{1-Q^{-n}} = Q^{b-n}(1-Q^{-b})\sum_{r\geq 0} Q^{-nr}. \tag{6.14.3}$$

This expansion shows that to prove (iii), it suffices to prove that for each divisor m of n, when one expands the product (6.6.2) defining $T'_m(Q;(X_i))$, the monomials occuring are of the form

$$(\text{power of } Q)\cdot\Big(\text{monomial dividing}\Big(\prod X_i\Big)^{n-1}\Big).$$

The largest power of X_i occuring is indeed

$$(m-1)+m\left(\frac{n}{m}-1\right)=n-1.$$

Expanding (6.6.2), one sees that the highest power with which Q occurs in $T'_m(Q;(X_i))$ is

$$|B|\cdot m\cdot \sum_{1\leq j<\frac{n}{m}} j = |B|\cdot m\cdot\frac{n}{m}\cdot\Big(\frac{n}{m}-1\Big)/2 = |B|\cdot n\cdot\Big(\frac{n}{m}-1\Big)/2.$$

In $\mu(a)\frac{Q^b-1}{Q^n-1}T'_m(Q;(X_i))$, it is the same plus $b-n$, which at a given m is maximum for $b=m$. This maximum is

$$|B|\cdot n\Big(\frac{n}{m}-1\Big)/2+m-n = \Big(|B|\frac{n^2}{2}\cdot\frac{1}{m}+m\Big)-\Big(|B|\frac{n}{2}+n\Big). \tag{6.14.4}$$

Define $A:=|B|\frac{n^2}{2}$. For $x\geq 0$, the function $\frac{A}{x}+x$ decreases from $x=0$ to its minimum at $x=\sqrt{A}=\Big(\frac{|B|}{2}\Big)^{1/2}\cdot n$. If $|B|\geq 2$, the divisor m of n at which $\frac{A}{m}+m$ takes its largest value is hence $m=1$, while for $|B|=1$ it is 1 or n. For $|B|=1$, the values of $\frac{A}{m}+m$ for $m=1$ and n are

$$\frac{n^2}{2}+1 \qquad \text{and} \qquad \frac{3}{2}n.$$

When $|B|=1$, we assumed that $n\geq 3$, and the value at $m=1$ is again the largest.

We conclude that in (6.10.1) only one term contributes the largest power of Q: the term $(a,b)=(1,1)$, and 6.14 is now easily checked. □

The product of the β in B is the determinant of Frob acting on $H^1_c(X-S)$. By (6.2.2), this determinant is the product of the following two factors: $\varepsilon(S)$, the determinant of Frob acting on $\mathbb{Z}^S$, that is, the signature of the permutation Frob of S, and the determinant of Frob acting on $H^1(X)$, equal to q^g:

$$\prod_{\beta\in B}\beta = \varepsilon(S)q^g. \tag{6.14.5}$$

Corollary 6.15. *If we exclude the cases* 6.14(i), *in the decomposition* (6.8.1) *of the second factor of* (6.6.1) *for* $(X_m, S_m)/\mathbb{F}_{q^m}$, *the eigenvalue* $\varepsilon(S)^{n-1}q^{D'/2}$, *with*

$$D' = (2g-2)(n^2-1) + N(n^2-n),$$

occurs with multiplicity $(-1)^{N(n-1)}$. *The other eigenvalues occuring with nonzero multiplicity have strictly smaller complex absolute values.*

Proof. One applies the description of the eigenvalues given in the proof of 6.8, applied to (6.10.1). The β in B have complex absolute value 1 or $q^{1/2}$, and by 6.14 the top eigenvalue is

$$\Big(\prod \beta_i\Big)^{n-1} q^d,$$

occuring with multiplicity $(-1)^{N(n-1)}$. By (6.14.5), it equals

$$\varepsilon(S)^{n-1} q^{g(n-1)+d},$$

and

$$\begin{aligned} g(n-1)+d &= [(g-1)(n-1)+(n-1)] + [(2g-2+N)\frac{n(n-1)}{2} - (n-1)] \\ &= (g-1)((n-1)+n(n-1)) + N\frac{n(n-1)}{2} \\ &= \frac{1}{2}[(2g-2)(n^2-1)+N(n^2-n)]. \end{aligned}$$

□

From now on, we again assume that $n \geq 2$.

6.16. Over $\mathbb{C}$, let us consider a compact Riemann surface $\sum$ of genus g, a set S of N points of Σ, the inclusion j of $\Sigma - S$ in Σ, and, on $\Sigma - S$, irreducible complex local systems of rank n, with trivial determinant and–at each s in S–with local monodromy which is unipotent with one Jordan block (principal unipotent). Let $\mathcal{M}$ be the moduli space of these local systems. In the cases 6.14(i) it is empty. We exclude these cases.

The deformation theory of local systems V as above is controlled by $H^*(\Sigma, j_*\mathcal{E}nd^0(V))$, where $\mathcal{E}nd^0(V)$ is the local system of trace zero endomorphisms of V. The local system $\mathcal{E}nd^0(V)$ is self dual, for the pairing $\mathrm{Tr}(uv)$. From this pairing we get a perfect pairing between H^0 and the H^2 of $j_*\mathcal{E}nd^0(V)$, and a symplectic form on the H^1. By Schur's lemma, the H^0 vanishes. By duality, so does H^2: the deformation theory is unobstructed, and $\mathcal{M}$ is smooth. Its tangent space at V is $H^1(\Sigma, j_*\mathcal{E}nd^0(V))$, and the autoduality of this H^1 turns $\mathcal{M}$ into a complex symplectic manifold, of dimension

$$\dim \mathcal{M} = \dim H^1(\Sigma, j_*\mathcal{E}nd^0(V)) = -\chi(j_*\mathcal{E}nd^0(V)).$$

The local system $\mathcal{E}nd^0(V)$ on $\Sigma - S$ is of rank n^2-1, and the fiber of $j_*\mathcal{E}nd^0(V)$ at each $s \in S$ is of rank $n-1$ (it is the centralizer of the local monodromy in a nearby fiber of $\mathcal{E}nd(V)^0$). By additivity of Euler-Poincaré, this gives

$$\begin{aligned}\chi(j_*\mathcal{E}nd^0 V) &= \chi(\Sigma - S)\cdot(n^2-1) + N(n-1) \\ &= \chi(\Sigma)(n^2-1) = N(n^2-n) = -(2g-2)(n^2-1) - N(n^2-n),\end{aligned}$$

showing that the complex dimension of $\mathcal{M}$ is D'. We do not understand why q at the power half the dimension of $\mathcal{M}$ appears in 6.15.

6.17. Let us write $R_k(B)$ for the value of the polynomial R_k at a point (x_i) where the x_i run over the multiset B. By (2.3.1), (6.6.1) and the definition 6.10 of the R_k, one has

$$T(X_1, S_1, n) = (-1)^{N_1(n-1)}[-f(1) + f(1)\sum_k q^k R_k(B)]. \tag{6.17.1}$$

By 6.13, the $k=0$ term in the sum cancels $-f(1)$. If a permutation σ of N letters has N_1 cycles, its signature is $(-1)^{N-N_1}$. Plugging this in (6.17.1), we get

Theorem 6.18 (second form of theorem 2.3)**.** *One has*

$$T(X_1, S_1, n) = (-1)^{N(n-1)}\varepsilon(S)^{n-1} f(1)\sum_{k\geq 1} q^k R_k(B). \tag{6.18.1}$$

In the exceptional cases 6.14(i), the right side vanishes. Otherwise, the sum over k ranges from 1 to d (6.14.1), and $d \geq 1$.

When we go from $(X_1, S_1)/\mathbb{F}_q$ to $(X_m, S_m)/\mathbb{F}_{q^m}$, the sign $\varepsilon(S) = \det(\mathrm{Frob}, \mathbb{Z}^S)$ is replaced by its m^{th} power. As a function of m, all factors $\varepsilon(S)$, $f(1) = \prod(1-\alpha)$, $\sum\limits_{k\geq 1} q^k R_k(B)$ of (6.18.1), and hence $T(X_m, S_m, n)$ have the form (6.8.1): a sum $\sum n_j \gamma_j^m$, reminiscent of a Lefschetz trace formula. If we exclude the cases 6.14(i), the top eigenvalue for $m \mapsto T(X_m, S_m, n)$ is the product of that for $m \mapsto f_m(1)$, that is q^g, of $\varepsilon(S)^{n-1}$, and of the top eigenvalue computed in 6.15, that is $\varepsilon(S)^{n-1}q^{D'/2}$. Its multiplicity is $(-1)^{N(n-1)}$ times the multiplicity $(-1)^{N(n-1)}$ of 6.15: it is one.

Corollary 6.19. (i) *As a function of m, the number $T(X_1, S_1, n, m) = T(X_m, S_m, n)$ of fixed points of* Fr^{m*} *has the form explained in* (6.8.1) :

$$\sum n_i \gamma_i^m. \tag{6.19.1}$$

The γ_i are products of a strictly positive power of q, of a root of unity, and of eigenvalues of Frob *acting on $H^1(X)$.*

(ii) *If $g = 0$, $N = 2$ or if $g = 0$, $N = 3$, $n = 2$, $T(X_1, S_1, n, m) = 0$. Otherwise, the eigenvalue $q^{D/2}$, with $D = D' + 2g = (2g-2)(n^2-1) + N(n^2-n) + 2g$, occurs in* (6.19.1) *with multiplicity one, and the other eigenvalues occuring with nonzero multiplicity have strictly smaller complex absolute values.* □

6.20. In parallel to 6.16, and excluding the cases 6.14(i), D is the dimension of the space of irreducible complex local systems of dimension n and principal unipotent local monodromy at each point of S.

A similar phenomenon occurs for $n = 2$ in the case of no ramification, studied by Drinfeld [D], and in cases of fixed and "generic" tame local ramifications, studied by Arinkin.

Corollary 6.21. *The number $T(X_1, S_1, n)$ is divisible by q.* □

6.22. Somewhat abusively, we will write "virtual object of $\mathcal{A}$" to mean "element of the Grothendieck group of the abelian category $\mathcal{A}$". The polynomial R_k, being symmetric and with integral coefficients, is the character of a virtual polynomial representation of $\mathrm{GL}(2g + N - 2)$, evaluated at the diagonal matrix with diagonal entries the X_i. We write R_k also for this virtual representation. It is a difference of two representations, say R_k^+ and R_k^-.

Example 6.23. The elementary symmetric polynomial σ_j is the character of $\overset{j}{\wedge} V$, for V the defining representation of the linear group. For $n = 2$ and $k \geq 1$, 6.11 tells that the virtual representation R_k is

$$R_k = (-1)^{k+1} \sum_{j \geq k+1} \overset{j}{\wedge} V \qquad (\text{for } n = 2,\ k \geq 1). \tag{6.23.1}$$

6.24. Representations of the linear group $\mathrm{GL}(M)$ can be identified (equivalence of categories) with functors from the category of vector spaces of dimension M, with isomorphisms as morphisms, to the category of vector spaces and linear maps. For instance, to the representation $\overset{j}{\wedge} V$, for V the defining representation of $\mathrm{GL}(M)$, corresponds the functor j-th exterior power. The equivalence is (functor T)$\mapsto$(representation $T(V)$). Such a functor T can be applied to smooth l-adic sheaves of rank M.

6.25. Let $H_c - 1$ be the quotient of $H_c^1(X - S)$ by a line fixed by Frob. The assumption $N_1 \geq 2$ ensures there is one. The eigenvalues of Frob acting on $H_c - 1$ are the β in B. It follows that $R_k(B)$ of 6.17 is

(6.25.1)
$$\begin{aligned} R_k(B) &= \mathrm{Tr}(\mathrm{Frob}, R_k(H_c - 1)) \\ &:= \mathrm{Tr}(\mathrm{Frob}, R_k^+(H_c - 1)) - \mathrm{Tr}(\mathrm{Frob}, R_k^-(H_c - 1)). \end{aligned}$$

The other pieces in (6.18.1) have a similar interpretation: q is the eigenvalue of Frob acting on $\mathbb{Q}_l(-1) = H^1(\mathbb{G}_m)$, $\varepsilon(S)$ is the eigenvalue of Frob acting on the top exterior power of $\mathbb{Q}_l^S$, and

$$f(1) = \sum\nolimits_i (-1)^i \mathrm{Tr}(\mathrm{Frob}, \overset{i}{\wedge} H^1(X)).$$

Let us write $W(-k)$ for $W \otimes \mathbb{Q}_l(-1)^{\otimes k}$, and $\varepsilon_l(S)$ for the top exterior power of $\mathbb{Q}_l^S$. We get

(6.25.2)
$$\begin{aligned} &T(X_1, S_1, n) \\ &= \mathrm{Tr}\left(\mathrm{Frob}, (-1)^{N(n-1)} \varepsilon_l(S)^{\otimes(n-1)} \otimes \sum\nolimits_i (-1)^i \overset{i}{\wedge} H^1(X) \otimes \sum_{k=1}^{d} R_k(H_c - 1)(-k)\right). \end{aligned}$$

Example 6.26. If $n = 2$, (6.23.1) gives

$$R_k(B) = (-1)^{k+1} \mathrm{Tr}\left(\mathrm{Frob}, \sum_{j \geq k+1} \overset{j}{\wedge}(H_c - 1)\right) \qquad (\text{for } k \geq 1).$$

Define $H_c := H_c^1(X - S)$. As $\overset{j+1}{\wedge} H_c$ is an extension of $\overset{j+1}{\wedge}(H_c - 1)$ by $\overset{j}{\wedge}(H_c - 1)$, one also has

(6.26.1)
$$R_k(B) = (-1)^{k+1} \mathrm{Tr}\left(\mathrm{Frob}, \sum_{j \geq 1} \overset{k+2j}{\wedge} H_c\right) \qquad (\text{for } n = 2,\ k \geq 1)$$

from which it follows that, under our standing assumption $N_1 \geq 2$,

(6.26.2) $T(X_1, S_1, 2) =$
$$\mathrm{Tr}\left(\mathrm{Frob}, (-1)^N \varepsilon_l(S) \otimes \sum\nolimits_i (-1)^i \overset{i}{\wedge} H^1(X) \otimes \sum_{k \geq 1} (-1)^{k+1} \sum_{j \geq 1} \left(\overset{k+2j}{\wedge} H_c\right)(-k)\right).$$

6.27. Let $\mathcal{M}_{g,[N]}$ be the moduli stack, over $\mathrm{Spec}(\mathbb{Z})$, of curves of genus g given with a set S of N distinct points. More precisely, a morphism $Y \to \mathcal{M}_{g,[N]}$, or equivalently an object of $\mathcal{M}_{g,[N]}$ over the scheme Y, is a proper and smooth morphism $a\colon X \to Y$ whose geometric fibers are irreducible curves of genus g, given with a relative divisor $S \subset X$, finite étale of degree N over Y.

The moduli stack $\mathcal{M}_{g,N}$ of curves of genus g given with an ordered set of N distinct points is a Galois covering of $\mathcal{M}_{g,[N]}$, with Galois group the symmetric group S_N. More precisely, it is an S_N-torsor over $\mathcal{M}_{g,[N]}$.

Fix a decomposition $N = N' + N''$, with $N', N'' \geq 1$. Corresponding to the subgroup $S_{N'} \times S_{N''}$ of S_N, we have between $\mathcal{M}_{g,N}$ and $\mathcal{M}_{g,[N]}$ the moduli stack $\mathcal{M}_{g,[N',N'']}$ of curves of genus g given with disjoint sets of N' and N'' distinct points. An object of $\mathcal{M}_{g,[N',N'']}$ over Y is $a\colon X \to Y$ as above, given with disjoint relative divisors S', S'', finite étale of degrees N', N'' over Y. We put $S := S' \cup S''$.

For each such stack $\mathcal{M}$, we denote by $\mathcal{M}[1/l]$ the open substack where l is invertible.

A $\mathbb{Q}_l$-*smooth sheaf on a stack* $\mathcal{M}$ is the data, for each $Y \to \mathcal{M}$, of a $\mathbb{Q}_l$-smooth sheaf whose formation is compatible with pullbacks by maps $Y' \to Y$. Here are examples of $\mathbb{Q}_l$-smooth sheaves on $\mathcal{M}_{g,[N]}[1/l]$, defined by giving their value on $a\colon (X,S) \to Y$, object of the stack over Y. By pullback, they give similar smooth sheaves on $\mathcal{M}_{g,[N',N'']}[1/l]$.

- $\mathcal{H}\colon R^1(a\colon X \to Y)_*\mathbb{Q}_l$. It is of rank $2g$.
- $\mathcal{S}\colon a(S \to Y)_*\mathbb{Q}_l$. It is of rank N.
- $\varepsilon_l(\mathcal{S}) := \overset{N}{\wedge}\mathcal{S}$, of rank one.
- $\mathcal{H}_c\colon R^1(a\colon X - S \to Y)_!\mathbb{Q}_l$. It is of rank $2g + N - 1$ and sits in an exact sequence whose geometric fibers are given by (6.2.2):

$$0 \to \mathbb{Q}_l \to \mathcal{S} \to \mathcal{H}_c \to \mathcal{H} \to 0. \tag{6.27.1}$$

On $\mathcal{M}_{g,[N',N'']}[1/l]$, $\mathcal{H}_c$ contains a copy of the constant sheaf $\mathbb{Q}_l$. Indeed, $\mathcal{S}$ decomposes into the sum of $\mathcal{S}'\colon (a\colon S' \to Y)_*\mathbb{Q}_l$ and $\mathcal{S}''\colon (a\colon S'' \to Y)_*\mathbb{Q}_l$, the map $\mathbb{Q}_l \to a_*a^*\mathbb{Q}_l$ embeds $\mathbb{Q}_l$ into both $\mathcal{S}'$ and $\mathcal{S}''$, and (6.27.1) embeds $\mathbb{Q}_l \simeq \mathbb{Q}_l \oplus \mathbb{Q}_l/(\text{diagonal } \mathbb{Q}_l)$ into $\mathcal{H}_c$. On $\mathcal{M}_{g,[N',N'']}[1/l]$, we define $\mathcal{H}_c/\mathbb{Q}_l$ to be the quotient of $\mathcal{H}_c$ by this copy of $\mathbb{Q}_l$.

If $\mathcal{W}$ is a $\mathbb{Q}_l$-smooth sheaf, or more generally a $\mathbb{Q}_l$-sheaf, on a stack $\mathcal{M}$, if $x\colon \mathrm{Spec}(\mathbb{F}_q) \to \mathcal{M}$ is an $\mathbb{F}_q$-point of $\mathcal{M}$, and $\bar{x}\colon \mathrm{Spec}(\mathbb{F}) \to \mathrm{Spec}(\mathbb{F}_q) \to \mathcal{M}$ a corresponding geometric point, the geometric Frobenius $\mathrm{Frob} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts on the fiber of $\mathcal{W}$ at $\bar{x}$. One defines

$$\mathrm{Tr}(\mathrm{Frob}_x, \mathcal{W}) := \mathrm{Tr}(\mathrm{Frob}, \mathcal{W}_{\bar{x}}).$$

This definition extends by additivity to the case of virtual smooth sheaves, that is of elements of the Grothendieck group of the category of $\mathbb{Q}_l$-smooth sheaves.

The formula (6.25.2) can now be translated as follows.

Proposition 6.28. *Let x be an $\mathbb{F}_q$-point of $\mathcal{M}_{g,[N',N'']}[1/l]$, that is (X_1, S_1', S_1'') over $\mathbb{F}_q$, and let $S_1 := S_1' \cup S_1''$. Then $T(X_1, S_1, n)$ is the trace of* Frob_x *on the virtual smooth sheaf*

$$(-1)^{N(n-1)}\varepsilon_l(\mathcal{S})^{\otimes(n-1)} \otimes \sum\nolimits_i (-1)^i \overset{i}{\wedge}\mathcal{H} \otimes \sum_{k=1}^{d} R_k(\mathcal{H}_c/\mathbb{Q}_l)(-k). \tag{6.28.1}$$

Conjecture 6.29. *Let us drop our standing assumption $N_1 \geq 2$. We conjecture that for any $g \geq 0$, $N \geq 0$ and $n \geq 2$, there exists a virtual $\mathbb{Q}_l$-smooth sheaf $\mathcal{W}^{(n)}$ on $\mathcal{M}_{g,[N]}[1/l]$ such that for x an $\mathbb{F}_q$-point of $\mathcal{M}_{g,[N]}[1/l]$, that is (X_1, S_1) over $\mathbb{F}_q$, one has*

$$T(X_1, S_1, n) = \mathrm{Tr}(\mathrm{Frob}_x, \mathcal{W}^{(n)}).$$

This implies a dependence on m of the form (6.8.1) for $T(X_m, S_m, n)$. An optimistic version of the conjecture would be that for some virtual polynomial representations $R_k^{(n)}$ of $\mathrm{GL}(2g + N - 1)$, $\mathcal{W}^{(n)}$ is of the form

$$\mathcal{W}^{(n)} = \varepsilon_l(\mathcal{S})^{\otimes(n-1)} \otimes \sum\nolimits_i (-1)^i \overset{i}{\wedge}\mathcal{H} \otimes \sum\nolimits_k R_k^{(n)}(\mathcal{H}_c)(-k).$$

If $N > 0$, one might also hope that the sum over k is over positive k's.

For $n = 2$ and $N = 0$, the conjecture (in its optimistic form) is a corollary of what Drinfeld proves in [D].

For $n = 2$ and $N \geq 1$, one of us (Y.F.) has checked, using the trace formula in its full gory, that (6.26.2) continues to hold for $N_1 = 1$. From this the optimistic version of the conjecture (for $n = 2$ and $N \geq 1$) readily follows.

For $N \geq 2$, and $N = N' + N''$ with $N', N'' \geq 1$, the conjectural $\mathcal{W}^{(n)}$ on $\mathcal{M}_{g,[N]}$ would have (6.28.1) as inverse image on $\mathcal{M}_{g,[N',N'']}$. For $n = 2$, this made (6.26.2) a plausible guess. One should, however, beware that there are virtual $\mathbb{Q}_l$-smooth sheaves on $\mathcal{M}_{g,[N]}$ all of whose inverse images to the $\mathcal{M}_{g,[N',N'']}$ ($N = N' + N''$, $N', N'' \geq 1$) are zero. Examples are $\sum(-1)^i \overset{i}{\wedge}\mathcal{H}_c$ and $\sum(-1)^i \overset{i}{\wedge}(\mathcal{S}/\mathbb{Q}_l)$. The latter boils down to the following fact. Given a finite group H, with representation ring $R(H)$, and a family of subgroups H_j, if $\cup H_j$ does not meet all conjugacy classes in H, the restriction map

$$R(H) \to \prod R(H_j)$$

is not injective. In the case of S_N, the family of subgroups $S_{N'} \times S_{N''}$ ($N = N' + N''$, $N', N'' \geq 1$) misses the conjugacy class of cycles of length N. If V is the representation

$\mathbb{C}^N/(\text{diagonal } \mathbb{C})$ of S_N, its restriction to $S_{N'} \times S_{N''}$ contains a copy of the trivial representation 1, and as

$$\overset{j}{\wedge}(W+1) = \overset{j}{\wedge} W \oplus \overset{j-1}{\wedge} W$$

(with $\overset{-1}{\wedge} W := 0$), the restriction of $\sum(-1)^j \overset{j}{\wedge} V$ to each $S_{N'} \times S_{N''}$ vanishes. The character of $\sum(-1)^j \overset{j}{\wedge} V$ vanishes outside of the conjugacy class of cycles of length N, where its value is N.

6.30. The conjecture 6.29 should not be specific to the case of principal unipotent local monodromy. It should hold, with other virtual $\mathbb{Q}_l$-smooth sheaves $\mathcal{W}$, when one imposes the local monodromy at each s in S. When the local monodromy imposed is tame (and makes sense in characteristic not dividing P), the relevant moduli stack $\mathcal{M}$ will be over $\mathbb{Z}[1/lP]$. It is the moduli stack of curves of genus g given with a set of N points decorated by the imposed monodromy. When the local monodromy imposed is not tame, one might have to stay in a specific characteristic, and consider points decorated by a local parameter given up to some order.

Question 6.31. The virtual $\mathbb{Q}_l$-smooth sheaf (6.28.1) is the formal difference of local systems built out of the local systems of cohomology of X, S' and S''. We used l-adic cohomology, but (6.28.1) would make sense for any of the standard cohomology theories. It is "motivic". Over $\mathbb{C}$, the same construction gives a virtual variation of Hodge structures on $\mathcal{M}_{g,[N',N'']}(\mathbb{C})$.

Over $\mathbb{C}$, to each curve X of genus g given with a set $S = S' \cup S''$ of $N = N' + N''$ points, one can attach the moduli space $\mathcal{T}_{X,S}$ of rank n irreducible complex local systems on $X - S$, with principal unipotent local monodromy at each s in S.

These spaces are the fibers of a morphism $a\colon \mathcal{T} \to \mathcal{M}_{g,[N',N'']}$. Can one relate the virtual variation (6.28.1) and this family of spaces? From the complex analytic point of view, this family is locally constant, because it can be interpreted in terms of representations of the fundamental group of $X - S$, itself locally constant. This should be related to the fact that $T(X_1, S_1, n)$ is controlled by a virtual $\mathbb{Q}_l$-smooth sheaf, rather than a virtual $\mathbb{Q}_l$-sheaf.

The analogue of 6.19(ii) is that the top weight part of the variation (6.28.1) is a $\mathbb{Q}(-D/2)$, of Hodge type $(D/2, D/2)$, where D is the (complex) dimension of the $\mathcal{T}_{X,S}$.

7. Example: $g = 0,\ N = 4,\ n = 2$

The case of rank 2 local systems over the projective line minus an étale divisor of degree four is among the simplest nontrivial cases. It has been investigated numerically by Kontsevich (cf. [Kn], 0.1). It might be a useful testing ground to tentative answers to question 6.31.

In this section, we do not assume $N_1 \geq 2$: X_1 is a projective line over $\mathbb{F}_q$ and the degree four reduced divisor S_1 is allowed to consist of one closed point of degree four.

Proposition 7.1. *With the above notations, one has*

$$T(X_1, S_1, 2) = q. \tag{7.1.1}$$

As $g = 0$, one has $f(1) = 1$. When $N_1 \geq 2$,

$$D = D' = (2g-2)(n^2-1) + N(n^2-n) = 2$$

and the dominant term q computed in 6.19 (ii) is the only term by 6.19 (i). This leaves out the case where S_1 consists of a single point of degree 4. We will explain in 7.8 how this case can be reduced to the case $N_1 \geq 2$.

7.2. Let E be a set with four elements. We denote by V_E the Vierergruppe of E, that is the subgroup of the symmetric group of E consisting of the identity and of the three $(2,2)$ permutations of E. The action of V_E on E is simply transitive: when viewed as a right action, it turns E to a V_E-torsor. When we need to emphasize this, we write $t(E)$ for E. The abelian group V_E being killed by 2, $t(E)$ is its own opposite: the torsor sum $t(E)+t(E)$ is the trivial V_E-torsor V_E. As V_E acts on E, the twist $E^{t(E)}$ of E by $t(E)$ is defined. It is just $t(E) + t(E) = V_E$. It has a canonical point 0.

7.3. Let P be a projective line over a field k. If (s_1, s_2, s_3, s_4) and (s'_1, s'_2, s'_3, s'_4) are two quadruples of distinct k-points of P with the same cross ratio, there is a unique automorphism of P over k mapping the first quadruple to the second. Special case: the cross ratio being invariant by a permutation of a quadruple of points which belongs to the Vierergruppe, if $S \subset P(k)$ consists of four points, the action of V_S on S extends (uniquely) to an action on P. A twist $P^{t(S)}$ of P by the torsor $t(S)$ is hence defined. It contains the twist $S^{t(S)} = V_S$ of S by $t(S)$. Concrete description: one takes four copies $P^{[s]}$ of P indexed by S; one has a transitive system of isomorphisms between the $P^{[s]}$: as isomorphism from $P^{[s]}$ to $P^{[t]}$, one takes the action of the unique element of V_S mapping s to t; the twist $P^{t(S)}$ is the "common value" (projective limit) of the $P^{[s]}$. In other words,

$$P^{t(S)} = (P \times S)/V_S$$

(diagonal action). The canonical point $0 \in V_S = S^{t(S)} \subset P^{t(S)}$ is the image of the diagonal S of $S \times S \subset P \times S$.

By étale descent, this construction continues to make sense for P a projective and smooth curve of genus 0 over k, and for S a divisor of degree 4, étale over k. The group V_S is now a group scheme étale over k. It acts on S, on P, on $P^{t(S)} = (P \times S)/V_S$ and $0 \in V_S = S^{t(S)} \subset P^{t(S)}$ is again the image of the diagonal of $S \times S$.

7.4. We now suppose that $k = \mathbb{C}$. Let $\mathcal{T}((P,S)/\mathbb{C})$ denote the set of isomorphism classes of irreducible rank 2 complex local systems on $P - S$, with principal unipotent local monodromy at each s in S (6.16). The group V_S acts on this set by transport of structures.

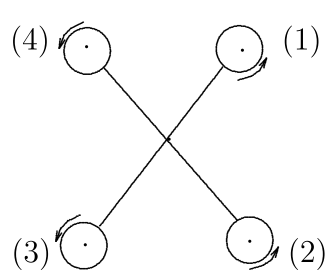
Proposition 7.5. *The action of V_S on $\mathcal{T}((P,S)/\mathbb{C})$ is trivial.*

Fix an involution σ in V_S, and label S by $\mathbb{Z}/4$, in such a way that σ is $s_i \mapsto s_{i+2}$. Proposition 7.5 results from the more general

Proposition 7.6. *Let $\mathcal{V}$ be an irreducible rank 2 complex local system on $P - S$. Assume that the local monodromy transformations are in $\mathrm{SL}(2)$, and that for each $s \in S$ the local monodromies around s and $\sigma(s)$ are conjugate. Then, $\mathcal{V}$ is isomorphic to $\sigma^*\mathcal{V}$.*

Proof. All involutions in $\mathrm{Aut}(P)$ are conjugate: we may and shall choose a coordinate z such that the automorphism σ of P is $z \mapsto -z$. The claim is invariant under deformation of S. We may assume that S consists of the points $\pm 1 \pm i$, labelled by $\mathbb{Z}/4$ as in the following picture.

(7.6.1)



Take the fixed point $z = 0$ of σ as base point. The fundamental group $\pi_1(P - S, 0)$ is generated by the loops γ_i ($i \in \mathbb{Z}/4$) pictured in (7.6.1), with $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$ as the only relation, and σ maps γ_i to γ_{i+2}.

To give a local system $\mathcal{V}$ on $P - S$ amounts to giving its fiber $\mathcal{V}_0$ at 0, and the action of $\pi_1(P - S, 0)$ on it. Let A_i be the image of γ_i. For our $\mathcal{V}$, if we choose an isomorphism of $\mathcal{V}_0$ with $\mathbb{C}^2$, the representation of π_1 is given by four A_i in $\mathrm{SL}_2(\mathbb{C})$, obeying

$$A_1A_2A_3A_4 = 1. \tag{7.6.2}$$

Our assumptions are that the representation is irreducible and that A_i is conjugate to A_{i+2}. Our claim is that the quadruples (A_1, A_2, A_3, A_4) and (A_3, A_4, A_1, A_2) are conjugate, in other words that the corresponding representations of π_1 are isomorphic. For this, it suffices (by irreducibility) to check that they have the same character: that for any word $\prod A_{i(k)}^{\varepsilon(k)}$ in the $A_i^{\pm}$,

$$\mathrm{Tr} \prod A_{i(k)}^{\varepsilon(k)} = \mathrm{Tr} \prod A_{i(k)+2}^{\varepsilon(k)}. \tag{7.6.3}$$

We are in rank $n = 2$. By Procesi [P], Theorem 3.4 (a) p.316, applied to the A_i and their inverses, it suffices to check (7.6.3) for words of length $\leq 2^n - 1 = 3$. Words can be viewed as circular words. By the identities $A_i^{-1} = \mathrm{Tr}(A_i) - A_i$ and $A_i^2 = \mathrm{Tr}(A_i)A_i - 1$, it suffices to consider words in the A_i, for which consecutive A_i have distinct indices. For circular words of length ≤ 3, this means all indices distinct. We now check (7.6.3) for these words. We will use that in $\mathrm{SL}(2)$, $\mathrm{Tr}(A) = \mathrm{Tr}(A^{-1})$.

We assumed $\mathrm{Tr}(A_i) = \mathrm{Tr}(A_{i+2})$. That $\mathrm{Tr}(A_iA_{i+2}) = \mathrm{Tr}(A_{i+2}A_i)$ is clear. By (7.6.2), one has

$$\mathrm{Tr}(A_3A_4) = \mathrm{Tr}((A_1A_2)^{-1}) = \mathrm{Tr}(A_1A_2)$$

and similarly for A_iA_{i+1}. The case of $A_1A_2A_3$ is reduced to that of A_4 by

$$\mathrm{Tr}(A_1A_2A_3) = \mathrm{Tr}(A_4^{-1}) = \mathrm{Tr}(A_4).$$

The case of $\mathrm{Tr}(A_1A_3A_2)$ follows, thanks to the identity

$$\begin{aligned} \mathrm{Tr}(ABC) + \mathrm{Tr}(ACB) - \mathrm{Tr}(A)\mathrm{Tr}(BC) - \mathrm{Tr}(B)\mathrm{Tr}(CA) \\ - \mathrm{Tr}(C)\mathrm{Tr}(AB) + \mathrm{Tr}(A)\mathrm{Tr}(B)\mathrm{Tr}(C) = 0, \end{aligned} \tag{7.6.4}$$

which follows from the vanishing of the antisymmetrization operator $\mathbf{a} = \sum \varepsilon(\tau)\tau$ of $(\mathbb{C}^2)^{\otimes 3}$: one expands

$$\mathrm{Tr}(A \otimes B \otimes C \circ \mathbf{a}) = 0.$$

Similarly for $(1, 2, 3)$ replaced by $(i, i + 1, i + 2)$. This concludes the required check. $\square$

Suppose now that k is any algebraically closed field, and consider $\overline{\mathbb{Q}}_l$-smooth sheaves in the étale sense of 1.1.

Corollary 7.7. *For $(P, S)/k$ as above,* 7.6 *remains valid provided the local monodromy is tame. As a consequence,* 7.5 *remains valid.*

Tameness is needed already to make sense of "conjugate local monodromy".

Proof for $k = \mathbb{C}$. The proof of 7.6 is algebraic, hence holds also for local systems of $\overline{\mathbb{Q}}_l$-vector spaces on $P(\mathbb{C}) - S$. It remains to observe that the functor

$$(\overline{\mathbb{Q}}_l\text{-smooth sheaves on } P - S) \to (\text{local systems of } \overline{\mathbb{Q}}_l\text{-vector spaces on } P(\mathbb{C}) - S)$$

is fully faithful. Indeed, both categories are 2-inductive limits of similar categories, with $\overline{\mathbb{Q}}_l$ replaced by a finite extension E_λ of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. This reduces us to the E_λ case. Let $\mathcal{O}_\lambda$ be the valuation ring of E_λ and fix a base point 0. Local systems on $P(\mathbb{C}) - S$ are representations of $\pi_1(P(\mathbb{C}) - S, 0)$ on an E_λ-vector space of finite dimension. The E_λ-smooth sheaves on $P - S$ are those representations V which extend to continuous representations of the profinite completion of π_1, that is for which V contains a lattice V^0 (a free $\mathcal{O}_\lambda$-submodule of V which generates V over E_λ) stable by the action. Morphisms are the same.

Proof in characteristic zero. This case follows from the case $k = \mathbb{C}$, by invariance of the algebraic π_1 upon extension of scalars from one algebraically closed field of characteristic zero to another one.

Proof in characteristic p. Grothendieck proved that the tame π_1 is a quotient of the characteristic 0 group π_1. We will show that his proof reduces the characteristic p case to the characteristic zero case. Let $W(k)$ be the ring of Witt vectors over k. Let $\overline{K}$ be an algebraic closure of the field of fractions K of $W(k)$. Consider a lifting (P_W, S_W) of (P, S) to $W(k)$. The action of V_S lifts. A $\overline{\mathbb{Q}}_l$-local system on P_S which is tamely ramified along S lifts uniquely to $P_W - S_W$, and pulls back to $(P_W, S_W) \otimes_W \overline{K}$. By Grothendieck, the resulting functor from tame local systems on $P - S$ to local systems on $P_{\overline{K}} - S_{\overline{K}}$ is fully faithful, reducing us to the characteristic zero case.

7.8. *End of Proof of* 7.1. Define (X_1', S_1') to be the twist of (X_1, S_1) by $t(S_1)$. Over $\mathbb{F}$, we have a natural system of four isomorphisms between (X, S) and (X', S'), exchanged by the Vierergruppe. By 7.5, they all induce the same bijection from $\mathcal{T}^{(2)}(X, S)$ to $\mathcal{T}^{(2)}(X', S')$. By transport of structure, this bijection is compatible with the action of Frob. It follows that

$$T(X_1, S_1, 2) = T(X_1', S_1', 2).$$

The divisor S_1' contains a rational point. For (X_1', S_1'), one hence has $N_1 \geq 2$, and (7.1.1) for (X_1, S_1) results from (7.1.1) for (X_1', S_1'). $\square$

Translating the generalization 7.7 of 7.5 using the global Langlands correspondence [L], one obtains the following result, of which we do not know a proof not using [L].

Proposition 7.9. *Let S_1 be an étale divisor of degree four of $\mathbb{P}^1/\mathbb{F}_q$, and σ an involutive automorphism of $(\mathbb{P}^1, S_1)$ which acts on S by an element of V_S. Suppose that the automorphic representation π of $\mathrm{GL}(2, \mathbb{A})$ is cuspidal, unramified outside of S_1, and that its local component at each s in S_1 is of the form*

$$\text{Steinberg } \otimes \chi(\det)$$

with χ unramified.

Under these assumptions, $\sigma(\pi)$ is an $\mathbb{F}_q$-twist of π : there exists a sign $\varepsilon = \pm 1$ such that $\sigma(\pi)$ is the twist of π by the character $a \mapsto \varepsilon^{\deg(a)}$ of the idèle class group. As a consequence, for any closed point $x \notin S_1$, the Hecke eigenvalue of π at x is $\varepsilon^{\deg(a)}$ times the Hecke eigenvalue of π at $\sigma(x)$.

The pair $(\mathbb{P}^1, S_1)$ admits an involution σ as in 7.9 as soon as S_1 does, that is, except in the case where S_1 contains a rational point and a point of degree 3.

Proof. Let $\mathcal{F}_1$ be the smooth l-adic sheaf on $(\mathbb{P}^1 - S_1)/\mathbb{F}_q$ attached to π. As π is cuspidal, $\mathcal{F}_1$ is irreducible. Its local monodromy at each $s \in S_1$ is principal unipotent. By 1.9(i), its inverse image $\mathcal{F}$ on $(\mathbb{P}^1 - S)/\mathbb{F}$ is still irreducible. By 7.7, $\sigma^*\mathcal{F}$ is isomorphic to $\mathcal{F}$. By 1.9(ii), $\sigma^*\mathcal{F}_1$ is an $\mathbb{F}_q$-twist of $\mathcal{F}_1$. It follows that $\sigma(\pi)$ is an $\mathbb{F}_q$-twist of π: for some character χ of the quotient $\mathbb{Z}$ of the idèle class group, $\sigma(\pi) = \pi\chi$. Applying σ, we get that $\pi = \sigma(\pi)\chi$, and $\pi = \pi\chi^2$. By 1.9(ii), $\chi^2 = 1$: χ is of the form $\varepsilon^{\deg(a)}$ for some sign ε. $\square$

1. APPENDIX. Transfer of special automorphic representations

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The purpose of this appendix is to extract the statement 11.3 from the literature.

The correspondence, relating discrete spectrum automorphic representations π' of any inner form G' of $G = \mathrm{GL}(n)$ (multiplicative group of a simple algebra of dimension n^2) with discrete spectrum representations of G, is known unconditionally when the base field is a number field F, by Arthur's work on the trace formula. The case where π and π' have a cuspidal ([BZ]) component at a place v where G'_v is $\mathrm{GL}(n, F_v)$ had been proven in [FK] (after previous work of [BDKV] and [F1;III] on π, π' with two such components), using the simple trace formula of [FK]. The latter method applies also when the base field F is a function field, but does not cover the case which we need, which concerns automorphic representations π' of $D^*(\mathbb{A})$, where D is a central division algebra over F, such that no component π'_v of π' corresponds to a cuspidal representation π_v of $G_v = \mathrm{GL}(n, F_v)$ by the local correspondence.

To establish the correspondence in the case stated in 1.13 we shall use the trace formula for $\mathrm{GL}(n)$ over a function field F as developed by Lafforgue [Laf], where the formula is proven for any inner form of $\mathrm{GL}(n)$. The case we need is where at two places $v = v_1$, v_2 (denoted 0 and ∞ in [Laf]) the test function $f = \otimes f_v$ (denoted h in [Laf]) has a discrete component f_v. A test (compactly supported locally constant) function f_v is called *discrete* if for every proper standard parabolic subgroup $P = MN$ of $G = \mathrm{GL}(n)$, with unipotent radical N and standard Levi subgroup M, it satisfies the identity

$$\int_{K_v} \int_{N_v} f_v(k_v^{-1} m_v n_v k_v) dn_v dk_v = 0$$

for every $m_v \in M_v$. A discrete pseudo coefficient of the Steinberg representation is constructed in [Lau], Theorem (5.1.3), p. 133, after previous work by Kottwitz. Replacing f_v by $g \mapsto \int_{K_v} f_v(k_v^{-1} g k_v) dk_v$, $K_v = \mathrm{GL}(n, \mathcal{O}_v)$, we may assume f_v satisfies $f_v(k_v^{-1} g k_v) = f_v(g)$ ($k_v \in K_v$, $g \in G_v$). For f with a discrete component f_v the truncation ([Laf], pp. 225-227) is trivial.

Theorem 10, p. 241, V.2.d, of [Laf] implies that for f with two discrete components the geometric side of the trace formula reduces to

$$\int_{G(F)\backslash G(\mathbb{A})/a^{\mathbb{Z}}} \sum_{\gamma} f(g^{-1}\gamma g) dg = \sum_{\{\gamma\}} \int_{Z_G(\gamma)(F)\backslash G(\mathbb{A})/a^{\mathbb{Z}}} \sum_{\gamma} f(g^{-1}\gamma g) dg$$

where the sum on the left ranges over the elements γ in $G(F)$ whose characteristic polynomial is a power of an irreducible polynomial, while the sum on the right ranges over a set of representatives for the conjugacy classes of such γ in $G(F)$, and $Z_G(\gamma)$ denotes the centralizer of γ in G. To simplify things we may choose f with a component f_{v_3} which vanishes on the singular set (the set of γ with at least two equal eigenvalues). As explained in the last paragraph below, this does not reduce the applicability of our techniques. Then the γ in the sum are elliptic regular (regular: distinct eigenvalues): $F[\gamma]$ is a separable field extension of F of degree n.

This sum is equal to the analogous sum for the inner form G', recorded in (2), p. 191, of [FK], for matching test functions $f = \otimes f_v$ on $G(\mathbb{A})$ and $f' = \otimes f'_v$ on $G'(\mathbb{A})$. In particular $f_v = f'_v$ at the v where $G_v \simeq G'_v$, and f_v, f'_v have matching orbital integrals (at all regular elements γ' in G'_v and the γ in G_v with the same characteristic polynomials; the orbital integral of f_v at the regular elements which do not come from G'_v in this sense are zero, for all v). Note that the usage of Theorem 10, p. 241, V.2.d, of [Laf] is made for convenience. The method of "n-admissible spherical functions" of [FK], p. 192, could be used too.

For a test function f with a cuspidal component (thus at some place v, for every proper parabolic subgroup $P_v = M_v N_v$ of G_v we have $\int_{N_v} f_v(xny)dn = 0$ for all x, $y \in G_v$), the convolution operator $r(f)$ splits through the projection to the cuspidal spectrum. The spectral side of the trace formula becomes the sum $\sum_\pi m(\pi)\operatorname{tr}\pi(f)$, where π ranges over the equivalence classes of the irreducible representations in the cuspidal spectrum of $G(\mathbb{A})$, and $m(\pi)$ denotes the multiplicity of π in the cuspidal spectrum ($m(\pi)$ is known to be 1 for $G = \mathrm{GL}(n)$). For a general test function f, which may not have a cuspidal component, one needs to use the spectral decomposition of the space of automorphic forms and compute the spectral side of the trace formula. This is done in [Laf], Theorem 12, p. 309, VI.2.f, in the case we need, namely $\mathrm{GL}(n)$ over a function field F.

In the number field case Arthur has shown ("a splitting property") that for a test function $f = \otimes f_v$ with two discrete components the spectral side reduces to a discrete sum $\sum_\pi m(\pi)\operatorname{tr}\pi(f)$, where π ranges over the equivalence classes of the irreducible representations π in the discrete spectrum of $G(\mathbb{A})$, and $m(\pi)$ is the multiplicity of π in the discrete spectrum. In the function field case this has not yet been done, so we proceed differently (and in fact deduce this result).

Fix a new place u ($\neq v_1$, v_2) of F. Let the component f_u be spherical (K_u-biinvariant). Denote by $f_u^\vee$ the Satake transform of f_u. Then the spectral side, described by Theorem 12, p. 309, VI.2.f, of [Laf], has the form

$$\sum_i c_i f_u^\vee(t_i) + \sum_j \int_{\tilde{T}_j} c_j(t) f_u^\vee(t) d_j t$$

where the c_i are complex numbers, the t_i lie in the compact Hausdorff space $\tilde{T}$ defined in the first lines of [FK], proof of Theorem 2, p. 197, the $\tilde{T}_j$ are compact submanifolds of $\tilde{T}$ (all irreducible components of a $\tilde{T}_j$ have the same dimension j ($1 \leq j < n$)), the $c_j(t)$ are complex valued functions on $\tilde{T}_j$ which are measurable with respect to a bounded measure $d_j t$ on $\tilde{T}_j$ which has the property that $\mathrm{vol}(\tilde{T}_\varepsilon(t))/\varepsilon$ is bounded uniformly in ε (see [FK], first paragraph in the proof of the Proposition, p. 198), and

$$\sum_i |c_i| + \sum_j \sup_{t\in\tilde{T}_j} |c_j(t)| + \sum_j \int_{\tilde{T}_j} |c_j(t)||d_j t|$$

is finite.

Now the trace formula asserts that the spectral side equals the geometric side of the trace formula. As we saw above, the geometric side is a sum of orbital integrals (for f with discrete f_{v_1}, f_{v_2}). For matching test functions f and f' the geometric sides of the trace formulae for f on $G(\mathbb{A}) = \mathrm{GL}(n,\mathbb{A})$ and for f' on $G'(\mathbb{A}) = D^*(\mathbb{A})$, are equal. Hence the spectral sides are equal. As is well known (see, e.g., [FK], Proposition, p. 191), the spectral side in the anisotropic case (of $G' = D^*$) is discrete: has the form $\sum_i c'_i f'^{\vee}_u(t'_i)$. We combine this last sum with the sum $\sum_i c_i f^{\vee}_u(t_i)$, for new c_i's. The Proposition on p. 198 of [FK], prepared precisely for a situation as the present one, implies that all (new) c_i are zero (namely $c'_i = c_i$ and $t'_i = t_i$).

We conclude that for a test function $f = \otimes f_v$ with two discrete components, the spectral side, as described in Theorem 12, p. 309, VI.2.f, of [Laf], reduces to a discrete sum. Then for matching test functions $f = \otimes f_v$ on $G(\mathbb{A}) = \mathrm{GL}(n,\mathbb{A})$ and $f' = \otimes f'_v$ on $G'(\mathbb{A}) = D^*(\mathbb{A})$ we have the identity

$$\sum_\pi m(\pi)\,\mathrm{tr}\,\pi(f) = \sum_{\pi'} m(\pi')\,\mathrm{tr}\,\pi'(f'),$$

where the sum on the left (resp. right) ranges over the equivalence classes of the irreducible representations π (resp. π') in the discrete spectrum of $G(\mathbb{A})$ (resp. $G'(\mathbb{A})$), and $m(\pi)$ (resp. $m(\pi')$) signifies the multiplicity of π (resp. π') in the discrete spectrum.

A standard argument of "generalized linear independence of characters" implies that on fixing a representation $\pi_0^S = \otimes_{v\notin S}\pi_{0v}$ of $G(\mathbb{A}^S) = G'(\mathbb{A}^S)$, where S is the set of places of F such that $G'_v \simeq G_v$ for all $v \notin S$, the sums over π and π' can be taken to range over the subsets of π with $\pi_v = \pi_{0v}$ and π' with $\pi'_v = \pi_{0v}$ for all $v \notin S$. Rigidity theorem ("strong multiplicity one theorem") implies that the sum over π reduces to at most one term (with $m(\pi) = 1$, by multiplicity one theorem for $\mathrm{GL}(n)$).

Since in our case D is a division algebra, a result of Godement-Jacquet can be used as in [F2] to show that the sum over π' is finite. As explained in [F2], using matching functions f_v on G_v and f'_v on G'_v for $v \in S$ which are *supported only on the regular set* (in particular we may use f_{v_3} as above), linear independence of characters implies character relations between π_v and π'_v for all $v \in S$. The character determines π_v and π'_v uniquely since it is locally integrable (a function field analogue of this result of Harish-Chandra in the characteristic 0 case was proven by Lemaire [Le]). This implies the correspondence which we need: if π is a cuspidal representation of $G(\mathbb{A})$ whose components at $v \in S$ are Steinberg twisted by an unramified character, there is precisely one (thus $m(\pi') = 1$) cuspidal representation π' of $G'(\mathbb{A})$ with $\pi'_v \simeq \pi_v$ for all $v \notin S$. Then π'_v corresponds to π_v by the local correspondence, thus π'_v is one dimensional unramified character, for all $v \in S$. Conversely, given cuspidal π' with $\dim \pi' > 1$ there exists a unique corresponding π.

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