AUTOMORPHIC REPRESENTATIONS OF LOW RANK GROUPS

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PREFACE

This volume concerns two related but independent topics in the theory of liftings of automorphic representations. These are the symmetric square lifting from the group SL(2) to the group PGL(3), and the basechange lifting from the unitary group $\mathrm{U}(3, E/F)$ to $\mathrm{GL}(3, E)$, where E/F is a quadratic extension of number fields. I initially considered these topics in preprints dated 1981 and 1982, and since then found reasonably simple proofs for many of the technical details, such as the fundamental lemma and the unrestricted equality of the trace formulae. The fruits of these efforts are the subject matter of the first two parts of this volume, which are independent of each other, while the third part concerns applications of the basechange theory for U(3) to the theory of Galois representations which occur in the cohomology of the Shimura variety associated with U(3).

The method used relies on a comparison of trace formulae, the same as in my *Automorphic Forms and Shimura Varieties of* PGSp(2), which concerns a rank-two situation. Both topics considered in this volume are lower, rank-one cases. They can be viewed as more elementary, certainly more complete. The last part of the volume on PGSp(2), entitled *Background*, contains many of the (standard) definitions used in this volume too. It is a brief exposition to the principle of functoriality, which predicts the liftings which concern us here, on a conjectural level, in terms of homomorphisms of dual groups. Thus here we consider two rank-one examples of this principle.

To describe the first topic, let F be a number field. Denote by $\mathbb{A}$ its ring of adèles. Let λ be the symmetric square (or adjoint) three-dimensional representation of the dual group $\widehat{H} = \mathrm{PGL}(2, \mathbb{C})$ of the F-group $\mathbf{H} = \mathrm{SL}(2)$ in the dual group $\widehat{G} = \mathrm{SL}(3, \mathbb{C})$ of $\mathbf{G} = \mathrm{PGL}(3)$. We study the lifting (or correspondence) of automorphic forms on $\mathrm{SL}(2, \mathbb{A})$ to those of $\mathrm{PGL}(3, \mathbb{A})$ which is compatible with λ. This lifting is defined by means of character relations. It is studied using a trace formula twisted by the outer automorphism σ of $\mathbf{G}$, which takes a representation to its contragredient. Complete results are obtained. We not only demonstrate the existence of the lifting but also describe its image and fibers. Main results include an intrinsic definition of packets of admissible and automorphic representations of $\mathrm{SL}(2, F_v)$ and $\mathrm{SL}(2, \mathbb{A})$, a proof of multiplicity one theorem for the cuspidal representations of $\mathrm{SL}(2, \mathbb{A})$ and of the rigidity theorem for packets of such cuspidal representations, and a determination of the selfadjoint automorphic representations of $\mathrm{PGL}(3, \mathbb{A})$.

Technical novelties include an elementary proof of the Fundamental Lemma, a simplification of the trace formula by means of regular functions, and a twisted analogue of Rodier's theorem capturing the number of Whittaker models of a (local) representation in the germ expansion of its character.

In the second part, locally we introduce packets and quasi-packets of admissible representations of the quasi-split unitary group $\mathrm{U}(3, E/F)$ in three variables, where E/F is a quadratic extension of local fields, and determine their structure. We determine the admissible representations of $\mathrm{GL}(3, E)$ which are invariant under the involution transpose-inverse-bar. These (quasi) packets are defined by means of both the basechange lifting from $\mathrm{U}(3, E/F)$ to $\mathrm{GL}(3, E)$ and the endoscopic lifting from $\mathrm{U}(2, E/F)$ to $\mathrm{U}(3, E/F)$. Globally, we introduce packets and quasi-packets of the discrete -spectrum automorphic representations of $\mathbf{U}(3, E/F)(\mathbb{A})$ where E/F is a quadratic extension of number fields, determine their structure, and determine the discrete-spectrum automorphic representations of $\mathrm{GL}(3, \mathbb{A}_E)$ fixed by the same involution. In particular we prove multiplicity one theorem for $\mathrm{U}(3, E/F)$, determine which members of a (quasi-) packet are automorphic, establish a rigidity theorem for (quasi-) packets of $\mathrm{U}(3, E/F)$, prove the existence of the global basechange and endoscopic liftings, as well as another twisted endoscopic lifting from $\mathrm{U}(2, E/F)$ to $\mathrm{GL}(3, E)$, and show that each packet of $\mathrm{U}(3, E/F)$ which lifts to a generic representation of $\mathrm{GL}(3, E)$ contains a unique generic member. Technical novelties include a proof of multiplicity one theorem and counting the generic members in packets, two elementary proofs of the Fundamental Lemma, and a simple proof of the unrestricted equality of trace formulae for all test functions by means of regular functions.

To emphasize, multiplicity one theorem was claimed as proved since 1982, but we noticed that the global proof was lacking and completed our local proof (for all noneven places) only a few years before this local proof appeared in 2004. For more details on the development of this area see the concluding remarks section at the end of part 2.

The third part concerns the cohomology $H_c^*(\mathcal{S}_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$ with compact supports and coefficients in any local system (ρ, V), of a Shimura variety $\mathcal{S}_{K_f}$ defined over its reflex field $\mathbb{E}$, associated with the quasi-split unitary group of similitudes $G = \mathrm{GU}(3, E/F)$, where E is a totally imaginary quadratic extension E of a totally real field F. It is a Hecke $\times$ Galois bi-module. We determine its decomposition. The Hecke modules which appear are the finite parts π_f of the discrete-spectrum representation $\pi_f \otimes \pi_\infty$ of $G(\mathbb{A}_F)$ such that π_∞ has nonzero Lie algebra cohomology. We determine the π_f-isotypic part $H_c^*(\pi_f)$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$-module in terms of the Hecke eigenvalues of π_f. In the stable case $\dim[H_c^*(\pi_f)]$ is $3^{[F:\mathbb{Q}]}$. The dimension is smaller in the unstable case. The cuspidal part of $H_c^*(\mathcal{S}_{K_f}\otimes_{\mathbb{E}}\overline{\mathbb{Q}},\mathbb{V})$ coincides with the cuspidal part of the intersection cohomology $IH^*(\mathcal{S}'_{K_f}\otimes_{\mathbb{E}}\overline{\mathbb{Q}},\mathbb{V})$ of the Satake Baily-Borel compactification $\mathcal{S}'_{K_f}$. Purity for the eigenvalues of the Frobenius acting on IH^*, using a computation of the Lie algebra cohomology of the π_∞, implies the Ramanujan conjecture for the π_f (with the exception of the obvious counter examples "$\pi(\mu)$"). More precisely we show that the Satake parameters of each local component π_v of π_f are algebraic, and if $\pi \neq \pi(\mu)$ that all of their conjugates lie on the unit circle in the complex plane. A description of the Zeta function of H_c^* formally follows.

This third part uses the results of the second part, and compares the trace formula with the Lefschetz-Grothendieck fixed point formula. This comparison is greatly simplified on using the (proven) Deligne conjecture on the form of the fixed point formula for a correspondence twisted by a sufficiently high power of the Frobenius. The underlying idea is used in the representation theoretic parts in the avatar of regular, Iwahori biinvariant functions. It leads to a drastic simplification of the proof of the comparison of trace formulae, on which the work of parts 1 and 2 is based. It was found while working with D. Kazhdan on applications of Drinfeld moduli schemes to the reciprocity law relating cuspidal representations of $\mathrm{GL}(n)$ over a function field (which have a cuspidal component) with n-dimensional Galois representations of this field (whose restriction to a decomposition group is irreducible). This work relied on Deligne's conjecture. First representation theoretic applications, inspired by Deligne's insight, were found in the proof with Kazhdan of the metaplectic correspondence, and then to prove basechange for $\mathrm{GL}(n)$. However, the higher-rank applications concern only cuspidal representations with a cuspidal component, while in the low-rank case considered here there are no restrictions. I then feel this idea has not yet been fully exploited. It may lead to significant simplifications in the use of the trace formula.

Yuval Flicker

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PART 1. ON THE SYMMETRIC SQUARE LIFTING

INTRODUCTION

Let F be a global field, F_v the completion at a place v, $\mathbb{A}$ the ring of adèles of F. Let $\mathbf{H}$, or $\mathbf{H}_0$, be the F-group SL(2), and $\mathbf{G}$ the F-group PGL(3). This part studies the lifting (or correspondence) of automorphic forms of $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2,\mathbb{A})$ to those of $\mathbf{G}(\mathbb{A}) = \mathrm{PGL}(3,\mathbb{A})$. It provides an intrinsic definition of packets of admissible and automorphic representations of $\mathrm{SL}(2,F_v)$ and $\mathrm{SL}(2,\mathbb{A})$. This definition is not based on relations to representations of $\mathrm{GL}(2,F_v)$ and $\mathrm{GL}(2,\mathbb{A})$, but rather on character relations and the lifting. This approach applies to groups other than $\mathrm{SL}(n)$. The work establishes multiplicity one theorem for cuspidal representations of $\mathrm{SL}(2,\mathbb{A})$, proves rigidity theorem for packets of these, computes the multiplicity of a cuspidal representation in a packet of a cuspidal representation, and determines the self-contragredient admissible representations of $\mathrm{PGL}(3,F_v)$ and the self-contragredient automorphic representations of $\mathrm{PGL}(3,\mathbb{A})$. The lifting is compatible with the symmetric square (or adjoint) three-dimensional representation of the dual group $\widehat{H} = \mathrm{PGL}(2,\mathbb{C})$ of $\mathbf{H}$ in $\widehat{G} = \mathrm{SL}(3,\mathbb{C})$. It is defined by means of twisted character relations. It is studied here by means of comparison of orbital integrals and of twisted trace formulae.

The interest in the symmetric square lifting originates from Shimura's work [Sm]. Let $f(z) = \sum_1^\infty c_n e^{2\pi i n z}$ be a holomorphic cusp form of weight k and character ω, denote by ψ a primitive Dirichlet character of $\mathbb{Z}$ with $\psi\omega(-1) = 1$, and suppose that

$$\sum_n c_n n^{-s} = \prod_p [(1 - a_p p^{-s})(1 - b_p p^{-s})]^{-1}.$$

Using Rankin's method Shimura [Sm] proved that the Euler product

$$\pi^{-3s/2}\Gamma(s/2)\Gamma((s+1)/2)\Gamma(\tfrac{1}{2}(s-k+2))$$

$$\times \prod_p [(1 - \psi(p)a_p^2 p^{-s})(1 - \psi(p)a_p b_p p^{-s})(1 - \psi(p)b_p^2 p^{-s})]^{-1}$$

is holomorphic everywhere except possibly at $s = k$ or $k - 1$.

Since f generates the space of a cuspidal representation π^* of $\mathrm{GL}(2,\mathbb{A})$ ($F=\mathbb{Q}$, with a discrete-series component $\pi_{0\infty}$ at ∞), this statement can be put in terms of a lifting of automorphic forms compatible with the above dual group homomorphism which takes the diagonal complex matrix $\mathrm{diag}(a_p,b_p)$ to $\mathrm{diag}(a_p^2,a_pb_p,b_p^2)$, or rather to $\mathrm{diag}(a_p/b_p,1,b_p/a_p)$ in a normalized (modulo the center) form.

To reformulate Shimura's result Gelbart and Jacquet [GJ] put

$$L_2(s,\pi_{0v},\chi_v)=L(s,\pi_{0v}\chi_v\times\check{\pi}_{0v})/L(s,\chi_v)$$

and

$$\varepsilon_2(s,\pi_{0v},\chi_v;\psi_v)=\varepsilon(s,\pi_{0v}\chi_v\times\check{\pi}_{0v};\psi_v)/\varepsilon(s,\chi_v;\psi_v)$$

for any representation π_{0v} of $\mathrm{GL}(2,F_v)$ and character χ_v of the multiplicative group $F_v^\times$ of the completion F_v of F at a place v. Here $\check{\pi}_{0v}$ denotes the contragredient of π_{0v}, and ψ_v is a nontrivial additive character of F_v. The representation π_{0v} is said in [GJ] to *L-lift* to a representation π_v of $G_v=\mathbf{G}(F_v)$ if π_v is self-contragredient, and for any χ_v,

$$L(s,\pi_v\chi_v)=L_2(s,\pi_{0v},\chi_v),\qquad \varepsilon(s,\pi_v\chi_v;\psi_v)=\varepsilon_2(s,\pi_{0v},\chi_v;\psi_v).$$

If π^* is an automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ and χ is a character of $\mathbb{A}^\times/F^\times$, put $L_2(s,\pi^*,\chi)=\prod_v L_2(s,\pi_{0v},\chi_v)$. The main theorem of [GJ] is obtained on adèlizing the method of [Sm]. It asserts that for any cuspidal representation π^* of $\mathrm{GL}(2,\mathbb{A})$ not of the form $\pi^*(\mathrm{Ind}_E^F(\mu^*))$, see below, the function $L_2(s,\pi^*,\chi)$ is entire for all χ. This refines the statement of [Sm], implies that each component π_{0v} of π^* L-lifts to some π_v, and that $\pi=\otimes_v\pi_v$ is a cuspidal representation of $\mathbf{G}(\mathbb{A})=\mathrm{PGL}(3,\mathbb{A})$.

Our approach to the lifting is different; it is motivated by the ideas of Saito, Shintani and Langlands in the basechange theory. Following Shintani, the local lifting is defined by means of character relations, and following Saito, the global (and local) lifting is studied by means of the (twisted) trace formula. It is shown that the above π^* (cuspidal, not of the form $\pi^*(\mathrm{Ind}_E^F(\mu^*))$, lifts to a cuspidal π. This implies the holomorphy of $L_2(s,\pi^*,\chi)=L(s,\pi\chi)$ for all χ. As obvious as it might be that the ideas of Saito and Shintani apply in our case too, the techniques required to carry out the work are less obvious. We describe them after we explain our results.

To describe our work, let $L(G)$ be the space of automorphic forms on $\mathbf{G}(\mathbb{A}) = \mathrm{PGL}(3, \mathbb{A})$. It consists of all right-smooth square-integrable complex-valued functions ϕ on $G\backslash\mathbf{G}(\mathbb{A})$, where $G = \mathbf{G}(F)$. The group $\mathbf{G}(\mathbb{A})$ acts on $L(G)$ by right translation: $(r(g)\phi)(h) = \phi(hg)$. The irreducible constituents π of $L(G)$ are called *automorphic* $\mathbf{G}(\mathbb{A})$-modules, or automorphic representations of $\mathbf{G}(\mathbb{A})$ (see, e.g., [BJ]).

Each such π is a restricted tensor product $\otimes_v \pi_v$ of irreducible admissible representations π_v (see [BZ1]) of the local groups $G_v = \mathbf{G}(F_v)$, which are *unramified* (contain a nonzero $K_v = \mathrm{PGL}(3, R_v)$-fixed vector) for almost all v. Each irreducible unramified G_v-module π_v is isomorphic to the unique unramified subquotient of a G_v-module $I((\mu_{iv}))$ normalizedly induced from an unramified character

$$(a_{ij}; i \le j) \mapsto \prod_i \mu_{iv}(a_{ii})$$

of the upper triangular subgroup. The character (μ_{iv}) is not uniquely determined. Yet we obtain a unique conjugacy class $t(\pi_v) = \mathrm{diag}(\mu_{iv}(\boldsymbol{\pi}))$ (where $\boldsymbol{\pi}$ denotes a generator of the maximal ideal in the ring R_v of integers in F_v) in the dual group $\widehat{G} = \mathrm{SL}(3, \mathbb{C})$ of $\mathbf{G}$. The map $\pi_v \mapsto t(\pi_v)$ is a bijection from the set of equivalence classes of irreducible unramified G_v-modules to the set of conjugacy classes in $\widehat{G}$.

Similar description holds in the case of $\mathbf{H} = \mathrm{SL}(2)$, where the automorphic representations $\pi_0 = \otimes_v \pi_{0v}$ have local components π_{0v} which are parametrized, in the unramified case, by conjugacy classes $t(\pi_{0v})$ in the dual group $\widehat{H} = \mathrm{PGL}(2, \mathbb{C})$ of $\mathbf{H}$. A π_{0v} is called *unramified* if it contains a nonzero $K_{0v} = \mathrm{SL}(2, R_v)$-fixed vector.

We study lifting of automorphic forms of $\mathbf{H}(\mathbb{A})$ to those of $\mathbf{G}(\mathbb{A})$, which is compatible with the symmetric square representation $\lambda_0 = \lambda = \mathrm{Sym}^2 \colon \widehat{H} \to \widehat{G}$ of $\widehat{H} = \mathrm{PGL}(2, \mathbb{C})$ in $\widehat{G} = \mathrm{SL}(3, \mathbb{C})$. This is the irreducible three-dimensional representation of $\widehat{H}$. It can be described also as the adjoint representation of $\widehat{H}$ on the Lie algebra of $\mathbf{H}$. It maps the diagonal matrix $\mathrm{diag}(a, b)$ to the diagonal matrix $\mathrm{diag}(a/b, 1, b/a)$. We say that the automorphic $\mathbf{H}(\mathbb{A})$-module $\pi_0 = \otimes_v \pi_{0v}$ *lifts* to the automorphic $\mathbf{G}(\mathbb{A})$-module $\pi = \otimes_v \pi_v$ if $t(\pi_v) = \lambda_0(t(\pi_{0v}))$ for almost all v (where π_{0v} and π_v are both unramified).

Our first global result asserts that *each cuspidal* $\mathbf{H}(\mathbb{A})$*-module lifts to an automorphic* $\mathbf{G}(\mathbb{A})$*-module.* This result is contained in [GJ].

We obtain more precise results. To state them, we prove a special case of the principle of functoriality, thus we prove the existence of monomial representations for SL(2) and GL(2).

Namely, let E be a quadratic extension of F, put $E^1 = \{z \in E^\times; z\overline{z} = 1\}$ and $\mathbb{A}_E^1 = \{z \in \mathbb{A}_E^\times; z\overline{z} = 1\}$ for the kernel of the norm map $N_{E/F}$ on $E^\times$ and $\mathbb{A}_E^\times$ — bar denotes here the conjugation of E over F — and let μ' be a character of $C_E^1 = \mathbb{A}_E^1/E^1$. Denote by W_F the Weil group ([D2], [Tt]) of F. Let $\mathrm{Ind}_E^F(\mu^*)$ be the two-dimensional complex representation of W_F induced from a character μ^* of $C_E = \mathbb{A}_E^\times/E^\times = W_E^{\mathrm{ab}} = W_{E/E}$. It factorizes through the quotient $W_{E/F}$ of W_F, an extension of $\mathrm{Gal}(E/F)$ by C_E. If the restriction of μ^* to C_E^1 is μ', the image of $\mathrm{Ind}_E^F(\mu^*)$ in $\mathrm{PGL}(2,\mathbb{C})$ depends only on μ'. We denote it by $\mathrm{Ind}_E^F(\mu')_0$. It is a two-dimensional projective representation of W_F.

At a place v of F where $E_v = F_v \oplus F_v$, μ_v^* is a pair (μ_{1v}, μ_{2v}) of characters of $C_{F_v} = F_v^\times$, the restriction of $\mathrm{Ind}_E^F(\mu^*)$ to W_{F_v} is the reducible $\mu_{1v} \oplus \mu_{2v}$, and we associate to it the normalizedly induced representation $I(\mu_{1v}, \mu_{2v})$ of $\mathrm{GL}(2, F_v)$, and to the restriction to W_{F_v} of $\mathrm{Ind}_E^F(\mu')_0$ the normalizedly induced representation $I_0(\mu_{1v}/\mu_{2v})$ of $\mathrm{SL}(2, F_v)$.

At a place v of F where E_v is a field and μ_v' is unramified, we associate to the restriction $\mathrm{Ind}_{E_v}^{F_v}(\mu_v')_0$ of $\mathrm{Ind}_E^F(\mu')_0$ to W_{F_v} the induced $I_0(\chi_{E_v})$ of $\mathrm{SL}(2, F_v)$, where χ_{E_v} is the character of $F_v^\times$ with kernel $N_{E_v/F_v}E_v^\times$. If μ_v^* is unramified, or more generally if $\mu_v^* = \overline{\mu}_v^*$, then there is a character μ_v of $F_v^\times$ with $\mu_v^*(z) = \mu_v(z\overline{z})$ $(z \in E_v^\times)$, and we associate $I(\mu_v, \chi_{E_v}\mu_v)$ to $\mathrm{Ind}_{E_v}^{F_v}(\mu_v^*)$.

We prove that *for each E and $\mu' \neq 1$ there exists a cuspidal representation $\pi_0(\mu')$*, more precisely $\pi_0(\mathrm{Ind}_E^F(\mu')_0)$, *of* $\mathrm{SL}(2,\mathbb{A})$, *with the indicated components*. From this we deduce that *for each E and $\mu^* \neq \overline{\mu}^*$* $(\overline{\mu}^*(z) = \mu^*(\overline{z}))$ *there exists a cuspidal representation* $\pi^*(\mu^*)$, or rather $\pi^*(\mathrm{Ind}_E^F \mu^*)$, *of* $\mathrm{GL}(2,\mathbb{A})$, *with the indicated components*. Further we prove the existence of analogous local objects. The representations $\pi_0(\mu')$ and $\pi^*(\mu^*)$ are called *monomial*.

The existence of the representation $\pi^*(\mathrm{Ind}_E^F \mu^*)$ was proven in [JL] by means of the converse theorem, and that of $\pi_0(\mathrm{Ind}_E^F(\mu')_0)$ was deduced from that in [LL]. As our work already contains this existence proof, we do not need to send the reader to study [JL].

It is clear that if π of $\mathrm{PGL}(3,\mathbb{A})$ is a lift from $\mathrm{SL}(2,\mathbb{A})$ then it is self-contragredient, or as we prefer to say: σ-invariant. Here σ is the involution of $\mathbf{G}$ given by $\sigma(g) = J{}^t g^{-1} J$, $J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$, and ${}^\sigma\pi(g) = \pi(\sigma(g))$ is the contragredient $\check{\pi}$ of π (see [BZ1]). A representation π is called *σ-invariant* if $\pi \simeq {}^\sigma\pi$.

Our next global result is a determination of the image of the lifting. Thus we prove that *if π is a cuspidal $\mathbf{G}(\mathbb{A})$-module which is σ-invariant then it is a lift of a cuspidal $\mathbf{H}(\mathbb{A})$-module π_0*. This π_0 is *not* of the form $\pi_0(\mathrm{Ind}_E^F(\mu')_0)$ for any E, μ'.

The cuspidal $\mathbf{H}(\mathbb{A})$-module $\pi_0(\mathrm{Ind}_E^F(\mu')_0)$, $\mu' \neq 1$, lifts to the normalizedly induced, noncuspidal, σ-invariant $\mathbf{G}(\mathbb{A})$-module $I(\pi^*(\mu''), \chi_E)$. Here $\mu''(z) = \mu'(z/\overline{z})$, $z \in C_E$. Note that the central character of $\pi^*(\mu'')$ is χ_E. If $\mu' = 1$ then $\pi_0(\mu')$ is the induced $I_0(\chi_E)$, and it lifts to the induced $I(\chi_E, 1, \chi_E)$. The trivial $\mathbf{H}(\mathbb{A})$-module lifts to the trivial $\mathbf{G}(\mathbb{A})$-module.

This gives a complete description of the image. Indeed, any σ-invariant automorphic $\mathbf{G}(\mathbb{A})$-module which is not in the above list, namely it does not have a trivial component, it is not cuspidal and it is not of the form $I(\pi^*(\mu''), \chi_E)$, must be of the form $I(\pi_1, 1)$, namely normalizedly induced from a discrete-spectrum $\mathrm{GL}(2, \mathbb{A})$-module π_1 with a trivial central character. Such $I(\pi_1, 1)$ are not obtained by the lifting.

The notion of lifting which we use is in fact a strong one, in terms of all places. Namely we define local lifting of irreducible H_v-modules to such G_v-modules, and show that if π_0 lifts to π, then π_{0v} lifts to π_v for all places v. The definition of local lifting is formulated in terms of identities of characters of representations. It generalizes the notion of lifting of unramified local representations described above.

The character relations compare the twisted character of π_v, which is a σ-stable function, with the sum of the characters of irreducible representations π_{0v}. This sum is a stable function, depending only on the stable conjugacy class of the element where the characters are evaluated. We define the *local packet* of π_{0v} to consist of those representations which occur in the sum. Thus the local lifting asserts that it is not a single H_v-module π_{0v} which lifts to π_v, but it is the packet of π_{0v} which lifts. This definition is inspired by our definition of packets of representations of the unitary group in three variables ([F3]) and of the projective symplectic group of similitudes of rank two ([F4]). The packet of an H_v-module π_{0v} coincides with the set of admissible irreducible H_v-modules of the form π_{0v}^g (g in $\mathrm{GL}(2,F_v)$), where $\pi_{0v}^g(h) = \pi_{0v}(g^{-1}hg)$ (h in H_v), and with the set of irreducibles in the restriction to $\mathrm{SL}(2,F_v)$ of a representation of $\mathrm{GL}(2,F_v)$.

Given local packets P_v for each place v of F such that P_v contains an unramified H_v-module π_{0v}^0 for almost all v, we define the global packet P to be the set of $\mathbf{H}(\mathbb{A})$-modules $\otimes_v \pi_{0v}$ with π_{0v} in P_v for all v and π_{0v} equivalent to π_{0v}^0 for almost all v. We say that the packet is *automorphic*, or *cuspidal*, if it contains such a representation of $\mathbf{H}(\mathbb{A})$. In the case of $\mathbf{G}(\mathbb{A})$, more generally for $\mathrm{GL}(n,\mathbb{A})$ and $\mathrm{PGL}(n,\mathbb{A})$, packets consist of a single term.

We are now in a position to state the main lifting theorem. *The lifting defines a bijection from the set of packets of cuspidal representations of* $\mathbf{H}(\mathbb{A})$ *to the set of* σ*-invariant representations of* $\mathbf{G}(\mathbb{A})$ *which are cuspidal or of the form* $I(\pi^*(\mu''),\chi_E)$, $\mu'' \neq \overline{\mu}''$.

This permits the transfer of two well-known theorems from the context of $\mathbf{G}(\mathbb{A}) = \mathrm{PGL}(3,\mathbb{A})$ to the context of $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2,\mathbb{A})$.

The first is a *rigidity theorem for cuspidal representations of* $\mathrm{SL}(2,\mathbb{A})$. It asserts that if $\pi_0 = \otimes_v \pi_{0v}$ and $\pi_0' = \otimes_v \pi_{0v}'$ are cuspidal representations of $\mathbf{H}(\mathbb{A})$ and $\pi_{0v} \simeq \pi_{0v}'$ for almost all v, then π_0 and π_0' define the same packet. The analogous statement for $\mathrm{GL}(n,\mathbb{A})$ is proven in [JS]. It does not hold for $\mathrm{SL}(n,\mathbb{A})$, $n \geq 3$ (see [Bla]).

The second application is *multiplicity one theorem for* $\mathrm{SL}(2,\mathbb{A})$. It asserts that each cuspidal representation of $\mathrm{SL}(2,\mathbb{A})$ occurs in the cuspidal spectrum of $L(H)$ with multiplicity one. The analogous statement for $\mathrm{GL}(n)$ is well known (see [Sl]). It holds for $\mathrm{PGL}(n,\mathbb{A}) = \mathrm{GL}(n,\mathbb{A})/\mathbb{A}^\times$, but not for $\mathrm{SL}(n)$, $n \geq 3$ (see [Bla]). Since the completion of our work other proofs of this result were claimed, but our technique of the trace formula still remains the most direct and transparent, being a part of a generalizable program.

The rigidity theorem holds for packets, but not for individual representations. There do exist two inequivalent cuspidal $\mathbf{H}(\mathbb{A})$-modules which are equivalent almost everywhere.

The packets partition the discrete spectrum of $\mathrm{SL}(2,\mathbb{A})$. The packets $\pi_0(\mu')$, or $\{\pi_0(\mu')\}$, form the *unstable spectrum*, and the other packets make the *stable spectrum*. The reason for these names is that *the multiplicity of each irreducible in a stable cuspidal packet is* 1. But the multiplicity is not constant on a packet $\{\pi_0(\mu')\}$. To describe our formula for the multiplicity, note that when E_v is a field, if $\mu'_v = 1$, $\{\pi_0(\mu'_v)\} = I_0(\chi_{E_v})$ has two constituents; if ${\mu'_v}^2 \neq 1$ the packet $\{\pi_0(\mu'_v)\}$ consists of two irreducibles; and if $\mu'_v \neq 1 = {\mu'_v}^2$ there are 3 quadratic extensions $E_{1v} = E_v$, E_{2v}, E_{3v}, and $\mu'_{iv} \neq 1 = {\mu'_{iv}}^2$ on E^1_{iv} with $\mu'_{1v} = \mu'_v$, and $\{\pi_0(\mu'_{1v})\} = \{\pi_0(\mu'_{2v})\} = \{\pi_0(\mu'_{3v})\}$ consists of 4 irreducibles. There are no other relations on the packets.

The character relations partition each packet into two subsets $\pi_0^+(\mu'_v)$ and $\pi_0^-(\mu'_v)$ of equal cardinality (note that this partition depends on the characters μ'_{iv} when $\mu'_v \neq 1 = {\mu'_v}^2$, and $\pi_0^+(\mu'_v)$ is unramified if μ'_v is unramified). Write $\varepsilon(\pi_{0v}, \mu'_v) = \pm 1$ if $\pi_{0v} \in \pi_0^{\pm}(\mu'_v)$, and $\varepsilon(\pi_0, \mu') = \prod_v \varepsilon(\pi_{0v}, \mu'_v)$. Almost all factors are 1. When $E_v = F_v \oplus F_v$, μ'_v is a character of $F_v^\times = \{(x, x^{-1}) \in E_v^\times\} = E_v^1$, $\{\pi_0(\mu'_v)\}$ is $I_0(\mu'_v)$, and unless $\mu'_v \neq 1 = {\mu'_v}^2$ this induced is irreducible, in which case π_{0v}^+ is $\pi_0(\mu'_v)$ and π_{0v}^- is zero, and $\varepsilon(\pi_{0v}, \mu'_v) = 1$.

The multiplicity of π_0 in $\pi_0(\mu')$, $\mu'^2 \neq 1$, *in the discrete spectrum is*

$$m(\pi_0) = \frac{1}{2}\Big(1 + \varepsilon(\pi_0, \mu')\Big).$$

If $\mu' \neq 1 = \mu'^2$ there are 3 quadratic extensions $E_1 = E$, E_2, E_3, and characters $\mu'_i \neq 1 = \mu_i^2$ on $C_E^1 = \mathbb{A}_E^1/E^1$ with $\mu'_1 = \mu'$ and $\{\pi_0(\mu'_1)\} = \{\pi_0(\mu'_2)\} = \{\pi_0(\mu'_3)\}$. We have $\prod_{1 \leq i \leq 3} \varepsilon(\pi_0, \mu'_i) = 1$, and an irreducible π_0 in such a packet has multiplicity

$$m(\pi_0) = \frac{1}{4}\left(1 + \sum_{1 \leq i \leq 3} \varepsilon(\pi_0, \mu'_i)\right)$$

in the cuspidal spectrum. There are no other relations among the packets.

Another corollary to the lifting theorem asserts that a σ-invariant cuspidal $\mathbf{G}(\mathbb{A})$-module cannot have a component of the form $I(\pi_{1v}, 1)$, where π_{1v} is a square-integrable representation of $\mathrm{GL}(2, F_v)$.

Further, if π_0 is a cuspidal $\mathrm{GL}(2, \mathbb{A})$-module with a local component $I(\mu_{1v}\nu_v^t, \mu_{2v}\nu_v^{-t})$, $t \geq 0$, normalizedly induced from the character $\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \mapsto \mu_{1v}(a)\mu_v(b)|a/b|_v^t$ of the upper triangular subgroup, μ_{1v}, μ_{2v} unitary, then we conclude (as in [GJ]) that $t < \frac{1}{4}$. The estimate $t < \frac{1}{2}$ follows from unitarity, and the equality $t = 0$ is asserted by the Ramanujan conjecture for $\mathrm{GL}(2, \mathbb{A})$.

As a final corollary we note that for cuspidal π_0 which is not of the form $\pi_0(\mathrm{Ind}_E^F(\mu')_0)$, since the L-function $L_2(s,\pi_0,\chi)$ is equal to $L(s,\pi\chi)$, where π is the lift of π_0, we conclude, as noted above, that it is entire for each character χ of $\mathbb{A}^\times/F^\times$.

An irreducible representation π of $\mathrm{GL}(3,F)$, F local, is said to be *essentially self-contragredient* if its contragredient $\check{\pi}$ is equivalent to the twist $\pi\chi$ of π by a character $\chi : \mathrm{GL}(3,F) \to F^\times$, $g \mapsto \chi(\det g)$. If the central character of such a π is denoted by ω, then $\pi(\omega\chi)^{-1}$ has trivial central character and is self-contragredient. Indeed $\check{\pi} \simeq \pi\chi$ implies that $\omega^{-1} = \omega\chi^3$, thus $\chi = (\omega\chi)^{-2}$, hence $\check{\pi} \simeq \pi(\omega\chi)^{-2}$ and $(\pi(\omega\chi)^{-1})\check{} \simeq \pi(\omega\chi)^{-1}$. The central character of this last representation is $\omega(\omega\chi)^{-3} = 1$. Thus the essentially self-contragredient representations of $\mathrm{GL}(3,F)$ are twists by characters of self-contragredient representations of $\mathrm{PGL}(3,F)$, characterized by our work.

The σ-invariant representations π_v of G_v not in the image of the λ_0-lifting are of the form $I(\pi_v,1)$, where π_v is a representation of $H_{1v} = \mathbf{H}_1(F_v)$, $\mathbf{H}_1 = \mathrm{PGL}(2)$. This lifting, λ_1, occurs naturally in our trace formulae comparison. In fact our two liftings, from H_v and from H_{1v} to G_v, are best described as liftings compatible with the natural embeddings of the two elliptic σ-endoscopic subgroups $\widehat{H} = \mathrm{PGL}(2,\mathbb{C})$ and $\widehat{H}_1 = \mathrm{SL}(2,\mathbb{C})$ of $\widehat{G} = \mathrm{SL}(3,\mathbb{C})$. These σ-endoscopic subgroups are simply the σ-centralizers of σ-semisimple elements in $\widehat{G}$.

The character relation which defines the lifting from H_v to G_v takes the form $\chi_\pi^\sigma(\delta) = \chi_{\{\pi_0\}}(N\delta)$. Here χ_π^σ indicates the twisted character of the local representation π. It is a function of σ-conjugacy classes δ in G_v. The $\chi_{\{\pi_0\}}$ is the character of the packet $\{\pi_0\}$ (sum of characters of the irreducibles in the packet). It is a function of the *stable* conjugacy classes in H_v. The character of a single representation of H_v is a function of conjugacy classes in H_v, but it may be nonconstant on the stable orbit (rational points in the orbit under $\mathrm{SL}(2,\overline{F}_v)$, or under $\mathrm{GL}(2,F_v)$ in our case).

To state the character relation we need a notion of a norm map N. It relates *stable* σ-conjugacy classes in G_v with *stable* conjugacy classes in H_v. It generalizes the natural norm map $\mathrm{diag}(a,b,c) \mapsto \mathrm{diag}(a/c,1,c/a)$. A consequence of the existence of the character relation is that the twisted character of the lift $\pi = \lambda_0(\pi_0)$ is a stable σ-conjugacy class function, namely it is constant on stable σ-conjugacy classes. Moreover the lifting relates a packet of H_v, not an individual representation.

The simple looking lifting $\lambda_1 : H_{1v} \to G_v$, $\pi_1 \mapsto \pi = I(\pi_1, 1)$, is also defined by means of a natural yet very interesting character relation, which takes the form $\Delta(\delta\sigma)\chi_\pi^\sigma(\delta) = \boldsymbol{\kappa}(\delta)\Delta_1(N_1\delta)\chi_{\pi_1}(N_1\delta)$. Here Δ and Δ_1 are some Jacobians (which appeared also in the case of λ_0 but were equal to each other in that case). The function $N_1 : G_v \to H_{1v}$ is a norm map, relating stable σ-conjugacy classes in G_v with conjugacy classes in H_{1v}. This N_1 generalizes the natural norm map $\operatorname{diag}(a, b, c) \mapsto \operatorname{diag}(a/c, 1, c/a)$ if H_{1v} is regarded as $\mathrm{SO}(3, F_v)$. A stable conjugacy class in H_{1v} consists of a single class. However, an elliptic σ-conjugacy class in G_v consists of two σ-conjugacy classes. The character $\boldsymbol{\kappa}$ assigns the values ± 1 to these two classes.

It follows from this character relation that the π which are λ_1-lifts (of elliptic π_1) are σ-unstable, that is, their σ-characters are not constant on the stable σ-conjugacy classes. This surprising fact is interesting and merits an independent local verification.

In the last chapter of this part we give an independent, direct computation of the very precise character calculation, by purely local means, not using the trace formula and global considerations. This gives another assurance of the validity of the trace formula approach to the lifting project.

The present volume is based on the series of papers [F2;II], ..., [F2;VI] in our Symmetric Square project, as well as on the papers [FK4] with D. Kazhdan and [FZ1] with D. Zinoviev. In these papers an attempt has been made to isolate different ideas or techniques and make them as independent as possible. The initial results and some of the techniques had been described in [F2;VIII], and the preliminary draft [F2;IX]. The publication of a series of papers could lead to confusion, as to what is the final outcome. Some techniques and results were not known or foreseen at the initial stages. Now that the work reached a stage of completeness, we rewrote it in a unified, updated form.

Not all the material in [F2;II], ..., [F2;VI], [FK4] and [FZ1] is used here. In addition to rearranging the material our foci of interest shifted. For example, §4 of the paper [F2;II] was made redundant by [F2;VII], so we use only the fundamental lemma of [F2;VII] in our section II.1 here. The second half of [FK4] is no longer needed, as it is replaced by [F2;VII], but its first half is used as the basis for [FZ1] in our chapter VI below.

In particular the chapters in the present part are labeled I to VI. They are not linearly related to the papers, but chapter I here is related to [F2;III], II to [F2;VII] and [F2;II], III to [F2;IV], IV to [F2;VI], V to [F2;V], and VI to [FK4] and [FZ1]. We refer to the current part as [F2;I].

The contents of the chapters are as follows. The basic definitions of local lifting of unramified and ramified representations are given in chapter I. To study the σ-invariant $\mathbf{G}(\mathbb{A})$-modules π not obtained by the lifting we introduce in section I.1 the the map $\lambda_1\colon \widehat{H}_1 \to \widehat{G}$, where $\widehat{H}_1 = \mathrm{SL}(2,\mathbb{C})$ is the dual group of $\mathbf{H}_1 = \mathrm{PGL}(2) = \mathrm{SO}(3)$, in addition to the symmetric square map $\lambda_0\colon \widehat{H}_0 \to \widehat{G}$. We then introduce the dual maps $\tilde{\lambda}_i^*\colon \mathbb{H}_G \to \mathbb{H}_i$ from the Hecke algebra $\mathbb{H}_G$ of spherical functions on G_v to the Hecke algebras $\mathbb{H}_i$ of H_{iv} $(i = 0, 1)$.

In I.2 we define a norm map $\gamma = N\delta$ from the set of stable σ-conjugacy classes of δ in G to the set of stable conjugacy classes of γ in H.

In section II.1 it is shown that the stable twisted orbital integral of the unit element of the Hecke algebra of $\mathrm{PGL}(3, F_v)$ is suitably related to the stable orbital integral of the unit element of the Hecke algebra of $\mathrm{SL}(2, F_v)$. Moreover, the unstable twisted orbital integral of the unit element on $\mathrm{PGL}(3, F_v)$ is matched with the orbital integral of the unit element on $\mathrm{PGL}(2, F_v)$. Thus these functions have matching orbital integrals. This statement is called *the Fundamental Lemma* (in the theory of automorphic forms (via the trace formula)). The direct and elementary proof of this fundamental lemma which is given here is based on a twisted analogue of Kazhdan's decomposition of compact elements into a commuting product of topologically unipotent and absolutely semisimple elements.

In section II.3 we transfer smooth compactly supported measures $f_v dg_v$ on G_v to such $f_{0v} dh_v$ on H_v. The definition is based on matching stable orbital integrals. Similar discussion is carried out for the transfer from G_v to H_{1v}.

In chapter III we give the global tool for the study of the lifting, an identity of trace formulae. First we compute the trace formula for $\mathbf{G}(\mathbb{A})$ twisted by the outer automorphism σ. Since σ does not leave all parabolic subgroups of G invariant, we introduced in [F2;IX] a modification of the truncation used by Arthur [A1] to obtain the trace formula. The subsequent computation of the twisted trace formula was carried out in [CLL], from which we quote (in section III.2) the contribution from the Eisenstein series. Thus in chapter III we compute explicitly all needed terms in the twisted formula, stabilize it, and compare it with a sum of trace formulae for $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2, \mathbb{A})$ and $\mathbf{H}_1(\mathbb{A}) = \mathrm{PGL}(2, \mathbb{A})$. The formulae in this chapter III are greatly simplified by the introduction of regular functions (see below).

In section V.1 we give an approximation argument to deduce from the global identity of trace formulae the local (hence also global) results. It is a new argument. It replaces the technique of [L5], which relies on the theory of spherical functions. The new argument is based on the usage of what we call *regular* functions, which are not spherical but in fact lie in the Hecke algebra with respect to an Iwahori subgroup. Their main property is that they both isolate the representations with a vector fixed by an Iwahori subgroup and their support is easy to control and work with, in contrast to that of a spherical function.

The approximation (or separation) argument given here applies in any rank-one situation (since there are only finitely many reducibility points of principal-series representations in this case) and does not use spherical functions at all, except the case where f_{0v} is the unit element f_{0v}^0 of $\mathbb{H}_0$ and f_v is the unit element in $\mathbb{H}_G$, which is proven in section II.1.

In deriving the main theorems in section V.2 we use the immediate twisted analogue of Kazhdan's fundamental study of characters [K2]. This is formulated in section I.4. It is not proven here since the proof is entirely parallel to that of [K2] and requires no new ideas (cf. [F1;II] in the case of any reductive group). The only nonimmediate result needed to twist [K2] is the analogue of [K2], Appendix. This is done in [F1;II], (I.4), in general; the special case needed in this chapter is done here in V.1.7.

In III.3.5, together with V.1.6.2, we give a new argument for the comparison of trace formulae for measures $fdg = \otimes_v f_v dg_v$ such that the transfer $f_{1u} dh_v$ of $f_u dg_v$ vanishes for some u. This new argument uses the regular functions mentioned above to annihilate the undesirable terms in the trace formula. It replaces the technique of [L5], which relies on the computations of singular and weighted orbital integrals and the study of their asymptotic behavior, and the correction technique of [F1;III]. In chapter IV this argument is pursued to give a simple proof of the comparison of trace formulae *for all test measures* fdg. Thus in chapter V we can deal with *all* *automorphic representations of* $\mathbf{H}(\mathbb{A})$.

The method of chapter IV establishes — by simple means — trace formulae comparisons also in other rank-one situations. This method may generalize to deal with groups of arbitrary rank and may give a simple proof of any trace formulae comparisons for general test functions, but we do not do this here. It affords a simple proof of the basechange lifting for GL(2) (see [F1;IV]), and its analogues for the quasi-split unitary groups $\mathrm{U}(2, E/F)$ and $\mathrm{U}(3, E/F)$. See [F3], where the automorphic and admissible representations of U(2) and U(3) are classified, and compared with those of the related general linear groups GL(2) and GL(3), and both rigidity and multiplicity one theorems for U(2) and U(3), are proven.

The approach of [F3] — reducing the study of the representation theory of $\mathrm{U}(3, E/F)$ to basechange lifting to $\mathrm{GL}(3, E)$ — was found by us by direct analogy with the techniques of the present part.

Our character relations, in V.2, take the form

$$\operatorname{tr} \pi_v(f_v dg_v \times \sigma) = (2m+1) \sum \operatorname{tr} \pi_{0v}(f_{0v} dh_v),$$

where the sum ranges over the π_{0v} in the packet $\pi_{0v}(\operatorname{Ind}_{E_v}^{F_v}(\mu'_v)_0)$, where $\pi_v = I(\pi_v^*(\mu''_v), \chi_{E_v})$, and m is a nonnegative integer. Multiplicity one theorem for $\mathrm{SL}(2, \mathbb{A})$ requires that and follows from: $m = 0$. We provide two independent proofs that $m = 0$.

One proof is global. It appeared already in [F2;V]. It is based on a remarkable result of [LL], 6.2 and 6.6, essentially derived only from properties of induction, that if π_0 is cuspidal and lies in a packet $\pi_0(\operatorname{Ind}_E^F(\mu')_0)$, it occurs with multiplicity one in the discrete spectrum. All other cuspidal representations in the packet of such π_0 are π_0^g, $g \in \mathrm{GL}(2, F)$, but those of the form π_0^g, $g \in \mathrm{GL}(2, \mathbb{A}) - \mathrm{GL}(2, F)G(\pi_0)$, $G(\pi_0) = \{g \in \mathrm{GL}(2, \mathbb{A}); \pi_0^g = \pi_0\}$, are not automorphic. The complete proof is given in V.2.3-V.2.4.

In V.2.5 we give a new, purely local proof that $m = 0$. It is based on a twisted analogue of a theorem of Rodier, proven in V.3, which encodes the number of Whittaker models of a representation in its character near the origin. Since π_v is generic, we conclude that $m = 0$ and only one π_{0v} in its packet is ψ_v-generic, for any character ψ_v.

The present work can be viewed as the first step in the study of the self-contragredient representations of $\mathrm{GL}(n)$. This would lead to liftings of representations of symplectic and orthogonal groups of the suitable index to the GL(n) in question. In the present work the twisted endoscopic groups are the symplectic group Sp(1) = SL(2) and the orthogonal group SO(3) = PGL(2). The next work in this project has recently been studied in [F4] in the case of PGL(4), and its twisted endoscopic groups PGp(2) and SO(4).

As noted above, chapter VI offers a new technique to compute a special twisted character. The approach of chapter VI is different from the well-known, standard techniques of trace formulae and dual reductive pairs. It will be interesting to develop this approach in other lifting situations. A first step in this direction was taken in the work [FZ2], where the twisted — by the transpose-inverse involution — character of a representation of PGL(4) analogous to the one considered in chapter VI, is computed. The situation of [FZ2] — see also [FZ3] — is new, dealing with the exterior product of two representations of GL(2) and the structure of representations of the rank-two symplectic group.

I. FUNCTORIALITY AND NORMS

Summary

The symmetric square lifting for admissible and automorphic representations, from the group $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$, to the group $\mathbf{G} = \mathrm{PGL}(3)$, is defined by means of character relations. Its basic properties are derived: the lifting is proven for induced, trivial and special representations, and both spherical functions and orthogonality relations of characters are studied. The definition is compatible with dual group homomorphisms

$$\lambda_0 = \mathrm{Sym}^2 : \widehat{H} = \mathrm{PGL}(2,\mathbb{C}) = \mathrm{SO}(3,\mathbb{C}) \hookrightarrow \widehat{G} = \mathrm{SL}(3,\mathbb{C})$$

and $\lambda_1 : \widehat{H}_1 = \mathrm{SL}(2,\mathbb{C}) \to \widehat{G}$, where $\mathbf{H}_1 = \mathrm{PGL}(2)$. Of course it will be compatible with the computation of orbital integrals (stable and unstable) in chapters II and III.

Introduction

In this chapter we define the symmetric square lifting in terms of character relations, and derive its basic properties. This work is required for the study of the lifting of automorphic forms of $\mathbf{H}(\mathbb{A})$ to $\mathbf{G}(\mathbb{A})$, where $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$ and $\mathbf{G} = \mathrm{PGL}(3)$, by means of the trace formula.

The lifting is suggested by the symmetric square, or adjoint, representation $\lambda_0 \colon \widehat{H} \to \widehat{G}$ of the dual group $\widehat{H} = \mathrm{PGL}(2,\mathbb{C})$ of $\mathbf{H}$ in $\widehat{G} = \mathrm{SL}(3,\mathbb{C})$. Put ${}^t g$ =transpose of g, and

$$\sigma(g) = J\,{}^t g^{-1} J, \qquad J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

The group $\mathbf{H}$ is a σ-endoscopic group of $\mathbf{G}$ (see [KS]). Indeed, $\widehat{H} = \mathrm{SO}(3,\mathbb{C})$ is the group $Z_{\widehat{G}}(\sigma) = \{g \in \widehat{G}; \sigma g = g\}$ of points fixed by σ in $\widehat{G}$. It is elliptic ($\widehat{H}$ is not contained in a σ-invariant proper parabolic subgroup of $\widehat{G}$). But $\mathbf{G}$ has another elliptic σ-endoscopic group, which is $\mathbf{H}_1 = \mathrm{PGL}(2)$:

$$\lambda_1 \colon \widehat{H}_1 = \mathrm{SL}(2,\mathbb{C}) = Z_{\widehat{G}}(s\sigma) = \{g \in \widehat{G}; s\sigma(g)s = g\} \hookrightarrow \widehat{G},$$

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto h_1 = \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}.$$

Via the Satake isomorphism, the maps λ_i formally define the lifting $\pi = \lambda_i(\pi_i)$ of unramified H_i-modules π_i to unramified G-modules π. Moreover, we introduce in section 1 (of this chapter I) the dual maps $\lambda_i^* \colon \mathbb{H} \to \mathbb{H}_i$ from the Hecke algebra $\mathbb{H}$ of G to the Hecke algebra $\mathbb{H}_i$ of H_i. It follows from the definitions that if $f_i = \lambda_i^*(f)$ then the spherical functions f and f_i have matching orbital integrals on the split tori.

In section 2 we define lifting, denoted $\pi_i = \lambda_i(\pi)$, of admissible representations π_i of H_i to such representations π of G, by means of character relations. The definition generalizes the spherical case, and uses packets rather than a single irreducible. Basic examples of the stable lifting λ_0 are given. These concern induced, trivial, and special representations.

Section 3 concerns orthogonality relations for characters, needed in our study of the local lifting. The cases of cuspidal G-modules and Steinberg π are standard but useful. We also record the twisted orthogonality relation for two tempered G-modules which are not relevant. The proof follows closely that of the nontwisted case by Kazhdan [K2]. It depends on the twisted analogue of the crucial appendix of [K2]; this is proven in [F1;II] for a general group, and in chapter V, (1.8), in our case.

I.1 Hecke algebra

1.1 Dual groups. Let F be a global or local field of characteristic zero. Put $\mathbf{G} = \mathrm{PGL}(3)$, $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$, and $\mathbf{H}_1 = \mathrm{PGL}(2) = \mathrm{SO}(3)$, viewed as $\mathbb{Z}$-groups. For any field k denote by $\mathbf{G}(k)$, $\mathbf{H}(k)$ and $\mathbf{H}_1(k)$ the group of k-rational points of $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{H}_1$. We write G' for the group $\mathbf{G}'(F)$ of F-rational points, for any algebraic group $\mathbf{G}'$ over F. Fix an algebraic closure $\overline{F}$ of F.

Let $\widehat{G} = \mathrm{SL}(3,\mathbb{C})$ be the connected dual group of G (for any reductive group G the connected dual group $\widehat{G}$ is defined in [Bo2], where it is denoted by ${}^LG^0$). Consider the semidirect product $\widehat{G}' = \widehat{G} \rtimes \langle\sigma\rangle$; $\langle\sigma\rangle$ denotes the group generated by the automorphism $\sigma(g) = J{}^tg^{-1}J$ of G of order 2.

The dual group $\widehat{H}$ of $\mathbf{H}$ is $\mathrm{PGL}(2,\mathbb{C}) \xrightarrow{\sim} \mathrm{SO}(3,\mathbb{C})$. It is isomorphic to the centralizer of $1\times\sigma$ in the connected component of 1 in $\widehat{G}'$, and to the σ-centralizer $\widehat{G}_1^\sigma = \{g \text{ in } \widehat{G}; g^{-1}\sigma(g) = 1\}$ of 1 in $\widehat{G}$. The isomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{x}\begin{pmatrix} a^2 & ab\sqrt{2} & b^2 \\ ac\sqrt{2} & ad+bc & bd\sqrt{2} \\ c^2 & cd\sqrt{2} & d^2 \end{pmatrix} \qquad (x = ad - bc).$$

This map will be denoted by λ and by $\lambda_0 \colon \widehat{H} \to \widehat{G}$.

The dual group $\widehat{H}_1$ of $\mathbf{H}_1 = \mathrm{PGL}(2)$ is $\mathrm{SL}(2,\mathbb{C})$, and the map

$$\lambda_1 \colon h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto h_1 = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}$$

embeds $\widehat{H}_1$ in $\widehat{G}$. The image is the centralizer of $s\times\sigma$ in $\widehat{G}$, where s is the diagonal matrix $\mathrm{diag}(-1,-1,1)$. Equivalently, it is the σ-centralizer $\widehat{G}_s^\sigma = \{g \in \widehat{G}; s\sigma(g)s^{-1} = g\}$ of s in $\widehat{G}$.

1.2 Hecke algebra. Let F be a p-adic field, $R = \{x \text{ in } F;\ |x| \le 1\}$ its ring of integers, and $K = \mathbf{G}(R)$ the standard maximal compact subgroup of G. Fix a Haar measure dg on G. The Hecke algebra $\mathbb{H} = \mathbb{H}_G$ is the convolution algebra $C_c(K\backslash G/K)$ of complex valued compactly supported K-biinvariant measures $f dg$ on G. Such $f dg$ are called *spherical*.

Let π be an admissible irreducible representation of G on a complex vector space V. A representation π is called *smooth* if each vector is fixed by an open subgroup of G. It is called *admissible* ([BZ1]) if it is smooth and if the subspace of V of vectors fixed by any open subgroup is finite dimensional. A smooth irreducible representation is admissible by a well-known theorem of Bernstein.

Put ${}^\sigma\pi(g) = \pi(\sigma g)$ (g in G). Then ${}^\sigma\pi$ is an admissible irreducible representation of G on V. We say that π is σ-*invariant* if π is equivalent to ${}^\sigma\pi$. In this case there is an invertible operator $A\colon V \to V$ with $\pi(\sigma g) = A\pi(g)A^{-1}$ (g in G). Since π is irreducible and A^2 intertwines π with itself, Schur's lemma ([BZ1]) implies that A^2 is a scalar. Multiplying A by $1/\sqrt{A^2}$, we assume that $A^2 = 1$. Then A is unique up to a sign. We put $\pi(\sigma) = A$, and define the operator $\pi(f dg\times\sigma) = \pi(f dg\sigma) = \pi(f dg)\pi(\sigma)$ to be the map $v \mapsto \int f(g)\pi(g)Av\, dg$.

If fdg is spherical (in $\mathbb{H}_G$) then $\pi(fdg)$ factorizes through the projection to the space π^K of K-fixed vectors in (π, V). If π is irreducible, $\dim_{\mathbb{C}} \pi^K \leq 1$. The representation π is called *unramified* if $\pi^K \neq 0$. Then $(k \in K)$ acts as the identity on π^K. If π is irreducible, $\pi(fdg) \neq 0$ implies that the image π^K of $\pi(fdg)$ is one dimensional.

If π is unramified, it lies in a representation $I = I(\eta)$ of G induced from an unramified character η of the upper triangular Borel subgroup $B = TN$ (e.g., [Bo3]). Here N denotes the unipotent upper triangular subgroup, and T denotes the diagonal subgroup. In fact π is the unique unramified constituent in the composition series of I.

Fix v in V so that $w = \pi(fdg \times \sigma)v$ is nonzero. Since $\sigma(K) = K$, Aw is also a K-fixed vector, and $Aw \neq 0$, since $A(Aw) = w \neq 0$. Hence there is a constant c with $Aw = cw$. As $A^2 = 1$, c is 1 or -1. We replace A by cA to have $Aw = w$. This normalization is compatible with the normalization for generic representations, see chapter V, (1.1.1).

The character η is given by

$$\eta(\delta) = \mu_1(a)\mu_2(b)\mu_3(c)$$

at an element $\delta = \operatorname{diag}(a, b, c)$ in the diagonal torus T of G. Here μ_i are characters of $F^\times$ with $\mu_1\mu_2\mu_3 = 1$. The induced representation $I = I(\eta)$ consists of all (right) smooth $\phi : G \to \mathbb{C}$ with

$$\phi(n\delta g) = \boldsymbol{\delta}^{1/2}(\delta)\eta(\delta)\phi(g), \qquad g \in G, \qquad n \in N, \qquad \delta \in T.$$

The action is by right translation: $(I(g)\phi)(h) = \phi(hg)$. The value of the factor

$$\boldsymbol{\delta}(\delta) = |\det(\operatorname{Ad}(\delta)|\operatorname{Lie} N)| \qquad \text{is} \qquad |a/c|^2.$$

Here $\operatorname{Lie} N$ denotes the Lie algebra of N.

Let $\boldsymbol{\pi}$ be a generator of the maximal ideal in the ring R of integers of F. Consider the element

$$t = \operatorname{diag}(\mu_1(\boldsymbol{\pi}), \mu_2(\boldsymbol{\pi}), \mu_3(\boldsymbol{\pi}))$$

in the diagonal torus $\widehat{T}$ of $\widehat{G}$. Then the equivalence class of the unramified representation π is uniquely determined by the conjugacy class in $\widehat{G}$ of t.

1.3 Orbital integrals. Fix a Haar measure da on the diagonal torus T. The normalized orbital integral

$$F(\delta, fdg) = \Delta(\delta) \int f(g\delta g^{-1}) \frac{dg}{da} \qquad (g \in G/T),$$

where

$$\Delta(\delta) = \boldsymbol{\delta}^{-1/2}(\delta)|\det(1 - \operatorname{Ad}(\delta))|\operatorname{Lie} N| = \left| \frac{a-b}{a} \frac{b-c}{b} \frac{a-c}{c} \right|,$$

depends only on the image of

$$\delta = \operatorname{diag}(a, b, c) \qquad \text{in} \qquad T/\mathbf{T}(R) \simeq X_*(\mathbf{T}) = \operatorname{Hom}(\mathbb{G}_m, \mathbf{T})$$

when fdg is spherical. Indeed, writing $g = an_1k$ (and $dg/da = dn\, dk$) and introducing n by $n_1^{-1}\delta n_1 = \delta n$, changing variables on n in the orbital integral gives the factor

$$|1 - \operatorname{Ad}(\delta^{-1})|^{-1} = |\operatorname{Ad}(\delta)||1 - \operatorname{Ad}(\delta)|^{-1}.$$

Hence

$$F(\delta, fdg) = \boldsymbol{\delta}^{1/2}(\delta) \int_N f^K(\delta n) dn, \qquad \text{where} \qquad f^K(g) = \int_K f(k^{-1}gk) dk.$$

We denote this value of the orbital integral by $F(\mathbf{n}, fdg)$, $\mathbf{n}$ being the image of δ in

$$X_*(\mathbf{T}) \simeq \{(n_1, n_2, n_3);\, n_i \in \mathbb{Z}\}/\{(n, n, n);\, n \in \mathbb{Z}\}.$$

For $t = \operatorname{diag}(t_1, t_2, t_3)$ in $\widehat{T}$ and $\mathbf{n} = (n_1, n_2, n_3)$ in $X_*(\mathbf{T})$, we put $\mathbf{n}(t) = t_1^{n_1} t_2^{n_2} t_3^{n_3}$.

The Satake transform $(fdg)\check{}$ of fdg is abbreviated to $\check{f}$ and is defined by

$$\check{f}(t) = |\mathbf{T}(R)| \sum_{\mathbf{n}} F(\mathbf{n}, fdg)\mathbf{n}(t) \qquad (\mathbf{n} \in X^*(\widehat{T}) \simeq X_*(\mathbf{T})),$$

where $|\mathbf{T}(R)|$ denotes the volume of $\mathbf{T}(R) = T \cap K$ with respect to dt. The map $fdg \mapsto \check{f}$ is an isomorphism from the algebra $\mathbb{H}_G$ to the algebra $\mathbb{C}[\widehat{T}]^W$ of finite Laurent series in $t \in \widehat{T}$ which are invariant under the action of the Weyl group W of $\widehat{T}$ in $\widehat{G}$.

Let $C_c^\infty(G)$ denote the space of all smooth compactly supported complex valued functions on G. If π is an admissible representation, for any f in $C_c^\infty(G)$ the operator $\pi(fdg) = \int_G f(g)\pi(g)dg$ has finite rank. We write $\operatorname{tr}\pi(fdg)$ for its trace. If π is irreducible but not equivalent to ${}^\sigma\pi$, then $\operatorname{tr}\pi(fdg\times\sigma)$ is zero. If π is irreducible and unramified, and fdg is spherical, then $\pi(fdg)$ is a scalar multiple of the projection on the K-fixed vector w. If, moreover, $\pi \simeq {}^\sigma\pi$, then $\pi(\sigma)$ acts as 1 on w, and $\operatorname{tr}\pi(fdg\times\sigma) = \operatorname{tr}\pi(fdg)$ is this scalar. Let us compute it.

1.4 LEMMA. *Suppose that π is unramified and $t = t(\eta) = t(\pi)$ is a corresponding element in $\widehat{T}$. If ${}^\sigma\eta = \eta$, then for any fdg in $\mathbb{H}_G$ we have*

$$\operatorname{tr}\pi(fdg\times\sigma) = \operatorname{tr}\pi(fdg) = \check{f}(t).$$

PROOF. Corresponding to $g = nak$ there is a measure decomposition $dg = \boldsymbol{\delta}^{-1}(a)dndadk$. For a test function $f \in C_c^\infty(G)$ the convolution operator $\pi(fdg) = \int_G \pi(g)f(g)dg$ maps $\phi \in \pi$ to

$$\begin{aligned}(\pi(fdg)\phi)(h) &= \int_G f(g)\phi(hg)dg = \int_G f(h^{-1}g)\phi(g)dg \\ &= \int_N\int_T\int_K f(h^{-1}n_1ak)(\boldsymbol{\delta}^{1/2}\eta)(a)\phi(k)\boldsymbol{\delta}^{-1}(a)dn_1dadk.\end{aligned}$$

The change of variables $n_1 \mapsto n$, where n is defined by $n^{-1}ana^{-1} = n_1$, has the Jacobian $|\det(1-\operatorname{Ad}a)|\operatorname{Lie}N|$. The trace of $\pi(fdg)$ is obtained on integrating the kernel of the convolution operator — viewed as a trivial vector bundle over K — on the diagonal $h = k \in K$. Hence

$$\begin{aligned}\operatorname{tr}\pi(fdg) &= \int_K\int_N\int_T (\Delta\eta)(a)f(k^{-1}n^{-1}ank)dndadk \\ &= \int_T \eta(a)\left[\Delta(a)\int_{G/T} f(gag^{-1})\frac{dg}{da}\right]da. \qquad \square\end{aligned}$$

1.5 DEFINITION. For δ in T put

$$\Phi(\delta\sigma, fdg) = \int_{T^\sigma\backslash G} f(g\delta\sigma(g)^{-1})\,\frac{dg}{da}.$$

Here $T^\sigma = \{a \in T; \sigma(a) = a\}$ is the group of σ-fixed points in T. Also put

$$\tilde{\delta} = J\delta J \, (= \operatorname{diag}(c,b,a) \text{ if } \delta = \operatorname{diag}(a,b,c)), \text{ and } T^{1-\sigma} = \{t\sigma(t)^{-1}; t \in T\}.$$

The involution σ defines (via differentiation) an involution, which we denote again by σ, on the Lie algebra $\operatorname{Lie} G$ of G. It stabilizes $\operatorname{Lie} N$. Define

$$F(\delta\sigma, f dg) = \Delta(\delta\sigma)\Phi(\delta\sigma, f dg)$$

where

$$\Delta(\delta\sigma) = \boldsymbol{\delta}^{-1/2}(\delta)|\det(1 - \operatorname{Ad}(\delta)\sigma)|\operatorname{Lie} N|.$$

Note that

$$|\det(1 - \operatorname{Ad}(\delta)\sigma)|\operatorname{Lie} N| = \left|\left(1 - \frac{a}{c}\right)\left(1 + \frac{a}{c}\right)\right|, \qquad \boldsymbol{\delta}^{1/2}(\delta) = |a/c|,$$

hence $\Delta(\delta\sigma) = \Delta_0(N\delta)$, where $N\delta = \operatorname{diag}(a/c, c/a)$. Here

$$\Delta_0(\operatorname{diag}(x,y)) = |(x-y)^2/xy|^{1/2}$$

is the usual Δ-factor on GL(2). We usually use indices 0, 1 or 2 for objects related to $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$, $\mathbf{H}_1 = \mathrm{PGL}(2)$, and GL(2), respectively.

1.6 Lemma. *For any character η of T we have*

$$\operatorname{tr} I(\eta; f dg \times \sigma) = \int_{T^{1-\sigma}\backslash T} \frac{1}{2}[\eta(a) + \eta(\tilde{a})] F(a\sigma, f dg)\, da.$$

Proof. For $\pi = I(\eta)$, we have

$$(\pi(\sigma f dg)\phi)(h) = \int_G f(g)\phi(\sigma(h)g)dg = \int_G f(\sigma(h)^{-1}g)\phi(g)dg$$
$$= \int_N \int_T \int_K f(\sigma(h^{-1})nak)(\boldsymbol{\delta}^{1/2}\eta)(a)\phi(k)\boldsymbol{\delta}^{-1}(a)dndadk.$$

Hence

$$\operatorname{tr} \pi(\sigma f dg) = \int_K \int_N \int_T f(\sigma(k)^{-1}n_1 ak)(\boldsymbol{\delta}^{-1/2}\eta)(a)dn_1 dadk.$$

We change variables $n_1 \mapsto n$, where $\sigma(n)^{-1}ana^{-1} = n_1$, which has the same Jacobian as if $na\sigma(n)^{-1}a^{-1} = n_1$, which is $|\det(1 - \operatorname{Ad}(a)\sigma)|\operatorname{Lie} N|$, to get

$$\operatorname{tr} \pi(f dg \times \sigma) = \operatorname{tr} \pi(\sigma f dg) = \int_{T/T^{1-\sigma}} \eta(a)\Delta(a\sigma)da \int_{T^\sigma\backslash G} f(\sigma(g)^{-1}ag)\frac{dg}{da}.$$

□

1.7 Cases of H and $\mathbf{H}_1$. Considerations analogous to (1.3), (1.4) apply in the cases of the groups $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$ and $\mathbf{H}_1 = \mathrm{PGL}(2) \simeq \mathrm{SO}(3)$, with respect to the maximal compact subgroups $K_i = \mathbf{H}_i(R)$. Unramified representations π_0, π_1 are associated with $I_0(\mu_1, \mu_2)$, $I_1(\mu, \mu^{-1})$ and their classes are represented by

$$t_0 = \mathrm{diag}(z_1, z_2), \qquad t_1 = \mathrm{diag}(z, z^{-1})$$

in $\widehat{H}_0$, $\widehat{H}_1$. Here $z_i = \mu_i(\boldsymbol{\pi})$, $z = \mu(\boldsymbol{\pi})$. For $f_i dh_i$ in the Hecke algebras $\mathbb{H}_i$ of compactly supported K_i-biinvariant measures on H_i, the Satake transform is

$$\begin{aligned} \check{f}_0(\mathrm{diag}(z_1, z_2)) =& |\mathbf{T}_0(R)| \sum_n F(n, f_0 dh_0)(z_1/z_2)^n, \\ \check{f}_1(\mathrm{diag}(z, z^{-1})) =& |\mathbf{T}_1(R)| \sum_n F(n, f_1 dh_1) z^n. \end{aligned}$$

The symbol $|\mathbf{T}_i(R)|$ denotes the volume of $\mathbf{T}_i(R) = T_i \cap K_i$ with respect to da_i. The expression $F(n, f_i dh_i)$ denotes the normalized orbital integral of $f_i dh_i$ at regular elements $\mathrm{diag}(a, b)$ in T_i (diagonal subgroup of H_i) with valuations $(n, -n)$ $(i = 0)$ and (m_1, m_2), $m_1 - m_2 = n$ $(i = 1)$. It depends on the choice of Haar measures dh_i, da_i on H_i, T_i; but $\check{f}_i$ depends only on dh_i.

The standard computation of (1.3) shows that for spherical $f_i dh_i$, π_i, we have

$$\mathrm{tr}\, \pi_i(f_i dh_i) = \check{f}_i(t_i) \qquad (t_i = t_i(\pi_i)).$$

Recall (1.1) that we have maps $\lambda_i \colon \widehat{H}_i \to \widehat{G}$ and ((1.2), (1.5)) classes t_i, t in $\widehat{H}_i$, $\widehat{G}$ for unramified representations π_i, π of H_i, H $(i = 0, 1)$.

1.8 Definition. The unramified representation π_i *lifts* to π through λ_i if $t = \lambda_i(t_i)$. In this case we write $\pi = \lambda_i(\pi_i)$.

The maps $\tilde{\lambda}_i^* \colon \mathbb{H} \to \mathbb{H}_i$ dual to λ_i are defined by $f_i dh_i = \tilde{\lambda}_i^*(f dg)$ if $\check{f}_i(t_i) = \check{f}(\lambda_i(t_i))$ for all t_i in $\widehat{T}_i$. Equivalently, $f_i dh_i = \tilde{\lambda}_i^*(f dg)$ if $\mathrm{tr}\, \pi_i(f_i dh_i) = \mathrm{tr}\, \pi(f dg \times \sigma)$ for all π_i and $\pi = \lambda_i(\pi_i)$. Note that $\pi = \lambda_i(\pi_i)$ if and only if $\check{f}_i(t_i) = \check{f}(t)$, where $t_i = t_i(\pi_i)$, $t = t(\pi)$, for all $f dg$ and $f_i dh_i = \tilde{\lambda}_i^*(f dg)$.

Note that $I_0(\mu) \overset{\mathrm{dfn}}{=} I_0(\mu, 1)$, $I_1(\mu) \overset{\mathrm{dfn}}{=} I_1(\mu, \mu^{-1})$ both lift (through λ_0, λ_1) to $I(\mu, 1, \mu^{-1})$.

There are several formal consequences concerning orbital integrals of measures fdg, $f_i dh_i$ related by $f_i dh_i = \tilde{\lambda}_i^*(fdg)$, as these integrals are the coefficients of $\check{f}$ and $\check{f}_i$.

1.9 LEMMA. *If*

$$\delta = \operatorname{diag}(a,b,c), \qquad \gamma = \operatorname{diag}(a/c, c/a), \qquad \textit{and} \qquad \gamma_1 = \operatorname{diag}(a,c),$$

then

$$F(\delta\sigma, fdg) = F(\gamma, f_0 dh_0) \qquad \textit{and} \qquad F(\delta\sigma, fdg) = F(\gamma_1, f_1 dh_1).$$

PROOF. If $t_1 = \operatorname{diag}(t, t^{-1})$ lies in $\widehat{T}_1$ then

$$|\mathbf{T}(R)| \sum_{\mathbf{m}=(m_1,m_2,m_3)} F(\mathbf{m}, fdg) t^{m_1-m_3} = \check{f}(\lambda_1(t_1))$$

$$= \check{f}_1(t_1) = |\mathbf{T}_1(R)| \sum_n F(n, f_1 dh_1) t^n.$$

Comparing coefficients of t^n we obtain

$$|\mathbf{T}_1(R)| F(n, f_1 dh_1) = \sum_{\{\mathbf{m}; m_1-m_3=n\}} |\mathbf{T}(R)| F(\mathbf{m}, fdg).$$

A simple change of variables shows that this is the product of $|\mathbf{T}^\sigma(R)|$, where

$$\mathbf{T}^\sigma(R) = \{t \in \mathbf{T}(R);\, t = \sigma(t)\},$$

and

$$F(n\sigma, fdg) = \Delta(\delta\sigma) \int f(g^{-1}\delta\sigma(g))\, dg,$$

where

$$\delta = \operatorname{diag}(a,b,c), \qquad \gamma = \operatorname{diag}(a/c, c/a), \qquad |a/c| = |\boldsymbol{\pi}|^n.$$

It is clear that the integral depends only on n, but not on the choice of δ.

In the case of $\mathbf{H}_0 = \mathrm{SL}(2)$, taking a representative $t_0 = (t, 1)$ in $\widehat{T}_0$ we have

$$|\mathbf{T}(R)| \sum_{\mathbf{m}} F(\mathbf{m}, fdg) t^{m_1-m_3} = \check{f}(\lambda_0(t_0))$$

$$= \check{f}_0(t_0) = |\mathbf{T}_0(R)| \sum_n F(n, f_0 dh_0) t^n.$$

Hence $F(n\sigma, f dg) = F(n, f_0 dh_0)$. □

REMARK. (1) We normalize the measures so that $|\mathbf{T}_i(R)| = |\mathbf{T}^\sigma(R)|$; the groups T_i and T^σ are isomorphic to the multiplicative group $\mathbb{G}_m$.
(2) Every $\check{f}_1$ is so obtained from some $\check{f}$, hence the $\check{f}_1$ separate the π_1. Every $\check{f}_0$ is so obtained from some $\check{f}$, hence the $\check{f}_0$ separate the π_0.

I.2 Norms

2.1 Stability. To extend the study of lifting from the unramified case to any admissible σ-invariant representation, we need to define norm maps N and N_1 to extend the definition suggested by the formal Lemma 1.9 on diagonal matrices. Thus for $\delta = \operatorname{diag}(a, b, c)$ we put:

$$N(\delta) = \operatorname{diag}(a/c, c/a) \qquad \text{and} \qquad N_1(\delta) = \operatorname{diag}(a, c).$$

These norm maps will be used to relate orbital integrals and characters, so they should be defined in terms of (twisted) conjugacy classes. More precisely, the norm will be defined to be a map from the set of regular stable σ-conjugacy classes in G to the sets of regular stable conjugacy classes in H and H_1. We begin with a description of these classes.

Let F be a local or global field of characteristic 0. Fix an algebraic closure $\overline{F}$ of F. Let $\mathbf{G}$ be a reductive group defined over F and $G = \mathbf{G}(F)$ the group of F-rational points of $\mathbf{G}$. Denote by σ an automorphism of $\mathbf{G}$ defined over F. The elements δ, δ' of G are called *σ-conjugate* if there is h in G with $\delta' = h\delta\sigma(h^{-1})$. They are called *stably σ-conjugate* if there is h in $\mathbf{G}(\overline{F})$ with $\delta' = h\delta\sigma(h^{-1})$. The term (stable) conjugacy (no mention of σ) is employed if σ is the trivial automorphism.

The stable σ-conjugates of δ in G are described by the set $A(\delta)$ of g in $\mathbf{G}(\overline{F})$ with $g\delta\sigma(g^{-1})$ in G. The map

$$A(\delta) \xrightarrow{\alpha'} H^1(F, Z_{\mathbf{G}}(\delta\sigma)), \qquad g \mapsto \{\tau \mapsto g_\tau = g^{-1}\tau(g)\},$$

where

$$Z_{\mathbf{G}}(\delta\sigma) = \{g \in \mathbf{G}; g\delta\sigma(g^{-1}) = \delta\},$$

factors through

$$1 \longrightarrow D(\delta) \xrightarrow{\alpha} H^1(F, Z_{\mathbf{G}}(\delta\sigma)) \longrightarrow H^1(F, \mathbf{G}),$$

where the double coset space $D(\delta) = G\backslash A(\delta)/Z_{\mathbf{G}}(\delta\sigma)(\overline{F})$ parametrizes the σ-conjugacy classes within the stable σ-conjugacy class of δ.

The definitions introduced above will be used with $\mathbf{G} = \mathrm{PGL}(3)$ and the (involution) outer automorphism $\sigma(g) = J{}^tg^{-1}J$, and also with $\mathbf{H} = \mathbf{H}_0 = SL(2)$, $\mathbf{H}_1 = \mathrm{PGL}(2) = \mathrm{SO}(3)$ and the trivial σ. If $\gamma \in H$, $Z_{\mathbf{H}}(\gamma)$ denotes the centralizer of γ in $\mathbf{H}$. Similarly, $Z_{\mathbf{H}_1}(\gamma_1)$ is the centralizer of $\gamma_1 \in H_1$ in $\mathbf{H}_1$.

Note that every conjugacy class of H_1 (and of $\mathrm{GL}(n,F)$ or $\mathrm{PGL}(n,F)$) is stable. Indeed, the centralizer of a semisimple element γ in $\mathrm{GL}(n,F)$ is a product $\prod_j E_j^\times$, where E_j are field extensions of F with $\sum_j [E_j : F] = n$. We have $H^1(F, \mathbb{G}_m) = \{0\}$, hence $D(\gamma)$ is trivial for $\mathrm{GL}(n,F)$ or $\mathrm{PGL}(n,F)$.

However, for $\mathbf{H} = \mathrm{SL}(2)$, the centralizer in H of a nonsplit γ is $E^1 = \ker N_{E/F}$, where $E = F(\gamma)$ is the extension generated by γ. Hence the set of conjugacy classes within the stable conjugacy class of a regular γ in H is parametrized by $F^\times/NE^\times$, which is $\mathbb{Z}/2\mathbb{Z}$ when F is local and γ is elliptic, and $\{0\}$ when the eigenvalues of γ are in $F^\times$. For this we need to compute $H^1(F, \mathbf{T}) = H^1(\mathrm{Gal}(E/F), E^\times)$ where $\mathbf{T}$ is $\mathbb{G}_m$ over E and $\sigma \neq 1$ in $\mathrm{Gal}(E/F)$ acts on $\mathbf{T}(E) = E^\times$ by $\sigma(x) = \overline{x}^{-1}$ ($\overline{x}$ is the conjugate of x in E over F). Then a cocycle is $b = b_\sigma \in E^\times$ with $1 = b_{\sigma^2} = b_\sigma \sigma(b_\sigma) = b/\overline{b}$, thus $b \in F^\times$. The coboundaries are $b/\sigma(b) = b\overline{b}$, thus $N_{E/F}E^\times$.

There is of course an easy way in the case of $\mathrm{SL}(2,F)$ (and more generally $\mathrm{SL}(n,F)$) to realize the stable conjugacy in $\mathrm{GL}(n,F)$. If $E = F(\sqrt{A})$, a γ in H splitting over E, thus with eigenvalues $a \pm b\sqrt{A}$, is equal to $\begin{pmatrix} a & bA \\ b & a \end{pmatrix}$ up to stable conjugacy. A γ' in H stably conjugate but not conjugate to γ has the same eigenvalues as γ, hence it is conjugate to γ in $\mathrm{GL}(2,F)$, thus it is conjugate in $\mathrm{SL}(2,F)$ to $\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & bA \\ b & a \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}$, where $D \in F^\times - N_{E/F}E^\times$. Indeed, if $D \in NE^\times$ then $\mathrm{diag}(D,1) \in T(F)\,\mathrm{SL}(2,F)$ where $T(F)$ is the centralizer of γ in $\mathrm{GL}(2,F)$.

To realize γ' as $g^{-1}\gamma g$, $g \in \mathrm{SL}(2,\overline{F})$, we solve $\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} x & yA \\ y & x \end{pmatrix}$, $x = x_1' + x_2'\sqrt{A} \in E$, $y = y_1' + y_2'\sqrt{A} \in E$, thus $x^2 - y^2A = D$. The solutions are $x = x_1(x_2+1) + x_2\sqrt{A}$, $y = x_2 + 1 + \frac{x_1x_2}{A}\sqrt{A}$, provided

$2x_2+1=\frac{D}{x_1^2-A}$. We take $x_1=0$. Then $x_2=-\frac{1}{2}(\frac{D}{A}+1)$, $x=x_2\sqrt{A}$, $y=x_2+1=\frac{1}{2}(1-\frac{D}{A})$. Then

$$g=\frac{1}{D}\begin{pmatrix} -\frac{1}{2}(\frac{D}{A}+1)\sqrt{A} & -\frac{A}{2}(1-\frac{D}{A}) \\ \frac{-1}{2}(1-\frac{D}{A}) & -\frac{1}{2}(\frac{D}{A}+1)\sqrt{A} \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies

$$g_\sigma=g\sigma(g)^{-1}=h_0^{-1}\begin{pmatrix} -A/D & 0 \\ 0 & -D/A \end{pmatrix}h_0^{-1}$$

where

$$h_0=\begin{pmatrix} 1 & \sqrt{A} \\ \frac{-1}{2\sqrt{A}} & \frac{1}{2} \end{pmatrix}, \qquad \begin{pmatrix} a & bA \\ b & a \end{pmatrix}=h_0^{-1}\begin{pmatrix} a+bA & 0 \\ 0 & a-bA \end{pmatrix}h_0,$$

$$h_0\sigma(h_0)^{-1}=\begin{pmatrix} 0 & 2\sqrt{A} \\ \frac{-1}{2\sqrt{A}} & 0 \end{pmatrix}, \qquad h_0g\sigma(h_0g)^{-1}=\begin{pmatrix} 0 & \frac{-2A\sqrt{A}}{D} \\ \frac{D}{2A\sqrt{A}} & 0 \end{pmatrix},$$

and

$$h_0g=\begin{pmatrix} \sqrt{A}/D & 0 \\ 0 & 1/\sqrt{A} \end{pmatrix}h_0\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}$$

so that g satisfies $\gamma'=g^{-1}\gamma g$.

2.2 The Norm. Let δ be an element of G. The set of eigenvalues of $\delta\sigma(\delta)$ is of the form $\{\lambda, 1, \lambda^{-1}\}$. Indeed, if λ is an eigenvalue of $\delta\sigma(\delta)$ then there is an eigenvector v with ${}^t(\delta\sigma(\delta))v=\lambda v$. Hence

$$\lambda^{-1}v={}^t(\delta\sigma(\delta))^{-1}v, \qquad \text{and} \qquad \lambda^{-1}(\delta Jv)=\delta J{}^t\delta^{-1}J(\delta Jv),$$

that is, λ^{-1} is also an eigenvalue. It is clear that $\lambda\in F^\times$ or that $[F(\lambda):F]=2$.

The element δ of G is called *σ-regular* if the eigenvalues λ, 1, λ^{-1} of $\delta\sigma(\delta)$ are distinct. In this case *let $N\delta$ be the class in H determined by the eigenvalues λ, λ^{-1}* and *$N_1\delta$ the class in H_1 with eigenvalues of ratio λ* if $\mathbf{H}_1$ is viewed as PGL(2), or *with eigenvalues λ, 1, λ^{-1}* if $\mathbf{H}_1$ is viewed as SO(3).

For any $h=\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ in $\mathrm{GL}(2,F)$ we put

$$h_1=\begin{pmatrix} x & 0 & y \\ 0 & 1 & 0 \\ z & 0 & t \end{pmatrix}, \qquad e=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad w=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assume that δ is σ-regular. Replacing $\delta\sigma(\delta)$ by a conjugate $g^{-1}\delta\sigma(\delta)g$, hence δ by a σ-conjugate $g^{-1}\delta\sigma(g)$, we may assume that $\delta\sigma(\delta)$ is of the form

h_1. Since δJ takes λ-eigenvectors of ${}^t(\delta\sigma(\delta))$ to λ^{-1}-eigenvectors of $\delta\sigma(\delta)$, the assumption $\delta\sigma(\delta) = h_1$ implies that δJ fixes the subspaces $\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ 0 \\ * \end{pmatrix}$. So does δ. Hence multiplying by a scalar we have $\delta = a_1$ for some a in $\mathrm{GL}(2, F)$.

Note that if $\delta = (ae)_1$, then $N\delta = h_1$; here

$$h = aew{}^ta^{-1}ew = \frac{-1}{\det a}a^2.$$

If $\delta' = (a'e)_1$ and $\delta' = \beta^{-1}\delta\sigma(\beta)$ [hence $\delta'\sigma(\delta') = \beta^{-1}\delta\sigma(\delta)\beta$ and $\beta = b_1$ for some b in $\mathrm{GL}(2, F)$], then $a'e = b^{-1}aew{}^tb^{-1}w$ and

$$a' = b^{-1}a(ew){}^tb^{-1}(ew)^{-1} = \frac{1}{\det b}b^{-1}ab.$$

Hence δ, δ' are (stably) σ-conjugate if and only if a, a' are projectively (stably) conjugate.

2.3 Lemma. *For any given regular γ in H there is a unique stable σ-conjugacy class of δ with $N\delta = \gamma$. The σ-conjugacy classes within such a stable class are parametrized by u in $F^\times/NE^\times$, $E = F(\delta\sigma(\delta))$. A set of representatives is given by $\delta = (uae)_1$.*

Proof. If the eigenvalues $\lambda, 1, \lambda^{-1}$ of $\delta\sigma(\delta)$ are distinct then they lie in a quadratic extension of F (or in F) and define a stable conjugacy class $N\delta$ in H with eigenvalues λ, λ^{-1}, and a conjugacy class $N_1\delta$ in H_1 with eigenvalues λ, 1, λ^{-1} in $\mathrm{SO}(3, F)$ or λ, 1 in $\mathrm{PGL}(2, F)$. Given λ there exist α, β in $F(\lambda)^\times$ with $\alpha/\beta = -\lambda$. Here $\beta = \overline{\alpha}$ and we use Hilbert Theorem 90 if $\lambda \notin F$. The pair α, β is determined up to a multiple by a scalar u in $F^\times$. The matrix $\delta\sigma(\delta)$ (where $\delta = (ae)_1$) has eigenvalues λ, 1, λ^{-1} iff a has eigenvalues α, β so that $\frac{-1}{\det a}a^2$ has eigenvalues $-\alpha/\beta$, $-\beta/\alpha$. Hence the norm map is onto the set of regular elements of H, and the δ in G with a fixed regular $N\delta$ make a single stable σ-conjugacy class, as a and ua (u in $F^\times$) are projectively stably conjugate.

But a and $a' = u^{-1}a$ are projectively conjugate only if $u^{-1}a = \frac{1}{\det b}b^{-1}ab$ for some b in $\mathrm{GL}(2, F)$. Then $u^2 = \det b^2$, and $u = \pm\det b$. If $u = -\det b$ then $-a = b^{-1}ab$, a has eigenvalues γ, $-\gamma$ and $h = I$ does not have eigenvalues different than 1. Hence $u = \det b$, $a = b^{-1}ab$ and $u = \det b$ lies in $N_{E/F}E^\times$, where $E = F(a)$. □

Thus the norm map has a particularly simple description in the case where $\delta\sigma(\delta)$ has distinct eigenvalues. Up to a σ-conjugacy such δ can be assumed to be of the form $\delta = (ae)_1$. Then $\gamma = N\delta = (-1/\det a)a^2$.

2.3.1 COROLLARY. *Let F be a global field, u a place of F, and δ, δ' stably σ-conjugate but non-σ-conjugate elements of $\mathbf{G}(F)$. Then there is a place $v \neq u$ of F such that δ, δ' are not σ-conjugate in $\mathbf{G}(F_v)$.*

2.4 DEFINITION. If $N\delta$ is regular put $\tilde{\delta} = \frac{1}{2}[\delta J + {}^t(\delta J)]J$. Note that $\tilde{\delta}\sigma(\tilde{\delta}) = 1$. Hence $\tilde{\delta}J$ is symmetric $(= {}^t(\tilde{\delta}J))$. Define $\boldsymbol{\kappa}(\delta)$ to be 1 if $\mathrm{SO}(3, \tilde{\delta}J)$ is split and -1 if not.

The function $\boldsymbol{\kappa}$ depends only on the σ-conjugacy class of δ. Indeed if δ is replaced by $\beta\delta J^t\beta J$ then $\delta J + {}^t(\delta J)$ is replaced by

$$\beta\delta J^t\beta + \beta J^t\delta^t\beta = \beta[\delta J + {}^t(\delta J)]^t\beta,$$

and the form $\delta J + {}^t(\delta J)$ splits if and only if $\beta[\delta J + {}^t(\delta J)]^t\beta$ does.

If δ, δ' are stably σ-conjugate with regular norm, but they are not conjugate, then the forms $\tilde{\delta}J$ and $\tilde{\delta}'J$ are not equivalent, and $\boldsymbol{\kappa}(\delta') = -\boldsymbol{\kappa}(\delta)$. Thus if $\delta = (ae)_1$ and $\delta' = (uae)_1$, then $\boldsymbol{\kappa}(\delta') = \chi(u)\boldsymbol{\kappa}(\delta)$, χ being the quadratic character of $F^\times$ trivial on $NE^\times$, $E = F(\delta\sigma(\delta))$.

If $N\delta = \gamma$ is regular in H then $Z_{\mathbf{G}}(\delta\sigma) \simeq Z_{\mathbf{H}}(\gamma)$. Indeed, if

$$g^{-1}\delta\sigma(g) = \delta \qquad \text{then} \qquad g^{-1}\delta\sigma(\delta)g = \delta\sigma(\delta);$$

if $\delta = (ae)_1$ then $g = b_1$ and $b^{-1}ab = a$, since $\delta\sigma(\delta) = h_1$, $h = \frac{-1}{\det a}a^2$. Hence

$$b^{-1}aew^tb^{-1}we = a, \qquad \text{namely} \qquad \frac{1}{\det b}b^{-1}ab = a,$$

so that $\det b = 1$. It is clear that $Z_{\mathbf{H}}(\gamma) = Z_{\mathbf{H}}(a)$.

The norm map can be extended to classes of δ in G which are not σ-regular. This is done next.

2.5 Identity. We now deal with the (two) cases where all eigenvalues of $\delta\sigma(\delta)$ are 1.

If $\delta\sigma(\delta) = 1$ we write $N\delta = 1$ and $N_1\delta = 1$. Then $\delta J = {}^t(\delta J)$ is symmetric, any two symmetric matrices are equivalent over F, hence for each δ' with $\delta'\sigma(\delta') = 1$ there is S in G with $\delta J = S\delta' J^tS$, so that $\delta = S\delta'\sigma(S^{-1})$, and the δ with $\delta\sigma(\delta) = 1$ form a single stable σ-conjugacy class.

For such δ the σ-centralizer

$$Z_{\mathbf{G}}(\delta\sigma) \qquad \text{is} \qquad (\mathrm{PO}(3,\delta J) =) \qquad \mathrm{SO}(3,\delta J),$$

the (projective =) special orthogonal group with respect to the form δJ. Replacing δ by a σ-conjugate $u\delta\sigma(u^{-1})$ or δJ by $u\delta J {}^t u$, implies replacing $Z_{\mathbf{G}}(\delta\sigma)$ by its conjugate $uZ_{\mathbf{G}}(\delta\sigma)u^{-1}$. Hence if F is $\mathbb{R}$ or p-adic then there are two σ-conjugacy classes in the stable σ-conjugacy class of the δ with $N\delta = 1$, corresponding to the split and nonsplit forms δJ. Put $\boldsymbol{\kappa}(\delta) = 1$ if $Z_{\mathbf{G}}(\delta\sigma) = \mathrm{SO}(3,\delta J)$ splits and $\boldsymbol{\kappa}(\delta) = -1$ if it is anisotropic. If we put $\gamma = N\delta$ $(= 1)$ then there is a natural surjection

$$\varphi : Z_{\mathbf{H}}(\gamma) = \mathrm{SL}(2) \to Z_{\mathbf{G}}(\delta\sigma) = \mathrm{SO}(3,\delta J)$$

with kernel $\{\pm 1\}$. The morphism φ is defined over F only if $\mathrm{SO}(3,\delta J)$ is split.

2.6 Unipotent. If $\delta\sigma(\delta)$ is unipotent but not 1 we check by matrix multiplication that it is a regular unipotent (not conjugate to $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$). Alternatively, $\delta\sigma(\delta)v = v$ if and only if $(\delta J - {}^t(\delta J))w = 0$, where $w = {}^t(\delta J)^{-1}v$. Thus the 1-eigenspace of $\delta\sigma(\delta)$ has the same dimension as the zero-eigenspace of the skew-symmetric matrix $\delta J - {}^t(\delta J)$, namely 1 or 3, and $\delta\sigma(\delta) \neq 1$ is regular unipotent. Up to stable σ-conjugacy we may assume that $\delta\sigma(\delta) = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, a σ-invariant matrix. Hence δ commutes with $\sigma(\delta)$ and $\delta\sigma(\delta)$, and it is unipotent of the form $\begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$. These make a single σ-conjugacy class. The σ-centralizer $Z_{\mathbf{G}}(\delta\sigma)$ is the additive group $\mathbb{G}_a$, $H^1(F,\mathbb{G}_a)$ is trivial, hence there is a unique σ-conjugacy class of δ with $\delta\sigma(\delta) =$ unipotent $\neq 1$, and we put $N\delta =$ unipotent in H.

If $\gamma = N\delta$ is unipotent then $Z_{\mathbf{H}}(\gamma) = \{\pm 1\} \times \mathbb{G}_a$ and there is a natural surjection $\varphi : Z_{\mathbf{H}}(\gamma) \to Z_{\mathbf{G}}(\delta\sigma)$ with kernel $\{\pm 1\}$.

2.7 Negative identity. It remains to deal with the case where two eigenvalues of $\delta\sigma(\delta)$ are -1. Here $Z_{\mathbf{G}}(\delta\sigma) \simeq Z_{\mathbf{H}}(\gamma)$, as we see next.

If $\delta\sigma(\delta) = h_1$ and $h = -I$ in $\mathrm{GL}(2,F)$ then $a^2 = \det a$ $(\delta = (ae)_1)$ and a is a scalar $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$. We put $N\delta = -I$, and note that all δ with $N\delta = -I$ form a single σ-conjugacy class, since

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \frac{\alpha}{\beta} \begin{pmatrix} \alpha/\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \beta/\alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

2.8 Negative unipotent. If $\delta\sigma(\delta) = h_1$ and $h = -\text{unipotent} \neq -I$ in $\mathrm{GL}(2, F)$, then up to conjugacy $h = -\begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix}$, hence $a = u^{-1}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ with $\alpha \in F^\times$, $u \in F^\times$. But a is equal to

$$\frac{1}{u}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & \alpha u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

hence it is projectively conjugate to

$$\begin{pmatrix} 1 & \alpha u \\ 0 & 1 \end{pmatrix}. \qquad \text{Now} \qquad \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \qquad (\alpha, \beta \in F^\times)$$

are (projectively) conjugate only if α/β is a square in $F^\times$; they are clearly stably conjugate. Hence the σ-conjugacy classes within the single stable σ-conjugacy class of our δ are parametrized by $F^\times/F^{\times 2}$. If

$$\delta = (ae)_1, \qquad a = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \qquad \alpha \neq 0,$$

we let $N\delta$ be the stable conjugacy class of h in H, and define $N_1\delta$ to be the conjugacy class in H_1 of elements which generate $F(\sqrt{\alpha})$ over F, and the quotient of whose eigenvalues is -1. Such an element of $\mathrm{GL}(2, F)$ is $\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$.

I.3 Local lifting

3.1 Orbital integrals. Let F be a local field. Fix a Haar measure dg on G. For any σ-regular δ, the σ-centralizer $Z_G(\delta\sigma)$ of δ in G is a torus. Fix a Haar measure dt on it. If δ' in G is stably σ-conjugate to δ, $Z_G(\delta\sigma)$ is isomorphic to $Z_G(\delta'\sigma)$. We choose dt and dt' on these groups to assign their maximal compact subgroups the same volumes. The measures dg, dt determine a measure on the quotient $G/Z_G(\delta\sigma)$. Let $f \in C_c^\infty(G)$ be a smooth compactly supported function on G. Put

$$\Phi(\delta\sigma, f dg) = \int_{G/Z_G(\delta\sigma)} f(g\delta\sigma(g)^{-1})\frac{dg}{dt}.$$

If δ is σ-regular, put

$$\Phi^{\mathrm{st}}(\delta\sigma, f dg) = \sum_{\delta'} \Phi(\delta'\sigma, f dg).$$

The sum is over a set of representatives for the σ-conjugacy classes in the stable σ-conjugacy class of δ.

If f_0 is a smooth compactly supported function on H define

$$\Phi(\gamma, f_0 dh) = \int_{H/Z_H(\gamma)} f_0(h\gamma h^{-1}) \frac{dh}{dt}.$$

Here dh is a Haar measure on H and dt on the centralizer $Z_H(\gamma)$. Also put

$$\Phi^{\mathrm{st}}(\gamma, f_0 dh) = \sum_{\gamma'} \Phi(\gamma', f_0 dh).$$

If $\gamma = N\delta$ is regular then $Z_H(\gamma) \simeq Z_G(\delta\sigma)$. The measures on the two groups are related by assigning the maximal compact subgroup the same volume.

The measures $f dg$ and $f_0 dh$ are said to have *matching orbital integrals* and we write $f_0 dh = \lambda^*(f dg)$ if for all γ, δ with regular $\gamma = N\delta$ they satisfy the relation

$$\Phi^{\mathrm{st}}(\gamma, f_0 dh) = \Phi^{\mathrm{st}}(\delta, f dg).$$

3.2 Weyl integration formula. Let $\{T_0\}$ denote a set of representatives for the conjugacy classes of tori of H over F. The regular set H^{reg} of H (distinct eigenvalues) is the union over $\{T_0\}$ of $\mathrm{Int}(H/T_0)(T_0^{\mathrm{reg}})$. The Jacobian of the morphism

$$T_0 \times H/T_0 \to H, \qquad (t, h) \mapsto \mathrm{Int}(h)t = hth^{-1},$$

is

$$D_0(t) = |\det(1 - \mathrm{Ad}\ t) | \,\mathrm{Lie}(H/T_0)|.$$

We have the Weyl integration formula

$$\int_H f_0(h)\, dh = \sum_{\{T_0\}} |W(T_0)|^{-1} \int_{T_0} \Delta_0(t)^2\, dt \int_{H/T_0} f_0(hth^{-1})\, \frac{dh}{dt}.$$

Here $W(T_0)$ is the Weyl group of T_0 (normalizer/centralizer), and $\Delta_0(t)^2 = D_0(t)$. It is $\mathbb{Z}/2$ if T_0 splits over F or -1 lies in $N_{E/F}E^\times$, and $\{0\}$ otherwise, as the normalizer of $T_0 \simeq E^1$ is $x \mapsto \overline{x}$, realized by $\mathrm{Int}(\mathrm{diag}(-1, 1))$ with the choices of section 2.1.

Let $\{T_0\}_s$ denote a set of representatives for the stable conjugacy classes of tori of H over F. It consists of a representative, say the diagonal torus, for the tori which split over F, and elliptic tori, which are parametrized by the quadratic field extensions E of F, where $T_0 = E^1$. The Weyl group of T_0 in $A(T_0)$ (see section 2.1) is $\mathbb{Z}/2$. Hence

$$\int_H f_0(h)\,dh = \frac{1}{2}\sum_{\{T_0\}_s}\int_{T_0}\Delta_0(t)^2\,dt\sum_{t'}\int_{H/Z_H(t')} f_0(ht'h^{-1})\,\frac{dh}{dt}.$$

The sum over t' ranges for a set of representatives for the conjugacy classes within the stable conjugacy class of t in T_0.

Next we write an analogue of the Weyl integration formula in the twisted case. We use the observation of (1.9) that each σ-regular element in G is σ-conjugate to an element $\delta = (ae)_1$ with a in $\mathrm{GL}(2,F)$. Recall that $\delta = (ae)_1$ and $\delta' = (a'e)_1$ are σ-conjugate if and only if $a' = (1/\det b)b^{-1}ab$. Hence we may take the a in $N\mathbf{Z}(E)\backslash T_E$, where T_E ranges over a set of representatives for the conjugacy classes of tori T_2 of $\mathrm{GL}(2)$ over F. If T_2 splits over E (= F or a quadratic extension of F), we denote it by T_E. We denote by $\mathbf{Z}$ the center of $\mathrm{GL}(2)$ and by N the norm map form E to F.

Every σ-regular element of G has the form

$$g\delta\sigma(g)^{-1}, \qquad \delta \in T = T(T_E/N\mathbf{Z}(E)), \qquad g \in G/Z_G(T\sigma),$$

for some E. Here

$$T(T_E/N\mathbf{Z}(E)) = \{\delta_a = (ae)_1; a \in T_E/N\mathbf{Z}(E)\}.$$

The σ-centralizer of T in G,

$$Z_G(T\sigma) = \{g \in G; g\delta\sigma(g)^{-1} = \delta, \forall\delta \in T\},$$

is isomorphic to $Z_H(NT)$ where $NT = N(T) = T_E^1 (= T_E \cap \mathrm{SL}(2,F))$.

The expression is unique up to the action of the σ-normalizer, which consists of the w with $w^{-1}\delta\sigma(w) = \delta' = \delta'(\delta) \in T$ for all δ in T. Then $w^{-1}\delta\sigma(\delta)w = \delta'\sigma(\delta')$. Modulo the centralizer there are two w's, $w = (e)_1$ represents the nontrivial one with the choices made in section 2.1.

The Jacobian of the morphism

$$T \times G/Z_G(T\sigma) \to G, \qquad (\delta, g) \mapsto g\delta\sigma(g)^{-1}$$

is

$$\Delta(\delta\sigma)^2 = |\det[1 - \mathrm{Ad}(\delta)\sigma]|\,\mathrm{Lie}(G/T^\sigma)|.$$

The twisted Weyl integration formula is then (put δ_a for $(ae)_1$)

$$\int_G f(g)\,dg = \frac{1}{2}\sum_E \int_{T_E/N\mathbf{Z}(E)} \Delta(\delta_a\sigma)^2 da \int_{G/Z_G(T\sigma)} f(g\delta_a\sigma(g)^{-1})\frac{dg}{da}.$$

Let us compute $\Delta(\delta\sigma)^2$ explicitly. We may assume δ is $\mathrm{diag}(a,b,c)$. $\mathrm{Lie}\,\mathbf{G}$ consists of $X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$ modulo center. Thus we assume that $x_5 = 0$ to fix representatives. Note that $\mathrm{Lie}\,Z_{\mathbf{G}}(\delta\sigma) = \{\mathrm{diag}(x,0,-x)\}$, since

$$-\sigma X = J^t X J = \begin{pmatrix} x_9 & -x_6 & x_3 \\ -x_8 & x_5 & -x_2 \\ x_7 & -x_4 & x_1 \end{pmatrix},$$

$$X - \mathrm{Ad}(\delta)\sigma X = \begin{pmatrix} x_1+x_9 & x_2-\frac{a}{b}x_6 & (1+\frac{a}{c})x_3 \\ x_4-\frac{b}{a}x_8 & 2x_5 & x_6-\frac{b}{c}x_2 \\ (1+\frac{c}{a})x_7 & x_8-\frac{c}{b}x_4 & x_1+x_9 \end{pmatrix}.$$

Recalling that $x_5 = 0$, and noting that in $\mathrm{Lie}\,\mathbf{G}/Z_{\mathbf{G}}(\delta\sigma)$ the $x_1 + x_9$ is a single variable (alternatively, in X we could replace x_9 by zero and x_1 by $x_1 + x_9$), we conclude that

$$\Delta(\delta\sigma)^2 = \left|\left(1 - \frac{a}{c}\right)\left(1 + \frac{a}{c}\right)\left(1 - \frac{c}{a}\right)\left(1 + \frac{c}{a}\right)\right|.$$

The 4 factors correspond to change of variables on: (x_2, x_6), x_3, (x_4, x_8), x_7. This $\Delta(\delta\sigma)^2$ is then equal to $\Delta_0(\gamma)^2$, $\gamma = N\delta$. Indeed we may assume that $\gamma = \mathrm{diag}(a/c, c/a)$, and then

$$\Delta_0(\gamma)^2 = \left|\left(\frac{a}{c} - \frac{c}{a}\right)^2\right| = \left|\frac{a^2 - c^2}{ac}\right|^2 = \Delta(\delta\sigma)^2.$$

3.3 Characters. Let F be a local (archimedean or not) field, f_i a compactly supported smooth function on H_i, π_i an admissible irreducible representation of H_i, and $\pi_i(f_i dh_i)$ the convolution operator $\int f_i(g)\pi_i(g)\,dg$. This operator has finite rank, see (1.3).

A well-known result of Harish-Chandra ([HC2]) asserts that there exists a complex-valued conjugacy-class function $\chi_i = \chi_{\pi_i}$ on H_i which is smooth on the regular set such that for all measures $f_i dh_i$ on the regular set

$$\operatorname{tr}\pi_i(f_i dh_i) = \int f_i(g)\chi_i(g)\,dg.$$

It is called the *character* of π_i. It is locally integrable on H_i.

The twisted analogue of [HC2] (see [Cl2]) asserts that given a σ-invariant admissible irreducible representation π of G, there exists a complex-valued σ-conjugacy class function $\chi_\pi^\sigma : g \mapsto \chi_\pi(g\sigma)$ on G which is smooth on the σ-regular set, such that

$$\operatorname{tr}\pi(f dg \times \sigma) = \int f(g)\chi_\pi^\sigma(g)\,dg$$

for all measures $f dg$ on the σ-regular set. It is called the *twisted character* of π. It is locally integrable on G, hence the identity extends to all measures $f dg$.

Note that χ_π^σ is the *twisted* character of π. It is not the character in the usual sense. We also write $\chi_\pi(g\sigma)$ for $\chi_\pi^\sigma(g)$. Note that the (twisted) character is defined only on the (σ-) regular set. We need the character and its properties for the orthogonality relations, as well as for the study of the approximation in section V.1, and lifting in section V.2.

A function χ on H is called a conjugacy *class function* if $\chi(h) = \chi(h')$ whenever h, h' are regular and conjugate in H. For example, characters of representations are class functions. We shall later show that characters are dense in the space of class functions. A class function is called a *stable* class function if $\chi(h) = \chi(h')$ whenever h, h' are regular and stably conjugate in H ($h' = ghg^{-1}$ for some $g \in \mathbf{H}(\overline{F})$).

Let $\{\pi_0\}$ be a set of irreducible admissible representations of H such that $\chi_{\{\pi_0\}}$, the sum of $\chi_{\pi_0'}$ where π_0' ranges over $\{\pi_0\}$, is a stable class function. We say that $\{\pi_0\}$ is a *stable* set. Similar definition can be made for a set with multiplicities. But in our case it turns out that the stable class functions that we need are all of the form $\chi_{\{\pi_0\}}$. Note that $\chi_{\{\pi_0\}}$ is the character of the reducible admissible representation $\oplus\pi_0'$, sum over the π_0' in $\{\pi_0\}$.

DEFINITION. The representation π_0 of H_0 *lifts* to the representation π of G if π is σ-invariant and there is a stable set $\{\pi_0\}$ including π_0 such that whenever $\gamma = N\delta$ is a regular element of H we have

$$\chi_\pi(\delta\sigma) = \chi_{\{\pi_0\}}(\gamma).$$

In this case we write $\pi = \lambda_0(\pi_0)$ or $\pi = \lambda_0(\{\pi_0\})$.

REMARK. This definition is based on the definition of the norm N in (2.2). The norm relates stable σ-conjugacy classes in G and stable conjugacy classes in H. To lift, $\gamma \mapsto \chi_{\{\pi_0\}}(\gamma)$ has to be a stable class function. To be a lift of π_0 the twisted character χ_π^σ of π has to be a stable σ-class function, namely $\chi_\pi^\sigma(\delta) = \chi_\pi^\sigma(\delta')$ if δ and δ' are stably σ-conjugate.

3.4 LEMMA. *We have $\pi = \lambda_0(\pi_0)$ if and only if for all fdg, f_0dh with $f_0dh = \lambda_0^*(fdg)$ we have $\operatorname{tr}\pi(fdg \times \sigma) = \operatorname{tr}\{\pi_0\}(f_0dh)$.*

PROOF. Suppose that $\operatorname{tr}\pi(fdg \times \sigma) = \operatorname{tr}\{\pi_0\}(f_0dh)$. We use the Weyl integration formula of (3.2) to write $\operatorname{tr}\pi(fdg \times \sigma) = \int f(g)\chi_\pi^\sigma(g)\,dg$ as

$$\sum_{\{E\}} \frac{1}{2} \int_{N\mathbf{Z}(E)\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi^\sigma((ae)_1)\Phi((ae)_1\sigma, fdg)da.$$

Fix a quadratic extension E of F. Denote by T_E the element of $\{T_2\}$ (i.e., a torus in $\mathrm{GL}(2,F)$) which splits over E. Take fdg so that its twisted orbital integral $\Phi(\delta\sigma, fdg)$ is supported on T_E, namely on the σ-orbits of the $\delta_a = (ae)_1$ with a in T_E. We claim that

$$\operatorname{tr}\pi(fdg \times \sigma) = \frac{1}{2} \int_{Z\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi^\sigma(\delta)\Phi^{\mathrm{st}}(\delta\sigma, fdg)da \qquad (\delta = (ae)_1),$$

where $\Phi^{\mathrm{st}}(\delta\sigma, fdg)$ denotes the stable twisted orbital integral of fdg at δ, as in (3.1). To show this, note that the trace $\operatorname{tr}\pi(fdg \times \sigma)$ depends only on the stable twisted orbital integral of fdg, since it is equal to $\operatorname{tr}\{\pi_0\}(f_0dh)$. If we take $f_0 = 0$, then for each a in T_E we have

$$\Phi((uae)_1\sigma, fdg) = -\Phi((ae)_1\sigma, fdg) \qquad (u \in F - N_{E/F}E).$$

Since $\operatorname{tr}\pi(fdg \times \sigma)$ vanishes for such fdg, we have

$$\int_{Z\backslash T_E} \Delta_0(\gamma)^2[\chi_\pi^\sigma((ae)_1) - \chi_\pi^\sigma((uae)_1)]\Phi((ae)_1\sigma, fdg)da = 0.$$

Choosing fdg so that the support of $\Phi((ae)_1\sigma, fdg)$ is small, we deduce that

$$\chi_\pi^\sigma((ae)_1) = \chi_\pi^\sigma((uae)_1)$$

depends only on the stable σ-conjugacy class of $(ae)_1$. Hence the claim follows.

On the other hand,

$$\begin{aligned}\operatorname{tr}\{\pi_0\}(f_0dh) &= \int f_0(g)\chi_{\{\pi_0\}}(g)\,dg \\ &= \sum_{\{T_0\}} [W(T_0)]^{-1} \int_{T_0} \Delta_0(\gamma)^2 \chi_{\{\pi_0\}}(\gamma)\Phi(\gamma, f_0dh)d\gamma \\ &= \frac{1}{2}\int_{T_{0E}} \Delta_0(\gamma)^2\chi_{\{\pi_0\}}(\gamma)\Phi^{\rm st}(\gamma, f_0dh)d\gamma.\end{aligned}$$

The last equality follows from our assumption on f_0: the stable orbital integral $\Phi^{\rm st}(\gamma, f_0dh)$ of f_0dh at γ is supported on (the stable conjugacy class of) the torus T_{0E} in $\{T_0\}$ which splits over E. Since the map $F^\times\backslash E^\times \to E^1$ by $z \mapsto z/\overline{z}$ is a bijection and serves to relate measures from $Z\backslash T_E$ to the torus T_{0E} of $\mathrm{SL}(2,F)$, and $f_0dh = \lambda_0^*(fdg)$ means $\Phi^{\rm st}(\delta\sigma, fdg) = \Phi^{\rm st}(\gamma, f_0dh)$ for all δ, γ with $N\delta = \gamma$, it follows that $\pi = \lambda_0(\pi_0)$.

The opposite direction is proven by reversing the above steps. □

3.5 Unstable characters. Recall that the norm map N_1 of (2.2) bijects the stable σ-regular σ-conjugacy classes in G with the regular conjugacy classes in $H_1 = \mathrm{SO}(3,F)$. In each stable σ-conjugacy class of elements δ such that $\delta\sigma(\delta)$ has distinct eigenvalues there are two σ-conjugacy classes (unless the eigenvalues of $\delta\sigma(\delta)$ lie in $F^\times$, in which case there is a single σ-conjugacy class). They differ by whether $Z_G(\delta'\sigma)$ is split or not for a representative δ, and we write $\boldsymbol{\kappa}(\delta) = 1$ or -1 accordingly. Here we put $\delta' = \frac{1}{2}(\delta + J^t\delta J)$ as in (2.4), and note that the σ-centralizer $Z_G(\delta'\sigma)$ of δ' depends only on the σ-conjugacy class of δ, up to conjugacy in G.

The twisted character χ_π is a *σ-class function* on the σ-regular set, namely,

$$\chi_\pi^\sigma(g\delta\sigma(g)^{-1}) = \chi_\pi^\sigma(\delta)$$

for all g in G. By an *unstable σ-class function* we mean a σ-class function which satisfies $\chi_\pi^\sigma(\delta) = -\chi_\pi^\sigma(\tilde{\delta})$ whenever δ, $\tilde{\delta}$ are stably σ-conjugate but not σ-conjugate.

Note that if $\tilde{\delta}$, δ are stably σ-conjugate, but not conjugate, then up to σ-conjugacy $\delta = (ae)_1$ and $\tilde{\delta} = (uae)_1$ with u in $F^\times$ but not in $N_{E/F}E^\times$, where E/F is a quadratic extension determined by δ.

Definition. The representation π_1 of $H_1 = \mathrm{SO}(3, F)$ *lifts* to the representation π of G if χ_π^σ is an unstable σ-class function and

$$|(1+\gamma')(1+\gamma'')|^{1/2}\chi_\pi^\sigma(\delta) = \chi_{\pi_1}(\gamma_1) \tag{3.5.1}$$

for all γ_1 in H_1 and δ in G such that $Z_G(\delta'\sigma)$ is split and $N_1\delta = \gamma_1$ has distinct eigenvalues as an element of $H_1 = \mathrm{SO}(3, F)$. Here γ', γ'' denote the eigenvalues of γ_1 which are not equal to 1. Note that $\chi_\pi^\sigma(\delta) = -\chi_\pi^\sigma(\delta')$ whenever δ, δ' are stably σ-conjugate but not σ-conjugate. We then write $\pi = \lambda_1(\pi_1)$.

We shall relate orbital integrals on G and on $H_1 = \mathrm{SO}(3, F)$.

3.6 Definition. If $\gamma_1 = N_1\delta$ has eigenvalues 1, γ', γ'' with $\gamma' \neq \gamma''$, put

$$\Phi^{\mathrm{us}}(\delta\sigma, fdg) = \sum_{\delta'} \boldsymbol{\kappa}(\delta')\Phi(\delta'\sigma, fdg).$$

If f_1 is a smooth compactly supported function on H_1 then for all regular semisimple γ_1 we put

$$\Phi(\gamma_1, f_1dh_1) = \int_{H_1/Z_{H_1}(\gamma_1)} f_1(h\gamma_1h^{-1})\frac{dh}{dt}.$$

We say that $f_1dh_1 = \lambda_1^*(fdg)$ if, when the measures $d\gamma_1$, $d\delta$ used in the definition of the orbital integrals assign the same volume to the maximal compact subgroups of $Z_{H_1}(\gamma_1)$ and $Z_G(\delta\sigma)$, we have

$$\Phi(\gamma_1, f_1dh_1) = |(1+\gamma')(1+\gamma'')|^{1/2}\Phi^{\mathrm{us}}(\delta\sigma, fdg)$$

for all $\gamma_1 = N_1\delta$ with distinct eigenvalues.

3.7 Lemma. *We have* $\operatorname{tr}\pi(fdg \times \sigma) = \operatorname{tr}\pi_1(f_1dh_1)$ *for all* fdg, f_1dh_1 *with* $f_1dh_1 = \lambda_1^*(fdg)$ *if and only if* $\pi = \lambda_1(\pi_1)$.

Proof. If $\operatorname{tr}\pi(fdg \times \sigma) = \operatorname{tr}\pi_1(f_1dh_1)$ for fdg, f_1dh_1 with $f_1dh_1 = \lambda_1^*(fdg)$, then $\operatorname{tr}\pi(fdg \times \sigma)$ is equal to $\int f_1(g)\chi_{\pi_1}(g)\,dg$, which by the Weyl integration formula of (3.2), is

$$\sum_{\{T_1\}} \frac{1}{2} \int_{T_1} \Delta_1(\gamma_1)^2 \chi_{\pi_1}(\gamma_1) \Phi(\gamma_1, f_1 dh_1)\, d\gamma_1$$
$$= \sum_{\{T_1\}} \frac{1}{2} \int_{T_1} \Delta_1(\gamma_1)^2 \chi_{\pi_1}(\gamma_1) |(1+\gamma')(1+\gamma'')|^{1/2} \Phi^{\mathrm{us}}(\delta\sigma, fdg)\, d\gamma_1.$$

We write Δ_1 to emphasize that the Δ-factor is on the group H_1. The sum is taken over a set of representatives for the conjugacy classes of tori T_1 of H_1 over F. Recall that $H_1 = \mathrm{SO}(3) \simeq \mathrm{PGL}(2)$, and in H_1 a stable conjugacy class is a conjugacy class.

The element δ, or rather its σ-conjugacy class, is uniquely determined by γ_1 and the requirement that $Z_G(\delta\sigma)$ be split over F. Moreover, $\Phi^{\mathrm{us}}(\tilde{\delta}\sigma, fdg)$ is $-\Phi^{\mathrm{us}}(\delta\sigma, fdg)$ if δ, $\tilde{\delta}$ are stably σ-conjugate but not σ-conjugate.

Define χ_π^σ by the equation (3.5.1) to be an unstable σ-conjugacy class function. Then our sum becomes

$$\sum_{\{T_2\}} \frac{1}{2} \int_{Z\backslash T_2} \Delta_0(\gamma)^2 \chi_\pi^\sigma(\delta) \Phi^{\mathrm{us}}(\delta\sigma, fdg) da.$$

The sum is over conjugacy classes of F-tori T_2 in $\mathrm{GL}(2,F)$, $\delta = (ae)_1$, $\gamma = (-1/\det a)a^2$, and $a \mapsto \gamma_1$ defines an isomorphism of $Z\backslash T_2$ and T_1 for tori T_2, T_1 which share their splitting field. Note that when the eigenvalues of a are u, v, then those of γ are $-u/v$, $-v/u$, we have

$$\Delta_0(\gamma) = \left|\left(\frac{u}{v} - \frac{v}{u}\right)^2\right|^{1/2} = \left|\left(1+\frac{u}{v}\right)\left(1-\frac{v}{u}\right)\right| = \left|\left(1-\frac{u}{v}\right)\left(1+\frac{u}{v}\right)\frac{v}{u}\right|$$

and

$$\Delta_1(\gamma_1) = \left|\frac{(u-v)^2}{uv}\right|^{1/2} = \left|\left(1-\frac{u}{v}\right)\left(1-\frac{v}{u}\right)\right|^{1/2} = \left|\left(1-\frac{u}{v}\right)\left(1-\frac{u}{v}\right)\frac{v}{u}\right|^{1/2}.$$

Hence

$$\Delta_0(\gamma)^2 = \Delta_1(\gamma_1)^2 |(1+\gamma')(1+\gamma'')|, \qquad \gamma' = \gamma''^{-1} = \frac{u}{v}.$$

The sum is equal to

$$\sum_{\{T_E\}} \frac{1}{2} \int_{N\mathbf{Z}(E)\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi^\sigma(\delta) \Phi(\delta\sigma, fdg) d\delta.$$

This is

$$\int f(g)\chi_\pi^\sigma(g)\,dg$$

by the twisted Weyl formula (3.2). Hence $\pi = \lambda_1(\pi_1)$ by the definition of χ_π^σ and λ_1. □

3.8 Induced. Let $\pi = I(\eta)$ denote the representation of G normalizedly induced from the character $\eta(\mathrm{diag}(a,b,c)) = \mu(a/c)$ of the Borel subgroup B, where μ is a character of $F^\times$. Denote by $\pi_0 = I_0(\mu)$ and $\pi_1 = I_1(\mu)$ the representations of H_0, H_1 normalizedly induced from the characters

$$\left(\begin{smallmatrix} a & * \\ 0 & a^{-1} \end{smallmatrix}\right) \mapsto \mu(a), \qquad \left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \mapsto \mu(a/b)$$

of the upper triangular Borel subgroups. Then the computation of (1.6) and the Weyl integration formulae of (3.2) show that the σ-character χ_π^σ of $\pi = I(\eta)$ vanishes at δ unless δ is diagonal (up to σ-conjugacy), where

$$\chi_\pi^\sigma(\delta) = \Delta_0(\gamma)^{-1}(\eta(\delta) + \eta(\tilde{\delta})) \qquad (\tilde{\delta} = J\delta J, \quad \gamma = N\delta).$$

Similar standard computations show that the χ_{π_i} are also supported on the (conjugacy classes of) diagonal elements of H_i. They are given there by

$$\chi_{\pi_0}(\gamma) = \Delta_0(\gamma)^{-1}(\mu(a) + \mu(a^{-1})), \qquad \gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

and

$$\chi_{\pi_1}(\gamma_1) = \Delta_1(\gamma_1)^{-1}(\mu(a) + \mu(a^{-1})), \qquad \gamma_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that if $\pi = I(\eta)$, $\pi_0 = I_0(\mu)$, $\pi_1 = I_1(\mu)$, then

Lemma. *$\pi = \lambda_0(\pi_0) = \lambda_1(\pi_1)$, namely $I_0(\mu)$ and $I_1(\mu)$ both lift to $I(\eta)$.*

Proof. The characters of π, π_i are supported on the split tori, and the stable σ-conjugacy class of an element where χ_π^σ does not vanish consists of a single σ-conjugacy class. □

Remark. Here the field F is any (archimedean or not) local field.

3.9 Special representation. Let F be nonarchimedean. Let ν denote the valuation character of $F^\times$, thus $\nu(x) = |x|$. The composition series of the induced representation $I_0 = I_0(\nu)$ of H consists of the one-dimensional representation $\mathbf{1}_0$ and of the special, or Steinberg, representation sp, of H.

Note that sp is irreducible. But by Lemma 3.8 I_0 lifts to the representation $\pi = I(\eta)$ of G, induced from the character $\eta = (\nu, 1, \nu^{-1})$ of the upper triangular Borel subgroup of G. The composition series of π consists of the trivial representation $\mathbf{1}_3$, the irreducible representation $\pi_{P_1}(\mathrm{sp}(\nu,1),\nu^{-1})$ normalizedly induced from the representation $\mathrm{sp}(\nu,1)\times\nu^{-1}$ of the maximal parabolic subgroup P_1 of type $(2,1)$, and the reducible representation $I_{P_2}(\nu,\mathrm{sp}(1,\nu^{-1}))$ induced from the maximal parabolic P_2 of type $(1,2)$. This last representation has composition series consisting of the irreducible $\pi_{P_2}(\nu,\mathrm{sp}(1,\nu^{-1}))$ and the Steinberg representation St. This result is due to Bernstein-Zelevinsky [BZ2]. Now $I_{P_2}(\nu,\mathrm{sp}(1,\nu^{-1}))$ is not σ-invariant, but St, being the unique square-integrable irreducible constituent of $I(\eta) \simeq {}^{\sigma}I(\eta)$, is σ-invariant. Hence, $\pi_{P_2}(\nu,\mathrm{sp}(1,\nu^{-1}))$, as well as $\pi_{P_1}(\mathrm{sp}(\nu,1),\nu^{-1})$ (for the same reason), is not σ-invariant. The one-dimensional representation $\mathbf{1}_3$ of G is clearly σ-invariant. Hence

$$\operatorname{tr} I(\eta)(f dg \times \sigma) = \operatorname{tr}\mathrm{St}(f dg\times\sigma) + \operatorname{tr}\mathbf{1}_3(f dg\times\sigma).$$

Lemma. *The trivial and special representations of H lift to the trivial and Steinberg representations of G, respectively.*

Proof. As the characters of both $\mathbf{1}_0$ and $\mathbf{1}_3$ are identically one, the lemma follows at once from the definition (3.3) of the lifting. □

Remark. The only σ-invariant one-dimensional representation π of G is the trivial one. Indeed, π is given by a character β of $F^\times$ (namely, $\pi(g) = \beta(\det g)$) of order 3, thus $\beta^3 = 1$. But π is σ-invariant only if $\beta = \beta^{-1}$. Hence $\beta = 1$ and π is trivial, as asserted.

I.4 Orthogonality

4.1 Orthogonality relations. For any conjugacy class functions χ, χ' on the elliptic set H_e of H put

$$\langle\chi,\chi'\rangle_e = \int_{H_e/\sim}\chi(h)\overline{\chi}'(h)dh$$

$$= \sum_{\{T_0\}}[W(T_0)]^{-1}|T_0|^{-1}\int_{T_0}\Delta_0(\gamma)^2\chi(\gamma)\overline{\chi}'(\gamma)d\gamma.$$

The sum ranges over a set of representatives T_0 for the conjugacy classes of elliptic tori of H over F. $[W(T_0)]$ is the cardinality of the Weyl group of T_0 (1 or 2). As usual, $|T_0|$ denotes the volume of T_0. We write $\gamma \sim \gamma'$ if γ, γ' are conjugate. The measure dh on $H_e/\sim$ is defined by the last displayed equality. The Hermitian bilinear form $\langle \chi, \chi' \rangle_e$ satisfies the Schwartz inequality

$$\langle \chi, \chi' \rangle_e^2 \leq \langle \chi, \chi \rangle_e \cdot \langle \chi', \chi' \rangle_e.$$

If χ, χ' are stable conjugacy class functions, $\langle \chi, \chi' \rangle_e^2$ is equal to

$$\langle \chi, \chi' \rangle_e = \frac{1}{2} \sum_{\{T_0\}_s} [D(T_0)] |T_0|^{-1} \int_{T_0} \Delta_0(\gamma)^2 \chi(\gamma) \overline{\chi}'(\gamma) d\gamma.$$

Here the sum is taken over a set of representatives T_0 for the stable conjugacy classes of elliptic tori of H over F. $[D(T_0)]$ is the number of conjugacy classes within the stable conjugacy class of T_0; it is 2 if T_0 is elliptic, 1 if T_0 is split.

Tempered (irreducible) representations π, π' of a reductive p-adic group G are called *relatives* if both are direct summands of the representation normalizedly induced from a tempered representation of a parabolic subgroup of G (which is trivial on the unipotent radical). The orthogonality relations for characters (see [K2], Theorems G, K) assert that $\langle \chi_\pi, \chi_{\pi'} \rangle_e$ is zero unless the tempered π, π' are relatives, and if one of them is square integrable then the result is 1 if $\pi \simeq \pi'$ and 0 if not. Then

4.1.1 LEMMA. *Let $\{\pi_0\}$ and $\{\pi_0'\}$ be stable finite sets of admissible irreducible tempered representations of H which are induced or square integrable. Then $\langle \chi_{\{\pi_0\}}, \chi_{\{\pi_0'\}} \rangle_e$ is equal to the number of square-integrable irreducible representations in $\{\pi_0\} \cap \{\pi_0'\}$.* □

4.2 Twisted orthogonality. Let π be a σ-invariant irreducible representation of G. As in (1.2) there is an intertwining operator A from the space of π to itself such that ${}^\sigma\pi(g) = \pi(\sigma(g))$ is equal to $A\pi(g)A^{-1}$. Since π is irreducible and A^2 intertwines π with itself, by Schur's lemma A^2 is a scalar, which we may normalize (by multiplying A with $1/\sqrt{A^2}$) to be 1. Extend π to a representation π' of $G' = G \rtimes \langle \sigma \rangle$ by setting $\pi(\sigma) = A$.

As noted in (3.3), the twisted character $\chi_{\pi'}^\sigma$ of π' is a σ-conjugacy class function which is locally integrable on G and is smooth on the subset of G

which consists of δ with regular $\gamma = N\delta$. Such δ is called σ-regular. Its σ-centralizer $Z_{\mathbf{G}}(\delta\sigma)$ in $\mathbf{G}$ is isomorphic to the centralizer $Z_{\mathbf{H}}(\gamma)$ of γ in $\mathbf{H}$.

For any two σ-conjugacy class functions χ^σ and χ'^σ on the σ-elliptic (δ with elliptic $N(\delta)$) subset G_e^σ of G define $\langle \chi^\sigma, \chi'^\sigma\rangle_e$ to be

$$\frac{1}{2}\sum_E |Z\backslash T_E|^{-1}\int_{T_E/N\mathbf{Z}(E)} \Delta_0(\gamma)^2\chi(\delta\sigma)\overline{\chi}'(\delta'\sigma)da.$$

We write $\chi(\delta\sigma)$ for $\chi^\sigma(\delta)$. The sum defines a measure dg on $G_e^\sigma/\sim$, where $\delta\sim\delta'$ if δ is σ-conjugate to δ', for which

$$\langle \chi^\sigma, \chi'^\sigma\rangle_e = \int_{G_e^\sigma/\sim} \chi^\sigma(g)\overline{\chi}'^\sigma(g)dg.$$

If $\delta\mapsto\chi(\delta\sigma)$ is a stable σ-conjugacy class function, the inner product can be written as

$$\frac{1}{2}\sum_E |Z\backslash T_E|^{-1}\int_{T_E/Z} \Delta_0(\gamma)^2\chi(\delta\sigma)\sum_{\delta'}\overline{\chi}'(\delta'\sigma)da.$$

The sum over δ' ranges over a set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of δ. For a in T_E we have $\delta = (ae)_1$, and there are two δ' in our case of δ with compact $Z_G(\delta\sigma)\simeq Z_H(\gamma)$, $\gamma = N(\delta)$.

4.2.1 LEMMA. *Given a stable conjugacy class function χ on H_e define $\chi_G(\delta) = \chi(N(\delta))$. Given a stable σ-conjugacy class function χ^σ on G_e^σ define $\chi_H(\gamma) = \chi(\delta\sigma)$ for $\gamma = N(\delta)$. Then*

$$\langle \chi^\sigma, \chi'_G\rangle_e = \langle \chi_H^\sigma, \chi'\rangle_e.$$

PROOF. This is clear from the definitions. Note that the inner product on the left is on G, while the one on the right is on H. □

Let π be a cuspidal σ-invariant representation. Such π do not exist unless the residual characteristic of F is 2. (This is proven in chapter V using the trace formula.). The orthogonality relations for characters assert in this case the following.

4.2.2 LEMMA. *Let π_2 be a σ-invariant irreducible admissible representation of G and π a σ-invariant cuspidal representation of G. Suppose that the function $\delta \mapsto \chi_\pi(\delta\sigma)$ is a stable σ-conjugacy class function on G_e^σ. Then $\langle \chi_\pi^\sigma, \chi_{\pi_2}^\sigma \rangle_e$ is equal to* 0 *unless π and π_2 are equivalent, in which case it is equal to* 1.

Thus for π which is cuspidal and σ-stable (by which we mean that $\chi_{\pi'}^\sigma$ is a stable σ-class function), $\langle \chi, \chi \rangle_e$ (inner product on $H_e/\sim$) is equal to 1, where χ is the stable class function on H defined by $\chi(N\delta) = \chi_{\pi'}(\delta\sigma)$.

PROOF. First suppose that π_2 is equivalent to π. Put $\pi'_i = \omega^i \pi'$ $(i = 0, 1)$, where ω is the character of G' which attains the value 1 on G and the value -1 at σ. The representations π'_0, π'_1 are inequivalent. Put

$$\overline{\phi}(g) = d(\pi)(\pi'(g)u, \tilde{u}), \qquad \pi'_i(\phi dg) = \int_{G'} \phi(g)\pi'_i(g)\, dg.$$

Here $d(\pi)$ denotes the formal degree of π; u, $\tilde{u}$ are vectors in the space of π and the contragredient of π, with $(u, \tilde{u}) = 1$. By the Schur orthogonality relations for the square-integrable representations π'_i we have

$$\operatorname{tr} \pi'_0(\phi dg) = 1, \qquad \operatorname{tr} \pi'_1(\phi dg) = 0.$$

Then

$$1 = \operatorname{tr} \pi'_0(\phi dg) - \operatorname{tr} \pi'_1(\phi dg) = 2 \int_G \phi(g\sigma)\chi_\pi(g\sigma) dg.$$

By the Weyl integration formula (3.2) this is equal to

$$2 \cdot \frac{1}{2} \sum_E \int_{N\mathbf{Z}(E)\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi(\delta\sigma) da \int_{G/Z_G(\delta\sigma)} \phi(g\delta\sigma(g)^{-1}) \frac{dg}{da}$$
$$= 2 \cdot \frac{1}{2} \sum_E \int_{Z\backslash T_E} \Delta_0(\gamma)^2 \chi_\pi(\delta\sigma) da \sum_{\delta'} \int_{G/Z_G(\delta'\sigma)} \phi(g\delta'\sigma(g)^{-1}) \frac{dg}{da}.$$

Harish-Chandra's "Selberg principle" [HC1], Theorem 29 implies the vanishing of the inner integral if $Z_G(\delta\sigma) \simeq Z_H(\gamma)$ is a torus of H which splits over F. If it is a compact torus of $\mathbf{H} = \mathrm{SL}(2)$ over F then the proof of [JL], Lemma 7.4.1, shows that

$$\chi_\pi(\delta\sigma) = d(\pi) \int_G [(\pi'(g \cdot \delta\sigma \cdot g^{-1})u, \tilde{u}) + (\pi'(g\sigma \cdot \delta\sigma \cdot (g\sigma)^{-1})u, \tilde{u})] dg$$

$$= 2\,d(\pi)|Z_G(\delta\sigma)| \int_{G/Z_G(\delta\sigma)} (\pi'(g\delta\sigma(g)^{-1}\cdot\sigma)u, \tilde{u})\frac{dg}{da}.$$

Note that $\delta\sigma(\delta)\sigma(\delta)^{-1} = \delta$ for the last equality. We obtain

$$\frac{1}{2}\sum_E |Z_G(\delta\sigma)|^{-1}\int_{Z_H(\gamma)} \Delta_0(\gamma)^2\chi_\pi(\delta\sigma)\sum_{\delta'}\overline{\chi}_\pi(\delta'\sigma)d\gamma.$$

We used the isomorphism $Z\backslash T_E \simeq Z_G(\delta\sigma) \simeq Z_H(\gamma)$, and the relation $d\delta$ $(= da) = d\gamma$ of measures on the groups $Z_G(\delta\sigma)$, $Z_H(\gamma)$.

It remains to deal with the case where π and π_2 are inequivalent. But then $(\omega^i\pi_2')(\phi) = 0$ for both i, and the lemma follows using the same argument. □

4.2.3 Lemma. *We have that $\langle\chi_\pi^\sigma, \chi_\pi^\sigma\rangle_e$ is 1 if π is the σ-invariant Steinberg representation.*

Proof. This follows from (4.1) and Lemma 3.9. The orthogonality relation (4.1) for sp follows from the orthogonality relation for the trivial representation of the group of elements of reduced norm 1 in the quaternion division algebra, and the correspondence of [JL]. □

To deal with π which are not cuspidal or Steinberg, we record a special case of a twisted analogue of [K2], Theorem G. The proof in the twisted case, for an arbitrary reductive not necessarily connected p-adic group, follows closely that of [K2], and will not be given here. Thus, let π, π' be σ-invariant, tempered representations with characters χ_π^σ, $\chi_{\pi'}^\sigma$. Each of π, π' defines a unique (up to association) parabolic subgroup and a square-integrable representation ρ, ρ' of its Levi factor, such that π is a subrepresentation of $I(\rho)$ and π' of $I(\rho')$. Then π, π' are called *relatives* if ρ is equivalent to ρ'. Recall that we have the inner product

$$\langle\chi^\sigma, \chi'^\sigma\rangle_e = \sum_E |T_0|^{-1}\int_{T_E/N\mathbf{Z}(E)} \Delta_0(\gamma)^2\chi(\delta\sigma)\overline{\chi}'(\delta\sigma)\,da.$$

4.2.4 Lemma ([K2]). *If π, π' are not relatives then $\langle\chi_\pi^\sigma, \chi_{\pi'}^\sigma\rangle_e = 0$.*

The same result holds also when F is the field of real numbers.

In our case of $\mathbf{G} = \mathrm{PGL}(3)$, a G-module normalizedly induced from a tempered one is irreducible, and we need only the following special case of the lemma.

4.2.5 Corollary. *If π, π' are inequivalent σ-invariant tempered G-modules, then $\langle \chi^\sigma_\pi, \chi^\sigma_{\pi'}\rangle_e$ is zero.*

The methods of [K2] do not afford computing the value $\langle \chi^\sigma_\pi, \chi^\sigma_{\pi'}\rangle_e$. But in the case of any (σ-stable) cuspidal π, we have $\langle \chi^\sigma_\pi, \chi^\sigma_\pi\rangle_e = 1$ by (4.2.2). In the local lifting theorem of chapter V we list all σ-stable elliptic π, and compute $\langle \chi^\sigma_\pi, \chi^\sigma_\pi\rangle_e$. It is equal to the cardinality of the set $\{\pi_0\}$ which lifts to π.

4.3 Definition. Let $\mathbf{J}$ be a reductive group over a local field, π a square-integrable irreducible J-module, and fdg a smooth compactly supported (modulo center) measure on J. Then fdg is called a *pseudo-coefficient* of π if $\operatorname{tr}\pi(fdg) = 1$ and $\operatorname{tr}\pi'(fdg) = 0$ for any irreducible tempered J-module π' inequivalent to π.

The existence of pseudo-coefficients for $H = \mathrm{SL}(2, F)$ is well known. Their existence for any p-adic group is proven in Kazhdan [K2], Theorem K. The orbital integral of fdg is equal to $|Z_J(\gamma)|^{-1}\overline{\chi}_\pi(\gamma)$ at an elliptic regular γ (whose centralizer $Z_J(\gamma)$ is a torus), and to zero on the regular nonelliptic set.

Pseudo-coefficients of σ-invariant representations are analogously defined: fdg is called a *pseudo-coefficient* of a σ-invariant (irreducible) representation π if $\operatorname{tr}\pi(fdg \times \sigma) = 1$ and $\operatorname{tr}\pi'(fdg \times \sigma) = 0$ for any irreducible tempered representation π' of G which is not a relative of π. In fact the name σ-pseudo-coefficient is more accurate, but too long, so we omit the prefix σ in the context of representations of G. The σ-orbital integral of fdg is equal to a nonzero multiple of $|Z_G(\gamma\sigma)|^{-1}\overline{\chi}_\pi(\delta\sigma)$ at any σ-elliptic σ-regular δ (whose σ-centralizer $Z_G(\gamma\sigma)$ is a torus), and to zero on the regular nonelliptic set.

4.3.1. Suppose that F is local, G is a reductive group over F, π is an admissible representation of G, C is a compact open subgroup of $F^\times$, fdg is the measure of volume 1 on G which is supported on C and is constant there.

Lemma. *The number $\operatorname{tr}\pi(fdg)$ is equal to the dimension of the space of C-fixed vectors in π, namely it is a nonnegative integer.*

Proof. The operator $\pi(fdg)$ is the projection on the space of C-fixed vectors in π. □

II. ORBITAL INTEGRALS

Summary. It is shown that the stable twisted orbital integral of the unit element of the Hecke algebra of $\mathrm{PGL}(3,F)$ is suitably related to the stable orbital integral of the unit element of the Hecke algebra of $\mathrm{SL}(2,F)$, while the unstable twisted orbital integral of the unit element on $\mathrm{PGL}(3,F)$ is matched with the orbital integral of the unit element on $\mathrm{PGL}(2,F)$. The direct and elementary proof of this *fundamental lemma* is based on a twisted analogue of Kazhdan's decomposition of compact elements into a commuting product of topologically unipotent and absolutely semisimple elements.

II.1 Fundamental lemma

Let F be a p-adic field ($p \neq 2$), and $\overline{F}$ a separable closure of F. Put

$$\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2), \quad \mathbf{G} = \mathrm{PGL}(3) = \mathrm{GL}(3)/\mathbf{Z}, \quad \mathbf{H}_1 = \mathrm{SO}(3, J)$$

where $\mathbf{Z}$ is the center of $\mathrm{GL}(3)$ and $J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$. Put

$$H = \mathbf{H}(F), \quad G = \mathbf{G}(F) \;\; (= \mathrm{GL}(3,F)/Z, \;\; Z = \mathbf{Z}(F)), \quad H_1 = \mathbf{H}_1(F).$$

Put $\sigma(g) = J \cdot {}^t g^{-1} \cdot J$ for $g \in \mathrm{GL}(3, \overline{F})$. The elements δ, δ' of G are called (*stably*) *σ-conjugate* if there is x in G (resp. $\mathbf{G}(\overline{F})$) with $\delta' = x\delta\sigma(x^{-1})$, or $\delta'\sigma = \mathrm{Int}(x)(\delta\sigma)$ in the semidirect product $G \rtimes \langle\sigma\rangle$. The elements γ, γ' of H are called (*stably*) *conjugate* if $\gamma' = \mathrm{Int}(x)\gamma$ for some x in H (resp. $\mathbf{H}(\overline{F})$); similar definitions apply to H_1.

A norm map N, from the set of stable σ-conjugacy classes in G, to the set of stable conjugacy classes in H, as well as such a map N_1 to the set of conjugacy classes in H_1, is defined in chapter I, (2.2)-(2.8). To recall this definition in the crucial, σ-regular case, note that for any $\delta \in G$, $(\delta\sigma)^2 = \delta\sigma(\delta) \in \mathrm{SL}(3,F)$ has an eigenvalue 1 (chapter I, end of (1.8)). Now if $\delta\sigma(\delta)$ is semisimple, with eigenvalues λ, 1, λ^{-1}, then $N\delta$ is the stable class in H with eigenvalues λ, λ^{-1}. If $\lambda \neq -1$ then $N_1\delta$ is the class in H_1 with eigenvalues λ, 1, λ^{-1}.

Denote by $Z_G(\delta\sigma)$ the group of x in G with $\delta\sigma = \mathrm{Int}(x)(\delta\sigma)$, by $Z_H(\gamma)$ the centralizer of γ in H, and by $Z_{H_1}(\gamma_1)$ the centralizer of γ_1 in H_1. For $f \in C_c^\infty(G)$, $f_0 \in C_c^\infty(H)$, $f_1 \in C_c^\infty(H_1)$, define the *orbital integrals*

$$\Phi(\delta\sigma, f dg) = \int_{G/Z_G(\delta\sigma)} f(\mathrm{Int}(x)(\delta\sigma)) \frac{dx}{dt},$$

$$\Phi(\gamma_i, f_i dh_i) = \int_{H_i/Z_{H_i}(\gamma_i)} f_i(\mathrm{Int}(x)(\gamma_i)) \frac{dx}{dt},$$

$(i = 0, 1)$, where we put $f(g\sigma) = f(g)$. These depend on choices of Haar measures, denoted dg or dx or dh_i depending on the context. In this section we mostly omit the measures from the notations. The measures on the centralizers are compatible with the isomorphisms $Z_G(\delta\sigma) \simeq Z_H(N\delta) \simeq Z_{H_1}(N_1\delta)$ when $\lambda \neq \pm 1$ (in this case $\delta\sigma, \gamma, \gamma_1$ are called *regular*).

Denote by $\{\delta'\}$ a set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of $\delta \in G$; it consists of one or two elements. Define the stable σ-orbital integral of f at δ with $\lambda \neq \pm 1$ by

$$\Phi^{\mathrm{st}}(\delta\sigma, f dg) = \sum_{\{\delta'\}} \Phi(\delta'\sigma, f dg).$$

Similarly put

$$\Phi^{\mathrm{st}}(\gamma, f_0 dh_0) = \sum_{\{\gamma'\}} \Phi(\gamma', f_0 dh_0).$$

Define

$$\Delta(\delta\sigma) = |(1+\lambda)(1+\lambda^{-1})|^{1/2}.$$

Put $\boldsymbol{\kappa}(\delta') = 1$ if

$$\mathrm{SO}\left(\frac{1}{2}[(\delta' J) + {}^t(\delta' J)]\right)$$

is split, and $\boldsymbol{\kappa}(\delta') = -1$ otherwise.

Define $\Phi^{\mathrm{us}}(\delta\sigma, f dg)$ to be $\Phi(\delta\sigma, f dg)$ if $\lambda \in F^\times$, but if $\lambda \notin F^\times$ it is

$$\sum_{\{\delta'\}} \boldsymbol{\kappa}(\delta')\Phi(\delta'\sigma, f dg).$$

Let R be the ring of integers of F. Put $K = \mathbf{G}(R)$, $K_0 = \mathbf{H}(R)$, $K_1 = \mathbf{H}_1(R)$. Denote by f^0 the function on G which is supported on K and whose value there is $1/\operatorname{vol}(K) = |K|^{-1}$. Denote by f_i^0 the quotient of the characteristic function ch_{K_i} of K_i in H_i by $\operatorname{vol}(K_i) = |K_i|, i = 0, 1$. Recall: $p \neq 2$.

Theorem. *For* $\lambda \neq \pm 1$ *we have* $\Phi^{\mathrm{st}}(\delta\sigma, f^0 dg) = \Phi^{\mathrm{st}}(N\delta, f_0^0 dh_0)$, *and* $\Delta(\delta\sigma)\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg) = \Phi(N_1\delta, f_1^0 dh_1)$.

This is the fundamental lemma for the symmetric square lifting from SL(2) to PGL(3) and the unit element of the Hecke algebra. A proof of the first assertion — due to Langlands, based on counting vertices on the Bruhat-Tits building associated with PGL(3) — is recorded in the paper [F2;II], §4, but it is conceptually difficult, hence not used in this work.

The current simpler proof is based on a twisted analogue of Kazhdan's decomposition [K1], p. 226, of a compact element into a commuting product of its absolutely semisimple and its topologically unipotent parts, on an explicit and elementary computation of orbital integrals of the unit element in the Hecke algebra of GL(2), and on the preliminary analysis of stable twisted conjugacy classes from section I.2. For an extension of the Theorem to general spherical functions, and for representation theoretic applications see chapter V.

We argue that the (twisted) Kazhdan decomposition of Proposition 2 already reduces all computations to GL(2), and we carry out explicitly these computations. This makes the proof of the fundamental lemma for the symmetric square lifting entirely elementary. Our elementary and purely computational proof extends to prove the fundamental lemma for the lifting from U(2) to U(3), see [F3;VIII], and for the lifting from GSp(2) to GL(4) twisted by an outer automorphism similar to the one considered here; see [F4;I].

We need a twisted analogue of the following definitions and results of [K1], p. 226.

Put $\mathbb{F}_q = R/\boldsymbol{\pi} R$, where $\boldsymbol{\pi}$ generates the maximal ideal in the local ring R.

Definition ([K1]). An element $k \in G = \mathrm{GL}(n, F)$ is called *absolutely semisimple* if $k^a = 1$ for some positive integer a which is prime to p (= residual characteristic of F). A $k \in G$ is called *topologically unipotent* if $k^{q^N} \to 1$ as $N \to \infty$.

1. Proposition ([K1]). *Any element* $k \in K = \mathrm{GL}(n, R)$ *has a unique decomposition* $k = su = us$, *where* s *is absolutely semisimple,* u *is topologically unipotent, and* s, u *lie in* K. *For any* $k \in K$ *and* $x \in G$, *if* $\mathrm{Int}(x)k$

($= xkx^{-1}$) is in K, then x lies in $KZ_G(s)$; here $Z_G(s)$ is the centralizer of s in G. □

Let σ be an automorphism of G of order $\ell, (\ell, p) = 1$, whose restriction to K is an automorphism of K of order ℓ. Denote by $\langle K, \sigma\rangle$ the group generated by K and σ in the semidirect product $G \rtimes \langle\sigma\rangle$.

Definition. The element $k\sigma$ of $G\sigma \subset G \rtimes \langle\sigma\rangle$ is called *absolutely semisimple* if $(k\sigma)^a = 1$ for some positive integer a indivisible by p.

2. Proposition. *Any $k\sigma \in K\sigma$ has a unique decomposition $k\sigma = s\sigma \cdot u = u \cdot s\sigma$ with absolutely semisimple $s\sigma$ and topologically unipotent u. Both s and u lie in K.*

Definition. This $s\sigma$ is called the *absolutely semisimple part* of $k\sigma$ and u is the *topologically unipotent part* of $k\sigma$.

Proof. For the uniqueness, if $s_1\sigma \cdot u_1 = s_2\sigma \cdot u_2$ then $u_1^a = u_2^a$ for $a = a_1a_2$. Since $(a, q) = 1$, there are integers α_N, β_N with $\alpha_N a + \beta_N q^N = 1$. Then

$$u_2u_1^{-1} = u_2^{\alpha_N a+\beta_N q^N} u_1^{-\alpha_N a-\beta_N q^N} = u_2^{\beta_N q^N} u_1^{-\beta_N q^N} \to 1$$

as $N \to \infty$. For the existence, recall that the prime-to-p part of the number of elements in $\mathrm{GL}(n, \mathbb{F}_q)$ is $c = \prod_{i=1}^{n}(q^i - 1)$. Let $\{(k\sigma)^{q^{m_i}}\}$ be a convergent subsequence in the sequence

$$\{(k\sigma)^{q^m}; q^m \equiv 1(\mathrm{mod}\ c\ell)\} \qquad \text{in} \qquad \langle K, \sigma\rangle.$$

Denote the limit by $s\sigma, s \in K$. Then $(s\sigma)^{c\ell} = 1$. Define $u = k\sigma(s\sigma)^{-1}$. Then $u^{q^{m_i}} \to 1$ as $m_i \to \infty$, and $u^{q^N} \to 1$ as $N \to \infty$. □

Corollary. *The centralizer Z_G $(s\sigma \cdot u)$ is contained in $Z_G(s\sigma)$.* □

3. Proposition. *Given $k \in K$, $k\sigma = s\sigma \cdot u$, put $\tilde{\sigma}(h) = s\sigma(h)s^{-1}$. This is an automorphism of order ℓ on $Z_K((s\sigma)^\ell)$. Suppose that the first cohomology set $H^1(\langle\tilde{\sigma}\rangle, Z_K((s\sigma)^\ell))$, of the group $\langle\tilde{\sigma}\rangle$ generated by $\tilde{\sigma}$, with coefficients in the centralizer $Z_K((s\sigma)^\ell)$ of $(s\sigma)^\ell$ in K, injects in*

$$H^1(\langle\tilde{\sigma}\rangle, Z_G((s\sigma)^\ell)).$$

Then, any $x \in G$ such that $k'\sigma = \mathrm{Int}(x)(k\sigma)$ is in $K\sigma$, must lie in $KZ_G(s\sigma)$.

Proof. Put $k'\sigma = s'\sigma \cdot u'$. Then

$$s'\sigma = \lim(k'\sigma)^{q^{m_i}} = \mathrm{Int}(x)\lim(k\sigma)^{q^{m_i}} = \mathrm{Int}(x)(s\sigma).$$

Hence $(s'\sigma)^\ell = \mathrm{Int}(x)(s\sigma)^\ell$, and by Proposition 1 there is $y \in K$ with

$$(s\sigma)^\ell = \mathrm{Int}(y)(s'\sigma)^\ell = (t\sigma)^\ell,$$

where $t = ys'\sigma(y^{-1})$. Replacing x by yx and k' by $yk'\sigma(y^{-1})$, we may assume that $y = 1$. Put $a(1) = 1$, and for $0 < r < \ell$,

$$a(\sigma^r) = s'\sigma(s')\cdots\sigma^{r-1}(s')\sigma^{r-1}(s)^{-1}\cdots\sigma(s)^{-1}s^{-1}.$$

Then $a(\sigma^r) \in Z_K((s\sigma)^\ell)$, and

$$a(\sigma^u)\tilde{\sigma}^u(a(\sigma^r)) = a(\sigma^{u+r}) \qquad (0 \le u, r < \ell).$$

Hence

$$a = \{\sigma^r \mapsto a(\sigma^r)\} \in H^1(\langle\tilde{\sigma}\rangle, Z_K((s\sigma)^\ell)).$$

Of course,

$$s' = xs\sigma(x^{-1}) = x\tilde{\sigma}(x^{-1})s$$

implies that $a(\sigma) = s's^{-1} = x\tilde{\sigma}(x^{-1})$, hence a is trivial in

$$H^1(\langle\tilde{\sigma}\rangle, Z_G((s\sigma)^\ell)).$$

The injectivity assumption then implies that $s's^{-1} = a(\sigma)$ is $b\tilde{\sigma}(b^{-1}) = bs\sigma(b^{-1})s^{-1}$, and $s' = bs\sigma(b^{-1})$, with $b \in Z_K((s\sigma)^\ell)$. It follows that

$$\mathrm{Int}(b)(s\sigma) = s'\sigma = \mathrm{Int}(x)(s\sigma).$$

Hence $b^{-1}x \in Z_G(s\sigma)$, and $x \in bZ_G(s\sigma) \subset KZ_G(s\sigma)$, as asserted. □

Remark. Let us verify the injectivity assumption of Proposition 3 in the case considered in the Theorem. We use the fact (chapter I, end of (2.1)) that if λ is an eigenvalue of $s\sigma(s)$ then so is λ^{-1}. Thus the semisimple element $s\sigma(s)$ in K is the identity, or has eigenvalues -1, 1, -1, or

λ, 1, λ^{-1}, $\lambda^2 \neq 1$. In the first case $Z_K(s\sigma s) = K$, and $I = k\tilde{\sigma}k$ implies $ksJ = {}^t(ksJ)$. This represents a quadratic form in 3 variables over R (= ring of integers in F), and these are parametrized by their discriminant, in $R^\times/R^{\times 2}$. If the form splits over F, thus the discriminant lies in $F^{\times 2}$, and in $R^\times$, then it lies in $R^{\times 2}$, and the form splits already over R. The injectivity follows.

In the second case, replacing s by a σ-conjugate (see (2.7)), we may assume that $s\sigma(s) = \mathrm{diag}(-1, 1, -1)$, and $s = \mathrm{diag}(-1, 1, -1)$. Then an element of $Z_G(s\sigma(s))$ has the form a_1 (a in $\mathrm{GL}(2, F)$, entries of a_1 indexed by (i, j), $i + j$ = odd, are 0), and $\tilde{\sigma}a_1 = ((\det a)^{-1}a)_1$. So $1 = a_1\tilde{\sigma}a_1$ means $a^2 = \det a$, and a is a scalar, in $R^\times$. Taking any $h \in \mathrm{GL}(2, R)$ with $\det h = a$, we get $h_1\tilde{\sigma}(h_1^{-1}) = a_1$.

In the third case, $H^1(\langle\tilde{\sigma}\rangle, Z_K((s\sigma)^\ell))$ is trivial (as in the second case) if $\lambda \in R^\times$, so let us consider the case where $F(\lambda)$ is a quadratic extension of F. As in chapter I, (2.2), we may assume that $T = Z_G(s\sigma(s))$ consists of b_1, $b \in \mathrm{GL}(2, F)$, and $s = (ae)_1$. Since $s\sigma(s) = (-(\det a)^{-1}a^2)_1$, a_1 lies in T, and $\tilde{\sigma}(t) = sJ{}^tb_1^{-1}Js^{-1}$

$$= (aew{}^tb^{-1}wea^{-1})_1 = ((\det b)^{-1}aba^{-1})_1 = ((\det b)^{-1}b)_1.$$

Hence $1 = t\tilde{\sigma}(t)$ means that b is a scalar, in $R^\times$. The image in

$$H^1(\langle\tilde{\sigma}\rangle, Z_G((s\sigma)^\ell))$$

is trivial when $b_1 = c_1\tilde{\sigma}(c_1^{-1}) = (\det c)_1$, where $c_1 \in T$, hence $b = \det c$ lies in the norm subgroup $N_{F(\lambda)/F}F(\lambda)^\times$, and in $R^\times$, hence in $N_{F(\lambda)/F}R(\lambda)^\times$, where $R(\lambda)$ denotes the ring of integers of $F(\lambda)$. We conclude that c can be taken in $\mathrm{GL}(2, R)$, and c_1 in $Z_K(s\sigma(s))$, as asserted. □

4. PROPOSITION. *If the elements $k\sigma = s\sigma \cdot u$ and $k'\sigma = s'\sigma \cdot u'$ of $K\sigma$ are stably conjugate, then $s\sigma$ and $s'\sigma$ are stably conjugate. If $s = s'$, then u, u' are stably conjugate in $Z_G(s\sigma)$.*

PROOF. Suppose that $k'\sigma = \mathrm{Int}(\overline{x})(k\sigma)$ for some $\overline{x} \in \overline{G} = \mathrm{GL}(n, \overline{F})$, where $\overline{F}$ is a finite Galois extension of F (in the course of this proof). We have the K-decomposition

$$s'\sigma \cdot u' = \mathrm{Int}(\overline{x})(s\sigma) \cdot \mathrm{Int}(\overline{x})u$$

in $\overline{G}$. The uniqueness of the K-decomposition in $\overline{G}$ implies that $s'\sigma = \text{Int}(\overline{x})(s\sigma)$, namely $s\sigma, s'\sigma$ are stably conjugate. If $s\sigma, s'\sigma$ are conjugate, we may assume that $s'\sigma = s\sigma$, then

$$s\sigma \cdot u' = \text{Int}(\overline{x})(s\sigma) \cdot \text{Int}(\overline{x})u$$

implies that $\overline{x} \in Z_{\overline{G}}(s\sigma)$ and $\text{Int}(\overline{x})u = u'$, as asserted. □

To prove the Theorem, decompose $k\sigma = s\sigma \cdot u$ (in our case $\sigma(x) = J \cdot {}^t x^{-1} \cdot J^{-1}$). Then $k\sigma(k) = s\sigma(s) \cdot u^2$. We shall consider three different cases, depending on whether $s\sigma(s)$ is the identity I, or it is $\text{diag}(-1, 1, -1)$, or it is regular (its eigenvalues $\lambda, 1, \lambda^{-1}$ are distinct). In all cases put

$$\begin{aligned} \tilde{f}^0_{s\sigma}(u) &= \int_{G/Z_G(s\sigma)} f^0(\text{Int}(x)(s\sigma \cdot u))dx \\ &= \int_{K/K\cap Z_G(s\sigma)} f^0(\text{Int}(x)(s\sigma \cdot u))dx = |K/K \cap Z_G(s\sigma)| f^0(s\sigma \cdot u), \end{aligned} \tag{4.1}$$

where the second equality follows from Proposition 3. Note that $\tilde{f}^0_{s\sigma}(1) = \Phi(s\sigma, f^0 dh)$. Then

$$\begin{aligned} \Phi(k\sigma, f^0 dg) &= \int_{G/Z_G(k\sigma)} f^0(\text{Int}(x)(k\sigma))dx \\ &= \int_{Z_G(s\sigma)/Z_G(s\sigma\cdot u)} \tilde{f}^0_{s\sigma}(\text{Int}(x)u)dx = \Phi(u, \tilde{f}^0_{s\sigma}dx). \end{aligned} \tag{4.2}$$

Here $\Phi(u, \tilde{f}^0_{s\sigma}dx)$ denotes the orbital integral of the characteristic function $\tilde{f}^0_{s\sigma}$ of the compact subgroup $Z_K(s\sigma) = K \cap Z_G(s\sigma)$ of $Z_G(s\sigma)$ (multiplied by $|Z_K(s\sigma)|^{-1}$) at the topologically unipotent element u in $Z_K(s\sigma)$.

As a useful example we compute explicitly the orbital integral of the characteristic function 1_K of the maximal compact subgroup $K = \text{GL}(2, R)$ in $G = \text{GL}(2, F)$, where — as usual — F is a local field of odd residual characteristic with ring R of integers. Normalize the Haar measure on G to assign K the volume $|K| = 1$. Put $\boldsymbol{\pi}$ for a generator of the maximal ideal in R, q for the cardinality of the residue field $R/\boldsymbol{\pi}R$, $|\cdot|$ for the normalized (by $|\boldsymbol{\pi}| = q^{-1}$) absolute value on F. Let E be a quadratic extension of F; then $E = F(\sqrt{\theta})$ for some θ with $|\theta|$ equals 1 or q^{-1}. The torus

$$T = \left\{ \gamma = \begin{pmatrix} a & b\theta \\ b & a \end{pmatrix} \in G \right\}$$

in G is isomorphic to $E^\times$, it subgroup $R_T = T \cap K$ is isomorphic to $R_E^\times$, the group of units in $E^\times$, via $\gamma \mapsto a + b\sqrt{\theta}$.

5. PROPOSITION. *For a regular ($b \neq 0$) γ in R_T, the orbital integral*

$$\int_{G/T} 1_K(\mathrm{Int}(x)\gamma)dx$$

is equal to

$$-\frac{2/e}{q-1} + \frac{q-1+2/e}{q-1}|b|^{-1}.$$

Here $e = e(E/F)$ is the ramification index of E over F. Note that $b = (\gamma - \overline{\gamma})/2\sqrt{\theta}$, where $\overline{\gamma} = a - b\sqrt{\theta}$.

PROOF. One has the disjoint decomposition $G = \bigcup_{m \geq 0} K \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-m} \end{pmatrix} T$, and

$$K \cap \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-m} \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{m} \end{pmatrix} = \left\{ \begin{pmatrix} a & \boldsymbol{\pi}^m b\theta \\ \boldsymbol{\pi}^{-m} b & a \end{pmatrix} \in K \right\} \simeq R_E(m)^\times.$$

Here

$$R_E(m) = \{a + b\sqrt{\theta}; |b| \leq |\boldsymbol{\pi}|^m, |a| \leq 1\} = R + \pi^m R_E = R + R\pi^m\sqrt{\theta}.$$

For any function $f \in C_c^\infty(G/T)$ we then have

$$\int_{G/T} f(g)dg = \sum_{m \geq 0} [R_E^\times : R_E(m)^\times] \int_K f\left(k \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-m} \end{pmatrix}\right) dk,$$

and so

$$\int_{G/T} 1_K(\mathrm{Int}(x)\gamma)dx = \sum_{m \geq 0} [R_E^\times : R_E(m)^\times] 1_K \begin{pmatrix} a & \boldsymbol{\pi}^m b\theta \\ b\boldsymbol{\pi}^{-m} & a \end{pmatrix}$$

$$= \sum_{0 \leq m \leq B} [R_E^\times : R_E(m)^\times],$$

if $|b| = |\boldsymbol{\pi}^B|$. Recall that $\boldsymbol{\pi} = \boldsymbol{\pi}_E^e$ and $q_E = q^{2/e}$ for the uniformizer $\boldsymbol{\pi}_E$ and residual cardinality q_E of E. Since

$$[R_E(m)^\times : 1 + \boldsymbol{\pi}^m R_E] = [R^\times : R^\times \cap (1 + \boldsymbol{\pi}^m R_E)] = (q-1)q^{m-1},$$

and

$$[R_E^\times : 1 + \boldsymbol{\pi}^m R_E] = (q_E - 1) q_E^{em-1},$$

we have that $[R_E^\times : R_E(m)^\times]$ is q^m if $e = 2$, while if $e = 1$ it is 1 when $m = 0$ and $(q+1)q^{m-1}$ when $m \geq 1$. The proposition follows on taking the sum over $0 \leq m \leq B$. □

PROOF OF THEOREM; STABLE CASE. We deal separately with the three cases, where the eigenvalues $\lambda, 1, \lambda^{-1}$ of $s\sigma(s)$ ($s\sigma$ is the absolutely semisimple part of $\delta\sigma \in K\sigma$) have: I. $\lambda \neq \pm 1$; II. $\lambda = -1$; III. $\lambda = 1$. Of course, if $\Phi^\sigma(\delta\sigma, f^0 dg) \neq 0$, then we may assume that $\delta \in K$.

Case I. Here $\delta\sigma = s\sigma \cdot u$, and $s\sigma(s)$ has distinct eigenvalues $\lambda, 1, \lambda^{-1}$. If $\delta\sigma, \delta'\sigma$ in $K\sigma$ are stably conjugate but not conjugate, then so are their absolutely semisimple parts $s\sigma$, $s'\sigma$. Indeed, if $s\sigma$, $s'\sigma$ are conjugate (in G), then they are so in K by Proposition 3, hence we may assume that $s = s'$. If $\delta'\sigma = \mathrm{Int}(\overline{x})\delta\sigma$ then $\overline{x} \in Z_G(s\sigma)$, and $u, u' \in Z_G(s\sigma)$. As $Z_G(s\sigma)$ is a torus, $u' = u$.

Since λ, λ^{-1} are absolutely semisimple and distinct, neither λ nor $-\lambda$ are topologically unipotent (as this would imply $\lambda = \pm 1$, and these are cases II, III). It follows that $F(\lambda)$ is not ramified over F. Indeed, if it is,

$$\lambda = a + b\sqrt{\theta}, \qquad \text{where} \qquad |\theta| = |\boldsymbol{\pi}|, \qquad |a| = 1, \qquad |b| \leq 1,$$

and

$$1 = \lambda\overline{\lambda} = a^2 - b^2\theta = a^2(1 - \theta(b/a)^2).$$

But

$$(1 - \theta(b/a)^2)^{q^N} \to 1 \qquad \text{as} \qquad N \to \infty.$$

Hence

$$a^{2q^N} \to 1, \qquad \text{and} \qquad \pm a = 1 + \boldsymbol{\pi} c, \qquad |c| \leq 1,$$

for some choice of a sign. Then $\pm a$, and consequently $\pm\lambda$, is topologically unipotent. For the same reason, if

$$\lambda = a + b\sqrt{\theta}, \qquad |\theta| = 1, \qquad \theta \in F - F^2,$$

and $F(\lambda)/F$ is unramified, then $|b| = 1$ and $|a| \leq 1$. A set of representatives for the set of σ-conjugacy classes within the stable σ-conjugacy class of δ is given (see (2.3)) by $\delta_y = (yhe)_1$,

$$e = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h \in \mathrm{GL}(2, R) \quad \left(\text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } g_1 = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}\right),$$

as y ranges over a set of representatives of $F^\times/NE^\times$, $E = F(\lambda)$. Note that $\delta\sigma(\delta) = (\frac{-1}{\det h}h^2)_1$. Take $y = 1$ to represent one class. When $F(\lambda)/F$ is unramified, the second representative y is not a unit, hence $\delta_y \notin K$, and the stable orbital integral is the sum of a single integral (same conclusion if $\lambda \in F^\times$):

$$\Phi^{\mathrm{st}}(\delta\sigma, f^0 dg) = \Phi(\delta\sigma, f^0 dg) = |K/K \cap Z_G(s\sigma)| f^0(s\sigma \cdot u)$$
$$= |K \cap Z_G(s\sigma)|^{-1} = |Z_K(s\sigma)|^{-1}.$$

The same reasoning implies in our case ($\lambda \neq \pm 1$) that $\Phi^{\mathrm{st}}(\gamma, f_0^0 dh) = \Phi(\gamma, f_0^0 dh)$, and $\lambda \in F^\times$ or $F(\lambda)/F$ is unramified, in which case γ can be taken to be represented by $\begin{pmatrix} a & b\theta \\ b & a \end{pmatrix}$, $|b| = 1 \geq |a|$. A stably conjugate, but not conjugate, element, is of the form $\gamma' = \mathrm{Int}\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}(\gamma)$, with $y \in F - NE$, $E = F(\lambda)$. In particular y is not a unit, and the conjugacy class of γ' does not intersect K_H (by Proposition 3, and since the eigenvalues of the absolutely semisimple part s_γ of γ are distinct). Hence

$$\Phi^{\mathrm{st}}(\gamma, f_0^0 dh) = \Phi(\gamma, f_0^0 dh) = \int_{H/Z_H(\gamma)} f_0^0(\mathrm{Int}(x)(s_\gamma u_\gamma))dx$$
$$= |K_0/K_0 \cap Z_H(s_\gamma)| f_0^0(s_\gamma u_\gamma) = |K_0 \cap Z_H(s_\gamma)|^{-1}.$$

Since $Z_G(s\sigma) \simeq Z_H(s_\gamma)$, and the measures are chosen in a compatible way, we conclude that $\Phi^{\mathrm{st}}(\delta\sigma, f^0 dg) = \Phi^{\mathrm{st}}(N\delta, f_0^0 dh)$ when $\delta\sigma = s\sigma \cdot u$, and $s\sigma(s)$ has distinct eigenvalues $\lambda, 1, \lambda^{-1}$. The stable assertion of the theorem is proven in case I. □

Case II. Here $\delta\sigma = s\sigma \cdot u$, $s\sigma(s)$ has eigenvalues $-1, 1, -1$. All such $s \in G$ make a single σ-conjugacy class. Suppose that

$$\delta'\sigma = s'\sigma \cdot u' = \mathrm{Int}(\overline{x})(\delta\sigma)$$

for some $\overline{x} \in \mathbf{G}(\overline{F})$, $\overline{F}$ finite extension of F, with $\delta\sigma$, $\delta'\sigma$ in $K\sigma$. Then $s'\sigma = \mathrm{Int}(g)(s\sigma)$ with g in K by Proposition 3. Replacing $\overline{x}$ by $g^{-1}\overline{x}$, we may assume that

$$s' = s = \mathrm{diag}(1, 1, -1).$$

Then u, u' are stably conjugate in H. Hence

$$\Phi^{\mathrm{st}}(\delta\sigma, f^0 dg) = \sum_{\{u'\}} \Phi(s\sigma \cdot u', f^0 dg) = \sum_{\{u'\}} \Phi(u', \tilde{f}^0_{s\sigma} dh) = \Phi^{\mathrm{st}}(u, \tilde{f}^0_{s\sigma} dh),$$

where $Z_G(s\sigma) = H$ and $\tilde{f}^0_{s\sigma} = f^0_0$. This we compare with $\Phi^{\rm st}(u^2, f^0_0 dh)$. Using the explicit computation of Proposition 5, it suffices to note that for topologically unipotent μ, the value of $|(\mu - \mu^{-1})^2|^{1/2}$ is equal to that of $|(\mu^2 - \mu^{-2})^2|^{1/2}$, since $|(\mu + \mu^{-1})^2| = 1$. This completes the proof of $\Phi^{\rm st}(\delta\sigma, f^0 dg) = \Phi^{\rm st}(N\delta, f^0_0 dh)$ in Case II. $\square$

Case III. By (2.5), there is one stable conjugacy class of $\delta \in G$ with $(\delta\sigma)^2 = I$, and it consists of two conjugacy classes, represented by σ and by $s'\sigma$ ($s' \in G$). The centralizer $Z_G(\sigma)$ of σ in G is the split form $\mathrm{SO}(2,1) = \mathrm{PGL}(2,F)$, while that of $s'\sigma$, $Z_G(s'\sigma)$, is the anisotropic form $\mathrm{SO}(3) = PD^\times$, $D =$ quaternion algebra over F.

6. Proposition. *The orbit* $\mathrm{Int}(G)(s'\sigma)$ *does not intersect* $K\sigma$.

Proof. The element $s'' = \begin{pmatrix} 0 & & 1 \\ & \varepsilon & \\ \boldsymbol{\pi} & & 0 \end{pmatrix}$, where ε is a nonsquare unit, lies in $\mathrm{Int}(G)(s'\sigma)$, since the Witt invariant of

$$s''J = \mathrm{diag}(1, -\varepsilon, \boldsymbol{\pi}) \qquad \text{is} \qquad (\varepsilon, \boldsymbol{\pi}) = -1.$$

Note that the quadratic form associated to $\mathrm{diag}(a_1, \dots, a_n)$ represents zero precisely when its Witt invariant

$$\prod_{j \le i} (a_i, a_j) \qquad \text{is} \qquad (-1, -1);$$

$(\cdot,\cdot)$ denotes the Hilbert symbol. If s' lies in K, and $s'J\,{}^t s'J = 1$, namely $s'J = {}^t(s'J)$, then there is $x \in K$ such that $xs'J\,{}^t x$ is diagonal, of the form $\mathrm{diag}(u_1, u_2, u_3)$, in K. Its Witt invariant is

$$\prod_{j \le i} (u_i, u_j) = 1 = (-1, -1).$$

Hence $s'J \neq zgs''J\,{}^t g$ for all $g \in G$. $\square$

We conclude that at $\delta\sigma = \sigma \cdot u, u \in K$ topologically unipotent, $u \in \mathrm{SO}(2,1) = Z_G(\sigma) \simeq \mathrm{PGL}(2,F)$, we have

$$\Phi^{\rm st}(\sigma u, f^0 dg) = \Phi(\sigma u, f^0 dg) = \Phi(u, \tilde{f}^0_\sigma dh_1).$$

Recall that the eigenvalues of $u\sigma(u) = u^2$ are $\mu, 1, \mu^{-1}$. Hence those of u are $\mu', 1, \mu'^{-1}$, where μ' is topologically unipotent in $R_E^\times$ with $\mu'^2 = \mu$.

Since $\mu'\overline{\mu}' = 1$, we have $\mu' = \nu/\overline{\nu}$ for some topologically unipotent ν in $R_E^\times$. Via the isomorphism $\mathrm{SO}(2,1) \simeq \mathrm{PGL}(2)$, u can be regarded as an element of $\mathrm{PGL}(2,R)$ with eigenvalues $\nu, \overline{\nu}$. The integral $\Phi(u, \tilde{f}_\sigma^0 dh)$ is then computed in Proposition 5. It has to be compared with the orbital integral $\Phi^{\mathrm{st}}(\nu, f_0^0 dh)$ on $\mathrm{SL}(2,F)$, where ν is an element of $K_0 = \mathrm{SL}(2,R)$ with eigenvalue μ, μ^{-1}. The stable orbital integral of a function f_0 on $\mathrm{SL}(2,F)$ coincides with its orbital integral over $\mathrm{GL}(2,F)$, where f_0 is extended to a C_c^∞-function on $\mathrm{GL}(2,F)$. This too is computed in Proposition 5. We are reduced then to comparing

$$|(\nu - \overline{\nu})^2/\nu\overline{\nu}|^{\frac{1}{2}} = |(1 - \overline{\nu}/\nu)(\nu/\overline{\nu} - 1)|^{\frac{1}{2}} = |(1 - \mu')(1 - \mu'^{-1})|^{\frac{1}{2}}$$

$$= |(1 - \mu)(1 - \mu^{-1})|^{\frac{1}{2}}$$

with

$$|(\mu - \mu^{-1})^2|^{\frac{1}{2}} = |(\mu^2 - 1)(\mu^{-2} - 1)|^{\frac{1}{2}}.$$

These are equal since ν, μ', μ are topologically unipotent.

This completes the proof of $\Phi^{\mathrm{st}}(\delta\sigma, f^0 dg) = \Phi^{\mathrm{st}}(N\delta, f_0^0 dh)$ in Case III, hence in all stable cases. □

Proof of theorem; unstable case. Note that if $\lambda, 1, \lambda^{-1}$ are the (distinct) eigenvalues of the regular $\delta\sigma(\delta)$, $\delta \in K$, then λ is a unit in $F(\lambda)$, and $(1+\lambda)(1+\lambda^{-1})$, which lies in F, is a unit in F in cases I and III ($-\lambda$ is not topologically unipotent). But in case II we have

$$|(1+\lambda)(1+\lambda^{-1})| < 1.$$

In *Case* I, as noted in the discussion of the stable case, $F(\lambda)$ is F or is unramified over F, the unstable integral is a sum of a single term, and since $\Delta(\delta\sigma) = 1$, if $N_1\delta = \gamma_1$ is the regular class in H_1 with eigenvalues $\lambda, 1, \lambda^{-1}$, and s_{γ_1} is its absolutely semisimple part, we have

$$\Delta(\delta\sigma)\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg) = \Phi(\delta\sigma, f^0 dg)$$

$$= |K \cap Z_G(s\sigma)|^{-1} = |K_1/K_1 \cap Z_{H_1}(s_{\gamma_1})| f_1^0(\gamma_1).$$

The tori $Z_G(s\sigma)$ and $Z_{H_1}(s_{\gamma_1})$ are isomorphic. The measures are chosen to be compatible with this isomorphism. □

In *Case* III, by Proposition 6 (and since $\Delta(\delta\sigma) = 1$) we have the first equality in

$$\Delta(\delta\sigma)\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg) = \Phi(\sigma u, f^0 dg) = \Phi(u, \tilde{f}^0_\sigma dh_1) = \Phi(u, f^0_1 dh_1).$$

Here $\delta\sigma = \sigma \cdot u$, u being topologically unipotent. The second equality follows from (4.2), and $f^0_1 = \tilde{f}^0_\sigma$ by (4.1). Note that f^0_1 is the characteristic function of $K_1 = K \cap Z_G(\sigma)$ in $H_1 = Z_G(\sigma) = \mathrm{SO}(2,1)$, divided by the volume of the maximal compact K_1 of H_1. Now $N_1\delta = u^2$. The eigenvalues of u, viewed as an element of $\mathrm{PGL}(2,R)$, are $\nu, \overline{\nu}$ (topologically unipotent), those of u^2 are $\nu^2, \overline{\nu}^2$, and $|(\nu^2 - \overline{\nu}^2)^2| = |(\nu - \overline{\nu})^2|$, hence Proposition 5 implies that $\Phi(u^2, f^0_1 dh_1) = \Phi(u, f^0_1 dh_1)$. Hence

$$\Delta(\delta\sigma)\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg) = \Phi(u, f^0_1 dg) = \Phi(u^2, f^0_1 dh_1) = \Phi(N_1\delta, f^0_1 dh_1).$$

□

In *Case* II, $\delta\sigma = s\sigma \cdot u \in K\sigma$,

$$s\sigma(s) = \mathrm{diag}(-1,1,-1), \qquad s = \mathrm{diag}(-1,1,1),$$

and $u \in \mathrm{SL}(2,R) = Z_K(s\sigma)$ has eigenvalues γ, γ^{-1}. Then $\delta\sigma(\delta)$ has eigenvalues $\lambda, 1, \lambda^{-1}$, where $\lambda = -\gamma^2$, as does $N_1\delta \in \mathrm{SO}(2,1)$. Also

$$\Delta(\delta\sigma) = |(1-\nu^2)(1-\nu^{-2})|^{1/2} = |(\nu - \nu^{-1})^2|^{1/2}.$$

If $\lambda \in F^\times$, as an element of $\mathrm{PGL}(2,F)$, γ_1 is represented by $\mathrm{diag}(1,\lambda)$, and

$$\Phi(\gamma_1, f^0_1 dh_1) = \int_F \mathrm{ch}_{K_1}\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx$$

$$= \int_F \mathrm{ch}_{K_1}\left(\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & (1-\lambda)x \\ 0 & 1 \end{pmatrix}\right) dx = 1,$$

where $f^0_1 = |K_1|^{-1}\,\mathrm{ch}_{K_1}$, ch_{K_1} is the characteristic function of K_1 in H_1. Indeed, $-\lambda = \nu^2$ is topologically unipotent, hence $1-\lambda$ (and λ) are units in R.

If $\lambda \notin F$, it lies in a quadratic extension $F(\sqrt{\theta})$, $\theta \in F - F^2$, and we may assume $|\theta| = 1$ in the unramified case, and $|\theta| = |\boldsymbol{\pi}|$ in the ramified case.

Since $\nu\overline{\nu} = 1$, we have $\nu = a + b\sqrt{\theta}$, with $a, b \in R$. Since ν is topologically unipotent, we have $a \equiv 1 \,(\operatorname{mod} \boldsymbol{\pi})$, and $|b^2\theta| < 1$. Then

$$\lambda = -\nu/\overline{\nu} = \nu\sqrt{\theta}/\overline{(\nu\sqrt{\theta})}, \qquad \nu\sqrt{\theta} = b\theta + a\sqrt{\theta},$$

and γ_1, as an element of $H_1 = \mathrm{PGL}(2, F)$, is represented by $\begin{pmatrix} b\theta & a\theta \\ a & b\theta \end{pmatrix}$, with eigenvalues $b\theta \pm a\sqrt{\theta}$. In the ramified case, the determinant $b^2\theta^2 - a^2\theta$ does not belong to $R^\times F^{\times 2}$, hence $\Phi(\gamma_1, f_1^0 dh_1) = 0$. In the unramified case, $\gamma_1 = s_1 u_1$, where the absolutely semisimple part $s_1 (\in \mathrm{PGL}(2, R))$ has eigenvalues whose quotient is -1. Hence

$$\Phi(\gamma_1, f_1 dh_1) = |K_1/Z_{K_1}(s_1)| f_1(\gamma_1) = |Z_{K_1}(s_1)|^{-1}$$

by Proposition 1 (the integral ranges over the quotient of $K_1 Z_{H_1}(s_1)$ by $Z_{H_1}(\gamma_1)$, and $Z_{H_1}(s_1) = Z_{H_1}(\gamma_1)$ is a torus in H_1).

Let us compare this with $\Delta(\delta\sigma)\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg)$. If $\nu \in R^\times$, then

$$\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg) = \Phi(s\sigma \cdot u, f^0 dg)$$

$$= \int_{G/Z_G(s\sigma\cdot u)} f^0(\mathrm{Int}(x)(s\sigma \cdot u))dx = \int_{H/Z_H(u)} f_0^0(\mathrm{Int}(x)u)dx.$$

Here $H = Z_G(s\sigma)$, and we used Proposition 3 in the last equality, noting that $f^0(1) = |K|^{-1}$ and $f_0^0(1) = |K_0|^{-1}$. We may represent u by $\mathrm{diag}(\nu, \nu^{-1})$, to get

$$\int_F \mathrm{ch}_{K_0}\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)dx$$

$$= \int_F \mathrm{ch}_{K_0}\left(\begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}\begin{pmatrix} 1 & (1-\nu^{-2})x \\ 0 & 1 \end{pmatrix}\right)dx = \Delta(\delta\sigma)^{-1},$$

since $|1 - \nu^{-2}| = |\nu - \nu^{-1}| = \Delta(\delta\sigma)$, and ν is a unit.

If $\nu \notin F^\times$, then the stable conjugacy class of u in H contains a second conjugacy class u', represented by $\mathrm{Int}(g)u$, where $g \in \tilde{H} = \mathrm{GL}(2, F)$ has $\det g \in F - NE$, $E = F(\nu)$; here $NE = \mathrm{Norm}_{E/F} E$. Then

$$\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg) = \Phi(s\sigma \cdot u, f^0 dg) - \Phi(s\sigma \cdot u', f^0 dg)$$

$$= \int_{H/Z_H(u)} f_0^0(\mathrm{Int}(x)u)dx - \int_{H/Z_H(u)} f_0^0(\mathrm{Int}(gx)u)dx$$

is zero when $F(\nu)$ is ramified over F, since g can be chosen in K_0, with $\det g$ in $R^\times - R^\times \cap NE$, in this case. When $F(\nu)$ is unramified over F, we have that $NE^\times = \boldsymbol{\pi}^{2\mathbb{Z}}R^\times \supset R^\times$. Since $H/Z_H(u)$ is open in $\tilde{H}/Z_{\tilde{H}}(u)$, the measure on $H/Z_H(u)$ defines one on $\tilde{H}/Z_{\tilde{H}}(u)$, and if κ denotes the character of $F^\times$ whose kernel is $NE^\times$ (this is the unramified character of $F^\times$ of order exactly two), then

$$\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg) = \int_{\tilde{H}/Z_{\tilde{H}}(u)} f_0^0(\mathrm{Int}(x)u)\kappa(\det x)dx.$$

We may represent the topologically unipotent element u by $\begin{pmatrix} a & b\theta \\ b & a \end{pmatrix}$, $\theta \in R^\times - R^{\times 2}$.

It is important to note that $\delta\sigma = s\sigma \cdot u = u \cdot s\sigma$ with

$$\delta J = usJ = \begin{pmatrix} b\theta & & -a \\ & -1 & \\ a & & -b \end{pmatrix}, \qquad \frac{1}{2}[(\delta J) + {}^t(\delta J)] = \mathrm{diag}(b\theta, -1, -b).$$

The quadratic form associated to $\mathrm{diag}(a_1, \dots, a_n)$ represents 0 precisely when $\prod_{j\le i}(a_i, a_j)$ is equal to $(-1,-1)$, $(\cdot,\cdot)$ is the Hilbert symbol. Hence $\boldsymbol{\kappa}(\delta)$ is 1, and $\mathrm{SO}(\mathrm{diag}(b\theta, -1, -b))$ splits, precisely when $(-b,\theta) = 1$. In our unramified case this happens precisely when $b \in \boldsymbol{\pi}^{2\mathbb{Z}}R^\times$. Hence $\kappa(b) = 1$. Note that

$$\Delta(\delta\sigma) = |(1-\nu^2)(1-\nu^{-2})|^{1/2} = |(\nu - \nu^{-1})^2|^{1/2}$$
$$= |(\nu - \overline{\nu})^2|^{1/2} = |4b^2\theta|^{1/2} = |b|.$$

Put $t = |K_0/Z_{K_0}(u)|$. Then, with $|b| = |\boldsymbol{\pi}^n|$,

$$\Delta(\delta\sigma)\Phi^{\mathrm{us}}(\delta\sigma, f^0 dg) = t\sum_{m=0}^{\infty} \delta_m \kappa(b\boldsymbol{\pi}^{-m})|b| f_0^0\left(\begin{pmatrix} a & b\theta\boldsymbol{\pi}^m \\ b\boldsymbol{\pi}^{-m} & a \end{pmatrix}\right)$$

$$= t\kappa(b)|b| f_0^0\left(\begin{pmatrix} a & b\theta \\ b & a \end{pmatrix}\right) + t(1+q^{-1})\sum_{m=1}^{\infty} \kappa(b\boldsymbol{\pi}^{-m})|b\boldsymbol{\pi}^{-m}| f_0^0\left(\begin{pmatrix} a & b\theta\boldsymbol{\pi}^m \\ b\boldsymbol{\pi}^{-m} & a \end{pmatrix}\right)$$

$$= [(-1)^n q^{-n} + (1+q^{-1})\sum_{m=1}^{n}(-1)^{n-m}q^{m-n}]|Z_{K_0}(u)|^{-1} = |Z_{K_0}(u)|^{-1}.$$

Since $Z_H(u)$ and $Z_{H_1}(\gamma_1)$ are isomorphic tori, and the measures are chosen in a compatible way, the theorem follows in the unstable case II as well, as asserted. □

II.1.1 SL(2) to tori

We shall use below the theory of endoscopy for $\mathbf{H} = \mathrm{SL}(2)$. We then prepare here the theory of transfer of orbital integrals from H to the proper endoscopic groups of H. For this, note that the connected component of the centralizer of a noncentral semisimple element s in $\widehat{H} = \mathrm{PGL}(2,\mathbb{C})$ is the diagonal subgroup $\widehat{A}$, up to conjugacy. The centralizer is connected, hence gives a nonelliptic endoscopic group, unless $s = \mathrm{diag}(1,-1)$, in which case $Z_{\widehat{H}}(s) = \widehat{A} \rtimes \langle w \rangle$, $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consequently the nonelliptic endoscopic groups of H are $\mathbf{T}_E$, where E is a quadratic extension of F, ${}^L T_E = \widehat{T}_E \rtimes W_{E/F}$, the Weil group $W_{E/F}$ acting via its quotient $\mathrm{Gal}(E/F)$ on $\widehat{T}_E = (\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{C}^\times$ ($\mathbb{C}^\times$ embeds diagonally in $\mathbb{C}^\times \times \mathbb{C}^\times$) by $\sigma(x,y) = (y,x)$. The embedding $e_E : {}^L T_E \to {}^L H$ is $(x,y) \mapsto \mathrm{diag}(x,y)$, $\sigma \mapsto w\sigma$.

The group $\mathbf{T}_E$ is the F-group $\{(x,y); xy = 1\}$ ($= \mathbb{G}_m$) with $\mathrm{Gal}(\overline{F}/F)$-action $\tau(x,y) = (\tau x, \tau y)$ if $\tau|E = 1$ and $\tau(x,y) = (\tau y, \tau x)$ if $\tau|E \neq 1$. Then $\mathbf{T}_E(E) = \{(x, x^{-1}); x \in E^\times\}$ and $T_E = \mathbf{T}_E(F) = \{(x, \sigma x); x\sigma x = 1, x \in E^\times\} = E^1$. The group T_E is isomorphic to an elliptic torus in H which we realize as $\gamma = \begin{pmatrix} a & b\theta \\ b & a \end{pmatrix}$ if $x = a + b\sqrt{\theta}$, where $E = F(\sqrt{\theta})$, $\theta \in R - R^2$ if E/F is unramified, θ is $\boldsymbol{\pi}$ if E/F is ramified.

A character $\mu' : T_E = E^1 \to \mathbb{C}^\times$ is parametrized by a map $W_{E/F} \to {}^L T_E$, $E^\times \ni z \mapsto (\mu'(z/\overline{z}), 1)$, $\sigma \mapsto \sigma$. Recall that the relative Weil group $W_{E/F}$ is generated by $z \in E^\times$ and σ with $\sigma^2 \in F - NE$ and $\sigma z = \overline{z}\sigma$. The composition with $e_E : {}^L T_E \to {}^L H$ is the image in $\mathrm{PGL}(2,\mathbb{C})$ of $z \mapsto \mathrm{diag}(\mu^*(z), \mu^*(\overline{z}))$, $\sigma \mapsto w\sigma$, namely the image $\mathrm{Ind}(\mu'; W_{E/F}, W_{E/E})_0$ in $\mathrm{PGL}(2,\mathbb{C})$ of the two-dimensional representation $\mathrm{Ind}(\mu^*; W_{E/F}, W_{E/E})$ induced from any extension μ^* to $W_{E/E} = E^\times$ of our μ'. We denote the two-dimensional representation also by $\mathrm{Ind}_E^F(\mu^*)$, and the image in $\mathrm{PGL}(2,\mathbb{C})$, which depends only on the restriction μ' of μ^* to E^1, by $\mathrm{Ind}_E^F(\mu')_0$.

This $\mathrm{Ind}_E^F(\mu^*)$ is reducible if $\mu^* = \overline{\mu}^*$ as a character of $E^\times$, that is $\mu' = 1$ on E^1, in which case there is a character μ of $F^\times$ with $\mu^*(z) = \mu(z\overline{z})$. Indeed in the direct product ${}^L H = \mathrm{PGL}(2,\mathbb{C}) \times W_{E/F}$ the image $e_E(t(\mu')) = \begin{pmatrix} \mu^*(z) & 0 \\ 0 & \mu^*(\overline{z}) \end{pmatrix} w\sigma$ of the class $t(\mu') = \begin{pmatrix} \mu^*(z) & 0 \\ 0 & \mu^*(\overline{z}) \end{pmatrix} \sigma$ of μ' is conjugate to $\begin{pmatrix} \mu^*(z) & 0 \\ 0 & -\mu^*(\overline{z}) \end{pmatrix} \sigma$, and $\mathrm{Ind}_E^F(\mu^*)$ is the reducible representation $\mu \oplus \mu\chi_E$ of $W_{F/F}$. Here χ_E is the character of $W_{F/F} = F^\times$ of order 2 whose kernel is the norm subgroup $N_{E/F}E^\times$. The character on $W_{E/F}$

with $E^\times \ni z \mapsto \mu^*(z) = \mu(z\overline{z})$ factorizes via $W_{E/F} \to W_{F/F}$, $z \mapsto Nz$, $\mathrm{Gal}(E/F) \ni \sigma \mapsto \sigma^2 \in F^\times - NE^\times$, and $\mu : W_{F/F} = F^\times \to \mathbb{C}^\times$, $x \mapsto \mu(x)$.

The group T_E is compact. Hence its spherical functions are the constants.

The unstable orbital integral of $f_0 dh$ in $C_c^\infty(H)$ at γ which generates the quadratic extension E over F is

$$\Phi^{\mathrm{us}}(\gamma, f_0 dh) = \int_{H/T_E} f_0(h\gamma h^{-1})dh - \int_{H/T'_E} f_0(h\gamma' h^{-1})dh$$

$$= \int_{\tilde{H}/T_E} f_0(g\gamma g^{-1})\kappa(g)dg.$$

Here γ' is stably conjugate but not conjugate to γ. Hence there is $g \in \tilde{H} = \mathrm{GL}(2,F)$ with determinant in $F - NE$, where $\kappa(g) = \kappa(\det g)$ and κ is the isomorphism of $F^\times/NE^\times$ with $\{\pm 1\}$.

Recall that $\Delta(\gamma) = |2b\sqrt{\theta}|$. It is $|b|$ if E/F is unramified and $p \neq 2$.

7. Lemma. *Let E/F be unramified. Then the normalized unstable orbital integral*

$$\kappa(b)\Delta(\gamma)\Phi^{\mathrm{us}}(\gamma, f_0 dh)$$

of the unit element $f_0 dh = f_0^0 dh$ of the Hecke algebra of H is equal to 1*.*

Proof. The computation is as in Proposition 5, except that in the sum we need to insert the factor $\kappa(\boldsymbol{\pi}^m) = (-1)^m$. We get

$$\kappa(b)|b|\left(1 - (q+1)\sum_{m=1}^{B}(-q)^{m-1}\right) = (-1)^B|b|[1 - (q+1)\frac{(-q)^B - 1}{-q-1}] = 1.$$

□

Other spherical functions of H (E/F unramified) are generated by

$$f_M = (-1)^M|\boldsymbol{\pi}^M|\,\mathrm{ch}(K\,\mathrm{diag}(\boldsymbol{\pi}^M, \boldsymbol{\pi}^{-M})K),$$

$M \geq 1$. Then

$$\int_{\tilde{H}/T_E} \kappa(x) f_M\left(\mathrm{Int}(x)\left(\begin{smallmatrix} a & b\theta \\ b & a \end{smallmatrix}\right)\right) dx$$

$$= \sum_{m \geq 1}[R_E^\times : R_E(m)^\times](-1)^m f_M\left(\left(\begin{smallmatrix} \boldsymbol{\pi}^m b^{-1} & 0 \\ 0 & \boldsymbol{\pi}^{-m} b \end{smallmatrix}\right)\right)$$

$$= (q+1)q^{-1}|\boldsymbol{\pi}^M||\boldsymbol{\pi}^{-M}b^{-1}|(-1)^B = \left(1+\frac{1}{q}\right)\Delta(\gamma)^{-1}\kappa(b)$$

as the only term in the sum is indexed by $m = M + B$.

For general measures $f_0 dh \in C_c^\infty(H)$, using the same decomposition it is easy to see that $\kappa(b)\Delta(\gamma)\Phi^{\mathrm{us}}(\gamma, f_0 dh)$ is a locally constant measure on T_E, and any locally constant measure on T_E is of such form for some $f_0 dh \in C_c^\infty(H)$.

For the global case we need to consider also places which split in the quadratic extension, namely $E = F \oplus F$. There $\kappa = 1$, $\gamma = \mathrm{diag}(a, a^{-1})$, its stable conjugacy class consists of a single conjugacy class,

$$F(\gamma, f_0 dh) = |a - a^{-1}| \int_N f_0^K(n^{-1}\gamma n)dn = |a| \int_N f_0^K(\gamma n)dn$$

implies that $f_A(\gamma) = F(\gamma, f_0 dh)$ is locally constant and compactly supported on the diagonal torus A, it is the characteristic function of $|a| = 1$ if $f_0 = f_0^0$, and spherical if f_0 is.

Globally, fix $\gamma_0 \in T_E$ with eigenvalue $x_0 = a_0 + b_0\sqrt{\theta}$. Then $|b_0|_v = 1$ for almost all v, and note that $\kappa(b) = \kappa(\frac{\gamma - \overline{\gamma}}{\gamma_0 - \overline{\gamma}_0})$.

The embedding $e_E : {}^L T_E \to {}^L H$ defines a lifting of representations in the unramified case. In this case the unramified character of T_E is $\mu' = 1$. The class parametrizing $\mu' = 1$ is $t(\mu') = (1,1)\sigma$, whose image in ${}^L H$ is $e(t(\mu')) = w\sigma$, which is conjugate to $\mathrm{diag}(1,-1)\sigma$ in the direct product ${}^L H = \mathrm{PGL}(2,\mathbb{C}) \times W_{E/F}$. Thus the endoscopic e_E-lift of $\mu' = 1$ is $\pi = I(\mu, \mu\chi_E)$ where $\mu = 1$. Working with $\mathrm{GL}(2,F)$, and the corresponding e_E and T_E in $\mathrm{GL}(2,F)$, if $\mu^*(z) = \mu(z\overline{z})$, namely $\mu^* = \overline{\mu}^*$, then $e_E(t(\mu^*))$ is conjugate to $t(I(\mu, \mu\chi_E))$ in $\mathrm{GL}(2,\mathbb{C})$. In terms of the Satake transform we have

$$\operatorname{tr}\mu^*(f_{T_E}dt) = f_{T_E}^\vee(t(\mu^*)) = f_0^\vee(e(t(\mu^*)))$$

$$= f_0^\vee\left(\begin{pmatrix} \mu(\boldsymbol{\pi}) & 0 \\ 0 & -\mu(\boldsymbol{\pi}) \end{pmatrix}\sigma\right) = \operatorname{tr} I(\mu, \mu\chi_E; f_0 dh).$$

Working with $\mathrm{SL}(2,F)$ we replace μ^* by its restriction μ' to E^1, and μ by 1.

At the places where the global quadratic extension E/F splits, the local component of the global character μ' of $\mathbb{A}_E^1/E^1$ is a pair of characters μ_1, $\mu_2 = \mu_1^{-1}$ of the local $F^\times$, and the endoscopic, e_E-lift to H is the induced representation $I(\mu_1, \mu_2)$. In the unramified case we have

$$\operatorname{tr}\mu'(f_{T_E}dt) = f_{T_E}^\vee(t(\mu')) = f_0^\vee(e(t(\mu')))$$

$$= f_0^\vee\left(\begin{pmatrix} \mu_1(\boldsymbol{\pi}) & 0 \\ 0 & \mu_2(\boldsymbol{\pi}) \end{pmatrix}\sigma\right) = \operatorname{tr} I(\mu_1, \mu_2; f_0 dh).$$

In section I.4 we considered orthogonality relations for characters χ on $H = \mathrm{SL}(2,F)$ and for twisted characters χ^σ on $G = \mathrm{PGL}(3,F)$, and their relationship. We need analogous investigation of the relations between character relations on H and on an elliptic torus $T_E = E^1$ of H, where F is a local field. Thus we view T_E as the group of $t = \begin{pmatrix} a & b\theta \\ b & a \end{pmatrix}$ with $\det(t) = 1$, $a, b \in F$. Denote by t' an element stably conjugate but not conjugate to t. Put $\bar{t} = \begin{pmatrix} a & -b\theta \\ -b & a \end{pmatrix}$. Let $f_{T_E} dt$ be a measure on T_E. Let μ' be a character on T_E.

8. Proposition. *If $f_{T_E}(t)dt = \kappa(b)\Delta_0(t)[\Phi(t, f_0 dh) - \Phi(t', f_0 dh)]$ is obtained from the measure $f_0 dh$ on H, then*

$$\mu'(f_{T_E} dt) = \int_{T_E} \mu'(t) f_{T_E}(t) dt$$

is equal to $\langle \chi_{\mu'}, {}'\Phi(f_0 dh)\rangle_e$ where ${}'\Phi(t, f_0 dh) = |Z_H(t)|^{-1}\Phi(t, f_0 dh)$ and $\chi_{\mu'}$ is the unstable ($\chi_{\mu'}(t') = -\chi_{\mu'}(t)$) function on H which is zero on the regular set of H except for the stable conjugacy classes of $t \in T_E$ where $\chi_{\mu'}(t) = \frac{\kappa(b)}{\Delta_0(t)}(\mu'(t) + \mu'(\bar{t}))$.

We have that $\langle \chi_{\mu'}, \chi_{\mu_1'}\rangle_e$ is zero unless μ', μ_1' are characters on the same E^1 and μ' equals μ_1' or ${\mu_1'}^{-1}$, in which case the inner product is 4 if $\mu'^2 = 1$ and 2 if $\mu'^2 \neq 1$.

Proof. Note that $\mu'(\bar{t}) = \mu'(t)^{-1} = \overline{\mu}'(t)$ where the first bar in conjugation in E over F, and the last is complex conjugation. Note that $\bar{t}$ is conjugate to t by $\mathrm{diag}(-1, 1)$.

We distinguish two cases. If -1 lies in $N_{E/F}E^\times$, then $\mathrm{diag}(-1,1)$ can be realized in H, in $\mathrm{Norm}_H(T_E) - T_E$ (it is in $HZ_{\mathrm{GL}(2,F)}(T_E)$). Hence $\kappa(-1) = 1$, $f_{T_E}(\bar{t}) = f_{T_E}(t)$ and the Weyl group $W(T_E)$ has $[W(T_E)] = 2$ elements. Then

$$\mu'(f_{T_E} dt) = \sum_{\{u\}} \int_{T_E} \Delta_0(t)^2 \cdot \frac{\mu'(t) + \mu'(\bar{t})}{2} \cdot \frac{\kappa(b)\boldsymbol{\kappa}(u)}{\Delta_0(t)} \Phi(t^u, f_0 dh) dt$$

$$= \sum_{E'} \sum_{\{u\}} [W(T_{E'})]^{-1} \int_{T_{E'}} \Delta_0(t)^2 \chi_{\mu'}(t^u) \Phi(t^u, f_0 dh) dt = \langle \chi_{\mu'}, {}'\Phi(f_0 dh)\rangle_e.$$

Here u ranges over the two-element group such that $\{t^u\}$ is $\{t, t'\}$, and $\boldsymbol{\kappa}$ is the nontrivial character on $\{u\}$.

If $-1 \notin N_{E/F}E^\times$ then $\bar{t}$ is stably conjugate but not conjugate to t, so we choose $t' = \bar{t}$. Then $\kappa(-1) = -1$, thus $\kappa(b(\bar{t})) = -\kappa(b(t))$, $f_{T_E}(\bar{t}) = f_{T_E}(t)$ and $[W(T_E)] = 1$. Then

$$\mu'(f_{T_E}dt) = \frac{1}{2}\int_{T_E} \Delta_0(t)^2 \cdot (\mu'(t) + \mu'(\bar{t}))\frac{\kappa(b(t))}{\Delta_0(t)}[\Phi(t, f_0dh) - \Phi(\bar{t}, f_0dh)]dt$$

$$= 2 \cdot \frac{1}{2}\sum_{E'}\int_{T_{E'}} \Delta_0(t)^2\chi_{\mu'}(t)\Phi(t, f_0dh)dt = \langle \chi_{\mu'}, {}'\Phi(f_0dh)\rangle_e.$$

We used:

$$\int_{T_E} (\mu'(t) + \mu'(\bar{t}))\kappa(b(\bar{t}))\Delta_0(t)\Phi(\bar{t}, f_0dh)dt$$

$$= \int_{T_E} (\mu'(t) + \mu'(\bar{t}))\kappa(b(t))\Delta_0(t)\Phi(t, f_0dh)dt.$$

For the last claim of the proposition, since $\kappa(b) \in \{\pm 1\}$ and $\boldsymbol{\kappa}(u) \in \{\pm 1\}$, we see that $\langle \chi_{\mu'}, \chi_{\mu'_1}\rangle_e$ is zero unless μ' and μ'_1 are characters on the norm one subgroup of the same quadratic extension $E = E'$ of F. Since $\chi_{\mu'}\overline{\chi}_{\mu'_1}$ is a stable function and $2 \cdot \frac{1}{2} = 1$, the inner product is

$$|T_E|^{-1}\int_{T_E} (\mu'(t) + \mu'(\bar{t}))(\overline{\mu}'_1(t) + \overline{\mu}'_1(\bar{t}))dt.$$

We are done by the first comment in this proof. □

II.2 Differential forms

2.1 The regular set of H. To compare orbital integrals on different groups we need to compare Haar measures, or invariant differential forms of highest degree. Let $\mathbb{G}_a$ be the additive group and $\zeta : \mathbf{H} \to \mathbb{G}_a$ the trace map. If $\gamma \in \mathbf{H}$ has distinct eigenvalues γ_1, $\gamma_2 = \gamma_1^{-1}$, then the differential $d\zeta$ of ζ at γ is given by

$$d\zeta = d\gamma_1 + d\gamma_2 = d\gamma_1 - \frac{d\gamma_1}{\gamma_1^2} = \gamma_1\frac{d\gamma_1}{\gamma_1} - \gamma_1^{-1}\frac{d\gamma_1}{\gamma_1} = (\gamma_1 - \gamma_2)\frac{d\gamma_1}{\gamma_1},$$

and it is nonzero. At a neighborhood of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\gamma_1 = \gamma_2$ we may assume that $a \neq 0$, $d = (1+bc)/a$ (or that $d \neq 0$, $a = (1+bc)/d$; this case is analogously treated). Then $\zeta(\gamma) = a + d$ has the differential

$$(1 - a^{-2}(1+bc))da + \frac{c}{a}db + \frac{b}{a}dc.$$

It vanishes only if $a^2 = 1 + bc$, $b = 0$, $c = 0$, namely at $\gamma = \pm I$. The subset $\mathbf{H}^{\mathrm{reg}}$ of $\mathbf{H}$ where $d\zeta$ is nonzero is called the *regular* set.

Fix (nonzero invariant) differential forms $\omega_{\mathbf{H}}$ and μ (of highest degrees 3 and 1) on $\mathbf{H}$ and on $\mathbb{G}_a$. Then μ defines a nonzero invariant form $\omega_\gamma(\mu)$ on $Z_{\mathbf{H}}(\gamma)$ (which is independent of $\omega_{\mathbf{H}}$). If $\mu = dx$ then $\omega_\gamma(\mu) = \frac{d\gamma_1}{\gamma_1}$ if γ is regular, or $= dx$ if $\gamma = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. If γ is stably conjugate to γ' then $\omega_{\gamma'}(\mu)$ is obtained from $\omega_\gamma(\mu)$ by the induced isomorphism of $Z_{\mathbf{H}}(\gamma)$ and $Z_{\mathbf{H}}(\gamma')$. The fibers of ζ are the stable conjugacy classes in $\mathbf{H}^{\mathrm{reg}}$. The quotient of $\omega_{\mathbf{H}}$ by $\omega_\gamma(\mu)$ defines an invariant form on the fibers of ζ in $\mathbf{H}^{\mathrm{reg}}$.

The trace map ζ extends to a map $\tilde{\zeta}$ from GL(2) to $X = \mathbb{G}_a^2$, defined by

$$\tilde{\zeta}(\gamma) = (\operatorname{tr}\gamma, \det\gamma) = (a+d, ad-bc).$$

It has 2×4 differential

$$\operatorname{diag}(da\ db\ dc\ dd) \cdot {}^t\begin{pmatrix} 1 & 0 & 0 & 1 \\ d & -c & -b & a \end{pmatrix},$$

which is nonsingular if one of $a-d$, b, c is nonzero. The singular set consists of the scalars.

2.2 The σ-regular set of G. Similarly, let $\xi : \mathbf{G} \to \mathbb{G}_a$ be defined by $\xi(\delta) = \operatorname{tr} N\delta$. To compute its differential note that $\xi(\delta) + 1 = \operatorname{tr}(\delta J {}^t\delta^{-1} J)$. Then $d\xi$ is the trace of the differential of the map $\delta \mapsto \delta J {}^t\delta^{-1} J$, which is

$$d\delta \cdot J {}^t\delta^{-1} J + \delta J \cdot d({}^t\delta^{-1}) \cdot J.$$

But

$$0 = dI = d(\delta\delta^{-1}) = d\delta \cdot \delta^{-1} + \delta \cdot d\delta^{-1},$$

hence

$$d\delta^{-1} = -\delta^{-1} \cdot d\delta \cdot \delta^{-1},$$

and

$$\operatorname{tr}[\delta J \cdot {}^t\delta^{-1} \cdot d({}^t\delta) \cdot {}^t\delta^{-1} \cdot J] = \operatorname{tr}[J\delta^{-1} \cdot d\delta \cdot \delta^{-1} J {}^t\delta] = \operatorname{tr}[d\delta \cdot \delta^{-1} \cdot J {}^t\delta J \delta^{-1}].$$

So

$$d\xi = \operatorname{tr} d\delta[\sigma(\delta) - \delta^{-1}\sigma(\delta^{-1})\delta^{-1}].$$

Then $d\xi$ is zero for all $d\delta$ only if $\delta\sigma(\delta) = (\delta\sigma(\delta))^{-1}$, thus $\delta\sigma(\delta)$ has square 1, hence has eigenvalues 1 or -1. Since $\delta\sigma(\delta)$ also has determinant 1, it is semisimple and $N\delta$ is $\pm I$. We conclude that the *σ-regular* set $\mathbf{G}^{\sigma\text{-reg}}$ of $\mathbf{G}$, defined to consist of the δ with $d\xi \neq 0$, consists of all δ with $N\delta \neq \pm I$.

The fibers of ξ on the regular set $\mathbf{G}^{\sigma\text{-reg}}$ are stable σ-conjugacy classes. We fix an invariant differential form $\omega_{\mathbf{G}}$ of highest degree on $\mathbf{G}$. As above μ determines an invariant form $\omega_\delta(\mu)$ of maximal degree on $Z_{\mathbf{G}}(\delta\sigma)$. If δ' is stably σ-conjugate to δ then $Z_{\mathbf{G}}(\delta'\sigma)$ is isomorphic to $Z_{\mathbf{G}}(\delta\sigma)$ over $\overline{F}$ and $\omega_\delta(\mu)$ transforms to a form $\omega_{\delta'}(\mu)$ of $Z_{\mathbf{G}}(\delta'\sigma)$ via this isomorphism.

2.3 Differential forms on G. Suppose that $\delta \times \sigma$ is semisimple in $G \rtimes \langle\sigma\rangle$ (namely $(\delta\sigma)^2 = \delta\sigma(\delta)$ is semisimple, hence $\gamma = N\delta$ and $\gamma_1 = N_1\delta$ are semisimple in H and H_1). Here F is a local field and as usual $G = \mathbf{G}(F)$. Choose a neighborhood $\mathbf{X}_\delta$ of the trivial coset $Z_{\mathbf{G}}(\delta\sigma)$ in $Z_{\mathbf{G}}(\delta\sigma)\backslash\mathbf{G}$, a section $s : Z_{\mathbf{G}}(\delta\sigma)\backslash\mathbf{G} \to \mathbf{G}$, and a σ-invariant neighborhood $\mathbf{Y}_\delta$ of the identity in $Z_{\mathbf{G}}(\delta\sigma)$ (all defined over F) so that the morphism

$$\mathbf{Y}_\delta \times \mathbf{X}_\delta \to \mathbf{G}, \qquad \text{by} \qquad (\varepsilon, g) \mapsto s(g)^{-1}\varepsilon\delta\sigma(s(g)),$$

is an immersion (its differential is nonsingular at each point). For the F-rational points we have that the map $Y_\delta \times X_\delta \to G$ is an analytic isomorphism onto an open subset of G. The neighborhoods $\mathbf{X}_\gamma$, $\mathbf{Y}_\gamma$, $\mathbf{X}_{\gamma_1}$, Y_{γ_1} can be introduced for γ in $\mathbf{H}$, γ_1 in $\mathbf{H}_1$. Let $\Theta(\varepsilon)$ be the determinant of the transformation $\operatorname{Ad}(\varepsilon\delta)\sigma - 1$ on the Lie algebra $\operatorname{Lie}(Z_{\mathbf{G}}(\delta\sigma)\backslash\mathbf{G})$ of $Z_{\mathbf{G}}(\delta\sigma)\backslash\mathbf{G}$.

LEMMA. *Locally the invariant form $\omega_{\mathbf{G}}$ on $\mathbf{G}$ can be taken to be $\Theta(\varepsilon)\omega_\delta^1 \wedge \omega^2$, where ω_δ^1 is an invariant form of maximal degree on $Z_{\mathbf{G}}(\delta\sigma)$, and ω^2 is a highest degree invariant form on $Z_{\mathbf{G}}(\delta\sigma)\backslash\mathbf{G}$.*

PROOF. To compute the differential we introduce an extension $F(\eta)$ of F, the quotient of the polynomial ring $F[x]$ by the ideal (x^2). For any algebraic group $\mathbf{J}$ over F there is an exact sequence

$$0 \to \operatorname{Lie}\mathbf{J} \to \mathbf{J}(F(\eta)) \to \mathbf{J} \to 1,$$

with maps $X \mapsto I + \eta X$, $h(I+\eta X) \mapsto h$. To study the map $(\varepsilon, h) \mapsto h^{-1} \cdot \varepsilon\delta \times \sigma \cdot h$ (ε in $Z_{\mathbf{G}}(\delta\sigma)$, h in $Z_{\mathbf{G}}(\delta\sigma)\backslash\mathbf{G}$), we replace h by $(I+\eta Y)h$, where Y is in $\mathrm{Lie}(Z_{\mathbf{G}}(\delta\sigma)\backslash\mathbf{G})$, and $\varepsilon\delta\times\sigma$ by $(I+\eta X)(\varepsilon\delta\times\sigma)$. Note that $(I+\eta Y)^{-1} = I - \eta Y$, and $aYa^{-1} = \mathrm{Ad}(a)Y$. Then $h^{-1}\cdot\varepsilon\delta\times\sigma\cdot h$ becomes

$$\begin{aligned} &h^{-1}(I-\eta Y)(I+\eta X)(\varepsilon\delta\times\sigma)(I+\eta Y)h \\ &\quad = h^{-1}(I+\eta(X-Y))(I+\eta\cdot\mathrm{Ad}(\varepsilon\delta)\sigma)Y\cdot\varepsilon\delta\times\sigma\cdot h \\ &\quad = h^{-1}[I+\eta(X-(I-\mathrm{Ad}(\varepsilon\delta)\sigma)Y)]\cdot\varepsilon\delta\times\sigma\cdot h. \end{aligned}$$

Then

$$\begin{aligned} \omega_{\mathbf{G}}(X+Y) &= \omega^1(X)\wedge\omega^2([\mathrm{Ad}(\varepsilon\delta)\sigma - I]Y) \\ &= \Theta(\varepsilon)\cdot\omega^1(X)\wedge\omega^2(Y), \end{aligned}$$

as asserted. □

2.4 Lemma. *Let $\mathbf{J}$ be a linear algebraic group defined over a local field F, contained in the matrix algebra $\mathbf{M}$. Suppose that δ is in J and ε in the centralizer $Z_J(\delta)$ of δ in J. If ε is near 1, then $Z_{\mathbf{J}}(\varepsilon\delta) \subset Z_{\mathbf{J}}(\delta)$.*

Proof. The group $\mathbf{J}$ acts on $\mathbf{M}$ by inner automorphisms. Enlarge F to include all eigenvalues λ of δ. Let $\mathbf{M}(\lambda)$ be the corresponding eigenspace. Then $\mathbf{M} = \oplus\mathbf{M}(\lambda)$. The group $Z_{\mathbf{J}}(\delta)$ is the intersection of $\mathbf{J}$ and $\mathbf{M}(1)$. Since ε lies in $Z_J(\delta)$, $\varepsilon\delta$ leaves each $\mathbf{M}(\lambda)$ invariant. If ε is near 1 all fixed vectors of $\varepsilon\delta$ lie in $\mathbf{M}(1)$. Indeed, if v lies in $\mathbf{M}(\lambda)$, then $v = \varepsilon\delta\cdot v = \lambda\varepsilon\cdot v$ and λ^{-1} is an eigenvalue of ε. This is impossible if $\lambda \neq 1$ and ε is near 1. But then $Z_{\mathbf{J}}(\varepsilon\delta) \subset \mathbf{J}\cap\mathbf{M}(1) = Z_{\mathbf{J}}(\delta)$, as asserted. □

Applying the lemma with $\mathbf{J} = \mathbf{G} \rtimes \{1,\sigma\}$ and δ in G, we have:

Corollary. *If ε is in $Z_G(\delta\sigma)$ near 1 then $Z_{\mathbf{G}}(\varepsilon\delta\sigma) \subset Z_{\mathbf{G}}(\delta\sigma)$.*

2.5 Lemma. *(i) If $N\delta = 1$, $\varepsilon \in Z_G(\delta\sigma)$ is near 1 and $N(\varepsilon\delta)$ has distinct eigenvalues, then $\boldsymbol{\kappa}(\varepsilon\delta) = \boldsymbol{\kappa}(\delta)$.*
(ii) If $N\delta = -I$; ε, ε' in $Z_{\mathbf{G}}(\delta\sigma) \simeq \mathbf{H}$ are stably conjugate but not conjugate, and $N(\varepsilon\delta)$ has distinct eigenvalues, then $\boldsymbol{\kappa}(\varepsilon\delta) = -\boldsymbol{\kappa}(\varepsilon'\delta)$.

Proof. (i) Note that

$$\varepsilon\delta J + {}^t(\varepsilon\delta J) = \varepsilon\delta J + {}^t(\delta J{}^t\varepsilon^{-1}) = \varepsilon\delta J + \varepsilon^{-1}{}^t(\delta J) = (\varepsilon+\varepsilon^{-1})\delta J.$$

Hence the value of $\boldsymbol{\kappa}(\varepsilon\delta)$ is 1 if and only if $Z_G(\frac{1}{2}(\varepsilon+\varepsilon^{-1})\delta\sigma)$ splits. But this is contained in $Z_G(\delta\sigma)$ by Corollary 2.4. Hence the two special orthogonal

groups split together.
(ii) We may assume that $\delta = e_1$, $e = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and then $\varepsilon = a_1$, $\varepsilon' = a'_1$, with a, a' in $\mathrm{SL}(2,F)$. The elements $\varepsilon\delta$ and $\varepsilon'\delta$ are σ-conjugate (and define equivalent forms) if and only if a and a' are conjugate (not only projectively conjugate, since $N(\varepsilon\delta)$ has distinct eigenvalues). □

2.6 Jacobians. Let $\xi' : Z_{\mathbf{G}}(\delta\sigma) \to \mathbb{G}_a$ be $\xi'(\varepsilon) = \xi(\varepsilon\delta) = \operatorname{tr} N(\varepsilon\delta)$ (ξ is defined in (2.2)). If $\varepsilon \in Z_G(\delta\sigma)$ is near 1 then ξ', μ and ω^1_δ can be used as above to define a form $\omega'_\varepsilon(\mu)$ on the centralizer of ε in $Z_{\mathbf{G}}(\delta\sigma)$. This centralizer is equal to $Z_{\mathbf{G}}(\varepsilon\delta\sigma)$ by Corollary 2.4. One has $\omega'_\varepsilon(\mu) = \Theta(\varepsilon)\omega_{\varepsilon\delta}(\mu)$.

Similarly we have

$$\omega_{\mathbf{H}} = \theta(\eta)\omega^1_\gamma \wedge \omega^2, \qquad \omega_{\mathbf{H}_1} = \theta_1(\eta_1)\omega^1_{\gamma_1} \wedge \omega^2,$$

where $\theta(\eta)$ and $\theta_1(\eta_1)$ are the functions

$$\det[\mathrm{Ad}(\eta\gamma) - I]_{\mathrm{Lie}\, Z_{\mathbf{H}}(\gamma)\backslash\mathbf{H}}, \qquad \det[\mathrm{Ad}(\eta_1\gamma_1) - I]_{\mathrm{Lie}\, Z_{\mathbf{H}_1}(\gamma_1)\backslash\mathbf{H}_1},$$

on $Z_{\mathbf{H}}(\gamma)$ and $Z_{\mathbf{H}_1}(\gamma_1)$. The maps

$$\zeta'(\eta) = \mathrm{tr}(\eta\gamma), \qquad \zeta'_1(\eta_1) = \mathrm{tr}(\eta_1\gamma_1)$$

are used to define $\omega'_\eta(\mu)$, $\omega'_{\eta_1}(\mu)$, and we have

$$\omega'_\eta(\mu) = \theta(\eta)\omega_{\eta\gamma}(\mu), \qquad \omega'_{\eta_1}(\mu) = \theta_1(\eta_1)\omega_{\eta_1\gamma_1}(\mu).$$

If $\gamma = N\delta$, $\gamma_1 = N_1\delta$ and ε is in $Z_G(\delta\sigma)$, then $\varepsilon\delta\sigma(\varepsilon\delta) = \varepsilon^2\delta\sigma(\delta)$ and ε commutes with $\delta\sigma(\delta)$, so that

$$N(\varepsilon\delta) = \eta\gamma \quad (\eta \in Z_H(\gamma)), \qquad N_1(\varepsilon\delta) = \eta_1\gamma_1 \quad (\eta_1 \in Z_{H_1}(\gamma_1)).$$

To compute $\Theta(\varepsilon)$, $\theta(\eta)$, $\theta_1(\eta_1)$, we may assume that ε, hence η, η_1 are semisimple, since these functions depend only on the semisimple parts of ε, η, η_1 in their Jordan decomposition. Further, we can work over the algebraic closure $\overline{F}$, and take δ to be the diagonal matrix $\mathrm{diag}(a,b,c)$. Then ε can also be taken to be diagonal; hence $\varepsilon = \mathrm{diag}(d,1,d^{-1})$ since it lies in $Z_G(\delta\sigma)$. If the eigenvalues of $N(\varepsilon\delta)$ are denoted by $\beta_1(= ad^2/c)$, $\beta_2 = \beta_1^{-1}$, then it can be checked that:

2.7 Twisted Jacobian. If $\gamma = I$ then $\theta(\eta) = 1$ and since a 3×3 matrix $X = \sigma X$ has the form

$$\begin{pmatrix} x_1 & x_2 & 0 \\ x_4 & 0 & x_2 \\ 0 & x_4 & -x_1 \end{pmatrix},$$

we have

$$\Theta(\varepsilon) = (1+d)(1+d^{-1})(1+d^2)(1+d^{-2}).$$

If $\gamma = -I$, take $\delta = \mathrm{diag}(-1,1,1)$, then $\theta(\eta) = 1$ and since a 3×3 matrix $X = \mathrm{Ad}(\delta)\sigma X$ has the form $\begin{pmatrix} x_1 & 0 & x_3 \\ 0 & 0 & 0 \\ x_7 & 0 & -x_1 \end{pmatrix}$, we have

$$\Theta(\varepsilon) = (1+d^2)(1+d^{-2}).$$

If $\gamma \neq \pm I$ then $\theta_1(\eta_1) = (1-\beta_1)(1-\beta_2)$, and since $X = \mathrm{Ad}(\delta)\sigma X$ has the form $\mathrm{diag}(x_1, 0, -x_1)$, we have

$$\Theta(\varepsilon) = (1-\beta_1^2)(1-\beta_2^2), \qquad \theta(\eta) = (1-\beta_1^2)(1-\beta_2^2).$$

2.8 Pullback. The map $\varphi : Z_{\mathbf{H}}(\gamma) \to Z_{\mathbf{G}}(\delta\sigma)$ of I.2 can be used to pull back the form $\omega_\delta(\mu)$ to a form $\varphi^*(\omega_\delta(\mu))$ on $Z_{\mathbf{H}}(\gamma)$. The comparison is given by

2.8.1 LEMMA. *The form $\varphi^*(\omega_\delta(\mu))$ is equal to $\omega_\gamma(\mu)$.*

The trace map $\zeta_1 : \mathbf{H}_1 = \mathrm{SO}(3) \to \mathbb{G}_a$ is smooth on the regular set $\mathbf{H}_1^{\mathrm{reg}}$ of γ_1 with distinct eigenvalues, and $\omega_{\gamma_1}(\mu)$ can be introduced for such γ_1. Note that the centralizer $Z_{\mathbf{H}_1}(\gamma_1)$ of γ_1 in $\mathbf{H}_1$ is isomorphic to $Z_{\mathbf{G}}(\delta\sigma)$. The pullback of $\omega_\delta(\mu)$ to $Z_{\mathbf{H}_1}(\gamma_1)$ is denoted again by $\omega_\delta(\mu)$.

2.8.2 LEMMA. *If $\gamma_1 = N_1\delta$ has distinct eigenvalues 1, γ', $\gamma'' = \gamma'^{-1}$ (see* I.2.3) *then*

$$\omega_{\gamma_1}(\mu) = (1+\gamma')(1+\gamma'')\omega_\delta(\mu).$$

PROOF. To verify the lemmas it suffices to take the standard form $\mu = dx$ on $\mathbb{G}_a$. If $N\delta$ has distinct eigenvalues then $Z_{\mathbf{G}}(\delta\sigma)$ is abelian, one-dimensional, and isomorphic to $Z_{\mathbf{H}}(\gamma)$ and to $Z_{\mathbf{H}_1}(\gamma_1)$. As in (2.1) we compute

$$(\xi')^*(\mu) = d\xi' = (\beta_1 - \beta_2)\frac{d\beta_1}{\beta_1}.$$

But $\omega_\delta^1 = e\frac{d\beta_1}{\beta_1}$ for some constant e. It is the product of $\omega'_\varepsilon(\mu)$ and the quotient $\omega_\delta^1/(\xi')^*(\mu) = e/(\beta_1 - \beta_2)$ of one-forms on $Z_{\mathbf{G}}(\delta\sigma)$ and $\mathbb{G}_a$. The same computation yields the same value for $\omega'_\eta(\mu)$ and $\omega'_{\eta_1}(\mu)$. So it remains to note that $\Theta(\varepsilon)/\theta(\eta) = 1$ and that

$$\Theta(\varepsilon)/\theta_1(\eta_1) = (1+\beta_1)(1+\beta_2),$$

and $\beta_i = \gamma_i$ when $\varepsilon = I$, to deduce the lemmas for δ with $N\delta \neq \pm I$.

It remains to complete the proof of lemma 2.8.1. If $\gamma = N\delta$ is I or $-I$ then the epimorphism $\varphi : Z_{\mathbf{H}}(\gamma) \to Z_{\mathbf{G}}(\delta\sigma)$, $\varphi(\eta_1) = \varepsilon$, satisfies $\eta = N(\varphi(\eta_1)) = \eta_1^m$ with $m = 4$ if $\gamma = I$ and $m = 2$ if $\gamma = -I$. Indeed, if $\gamma = I$ we may take $\delta = I$ and

$$\eta_1 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in Z_{\mathbf{H}}(\gamma) = \mathrm{SL}_2 \overset{\varphi}{\mapsto} \varepsilon = \begin{pmatrix} a^2 & & 0 \\ & 1 & \\ 0 & & a^{-2} \end{pmatrix} \in Z_{\mathbf{G}}(\delta\sigma) = \mathrm{SO}(3)$$

$$\overset{N}{\mapsto} \eta = \begin{pmatrix} a^4 & 0 \\ 0 & a^{-4} \end{pmatrix}.$$

If $\gamma = -I$ we may take $\delta = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$ and

$$\eta_1 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in Z_{\mathbf{H}}(\gamma) = \mathrm{SL}_2 \overset{\varphi}{\mapsto} \varepsilon = \begin{pmatrix} a & & 0 \\ & 1 & \\ 0 & & a^{-1} \end{pmatrix} \in Z_{\mathbf{G}}(\delta\sigma)$$

$$\overset{N}{\mapsto} \eta = \begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}.$$

Given ε near 1 we may choose η_1 near 1. Then $Z_{\mathbf{H}}(\eta_1\gamma) = Z_{\mathbf{H}}(\eta\gamma)$ and $Z_{\mathbf{G}}(\varepsilon\delta\sigma) = \varphi(Z_{\mathbf{H}}(\eta_1\gamma))$.

It remains to show that $\varphi^*(\omega'_\varepsilon(\mu)) = m^2\omega'_\eta(\mu)$ at a unipotent ε in $Z_{\mathbf{G}}(\delta\sigma)$, for then

$$\Theta(\varepsilon)\varphi^*(\omega_{\varepsilon\delta}(\mu)) = m^2\theta(\eta)\omega_{\eta\gamma}(\mu)$$

and at $\varepsilon = 1$, $\varphi^*(\omega_\delta(\mu)) = \omega_\gamma(\mu)$ (since $\theta(\eta) = 1$ and $\Theta(\varepsilon) \to m^2$ as $\varepsilon \to 1$).

Let O_η, O_{η_1}, O_ε be the conjugacy classes of η, η_1, ε. Since we have a commutative diagram

$$\begin{array}{ccc} Z_{\mathbf{H}}(\eta_1\gamma)\backslash Z_{\mathbf{H}}(\gamma) \simeq O_{\eta_1} & \hookrightarrow & Z_{\mathbf{H}}(\gamma) \\ \simeq\downarrow \varphi & & \downarrow \varphi\,, \\ Z_{\mathbf{G}}(\varepsilon\delta\sigma)\backslash Z_{\mathbf{G}}(\delta\sigma) \simeq O_\varepsilon & \hookrightarrow & Z_{\mathbf{G}}(\delta\sigma) \end{array}$$

the pullback $\varphi^*(\omega'_\varepsilon(\mu))$ of the form $\omega'_\varepsilon(\mu)$ on $Z_{\mathbf{G}}(\varepsilon\delta\sigma)$ is a form on $Z_{\mathbf{H}}(\eta_1\gamma)$ defined by the function $\xi'\circ\varphi : Z_{\mathbf{H}}(\gamma) \to \mathbb{G}_a$ and the form $\varphi^*(\omega^1_\delta)$ on $Z_{\mathbf{H}}(\gamma)$. Define $\psi(\eta_1) = \eta_1^m$. Then

$$\xi'(\varphi(\eta_1)) = \operatorname{tr} N(\varepsilon\delta) = \operatorname{tr} \eta\gamma = \operatorname{tr}(\eta_1^a\gamma) = \zeta'(\psi(\eta_1)).$$

There is also a commutative diagram

$$\begin{array}{ccc} Z_{\mathbf{H}}(\eta_1\gamma)\backslash Z_{\mathbf{H}}(\gamma) \simeq O_{\eta_1} & \hookrightarrow & Z_{\mathbf{H}}(\gamma) \\ \simeq\downarrow \psi & & \downarrow \psi, \\ Z_{\mathbf{H}}(\eta\gamma)\backslash Z_{\mathbf{H}}(\gamma) \simeq O_{\eta} & \hookrightarrow & Z_{\mathbf{H}}(\gamma) \end{array}$$

hence $\varphi^*(\omega'_\varepsilon(\mu)) = \psi^*(\omega'_\eta(\mu))$. But

$$\psi^*(\omega'_\eta(\mu))/\omega'_\eta(\mu) = \psi^*(\varphi^*(\omega^1_\delta))/\varphi^*(\omega^1_\delta)$$

$$= \theta(\eta)/\theta(\eta_1) = \frac{(1-\beta_1^{2m})(1-\beta_2^{2m})}{(1-\beta_1^2)(1-\beta_2^2)}$$

is equal to m^2 as $\beta_1 \to 1$. This completes the proof of lemma 2.8.1. $\square$

II.3 Matching orbital integrals

3.1 Definitions. Let F be a local field. All objects below are defined over F. A highest degree invariant differential form $\omega_{\mathbf{G}}$ determines a Haar measure $dg = d_G g = d_G$ on G. A maximal degree F-rational invariant form ω_δ on $Z_{\mathbf{G}}(\delta\sigma)$ determines a measure $d_\delta = d_\delta t$ on $Z_G(\delta'\sigma)$ for any δ' in G stably σ-conjugate to δ. The two measures dg, $d_{\delta'}t$ determine a quotient measure on the quotient $Z_G(\delta'\sigma)\backslash G$. Let f be a smooth compactly supported function on G, and put

$$\Phi(\delta\sigma, f dg) = \Phi(\delta\sigma, f; d_\delta, d_G) = \int_{Z_G(\delta\sigma)\backslash G} f(g^{-1}\delta\sigma(g))\frac{dg}{d_\delta t}.$$

If $N\delta \neq 1$ put

$$\Phi^{\mathrm{st}}(\delta\sigma, f dg) = \Phi^{\mathrm{st}}(\delta\sigma, f; d_\delta, d_G) = \sum_{\delta'} \Phi(\delta'\sigma, f dg).$$

The sum is over a set of representatives for the σ-conjugacy classes in the stable σ-conjugacy class of δ. If $N\delta = 1$ put

$$\Phi^{\mathrm{st}}(\delta\sigma, fdg) = \sum_{\delta'} \boldsymbol{\kappa}(\delta')\Phi(\delta'\sigma, fdg).$$

If f_0 is a smooth compactly supported function on H define

$$\Phi(\gamma, f_0 dh) = \Phi(\gamma, f_0; d_\gamma, d_H) = \int_{Z_H(\gamma)\backslash H} f_0(g^{-1}\gamma g)\frac{d_H h}{d_\gamma t_0}$$

and

$$\Phi^{\mathrm{st}}(\gamma, f_0 dh) = \Phi^{\mathrm{st}}(\gamma, f_0; d_\gamma, d_H) = \sum_{\gamma'} \Phi(\gamma', f_0 dh).$$

Here d_γ is a measure on $Z_H(\gamma)$, and d_H is a measure on H. If $\gamma = N\delta$ then there is $\varphi : Z_H(\gamma) \twoheadrightarrow Z_G(\delta\sigma)$, and we take $d_\gamma = |[\ker\varphi]|^{-1}\varphi^*(d_\delta)$. Thus the d_γ-volume of the maximal compact subgroup of $Z_H(\gamma)$ is $|[\ker\varphi]|^{-1}$ times the d_δ-volume of the maximal compact subgroup of $Z_G(\delta\sigma)$, $\gamma = N\delta$.

If the functions f and f_0 satisfy the relation

$$\Phi^{\mathrm{st}}(\gamma, f_0; d_\gamma, d_H) = \Phi^{\mathrm{st}}(\delta\sigma, f; d_\delta, d_G)$$

for all γ, δ with $\gamma = N\delta$, we write $f_0 dh = \lambda^*(fdg)$.

3.2 Proposition. *For each fdg there is $f_0 dh$ with $f_0 dh = \lambda^*(fdg)$. For each $f_0 dh$ there is fdg with $f_0 dh = \lambda^*(fdg)$.*

Proof. Applying partition of unity and translating, when passing from f to f_0 (resp. f_0 to f) we may assume that f (resp. f_0) is supported on a small neighborhood of a fixed semisimple element δ_0 (resp. γ_0). The proposition is proved by dealing with the various possible γ_0, δ_0. If δ_0 and γ_0 are such that $\gamma_0 = N\delta_0$ is nonscalar then the proof is simple, and it remains to deal with $\gamma_0 = -I$ and $\gamma_0 = I$.

Suppose that $\gamma_0 = -I$. Fix a section s of $Z_G(\delta_0\sigma)\backslash G$ in G. Given f and η_1 in $Z_H(\gamma_0) = H$, put $\varepsilon = \varphi(\eta_1)$. For η in some fixed neighborhood of I define

$$f_0(\eta\gamma_0) = f_0'(\eta_1), \qquad f_0'(\eta_1) = \int_{Z_G(\delta_0\sigma)\backslash G} f(g^{-1}\varepsilon\delta_0\sigma(g))\frac{d_G}{d_{\delta_0}}. \tag{3.2.1}$$

Here $\psi : H \to H$, $\eta_1 \mapsto \eta = \eta_1^m$ $(m = 2)$ has analytic inverse for η near 1, and we put $\eta_1 = \psi^{-1}(\eta)$. Put $f_0(\eta\gamma_0) = 0$ otherwise. Note that $\varphi(H) = Z_G(\delta_0\sigma)$, that $\varphi(Z_H(\eta_1')) = Z_{Z_G(\delta_0\sigma)}(\varepsilon) = Z_G(\varepsilon'\delta_0\sigma)$ if η_1' is near 1 and $\varepsilon' = \varphi(\eta_1')$. Further, $d_H = d_{\gamma_0} = \varphi^*(d_{\delta_0})$, $d_{\eta_1} = \varphi^*(d_{\varepsilon\delta_0})$, $d_\eta = d_{\eta_1}$. Hence

$$\begin{aligned}\Phi^{\mathrm{st}}(\eta\gamma_0, f_0; d_\eta, d_{\gamma_0}) &= \sum_{\eta'} \int_{Z_H(\eta')\backslash H} f_0(h^{-1}\eta'\gamma_0 h)\frac{d_{\gamma_0}}{d_\eta} \\ &= \sum_{\eta_1'} \int_{Z_H(\eta_1')\backslash H} f_0'(h^{-1}\eta_1' h)\frac{d_{\gamma_0}}{d_{\eta_1'}} = \Phi^{\mathrm{st}}(\eta_1, f_0'; d_{\eta_1}, d_{\gamma_0}) \\ &= \sum_{\eta_1'} \int_{Z_H(\eta_1')\backslash H} \int_{Z_G(\delta_0\sigma)\backslash G} f(g^{-1}\varphi(h^{-1}\eta_1' h)\delta_0\sigma(g))\frac{d_{\gamma_0}}{d_{\eta_1}}\frac{d_G}{d_{\delta_0}} \\ &= \sum_{\varepsilon'} \int_{Z_G(\varepsilon'\delta_0\sigma)\backslash G} f(g^{-1}\varepsilon'\delta_0\sigma(g))\frac{d_G}{d_{\varepsilon\delta_0}} = \Phi^{\mathrm{st}}(\varepsilon\delta_0\sigma, f; d_{\varepsilon\delta_0}, d_G).\end{aligned}$$

Here $\eta \in H$ is near 1, and $\eta' \in H$ ranges over a set of representative for the conjugacy classes within the stable conjugacy class of η. The element η' can be taken to be near 1. The same comment applies to $\eta_1' = \psi^{-1}(\eta')$. Then $\varepsilon'\delta_0$ ($\varepsilon' = \varphi(\eta_1') \in Z_G(\delta\sigma)$) ranges over a set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of $\varepsilon\delta_0$. Note that $\eta\gamma_0 = N(\varepsilon\delta_0)$, so that f_0 is the desired function.

Conversely, given f_0 with support near γ_0, (3.2.1) defines f_0' for $\eta_1 \in H$ near 1, and f is defined by

$$f(s(g)^{-1}\varepsilon\delta_0\sigma(s(g))) = f_0'(\eta_1)\beta(g),$$

where β is a smooth compactly supported function on $Z_G(\delta_0\sigma)\backslash G$ with

$$\int_{Z_G(\delta_0\sigma)\backslash G} \beta(g)dg = 1.$$

Next we deal with the case where $\gamma = I$. We replace H by an inner form H' if necessary, so that $\varphi : \mathbf{H'} \to Z_{\mathbf{G}}(\delta\sigma)$ be defined over F. Then $\varphi : H' \to Z_G(\delta\sigma)$ is a local isomorphism and (3.2.1) defines a function f_0' on H'. If $\eta_1 \neq I$ then φ restricted to $Z_H(\eta_1) = Z_{H'}(\eta_1)$ is not $\varphi_{\eta_1} : Z_H(\eta_1) \to Z_{Z_G(\delta\sigma)}(\varepsilon) = Z_G(\varepsilon\delta\sigma)$, but its square. Here we take η_1 near $\pm I$.

Hence $d_{\eta_1} = \frac{1}{|2|}\varphi^*(d_{\varepsilon\delta_0})$. We have taken $d_{\gamma_0} = \frac{1}{|2|}\varphi^*(d_{\delta_0})$. As in the case of $\gamma = -I$ above, we have

$$\Phi^{\mathrm{st}}(\eta_1, f_0'; d_{\eta_1}, d_{\gamma_0}) = \Phi^{\mathrm{st}}(\varepsilon\delta_0\sigma, f; d_{\varepsilon\delta_0}, d_G).$$

Both sides are 0 when η_1 is not close to $\pm I$. Since $\psi : \eta_1 \mapsto \eta' = \eta_1^m$ $(m = 4)$ has an analytic inverse on H' in a neighborhood of I, we may define a function f_0'' on H' by $f_0''(\eta') = f_0'(\eta_1)$.

As is well known, the orbital integrals of f_0'' can be transferred to H. This is clear if H' is isomorphic to H over F. Otherwise there exists f_0 on H with

$$\Phi^{\mathrm{st}}(\eta, f_0; d_\eta, d_H) = \Phi^{\mathrm{st}}(\eta', f_0''; d_{\eta'}, d_{H'})$$

when η in H is regular and corresponds to η' in H', and with

$$\Phi^{\mathrm{st}}(\eta, f_0; d_\eta, d_H) = 0$$

if η has distinct eigenvalues in $F^\times$ or it is a scalar multiple of a nontrivial unipotent. In this case $f_0(\pm I) = -f_0''(\pm I)$. This is the required f_0. The passage back from f_0 to f is done as in the case of $\gamma = -I$, but we have to choose δ_0 with $N\delta_0 = I$ and $\boldsymbol{\kappa}(\delta_0) = 1$. □

3.3 Corollary. *If f, f_0 are compactly supported smooth functions on G, H with*

$$\Phi^{\mathrm{st}}(\gamma, f_0; d_\gamma, d_H) = \Phi^{\mathrm{st}}(\delta\sigma, f; d_\delta, d_G)$$

for all $\gamma = N\delta$ with distinct eigenvalues, then $\lambda^*(f) = f_0$.

Proof. Choose f_0' with $f_0' = \lambda^*(f)$. Then the stable orbital integrals of $f_0 - f_0'$ are 0 on the regular semisimple set, hence identically 0, since the germs of $\Phi^{\mathrm{st}}(f)$ at $u = \pm I$ are scalar multiples of $f_0(u)$ and $\Phi^{\mathrm{st}}\left(u\begin{pmatrix}1&1\\0&1\end{pmatrix}, f_0\right)$. □

3.4 Unstable lifting. Analogous discussion has to be carried out for the transfer of functions from G to $H_1 = \mathrm{SO}(3, F)$. If $\gamma = N_1\delta$ has eigenvalues 1, γ', γ'' with $\gamma' \neq \gamma''$, put

$$\Phi^{\mathrm{us}}(\delta\sigma, f) = \Phi^{\mathrm{us}}(\delta\sigma, f; d_\delta, d_G) = \sum_{\delta'} \boldsymbol{\kappa}(\delta')\Phi(\delta'\sigma, f; d_\delta, d_G).$$

If f_1 is a smooth compactly supported function on H_1 then

$$\Phi(\gamma, f_1 dh_1) = \Phi(\gamma, f_1; d_\gamma, d_{H_1}) = \int_{Z_{H_1}(\gamma)\backslash H_1} f_1(h^{-1}\gamma h)\frac{dh}{d_\gamma t},$$

for all regular semisimple γ. We say that $f_1 = \lambda_1^*(f)$ if

$$\Phi(\gamma, f_1 dh_1) = |(1+\gamma')(1+\gamma'')|^{1/2}\Phi^{\mathrm{us}}(\delta\sigma, f dg)$$

for all $\gamma = N_1\delta$ with distinct eigenvalues, where $d_\gamma = \varphi^*(d_\delta)$ and $\varphi : Z_{H_1}(\gamma)\tilde{\rightarrow} Z_G(\delta\sigma)$.

Proposition. *For each $f dg$ there is $f_1 dh_1$, and for each $f_1 dh_1$ there is $f dg$, with $f_1 dh_1 = \lambda_1^*(f dg)$.*

Proof. This is easily verified for a function f with support near δ_0 and a function f_1 with support near a fixed element γ_0, if $\gamma_0 = N_1\delta_0$ has distinct eigenvalues, due to Lemma 2.8.2. The difficulty is when $N\delta_0$ is $-I$, for then there are several conjugacy classes in H_1 of elements γ_0 with eigenvalues 1, -1, -1. For each quadratic extension of F there is such a γ_0 in H_1 (with representative $\begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix}$ in $\mathrm{GL}(2,F)$, θ in F but not in F^2). The proposition defines $\Phi(\gamma, f_1; d_\gamma, d_{H_1})$ at any γ in H_1 with distinct eigenvalues; it is 0 unless the eigenvalues of γ are close to those of γ_0. It has to be shown that the function $\Phi(\gamma, f_1 dh_1)$ is smooth at γ_0 to use the classification theorem of orbital integral on H_1 to deduce the existence of f_1. Namely, we have to establish the smoothness at γ_0 of the sum

$$\sum_{\varepsilon'} \boldsymbol{\kappa}(\varepsilon'\delta)\Phi(\varepsilon'\delta_0\sigma, f; d_\delta, d_G) = \sum_{\eta_1'} \boldsymbol{\kappa}(\varepsilon'\delta_0)\Phi(\eta_1', f_0'; d_{\eta_1}, d_{\gamma_0})$$

of the proof of (3.2), multiplied by

$$|(1+\gamma')(1+\gamma'')|^{1/2} = |\gamma''|^{1/2}|1+\gamma'|.$$

Here $\varphi(\eta_1') = \varepsilon'$, $\varphi : H \to Z_G(\delta\sigma)$, and the product is smooth, since the eigenvalues γ', γ'^{-1} of $\gamma = N(\varepsilon\delta_0)$ are near -1. □

3.5 Unstable germs. It was noted above that there is a natural bijection between the conjugacy classes of γ in H_1 with eigenvalues 1, -1, -1 and the quotient $F^\times/F^{\times 2}$. The σ-conjugacy classes of δ in G with $N\delta$ equals the product of -1 and a nontrivial unipotent are also parametrized by $F^\times/F^{\times 2}$. The Hilbert symbol defines a pairing, which we denote by $\langle\gamma, \delta\rangle$.

PROPOSITION. *If γ in H_1 has eigenvalues 1, -1, -1 and $f_1 dh_1 = \lambda_1^*(f dg)$, then*

$$\lim_{\gamma_1 \to \gamma} |(1+\gamma_1')(1+\gamma_1'')|^{1/2} \Phi(\gamma_1, f_1; d_{\gamma_1}(\mu), d_{H_1})$$

$$= \sum_\delta \langle \gamma, \delta \rangle \Phi(\delta\sigma, f; d_\delta(\mu), d_G).$$

The sum is over σ-conjugacy classes of δ in G with $N\delta = -1$ times a nontrivial unipotent. The eigenvalues of γ_1 are 1, γ_1', γ_1''.

PROOF. As in (3.4) the expression on the left is

$$|(1+\gamma_1')(1+\gamma_1'')|^{1/2} \Phi(\gamma_1, f_1; d_{\gamma'}(\mu), d_{H_1}) = \Phi^{\mathrm{us}}(\delta_1\sigma, f; d_{\delta'}(\mu), d_G)$$

where $\delta_1 = \varepsilon\delta_0$ and $N\delta_1 = \gamma_1$. If $\varphi(\eta_1') = \varepsilon'$, $\varphi : H \tilde{\to} Z_G(\delta_0\sigma)$, by Lemma 2.8.1 this is equal to (the sum is over the conjugacy classes η_1' in the stable class)

$$\sum_{\eta_1'} \boldsymbol{\kappa}(\varphi(\eta_1')\delta_0) \Phi(\eta_1', f_0'; d_{\eta_1'}(\mu), d_H).$$

Here η_1' is a regular element of H, and lies in some torus T.

The right side

$$\sum_{\{\delta; N\delta = -\text{ unip} \neq -I\}} \langle \gamma, \delta \rangle \Phi(\delta\sigma, f; d_\delta(\mu), d_G)$$

is equal to

$$\sum_{\eta_1} \langle \gamma, \varphi(\eta_1)\delta_0 \rangle \Phi(\eta_1, f_0'; d_{\eta_1}(\mu), d_H),$$

where $\delta = \varphi(\eta_1)\delta_0$, and the sum ranges over the nontrivial unipotent classes η_1 in H. It suffices to show the equality of the two sums only for f supported on a small neighborhood of $\delta' = \varphi(\eta_1')\delta_0$, where δ' is close to $\delta = \varphi(\eta_1)\delta_0$, where η_1 is a nontrivial unipotent in H.

So we may assume that

$$\delta_0 = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}, \qquad \delta = \begin{pmatrix} 1 & & x \\ & 1 & \\ 0 & & 1 \end{pmatrix} \delta_0, \qquad \delta_1 = \begin{pmatrix} \alpha & & \alpha x \\ & 1 & \\ \alpha\varepsilon & & \alpha \end{pmatrix} \delta_0,$$

where $x \in F^\times$, $\eta_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, ε is near 0, $\eta_1' = \begin{pmatrix} \alpha & \alpha x \\ \alpha\varepsilon & \alpha \end{pmatrix}$ where $\alpha^2(1-\varepsilon x) = 1$ since $1 - \varepsilon x \in F^{\times 2}$ as ε is small; we may assume that α is also a square, since it is close to 1. It has to be shown that: when $N\delta_1 = \gamma_1 \to \gamma$, and δ_1 is near δ, namely η'_1 lies in the centralizer $Z_H(\gamma)$ of γ in H (as $N\delta_1 = \frac{-1}{\det \eta'_1}\eta'^2_1$), and it is near η_1, then $\boldsymbol{\kappa}(\delta') = \langle \gamma, \delta \rangle$. But

$$\frac{1}{2}[\delta' J + {}^t(\delta' J)] = \begin{pmatrix} x\alpha & & 0 \\ & -1 & \\ 0 & & -\varepsilon\alpha \end{pmatrix},$$

hence $\boldsymbol{\kappa}(\delta') = (x, -\varepsilon)$. The centralizer $Z_H(\gamma)$ of γ splits over $F(\lambda)$ with $\lambda^2 - c = 0$ for some c in $F^\times$, hence $\langle \gamma, \delta \rangle = (c, x)$. But η'_1 lies in $Z_H(\gamma)$ only if $(\lambda - 1)^2 - \varepsilon x = 0$ splits in $F(\lambda)$, namely if $\varepsilon x / c$ is a square in $F^\times$. Hence

$$\langle \gamma, \delta \rangle = (x, c) = (x, \varepsilon x) = (x, -\varepsilon) = \boldsymbol{\kappa}(\delta_1),$$

as asserted. □

3.6 Proposition. *If $\lambda_1^*(f dg) = f_1 dh_1$ then $f_1(1) = |2| \sum \Phi(\delta\sigma, f dg)$, where the sum is over the σ-conjugacy classes of δ with $N\delta = 1$. If $\gamma = N\delta$ is a nontrivial unipotent then*

$$\Phi(\gamma, f_1; d_\gamma(\mu), d_{H_1}) = |2| \Phi(\delta\sigma, f; d_\delta(\mu), d_G). \tag{3.6.1}$$

Proof. If $N\delta = 1$ and f'_0 is defined by (3.2.1) then

$$\Phi^{\mathrm{us}}(\varepsilon\delta\sigma, f; d_{\varepsilon\delta}, d_G) = \boldsymbol{\kappa}(\delta) \Phi(\eta_1, f'_0; d_{\eta_1}, d_H)$$

where $\varphi : H \to Z_G(\delta_0 \sigma)$, η_1 is near 1 with $\varphi(\eta_1) = \varepsilon$, hence $\boldsymbol{\kappa}(\varepsilon\delta) = \boldsymbol{\kappa}(\delta)$ by Lemma 2.5. The factor $|(1 + \gamma')(1 + \gamma'')|^{1/2}$ is smooth for γ' near 1, the asymptotic behavior permits the application of [L5], Lemma 6.1, hence f_1 satisfies $f_1(1) = \boldsymbol{\kappa}(\delta)|2| f'_0(1)$. When $\boldsymbol{\kappa}(\delta) = 1$ the right side of (3.6.1) is the limit of $\Delta_1(\eta_1)\Phi(\eta_1, f'_0 dh)$ as $\eta_1 \to 1$, and the left side is the corresponding limit of $\Delta_1 \Phi(f_1)$ as

$$N(\varepsilon\delta) = \varepsilon^2 N\delta = \varepsilon^2 = \eta_1^4 \to 1;$$

η_1 can be taken in the split set. □

II.4 Germ expansion

This section is not used anywhere else in this work. We sketch the well-known germ expansion of orbital integrals (cf. Shalika [Sl], Vigneras [Vi]), from which one can deduce that the fundamental lemma of II.1 implies the matching result of II.3.

For any g in G, the centralizer $Z_G(g)$ of g in G is unimodular (see, e.g., Springer-Steinberg [SS], III, (3.27b), p. 234). By Bernstein-Zelevinski [BZ1], (1.21), it follows that there is a unique (up to a scalar multiple) nonzero measure (positive distribution) on every $\mathrm{Int}(G)$-orbit $\mathcal{O}$. By Rao [Ra] for a general G in characteristic zero, and Bernstein [B], (4.3), p. 70, for $G = \mathrm{GL}(n)$ in any characteristic, this extends to a unique (nonzero) $\mathrm{Int}(G)$-invariant measure $\Phi_{\mathcal{O}}$ on G whose support is the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in G ($\Phi_{\mathcal{O}}$ is the orbital integral over $\mathcal{O}$; it is a linear form on $C_c^\infty(G)$ — not only $C_c^\infty(\mathcal{O})$ — which takes positive values at positive valued functions).

Let s be a semisimple element in a p-adic reductive group G. Its centralizer $Z_G(s)$ in G is reductive, and also connected when the derived group of G is simply connected ([SS], II, (3.19), p. 201). Lemma 19 of Harish-Chandra [HC1], p. 52, can be used to reduce the G-orbital integrals near s to $Z_G(s)$-orbital integrals near the identity.

The set X of the elements in G whose semisimple part is in $\mathrm{Int}(G)s$ is closed (see, e.g., [SS], III, Theorem 1.8(a), p. 217). There are only finitely many $\mathrm{Int}(G)$-orbits $\mathcal{O}$ in X (see Richardson [Ri], Proposition 5.2, and Serre [Se], III, 4.4, Cor. 2). Since $\mathcal{O}$ is open in $\overline{\mathcal{O}}$, and $\dim \mathcal{O}' < \dim \mathcal{O}$ for every orbit $\mathcal{O}' \subset \overline{\mathcal{O}}$, $\mathcal{O}' \neq \mathcal{O}$ (see Borel [Bo1], I.1.8 ("Closed Orbit Lemma"), and Harish-Chandra [HC1], Lemma 31, p. 71), there are $f_{\mathcal{O}} \in C_c^\infty(G)$ with $\Phi_{\mathcal{O}}(f_{\mathcal{O}'}) = \delta_{\mathcal{O},\mathcal{O}'}$ for all orbits $\mathcal{O}, \mathcal{O}'$ in X. In fact, the $\mathcal{O}$ can be numbered $\mathcal{O}_i$ $(1 \leq i \leq k)$, with $\mathcal{O}_1 = \mathrm{Int}(G)s$, $\mathcal{O}^j = \underset{i \leq j}{\cup} \mathcal{O}_i$ closed in G, and $\mathcal{O}_j$ open in $\mathcal{O}^j$ for all j. The $f_{\mathcal{O}_j}$ can then be chosen to be zero on $\mathcal{O}_i$ $(i < j)$. We may subtract a multiple of $f_{\mathcal{O}_i}$ $(i > j)$ to have $\Phi_{\mathcal{O}_i}(f_{\mathcal{O}_j}) = 0$ also for $i > j$.

Lemma. *For every $f \in C_c^\infty(G)$ there exists a G-invariant neighborhood V_f of the identity in G, such that the orbital integral $\Phi(\gamma, f)$ of f is equal to $\sum_{\mathcal{O}} \Phi_{\mathcal{O}}(f)\Phi(\gamma, f_{\mathcal{O}})$ for all γ in V_f. The germ $\Gamma_{\mathcal{O}}(\gamma)$ of $\Phi(\gamma, f_{\mathcal{O}})$ at the identity in G is independent of the choice of $f_{\mathcal{O}}$.*

Proof. The function $f' = f - \sum_{\mathcal{O}} \Phi_{\mathcal{O}}(f) f_{\mathcal{O}}$ satisfies $\Phi_{\mathcal{O}}(f') = 0$ for all $\mathcal{O} \subset X$. Denote by $C_c^\infty(X)^*$ the space of distributions on X, and by $C_c^\infty(X)^{*G}$ the subspace of Int(G)-invariant ones. Denote by $C_c^\infty(X)_0$ the span of $\hbar - g \cdot \hbar$ ($\hbar \in C_c^\infty(X), g \in G$), where $g \cdot \hbar(x) = \hbar(\mathrm{Int}(g^{-1})x)$. Then $C_c^\infty(X)^{*G} = (C_c^\infty(X)/C_c^\infty(X)_0)^*$. The $\Phi_{\mathcal{O}}$ span $C_c^\infty(X)^{*G}$. Hence f' is annihilated by any element of $(C_c^\infty(X)/C_c^\infty(X)_0)^*$. Then the restriction $\overline{f}'$ of f' to the closed subset X (see [BZ1], (1.8)) is in $C_c^\infty(X)_0$. Hence there are finitely many $\hbar_i$ in $C_c^\infty(X)$, and $g_i \in G$, with $\overline{f}' = \sum_i (\hbar_i - g_i \cdot \hbar_i)$. Extend (by [BZ1], (1.8)) $\hbar_i$ to elements h_i of $C_c^\infty(G)$. Then

$$f - \sum_{\mathcal{O}} \Phi_{\mathcal{O}}(f) f_{\mathcal{O}} - \sum_i (h_i - g_i \cdot h_i)$$

is (compactly) supported in the (G-invariant) open set $G - X$. Hence there is a (G-invariant) neighborhood V_f of the identity in G where

$$f = \sum_{\mathcal{O}} \Phi_{\mathcal{O}}(f) f_{\mathcal{O}} + \sum_i (h_i - g_i \cdot h_i),$$

and the lemma follows. □

The fundamental lemma of II.1 can be deduced from the matching theorem of II.3 on using the following homogeneity result of Waldspurger.

Let G be any of the groups considered in [W2] (these include all the groups considered here) $\mathfrak{g}$ its Lie algebra, K a standard maximal compact subgroup (i.e. the fixer of each point of a fixed face of minimal dimension in the building of the reductive connected F-group $\mathbf{G}$ whose group of F-points is G), and $\mathfrak{k}$ its Lie algebra (which is a sub-R-algebra of $\mathfrak{g}$). Denote by ch_K and $\mathrm{ch}_{\mathfrak{k}}$ the characteristic functions of K in G and $\mathfrak{k}$ in $\mathfrak{g}$. Then [W2] defines an isomorphism $e : \mathfrak{g}_{tn} \to G_{tu}$ from the set $\mathfrak{g}_{tn} = \{X \in \mathfrak{g}; \lim_{N \to \infty} X^N = 0\}$ of topologically nilpotent elements of $\mathfrak{g}$ to the set $G_{tu} = \{u \in G; \lim_{N \to \infty} u^{q^N} = 1\}$ of topologically unipotent elements in G, named the truncated exponential map. Let $\mathcal{O}_{\mathrm{nil}}$ denote the set of nilpotent orbits in $\mathfrak{g}$. For each $\mathcal{O} \in \mathcal{O}_{\mathrm{nil}}$ fix a G-invariant measure on $\mathcal{O}$, and denote by $\Phi_{\mathcal{O}}(f)$ the orbital integral of $f \in C_c^\infty(\mathfrak{g})$ over $\mathcal{O}$. Fix a maximal F-torus T, let $\mathfrak{t}$ be its Lie algebra, and denote by T_{reg} and $\mathfrak{t}_{\mathrm{reg}}$ their regular subsets. For each $\mathcal{O} \in \mathcal{O}_{\mathrm{nil}}$ there exists a unique real positive valued function $\Gamma_{\mathcal{O}}^T$ on $\mathfrak{t}_{\mathrm{reg}}$ satisfying the homogeneity relation

$$\Gamma_{\mathcal{O}}^T(\mu^2 H) = |\mu|^{-\dim \mathcal{O}} \Gamma_{\mathcal{O}}^T(H)$$

for all $\mu \in F^{\times}, H \in \mathfrak{t}_{\mathrm{reg}}$, and such that for each $f \in C_c^{\infty}(\mathfrak{g})$ one has that the orbital integral

$$\Phi_f(H) = \int_{G/Z_G(H)} f(\mathrm{Int}(x)H)$$

is equal to $\sum_{\mathcal{O} \in \mathcal{O}_{\mathrm{nil}}} \Gamma_{\mathcal{O}}^T(H)\Phi_{\mathcal{O}}(f)$ for each H in a neighborhood of 0 in $\mathfrak{t}_{\mathrm{reg}}$. Waldspurger's fundamental coherence result — which is not used in our proof — is the following (see [W2], Proposition V.3 and V.5).

PROPOSITION ([W2]). *For a sufficiently large p, for any H in $\mathfrak{t}_{\mathrm{reg}} \cap \mathfrak{g}_{tn}$, we have*

$$\Phi(e(H), \mathrm{ch}_K) = \sum_{\mathcal{O} \in \mathcal{O}_{\mathrm{nil}}} \Gamma_{\mathcal{O}}^T(H)\Phi_{\mathcal{O}}(\mathrm{ch}_{\mathfrak{k}}).$$

III. TWISTED TRACE FORMULA

Summary. A trace formula — for a smooth compactly supported measure fdg on the adèle group $\mathrm{PGL}(3, \mathbb{A})$ — twisted by the outer automorphism σ — is computed. The resulting formula is then compared with trace formulae for $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$ and $\mathbf{H}_1 = \mathrm{PGL}(2)$, and matching measures $f_0 dh$ and $f_1 dh_1$ thereof. We obtain a trace formula identity which plays a key role in the study of the symmetric square lifting from $\mathbf{H}(\mathbb{A})$ to $\mathbf{G}(\mathbb{A})$. The formulae are remarkably simple, due to the introduction of a new concept, of a regular function. This eliminates the singular and weighted integrals in the trace formulae.

Introduction

The purpose of this chapter is to compute explicitly a trace formula for a test measure $fdg = \otimes_v f_v dg_v$ on $\mathbf{G}(\mathbb{A})$, where $\mathbf{G} = \mathrm{PGL}(3)$ and $\mathbb{A}$ is the ring of adèles of a number field F. This formula is twisted with respect to the outer twisting

$$\sigma(g) = J{}^t g^{-1} J, \qquad J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix},$$

and plays a key role in the study of the symmetric square lifting. We also stabilize the formula and compare it with the stable trace formula for a matching test measure $f_0 dh = \otimes_v f_{0v} dh_v$ on $\mathbf{H}(\mathbb{A})$, $\mathbf{H} = \mathrm{SL}(2)$, and the trace formula for a matching test function $f_1 dh_1 = \otimes_v f_{1v} dh_{1v}$ on $\mathbf{H}_1(\mathbb{A})$, $\mathbf{H}_1 = \mathrm{PGL}(2)$. The final result of this section concerns a distribution $\mathcal{I}$ in fdg, $f_0 dh$, $f_1 dh_1$ of the form

$$\mathcal{I} = I + \frac{1}{2} I' + \frac{1}{4} I'' + \frac{1}{2} I_1' - \left[I_0 + \frac{1}{4} \sum_E I_E'' - \frac{1}{2} \sum_E I_E' - \frac{1}{2} \sum_E I_E + \frac{1}{2} I_1 \right],$$

where each I is a sum of traces of convolution operators. The result asserts: (3.5(1)) $\mathcal{I} = 0$ if fdg has two discrete components;

(3.5(2)) $\mathcal{I}$ is equal to a certain integral if fdg has (i) a discrete component or (ii) a component which is sufficiently regular with respect to all other components.

The result (3.5(1)) is used in the study of the local symmetric square lifting in chapter V. The result (3.5(2)) can be used to show that $\mathcal{I} = 0$ and to establish the global symmetric square lifting for automorphic forms with an elliptic component.

The vanishing of $\mathcal{I}$ for general matching functions is proven in chapter IV.

Our formulae here are essentially those of the unpublished manuscript [F2;IX], where we suggested, in the context of the (first nontrivial) symmetric square case, a truncation with which the trace formula, twisted by an automorphism σ, can be developed. This formula was subsequently computed in [CLL] to which we refer for proofs of the general form of the twisted trace formula. Our formulae here are considerably simpler than those of [F2;IX]. This is due to the fact that we introduce here a new notion, of a regular function, and compute only an asymptotic form of the formula for a test function with a component which is sufficiently regular with respect to all other components. For such a function f the truncation is trivial; in fact f vanishes on the $\mathbf{G}(\mathbb{A})$-σ-orbits of the rational elements (in G) which are not σ-elliptic regular, and no weighted orbital integrals appear in our formulae. In chapters V and IV we show that this simple, asymptotic form of the formula suffices to establish the symmetric square lifting, unconditionally. Similar ideas are used in [F1;IV] to give a simple proof of basechange for GL(2), and in our work on basechange for U(3) (see [F3]) and other lifting problems.

III.1 Geometric side

1.1 The kernel. Let F be a number field, $\mathbb{A}$ its ring of adèles, $\mathbf{G}$ a reductive group over F with an anisotropic center, and L the space of complex valued square-integrable functions φ on $G\backslash\mathbf{G}(\mathbb{A})$. The group $\mathbf{G}(\mathbb{A})$ acts on L by right translation, thus $(r(g)\varphi)(h) = \varphi(hg)$. Each irreducible constituent of the $\mathbf{G}(\mathbb{A})$-module L is called an *automorphic* $\mathbf{G}(\mathbb{A})$-module (or representation). Let σ be an automorphism of $\mathbf{G}$ of finite order, and

$\mathbf{G}' = \mathbf{G} \rtimes \langle\sigma\rangle$ the semidirect product of $\mathbf{G}$ and the group $\langle\sigma\rangle$ generated by σ. Extend r to a representation of $\mathbf{G}'(\mathbb{A})$ on L by putting $(r(\sigma)\varphi)(h) = \varphi(\sigma^{-1}(h))$. Fix a Haar measure $dg = \otimes_v dg_v$ on $\mathbf{G}(\mathbb{A})$. Let f be any smooth complex valued compactly supported function on $\mathbf{G}(\mathbb{A})$. Let $r(fdg)$ be the (convolution) operator on L which maps φ to

$$(r(fdg)\varphi)(h) = \int f(g)\varphi(hg)dg \qquad (g \in \mathbf{G}(\mathbb{A})).$$

Then $r(fdg)r(\sigma)$, which we also denote by $r(fdg \times \sigma)$, is the operator on L which maps φ to

$$h \mapsto \int_{\mathbf{G}(\mathbb{A})} f(g)\varphi(\sigma^{-1}(hg))dg$$

$$= \int_{\mathbf{G}(\mathbb{A})} f(h^{-1}\sigma(g))\varphi(g)dg = \int_{G\backslash\mathbf{G}(\mathbb{A})} K(h,g)\varphi(g)dg,$$

where

$$K(h,g) = K_f(h,g) = \sum_{\gamma\in G} f(h^{-1}\gamma\sigma(g)). \tag{1.1.1}$$

The theory of Eisenstein series provides a direct sum decomposition of the $\mathbf{G}(\mathbb{A})$-module L as $L_d \oplus L_c$, where L_d, the "discrete spectrum", is a direct sum with finite multiplicities of irreducibles, and L_c, the "continuous spectrum", is a direct integral of such. This theory also provides an alternative formula for the kernel. The Selberg trace formula is an identity obtained on (essentially) integrating the two expressions for the kernel over the diagonal $g = h$. To get a useful formula one needs to change the order of summation and integration. This is possible if $\mathbf{G}$ is anisotropic over F or if f has some special properties (see, e.g., [FK2]). In general one needs to truncate the two expressions for the kernel in order to be able to change the order of summation and integration.

When σ is trivial, the truncation introduced by Arthur [A1] involves a term for each standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. For $\sigma \neq 1$ it was suggested in [F2;IX] (in the context of the symmetric square) to truncate only with the terms associated with σ-invariant $\mathbf{P}$, and to use a certain normalization of a vector which is used in the definition of truncation. The consequent (nontrivial) computation of the resulting twisted (by σ) trace formula is carried out in [CLL] for general G and σ. In (2.1) we record the expression, proven in [CLL], for the analytic side of the trace formula, which involves Eisenstein series. In (2.2) and (2.3) we write out the various terms in our case of the symmetric square.

In this section we compute and stabilize the "elliptic part" of the geometric side of the twisted formula in our case. Namely we take $\mathbf{G} = \mathrm{PGL}(3)$ and $\sigma(g) = J{}^tg^{-1}J$, and consider

$$\int_{G\backslash \mathbf{G}(\mathbb{A})} \left[\sum_{\delta \in G} f(g^{-1}\delta\sigma(g))\right] dg, \tag{1.1.2}$$

where the sum ranges over the δ in G whose norm $\gamma = N\delta$ in H, $\mathbf{H} = \mathrm{SL}(2, F)$, is elliptic. Here we use freely the norm map N of section I.2, and its properties.

In [F2;IX] the integral of the truncated $\sum_{\delta \in G} f(g^{-1}\delta\sigma(g))$ was explicitly computed, and the correction argument of [F1;III] was applied to the hyperbolic weighted orbital integrals, to show that their limits on the singular set equal the integrals obtained from the δ with unipotent $N\delta$. These computations are not recorded here for the following reasons. We need the trace formula only for a function f which has a regular component or two discrete components (the definitions are given below). In the first case $f(g^{-1}\delta\sigma(g)) = 0$ for every g in $\mathbf{G}(\mathbb{A})$ and δ in G such that $N\delta$ is not elliptic regular in H; hence the geometric side of the trace formula (twisted by σ) is (1.1.2). In the second case the computations of [CLL], which generalize those of [F2;IX], suffice to show the vanishing of all terms in the geometric side, other than those obtained from (1.1.2).

1.2 Elliptic part. To compute and stabilize (1.1.2) let $Z_{\mathbf{G}}(\delta\sigma) = \{g \in \mathbf{G}; g^{-1}\delta\sigma(g) = \delta\}$ be the σ-centralizer of δ, and

$$\Phi(\delta\sigma, f dg) = \int_{Z_{\mathbf{G}}(\delta\sigma)(\mathbb{A})\backslash \mathbf{G}(\mathbb{A})} f(g^{-1}\delta\sigma(g)) \frac{dg}{dt}$$

the σ-orbital integral of $f dg$ at δ. Implicit is a choice of a Haar measure dt on $Z_{\mathbf{G}}(\delta\sigma)(\mathbb{A})$, which is chosen to be compatible with isomorphisms (of $Z_{\mathbf{G}}(\delta\sigma)$ with $Z_{\mathbf{G}}(\delta'\sigma)$, or $Z_{\mathbf{H}}(N\delta)$, etc.). Let $\{\delta\}$ denote the set of σ-conjugacy classes in G of elements δ such that $N\delta$ is elliptic in H. Then

(1.1.2) is equal to

$$\sum_{\{\delta\}} \int_{Z_G(\delta\sigma)\backslash \mathbf{G}(\mathbb{A})} f(g^{-1}\delta\sigma(g))dg = \sum_{\{\delta\}} c(\delta)\Phi(\delta\sigma, fdg). \tag{1.2.1}$$

The volume

$$c(\delta) = |Z_G(\delta\sigma)\backslash Z_{\mathbf{G}}(\delta\sigma)(\mathbb{A})|$$

is finite since $N\delta$ is elliptic in H. It is equal to $|Z_H(\gamma)\backslash Z_{\mathbf{H}}(\gamma)(\mathbb{A})|$ if $\gamma = N\delta$ is elliptic regular (in H). For completeness we deal also with δ such that $N\delta = \gamma$ is $\pm I$. Then $c(\delta)$ is $|H\backslash \mathbf{H}(\mathbb{A})|$ if $\gamma = -I$, and $|H_1\backslash \mathbf{H}_1(\mathbb{A})|$ if $\gamma = I$, where $\mathbf{H}_1 = \mathrm{PGL}(2)$.

Recall from section I.2 that $D(\delta/F)$ denotes the set of σ-conjugacy classes within the stable σ-conjugacy class of δ in G. Thus $D(\delta/F_v)$ denotes the local analogue for any place v of F. For any local or global field, $D(\delta/F)$ is a pointed set, isomorphic to $H^1(F, Z_{\mathbf{G}}(\delta\sigma))$, and we put

$$D(\delta/\mathbb{A}) = \oplus_v D(\delta/F_v) \qquad \text{and} \qquad H^1(\mathbb{A}, Z_{\mathbf{G}}(\delta\sigma)) = \oplus_v H^1(F_v, Z_{\mathbf{G}}(\delta\sigma))$$

(pointed direct sums). If $\gamma = N\delta$ is $-I$, we have $Z_{\mathbf{G}}(\delta\sigma) = \mathbf{H} = \mathrm{SL}(2)$ and $H^1(F, Z_{\mathbf{G}}(\delta\sigma))$ and $H^1(\mathbb{A}, Z_{\mathbf{G}}(\delta\sigma))$ are trivial. If $\gamma = N\delta$ is I or elliptic regular then $H^1(F, Z_{\mathbf{G}}(\delta\sigma))$ embeds in $H^1(\mathbb{A}, Z_{\mathbf{G}}(\delta\sigma))$ and the quotient is a group of order two. Denote by $\boldsymbol{\kappa}$ the nontrivial character of this group.

Denote by $\Phi(\delta\sigma, f_v dg_v)$ the σ-orbital integral at δ in $G_v = \mathbf{G}(F_v)$ of a smooth compactly supported complex valued measure $f_v dg_v$ on G_v. If F_v is nonarchimedean, denote its ring of integers by R_v. Let $f_v^0 dg_v$ be the unit element in the Hecke algebra $\mathbb{H}_v$ of compactly supported $K_v = \mathbf{G}(R_v)$-biinvariant measures on G_v. Consider $fdg = \otimes_v f_v dg_v$, product over all places v of F, where $f_v dg_v = f_v^0 dg_v$ for almost all v. Then, for every δ in G we have $\Phi(\delta\sigma, fdg) = \prod_v \Phi(\delta\sigma, f_v dg_v)$, where the product is absolutely convergent. Since fdg is compactly supported the sum

$$\sum_{\delta' \in D(\delta/F)} \Phi(\delta'\sigma, fdg) = \sum_{\delta' \in \mathrm{Im}[D(\delta/F)\to D(\delta/\mathbb{A})]} \prod_v \Phi(\delta'\sigma, f_v dg_v)$$

is finite for each fdg and δ. If $\gamma = N\delta$ is elliptic regular or the identity and $\boldsymbol{\kappa}_v$ is the component at v of the associated quadratic character $\boldsymbol{\kappa}$ on

$D(\delta/\mathbb{A})/D(\delta/F)$, then the sum can be written in the form

$$\frac{1}{2}\prod_v\left[\sum_{\delta'\in D(\delta/F_v)}\Phi(\delta'\sigma,f_v dg_v)\right] + \frac{1}{2}\prod_v\left[\sum_{\delta'\in D(\delta/F_v)}\boldsymbol{\kappa}_v(\delta')\Phi(\delta'\sigma,f_v dg_v)\right]. \quad (1.2.2)$$

Note that for a given fdg and δ, for almost all v, the integral $\Phi(\delta'\sigma, f_v dg_v)$ vanishes unless δ' and δ are equal σ-conjugacy classes in G_v.

Denote by $f_{0v}^0 dh_v$ the unit element of the Hecke algebra $\mathbb{H}_{0v}$ of $H_v = \mathbf{H}(F_v)$ with respect to $K_{0v} = \mathbf{H}(R_v)$. Similarly introduce K_{1v}, $\mathbb{H}_{1v}$, and $f_{1v}^0 dh_{1v}$. Recall that the norm maps N, N_1 from the set of σ-stable conjugacy classes in G to the set of stable conjugacy classes in H, H_1 are defined in section I.2.

To rewrite (1.2.2) we recall the following

1.3 Proposition. (1) *For each smooth compactly supported $f_v dg_v$ on G_v there exist smooth compactly supported $f_{0v} dh_v$ on H_v and $f_{1v} dh_{1v}$ on H_{1v} such that for all δ with regular $\gamma = N\delta$*

$$\Phi^{\mathrm{st}}(N\delta, f_{0v} dh_v) = \sum_{\delta'\in D(\delta/F_v)}\Phi(\delta'\sigma, f_v dg_v) \quad (1.3.1)$$

and

$$\Phi(N_1\delta, f_{1v} dh_{1v}) = |(1+a)(1+b)|_v^{1/2}\sum_{\delta'\in D(\delta/F_v)}\boldsymbol{\kappa}_v(\delta')\Phi(\delta'\sigma, f_v dg_v). \quad (1.3.2)$$

Here a, b denote the eigenvalues of $N\delta$.

(2) *Moreover, if $\delta = I$ then*

$$f_{0v}(I) = \sum\boldsymbol{\kappa}_v(\delta')\Phi(\delta'\sigma, f_v dg_v) \quad \textit{and} \quad f_{1v}(I) = \sum\Phi(\delta'\sigma, f_v dg_v),$$

where the sums are taken over δ' in $D(\delta/F_v)$. If $N\delta = -I$ then $f_{0v}(-I) = \Phi(\delta\sigma, f_v dg_v)$.

(3) *If F_v has odd residual characteristic, then the triple $f_{0v} dh_v = f_{0v}^0 dh_v$, $f_v dg_v = f_v^0 dg_v$, $f_{1v} dh_{1v} = f_{1v}^0 dh_{1v}$ satisfies* (1.3.1) *and* (1.3.2).

Proof. (3) is proven in section II.1. (1) and (2) follow from this by a theorem of Waldspurger [W3]. They are proven directly in section II.3. □

Definition. The measures $f_v dg_v$, $f_{0v} dh_v$ (resp. $f_v dg_v$, $f_{1v} dh_{1v}$) are called *matching* if they satisfy (1.3.1) (resp. (1.3.2)) for all δ such that $\gamma = N\delta$ is regular.

Corollary. *Put $f_0 dh = \otimes_v f_{0v} dh_v$ and $f_1 dh_1 = \otimes_v f_{1v} dh_{1v}$, where $f_v dg_v$, $f_{0v} dh_v$ and $f_v dg_v$, $f_{1v} dh_{1v}$ are matching for all v, and $f_{0v} dh_v = f^0_{0v} dh_v$ and $f_{1v} dh_{1v} = f^0_{1v} dh_{1v}$ for almost all v. Then* (1.1.2) = (1.2.1) *is the sum of*

$$\begin{aligned}\tilde{I}_0 &= |H\backslash \mathbf{H}(\mathbb{A})|[f_0(I) + f_0(-I)] \\ &+ \frac{1}{2}\sum_{\{T\}_{\text{st}}} \frac{1}{2}|T\backslash \mathbf{T}(\mathbb{A})| \sum_{\gamma\in T} \Phi^{\text{st}}(\gamma, f_0 dh)\end{aligned} \tag{1.3.3}$$

and $\frac{1}{2}$ times

$$\tilde{I}_1 = |H_1\backslash \mathbf{H}_1(\mathbb{A})| f_1(I) + \frac{1}{2}\sum_{\{T\}} |T\backslash \mathbf{T}(\mathbb{A})| {\sum}'_{\gamma\in T} \Phi(\gamma, f_1 dh_1). \tag{1.3.4}$$

In (1.3.3) $\{T\}_{\text{st}}$ *indicates the set of stable conjugacy classes of elliptic F-tori $\mathbf{T}$ in $\mathbf{H}$.*

In (1.3.4) $\{T\}$ *is the set of conjugacy classes of elliptic F-tori $\mathbf{T}$ in $\mathbf{H}_1 = \mathrm{SO}(3)$.*

The sum $\sum'$ in (1.3.4) *ranges over the γ in $T \subset \mathrm{SO}(3, F)$ whose eigenvalues are distinct* (*not* -1). *The sums are absolutely convergent.*

Proof. (1.2.1) is a sum over σ-stable conjugacy classes δ which are equal to $c(\delta)$ times (1.2.2) if $N\delta$ is I or elliptic regular. If $N\delta$ is elliptic regular then the first term in (1.2.2) makes a contribution in the sum of (1.3.3) by (1.3.1), and the second term in (1.2.2) contributes to (1.3.4) by (1.3.2). If $N\delta = I$ then the order is reversed, by (2) in the proposition. The single σ-conjugacy class δ in G with $N\delta = -I$ makes the term of $f_0(-I)$ in (1.3.3). The coefficient of $f_0(I)$ in (1.3.3) is $|H\backslash \mathbf{H}(\mathbb{A})|$ since the Tamagawa number of $\mathrm{SO}(3) = \mathrm{PGL}(2)$ is twice that of $\mathrm{SL}(2)$. The first one-half which appears in (1.3.3) and (1.3.4) exists since the number of regular γ in T which share the same set of eigenvalues is two. The sums in (1.3.3) and (1.3.4) are absolutely convergent since they are parts of the trace formula for f_0 on $\mathbf{H}(\mathbb{A})$ and f_1 on $\mathbf{H}_1(\mathbb{A})$. □

III.2 Analytic side

2.1 Spectral side. As suggested in (1.1) we shall now record the expression of [CLL] for the analytic side, which involves traces of representations, in the twisted trace formula. Let $\mathbf{P}_0$ be a minimal σ-invariant F-parabolic subgroup of $\mathbf{G}$, with Levi subgroup $\mathbf{M}_0$. Let $\mathbf{P}$ be any standard (containing $\mathbf{P}_0$) F-parabolic subgroup of $\mathbf{G}$; denote by $\mathbf{M}$ the Levi subgroup which contains $\mathbf{M}_0$ and by $\mathbf{A}$ the split component of the center of $\mathbf{M}$. Then $\mathbf{A} \subset \mathbf{A}_0 = \mathbf{A}(\mathbf{M}_0)$. Let $X^*(\mathbf{A})$ be the lattice of rational characters of $\mathbf{A}$, $\mathcal{A}_M = \mathcal{A}_P$ the vector space $X_*(\mathbf{A}) \otimes \mathbb{R} = \operatorname{Hom}(X^*(\mathbf{A}), \mathbb{R})$, and $\mathcal{A}^*$ the space dual to $\mathcal{A}$. Let $W_0 = W(A_0, G)$ be the Weyl group of A_0 in G. Both σ and every s in W_0 act on $\mathcal{A}_0$. The truncation and the general expression to be recorded depend on a vector T in $\mathcal{A}_0 = \mathcal{A}_{M_0}$. In the case of (2.2) below, this T becomes a real number, the expression is linear in T, and we record in (2.2) only the value at $T = 0$.

Proposition [CLL]. *The analytic side of the trace formula is equal to a sum over*
(1) *The set of Levi subgroups $\mathbf{M}$ which contain $\mathbf{M}_0$ of F-parabolic subgroups of $\mathbf{G}$.*
(2) *The set of subspaces $\mathcal{A}$ of $\mathcal{A}_0$ such that for some s in W_0 we have $\mathcal{A} = \mathcal{A}_M^{s\times\sigma}$, where $\mathcal{A}_M^{s\times\sigma}$ is the space of $s \times \sigma$-invariant elements in the space $\mathcal{A}_M$ associated with a σ-invariant F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$.*
(3) *The set $W^{\mathcal{A}}(\mathcal{A}_M)$ of distinct maps on $\mathcal{A}_M$ obtained as restrictions of the maps $s \times \sigma$ (s in W_0) on $\mathcal{A}_0$ whose space of fixed vectors is precisely $\mathcal{A}$.*
(4) *The set of discrete-spectrum representations τ of $\mathbf{M}(\mathbb{A})$ with $(s\times\sigma)\tau \simeq \tau$, $s \times \sigma$ as in (3).*

The terms in the sum are equal to the product of

$$\frac{[W_0^M]}{[W_0]}(\det(1 - s\times\sigma)|_{\mathcal{A}_M/\mathcal{A}})^{-1} \tag{2.1.1}$$

and

$$\int_{i\mathcal{A}^*} \operatorname{tr}[\mathcal{M}_{\mathcal{A}}^T(P,\lambda) M_{P|\sigma(P)}(s,0) I_{P,\tau}(\lambda; f dg \times \sigma)]|d\lambda|.$$

Here $[W_0^M]$ is the cardinality of the Weyl group $W_0^M = W(A_0, M)$ of A_0 in M; $\mathbf{P}$ is an F-parabolic subgroup of $\mathbf{G}$ with Levi component $\mathbf{M}$; $M_{P|\sigma(P)}$

is an intertwining operator; $\mathcal{M}^T_{\mathcal{A}}(P,\lambda)$ is a logarithmic derivative of intertwining operators, and $I_{P,\tau}(\lambda)$ is the $\mathbf{G}(\mathbb{A})$-module normalizedly induced from the $\mathbf{M}(\mathbb{A})$-module $m \mapsto \tau(m)e^{\langle\lambda, H(m)\rangle}$ (in standard notations).

Remark. The sum of the terms corresponding to $\mathbf{M} = \mathbf{G}$ in (1) is equal to the sum $I = \sum \operatorname{tr} \pi(fdg \times \sigma)$ over all discrete-spectrum representations π of $\mathbf{G}(\mathbb{A})$, counted with their multiplicity.

2.2 Case of PGL(3). We shall now describe, in our case of $\mathbf{G} = \mathrm{PGL}(3)$ and $\sigma(g) = J{}^tg^{-1}J$, the terms corresponding to $\mathbf{M} \neq \mathbf{G}$ in (1) of Proposition 2.1. There are three such terms. Let $\mathbf{M}_0 = \mathbf{A}_0$ be the diagonal subgroup of $\mathbf{G}$.

(a) For the three Levi subgroups $\mathbf{M} \supset \mathbf{A}_0$ of maximal parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$ we have $\mathcal{A} = \{0\}$. The corresponding contribution is

$$\sum_{\mathbf{M}} \sum_{\tau} \frac{2}{6} \cdot \frac{1}{2} \operatorname{tr} M(s,0) I_{P,\tau}(0; fdg \times \sigma)$$
$$= \frac{1}{2} \sum_{\tau} \operatorname{tr} M(\alpha_2\alpha_1, 0) I_{P_1}(\tau; fdg \times \sigma). \qquad (2.2.1)$$

Here $\mathbf{P}_1$ denotes the upper triangular parabolic subgroup of $\mathbf{G}$ of type (2,1). We write $\alpha_1 = (12), \alpha_2 = (23), J = (13)$ for the transpositions in the Weyl group W_0.

(b) The contribution corresponding to $\mathbf{M} = \mathbf{M}_0$ and $\mathcal{A} = \{0\}$ is

$$\frac{1}{6} \cdot \frac{1}{4} \sum_{\tau} \operatorname{tr} M(J,0) I_{P_0}(\tau; fdg \times \sigma)$$
$$+ \frac{1}{6} \sum_{\tau} \operatorname{tr} M(\alpha_1, 0) I_{P_0}(\tau; fdg \times \sigma) + \frac{1}{6} \sum_{\tau} \operatorname{tr} M(\alpha_2, 0) I_{P_0}(\tau; fdg \times \sigma).$$
$$(2.2.2)$$

(c) Corresponding to $\mathbf{M} = \mathbf{M}_0$ and $\mathcal{A} \neq \{0\}$ we obtain three terms, with $\mathcal{A} = \{(\lambda, 0, -\lambda)\}$ and $s = 1$, with $\mathcal{A} = \{(\lambda, -\lambda, 0)\}$ and $s = \alpha_2\alpha_1$, and with $\mathcal{A} = \{(0, \lambda, -\lambda)\}$ and $s = \alpha_1\alpha_2$. The value of (2.1.1) is $\frac{1}{12}$. It is easy to see that the three terms are equal and that their sum is

$$\frac{1}{4} \sum_{\tau} \int_{i\mathbb{R}} \operatorname{tr}[\mathcal{M}(\lambda, 0, -\lambda) I_{P_0,\tau}((\lambda, 0, -\lambda); fdg \times \sigma)] |d\lambda|. \qquad (2.2.3)$$

The operator $\mathcal{M}$ is a logarithmic derivative of an operator $M = m \otimes_v R_v$. Here R_v denotes a normalized local intertwining operator. It is normalized as follows. If $I(\tau_v)$ is unramified, its space of K_v-fixed vectors is one dimensional, and R_v acts trivially on this space. In particular $R'_{\tau_v}(\lambda) I_{\tau_v}(\lambda; f_v dg_v \times \sigma)$ is zero if $f_v dg_v$ is spherical, where $R'_{\tau_v}(\lambda)$ is the derivative of $R_{\tau_v}(\lambda)$ with respect to λ.

The τ in (2.2.3) are unitary characters (μ_1, μ_2, μ_3) of $\mathbf{M}_0(\mathbb{A})/M_0$, which are σ-invariant; thus $\mu_2 = 1$ and $\mu_1, \mu_3 = 1$. According to [Sh], where the R_v are studied, the normalizing factor $m = m(\lambda)$ is the quotient

$$L(1 - 2\lambda, \mu_3/\mu_1)/L(1 + 2\lambda, \mu_1/\mu_3)$$

of L-functions. In this case the logarithmic derivative $\mathcal{M}$ has the form

$$m'(\lambda)/m(\lambda) + (\otimes_v R_v^{-1}) \frac{d}{d\lambda}(\otimes_v R_v).$$

Hence (2.2.3) is equal to $\frac{1}{4}(S + S')$, where

$$S = \sum_{\tau} \int_{i\mathbb{R}} \frac{m'(\lambda)}{m(\lambda)} \left[\textstyle\prod_v \operatorname{tr} I_{\tau_v}(\lambda; f_v dg_v \times \sigma) \right] |d\lambda| \tag{2.2.4}$$

and

$$S' = \sum_{\tau} \sum_{v} \int_{i\mathbb{R}} [\operatorname{tr} R_{\tau_v}(\lambda)^{-1} R_{\tau_v}(\lambda)' I_{\tau_v}(\lambda; f_v dg_v \times \sigma)] \cdot \prod_{w \neq v} \operatorname{tr} I_{\tau_w}(\lambda; f_w dg_w \times \sigma) \cdot |d\lambda|. \tag{2.2.5}$$

In view of the normalization of the $R_v = R_{\tau_v}(\lambda)$, the inner sum in S' extends only over the places v where f_v is not spherical.

The terms (2.2.1) and (2.2.2) contain arithmetic information which is crucial for the study of the symmetric square. They are analyzed in (2.3) and (2.4) below.

2.3 Contribution from maximal parabolics. We shall now study the representations τ which occur in (2.2.1). Such a τ is a discrete-spectrum representation of the Levi component $\mathbf{M}(\mathbb{A})$ of a maximal parabolic subgroup of $\mathbf{G}(\mathbb{A})$. Hence τ has the form $(\tilde{\pi}, \chi)$, where $\tilde{\pi}$ is a discrete-spectrum representation of $\mathrm{GL}(2,\mathbb{A})$ and χ is a (unitary) character of $\mathbb{A}^\times/F^\times$. The central character of $\tilde{\pi}$ is χ^{-1} since $\mathbf{G}$ is the projective group $\mathrm{PGL}(3)$. Since $I(\tau) \simeq {}^\sigma I(\tau) \simeq I(\sigma\tau)$ implies $\tau \simeq \sigma\tau$, the representation $\tau = (\tilde{\pi},\chi)$ is σ-invariant. Hence $\chi = \chi^{-1}$, and $\tilde{\pi}$ is equivalent to its contragredient $\tilde{\pi}^\vee$ which is $\tilde{\pi}\chi^{-1}$.

If $\chi = 1$, then $\tilde{\pi}$ is a representation π_1 of $\mathrm{PGL}(2,\mathbb{A})$.

If $\chi \neq 1$ then χ is quadratic. Its kernel is $F^\times N_{E/F}\mathbb{A}_E^\times$ where E is a quadratic extension of F. We conclude that (2.2.1) is equal to $\frac{1}{2}(I_1' + I')$. Here

$$I_1' = \sum_{\pi_1} \mathrm{tr}\, I_{P_1}((\pi_1, 1); fdg \times \sigma) \tag{2.3.1}$$

where π_1 ranges over the discrete spectrum of $\mathbf{H}_1(\mathbb{A})$, and

$$I' = \sum_{\chi}\sum_{\pi_2} \mathrm{tr}\, I_{P_1}((\pi_2, \chi); fdg \times \sigma). \tag{2.3.2}$$

The first sum of I' ranges over all quadratic characters $\chi(\neq 1)$ of $\mathbb{A}^\times/F^\times$. The second sum of I' ranges over all discrete-spectrum representation π_2 of $\mathrm{GL}(2,\mathbb{A})$ with central character χ and $\pi_2 = \chi\pi_2$. Such π_2 is cuspidal, as it cannot be one-dimensional. The intertwining operator $M(s,\pi)$ of (2.2.1), $\pi = I(\tau)$, is equal to $\otimes_v R(s,\pi_v)$, where $R(s,\pi_v)$ takes $I(\tau)$, $\tau = (\tilde{\pi},\chi)$, to $I(\chi,\tilde{\pi})$, which is then taken by σ to $I(\tilde{\pi}^\vee,\chi^{-1})$. To simplify the notations we write $\mathrm{tr}\, I_{P_1}(\tau; fdg \times \sigma)$ for $\mathrm{tr}\, R(s,\pi_v) I_{P_1}(\tau; fdg \times \sigma)$.

2.4 Contribution from minimal parabolics. The representations τ which appear in (2.2.2) are (unitary) characters $\eta = (\mu_1,\mu_2,\mu_3)$, μ_i being a character of $\mathbb{A}^\times/F^\times$, and $\mu_1\mu_2\mu_3 = 1$. In the first sum appear all η with $\mu_i^2 = 1$, but in the other two sums appear only the η with $(s\times\sigma)\eta = \eta$, namely $\eta = (1,1,1)$. Since all representations which appear here are irreducible, the intertwining operators $M(s,\eta)$ are scalars. They can be seen to be equal to -1, as in the case of $\mathrm{GL}(2)$, unless μ_i are all distinct, where they are equal to 1. It remains to note that in the first sum each representation $I(\eta)$ with $\mu_i \neq 1$ $(i = 1,2,3)$ occurs six times, three times if $\mu_i = 1$ for a single i, and once if $\mu_i = 1$ for all i. Then (2.2.2) takes the form $\frac{1}{4}I'' - \frac{3}{8}I^* - \frac{1}{8}I^{**}$, where

$$I'' = \sum_{\eta=\{\chi,\mu\chi,\mu\}} \mathrm{tr}\, I(\eta; fdg \times \sigma) \tag{2.4.1}$$

and

$$I^* = \operatorname{tr} I(1; f dg \times \sigma), \qquad I^{**} = \sum_{\eta=(\mu,1,\mu)} \operatorname{tr} I(\eta; f dg \times \sigma). \quad (2.4.2)$$

The χ and μ are characters of $\mathbb{A}^\times/F^\times$ of order exactly two. The symbol $\{\chi, \mu\chi, \mu\}$ means an unordered triple of distinct characters.

III.3 Trace formulae

3.1 Twisted trace formula. We shall next state the twisted trace formula. This can be done for a general test function f on using the computations of [F2;IX] of the weighted orbital integrals on the nonelliptic σ-orbits. However, we shall use the formula only for f with a regular component or two discrete components (definitions soon to follow). For such f the formula simplifies considerably, and we consequently state the formula only in this case.

Definition. The function $f = \otimes_v f_v$ on $\mathbf{G}(\mathbb{A})$ is of *type* E if for every δ in G and g in $\mathbf{G}(\mathbb{A})$ we have $f(g^{-1}\delta\sigma(g)) = 0$ unless $N\delta$ is elliptic regular in H.

Example. If f has a component f_v which is supported on the set of g in G_v such that Ng is elliptic regular in H_v, then f is of type E.

If f is of type E then $K(g, g)$ of (1.1.1) is equal to the integrand of (1.1.2), and the truncation which is applied to $K(g, g)$ in [CLL] is trivial (it does not change $K(g, g)$). Hence the computations of sections 1 and 2 (in this chapter III) imply the following form of the twisted trace formula. Put

$$I = \sum_\pi \operatorname{tr} \pi(f dg \times \sigma), \qquad (3.1.1)$$

where π ranges over all discrete-spectrum (cuspidal or one-dimensional) $\mathbf{G}(\mathbb{A})$-modules which are σ-invariant: π is called σ-invariant if $\pi \simeq {}^\sigma\pi$, where ${}^\sigma\pi(g) = \pi(\sigma(g))$. By multiplicity one theorem for $\mathrm{GL}(n)$ the sum ranges over π up to equivalence.

PROPOSITION. *Suppose that f is a function of type E. Then we have*

$$\widetilde{I}_0+\frac{1}{2}\widetilde{I}_1=I+\frac{1}{2}I_1'+\frac{1}{2}I'+\frac{1}{4}I''-\frac{3}{8}I^*-\frac{1}{8}I^{**}+\frac{1}{4}S+\frac{1}{4}S'.$$

$\widetilde{I}_0$ *is defined in* (1.3.3), $\widetilde{I}_1$ *in* (1.3.4), I *in* (3.1.1), I_1' *in* (2.3.1), I' *in* (2.3.2), I'' *in* (2.4.1), I^* *and* I^{**} *in* (2.4.2), S *in* (2.2.4), *and* S' *in* (2.2.5). *These are distributions in* fdg.

3.2 Regular functions. We shall next introduce a class of functions f of type E which suffices to establish in chapters V and IV the symmetric square lifting. Fix a nonarchimedean place u of F. Denote by ord_u the normalized additive valuation on F_u; thus $\mathrm{ord}_u(\boldsymbol{\pi}_u)=1$ for a uniformizer $\boldsymbol{\pi}_u$ in R_u. Put q_u for the cardinality of the residue field $R_u/(\boldsymbol{\pi}_u)$. Given an element δ of G_u, denote by a, a^{-1} the eigenvalues of $N\delta$ and put

$$F(\delta\sigma, f_u dg_u)=|a-a^{-1}|_u^{1/2}\ \ \Phi(\delta\sigma, f_u dg_u);$$

here $|\cdot|_u$ is the valuation on F_u which is normalized by $|\boldsymbol{\pi}_u|_u=q_u^{-1}$.

DEFINITION. Let n be a positive integer. The function f_u on G_u is called *n-regular* if it is (compactly) supported on the set of δ with $|\mathrm{ord}_u(a)|=\pm n$, and $F(\delta, f_u dg_u)=1$ there.

3.2.1 PROPOSITION. *For every $f^u=\otimes_v f_v$ (product over $v\neq u$) there exists $n'>0$, such that $f=f_u\otimes f^u$ is of type E if f_u is n-regular with $n\geq n'$.*

PROOF. Given f^u there exists $C_v\geq 1$ for each $v\neq u$, with $C_v=1$ for almost all v (C_v depends only on the support of f_v) with the following property. Let $\mathbb{A}^u$ be the ring of adèles of F without component at u. If δ is an element of G such that the eigenvalues a, a^{-1} of $N\delta$ lie in $F^\times$, then $C_v^{-1}\leq |a|_v\leq C_v$ ($v\neq u$). Put $C_u=\prod_{v\neq u}C_v$. The product formula $\prod_v|a|_v=1$ on $F^\times$ implies that $C_u^{-1}\leq |a|_u\leq C_u$. The least integer n' with $q_u^{n'}>C_u$ has the property asserted by the proposition. □

Let μ_u be a σ-invariant character of the diagonal subgroup $\mathbf{A}(F_u)$. Then there is a character μ_{0u} of F_u with $\mu_u(\mathrm{diag}(a,b,c))=\mu_{0u}(a/c)$. Denote by $I(\mu_u)$ the G_u-module normalizedly induced from the associated character μ_u of the upper triangular subgroup, and by $I_0(\mu_{0u})$ the H_u-module normalizedly induced from $\left(\begin{smallmatrix} a & b\\ 0 & a^{-1}\end{smallmatrix}\right)\mapsto\mu_{0u}(a)$. A standard computation (I.3.10) implies that if $f_u dg_u$, $f_{0u} dh_u$ are matching then

$$\operatorname{tr} I(\mu_u; f_u dg_u \times \sigma) = \operatorname{tr} I_0(\mu_{0u}; f_{0u} dh_u). \tag{3.2.2}$$

If f_u is n-regular, then f_{0u} is n-regular: it is supported on the orbits of

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \qquad \text{with} \qquad |\operatorname{ord}_u(a)| = n,$$

and $F(\gamma, f_{0u} dh_u) = 1$ there. If now (3.2.1) is nonzero, then μ_{0u} and μ_u are unramified. Put $z = \mu_{0u}(\boldsymbol{\pi}_u)$. We conclude

3.2.3 LEMMA. *If f_u is n-regular then* (3.2.2) *is zero unless μ_u is unramified, in which case we have* $\operatorname{tr} I(\mu_u; f_u dg_u \times \sigma) = z^n + z^{-n}$.

DEFINITION. The function f_v on G_v is called *discrete* if $\Phi(\delta\sigma, f_v dg_v)$ is zero for every δ such that the eigenvalues a, a^{-1} of $N\delta$ are distinct and lie in $F_v^\times$.

EXAMPLE. If f_v is supported on the σ-elliptic regular set then it is discrete.

3.2.4 COROLLARY. *Fix a finite place u of F. For every $f^u = \otimes_{v \neq u} f_v$ which has a discrete component* (*at* $u' \neq u$) *there exists a bounded integrable function $d(z)$ on the unit circle in the complex plane with the following property. For every $n \geq n'(f^u)$ and n-regular f_u, we have*

$$\widetilde{I}_0 + \frac{1}{2}\widetilde{I}_1 = I + \frac{1}{2}I' + \frac{1}{4}I'' + \frac{1}{2}I_1' + \int_{|z|=1} d(z)(z^n + z^{-n})|d^\times z|.$$

PROOF. Recall that the I are linear functionals in $f = f_u \otimes f^u$. Since f^u, hence also f, has a discrete component, it is clear (from (3.2.2)) that $I^* = I^{**} = S = 0$, and that the sum over v in (2.2.5) (where S' is defined) ranges over $v = u'$ only. The sum over τ in (2.2.5) ranges over a set of representatives for the connected components of the one-dimensional complex manifold of σ-invariant characters of $\mathbf{A}(\mathbb{A})/A$ whose component τ_u at u is unramified. We may choose τ with $\tau_u = 1$. Put $z = q_u^\lambda$ for λ in $i\mathbb{R}$. Then $\operatorname{tr} I_{\tau_u}(\lambda; f_u dg_u \times \sigma) = z^n + z^{-n}$ by Lemma 3.2.3. Of course, z depends on λ only modulo $2\pi i\mathbb{Z}/\log q_u$. Since the sum over τ, the integral over $i\mathbb{R}$, and product over $w \neq u, u'$ in (2.2.5) are absolutely convergent, the function

$$d(z) = \sum_{\tau} \sum_{k \in \mathbb{A}} \left[\operatorname{tr} R_{\tau_{u'}}(\lambda + k')^{-1} R_{\tau_{u'}}(\lambda + k') I_{\tau_{u'}}(\lambda + k'; f_{u'} dg_{u'} \times \sigma) \right] \cdot \prod_{w \neq u, u'} \operatorname{tr} I_{\tau_w}(\lambda + k'; f_w dg_w \times \sigma),$$

where $k' = k2\pi i / \log q_u$, has the required properties. □

This corollary can be used to prove the symmetric square lifting for automorphic representations with an elliptic component. However, in chapter IV we prove an identity of trace formulae for sufficiently many test measures to deal with all automorphic representations. For the local work in chapter V we use also a simpler form of the formula, as follows.

3.2.5 Proposition. *If $f = \otimes_v f_v$ has two discrete components then*

$$\widetilde{I}_0 + \frac{1}{2}\widetilde{I}_1 = I + \frac{1}{2}I' + \frac{1}{4}I'' + \frac{1}{2}I'_1.$$

Proof. The terms in the geometric side of the twisted trace formula which are associated with nonelliptic σ-conjugacy classes are computed explicitly in [F2;IX] and also in [CLL]. They are similar to those obtained in the trace formulae of groups of rank one. In particular, they vanish if f has two discrete components. As noted in (3.2.4) we have $I^* = I^{**} = S = 0$ if f has a single discrete component. It is clear that $S' = 0$ if f has two discrete components, and the proposition follows. □

Remark. If f has a discrete component and a component as in Example (3.2.3) then f is of type E and Proposition 3.2.5 follows at once from Proposition 3.1.

3.3 Trace formula for H. The twisted trace formula for a function f on $\mathbf{G}(\mathbb{A})$ is analogous to the familiar trace formula for a function f_0 on $\mathbf{H}(\mathbb{A})$. We briefly recall this formula. Again we use only a function of type E, for which the weighted and singular orbital integrals vanish. The elliptic regular part, computed analogously to (1.1.2) and 1.2, has the form

$$\int_{\mathbf{H}(\mathbb{A})/H} \sum_{\gamma \in H'} f_0(h\gamma h^{-1}) dh = \sum_{\gamma \in H'} c(\gamma) \Phi(\gamma, f_0 dh)$$

$$= \frac{1}{2} \sum_{\gamma \in H'} c(\gamma) \Phi^{\mathrm{st}}(\gamma, f_0 dh) + \frac{1}{2} \sum_E \frac{c(E)}{2} \sum_{\gamma \in T'_E} \Phi^{\mathrm{us}}(\gamma, f_0 dh).$$

Here H' denotes the set of regular elliptic elements in H; E ranges over the quadratic field extensions of F; T'_E indicates the regular elements in T_E (thus $\gamma \neq \pm 1$); $c(\gamma) = c(E) = |Z_{\mathbf{H}}(\gamma, \mathbb{A})/Z_H(\gamma)| = |\mathbb{A}_E^1/E^1| = 1$. The 2nd $\frac{1}{2}$ in the sum over E is there since $\Phi^{\mathrm{us}}(\overline{\gamma}) = \Phi^{\mathrm{us}}(\gamma)$, so γ and $\overline{\gamma}$ are counted twice.

By Lemma II.1.7 we introduce

$$f_{T_E,v}(\gamma) = \kappa_v(b) \Delta_v(\gamma) \Phi^{\mathrm{us}}(\gamma, f_{0v} dh_v)$$

for $\gamma \in T_{E,v}$. Note that f_{T_E} depends on the choice of measure dt on $Z_{\mathbf{H}}(\gamma, \mathbb{A})$ which has $c(E) = 1$. By the product formula

$$f_{T_E}(\gamma) = \prod_v f_{T_E,v}(\gamma)$$

is equal to $\Phi^{\mathrm{us}}(\gamma, f_0 dh)$. The trace formula for $\mathbf{T}_E(\mathbb{A}) = \mathbb{A}_E^1$, which is the Poisson summation formula, expresses $\sum_{\gamma \in T_E} f_{T_E}(\gamma)$ as $\sum_{\mu'} \mu'(f_{T_E} dt)$, where μ' ranges over the characters $\mathbb{A}_E^1/E^1 \to \mathbb{C}^\times$. Note that with $\overline{\mu}'(t) = \mu'(\overline{t})$ $(= \mu'(t)^{-1})$ we have

$$\mu'(f_{T_E} dt) = \int_{\mathbb{A}_E^1/E^1} \mu'(t) f_{T_E}(t) dt = \int_{\mathbb{A}_E^1/E^1} \mu'(t) f_{T_E}(\overline{t}) dt = \overline{\mu}'(f_{T_E} dt).$$

Hence $\frac{1}{4} \sum_{\mu'} \mu'(f_{T_E} dt) = \frac{1}{2} I'_E + \frac{1}{4} I_E$,

$$I'_E = {\sum_{\mu' \neq \overline{\mu}'}}' \mu'(f_{T_E} dt), \qquad I_E = \sum_{\mu' = \overline{\mu}'} \mu'(f_{T_E} dt),$$

where $\sum'$ means here a sum over a set of representatives of equivalence classes $\mu' \sim \overline{\mu}'$. Also note that $\mu'(f_{T_E} dt) = \operatorname{tr} \mu'(f_{T_E} dt)$.

On the other hand the geometric side of the trace formula is equal to the spectral side, which is $I_0 + \frac{1}{4} \sum_E I''_E + \frac{1}{2} S_0 + \frac{1}{2} S'_0$. Here

$$I_0 = \sum_{\pi_0} m(\pi_0) \operatorname{tr} \pi_0(f_0 dh).$$

The sum over π_0 ranges over all equivalence classes of discrete-spectrum irreducible representations of $\mathbf{H}(\mathbb{A})$, and $m(\pi_0)$ indicates the multiplicity of π_0 in the discrete spectrum. Further, in standard notations, $I''_E = \operatorname{tr} M(\chi_E) I_0(\chi_E, f_0 dh)$,

$$S_0 = \int_{i\mathbb{R}} \sum_\eta \frac{m(\eta)'}{m(\eta)} \operatorname{tr} I_0(\eta, f_0 dh)|d\lambda|$$

and

$$S'_0 = \int_{i\mathbb{R}} \sum_\eta \sum_v \operatorname{tr}\{R_v(\eta)^{-1} R_v(\eta)' I_0(\eta; f_{0v} dh_v)\} \cdot \prod_{w \neq v} \operatorname{tr} I(\eta_w; f_{0w} dh_w) \cdot |d\lambda|.$$

We conclude

PROPOSITION. (1) *For every $f_0^u dh = \otimes_v f_{0v} dh_v$ $(v \neq u)$ there is $n' > 0$ such that for every n-regular f_{0u} with $n \geq n'$ we have*

$$\widetilde{I}_0 = I_0 + \frac{1}{4}\sum_E I''_E - \frac{1}{2}\sum_E I'_E - \frac{1}{4}\sum_E I_E - \frac{1}{4} I_0^* + \frac{1}{2} S_0 + \frac{1}{2} S'_0.$$

(2) *If in addition f_0^u has a discrete component $f_{0u'}$ then there is a function $d_0(z)$, bounded and integrable on $|z| = 1$, depending only on f_0^u, such that*

$$\widetilde{I}_0 = I_0 + \frac{1}{4}\sum_E I''_E - \frac{1}{2}\sum_E I'_E - \frac{1}{4}\sum_E I_E + \int_{|z|=1} d_0(z)(z^n + z^{-n})|d^\times z|$$

for every n-regular f_{0u} with $n \geq n'$.

(3) *If $f_0 = \otimes_v f_{0v}$ has two elliptic components then*

$$\widetilde{I}_0 = I_0 + \frac{1}{4}\sum_E I''_E - \frac{1}{2}\sum_E I'_E - \frac{1}{4}\sum_E I_E.$$

PROOF. It remains to recall that $\widetilde{I}_0$ is defined in (1.3.3) and $I_0^* = \operatorname{tr} I(1, f_0 dh)$ is equal to I^* of (2.4.2) for $f dg$ matching $f_0 dh$. □

3.4 Trace formula for $\boldsymbol{H_1}$. We also need the trace formula for a test function $f_1 = \otimes_v f_{1v}$ on $\mathbf{H}_1(\mathbb{A}) = \mathrm{PGL}(2, \mathbb{A})$. It suffices to consider f_1 analogous to the f_0 of (3.3). We first state the formula and then explain the notations.

Proposition. (1) *For every* $f_1^u = \otimes_{v\neq u} f_{1v}$ *there is* $n' > 0$ *such that for every* n*-regular* f_{1u} *with* $n \geq n'$ *we have*

$$\widetilde{I}_1 = I_1 - \frac{1}{4}I_1^* - \frac{1}{4}I_1^{**} + \frac{1}{2}S_1 + \frac{1}{2}S_1'.$$

(2) *If in addition* f_1^u *has a discrete component* $f_{1u'}$, *then there is a function* $d_1(z)$, *bounded and integrable on* $|z| = 1$, *depending only on* f_1^u, *such that*

$$\widetilde{I}_1 = I_1 + \int_{|z|=1} d_1(z)(z^n + z^{-n})|d^\times z|$$

for every n*-regular* f_{1u} *with* $n \geq n'$.
(3) *If* $f_1 = \otimes_v f_{1v}$ *has two elliptic components then* $\widetilde{I}_1 = I_1$.

Proof. Here $I_1 = \sum \operatorname{tr} \pi_1(f_1 dh_1)$. The sum ranges over all cuspidal and one-dimensional $\mathbf{H}_1(\mathbb{A})$-modules. Multiplicity one theorem for PGL(2) implies that π_1 ranges over equivalence classes of representations. The sums I_1^* and I_1^{**} are defined analogously to I^* and I^{**} of (2.4.2). They are equal to I^* and I^{**} for fdg matching $f_1 dh_1$. Their sum is

$$I_1^* + I_1^{**} = \sum_{w\eta=\eta} \operatorname{tr} I_1(\eta; f_1 dh_1);$$

for a character η of the diagonal subgroup of $\mathbf{H}_1(\mathbb{A})$ we put $w\eta(\operatorname{diag}(a,b)) = \eta(\operatorname{diag}(b,a))$. As usual,

$$S_1 = \int_{i\mathbb{R}} \frac{m(\eta)'}{m(\eta)} \operatorname{tr} I_1(\eta; f_1 dh_1)|d\lambda|$$

and S_1' is

$$\int_{i\mathbb{R}} \sum_\eta \sum_v \operatorname{tr}[R_v(\eta)^{-1} R_v(\eta)' I_1(\eta; f_{1v} dh_{1v})] \cdot \prod_{w\neq v} \operatorname{tr} I_1(\eta_w; f_{1w} dh_{1w}) \cdot |d\lambda|.$$

□

3.5 Comparison. Finally we compare the formulae of (3.2), (3.3), (3.4) for measures $fdg = \otimes_v f_v dg_v$ on $\mathbf{G}(\mathbb{A})$, $f_0 dh = \otimes_v f_{0v} dh_v$ on $\mathbf{H}(\mathbb{A})$, and $f_1 dh_1 = \otimes_v f_{1v} dh_{1v}$ on $\mathbf{H}_1(\mathbb{A})$, such that $f_{0v} dh_v$ matches $f_v dg_v$ for all v, and $f_{1v} dh_{1v}$ matches $f_v dg_v$ for all v. (Had we not known that $f_{1v}^0 dh_{1v}$ and

$f_v^0 dg_v$ match we could work with f which has a component f_v such that $f_{1v} = 0$ matches $f_v dg_v$ and $f_1 = 0$). Define $\mathcal{I}$ to be the difference

$$\mathcal{I} = I + \frac{1}{2}I' + \frac{1}{4}I'' + \frac{1}{2}I_1' - \left[I_0 + \frac{1}{4}\sum_E I_E'' - \frac{1}{2}\sum_E I_E' - \frac{1}{4}\sum_E I_E + \frac{1}{2}I_1\right].$$

It is an invariant distribution in $f dg$, depending only on the orbital integrals of $f dg$.

PROPOSITION. (1) *If f has two discrete components then $\mathcal{I} = 0$.*

(2) *Suppose that $f^u = \otimes_{v \neq u} f_v$ has a discrete component. Then there exists an integer $n' \geq 1$ and a bounded integrable function $d(z)$ on $|z| = 1$, depending only on f^u, f_0^u, f_1^u, such that for all n-regular functions f_u, f_{1u}, and f_{0u} with $n > n'$ we have*

$$\mathcal{I} = \int_{|z|=1} d(z)(z^n + z^{-n})|d^\times z|.$$

PROOF. This follows at once from (3.2.4), (3.2.5), (3.3), and (3.4). □

Concluding remarks. (1) is used in the local study of chapter V. In chapter IV we prove (2) without the assumption that f^u has a discrete component. This is used in chapter V to show that $\mathcal{I} = 0$ for any matching $f dg$, $f_0 dh$, $f_1 dh_1$. This is used in chapter V to establish the symmetric square lifting for all automorphic representations.

IV. TOTAL GLOBAL COMPARISON

Summary. The techniques of chapter III, based on the usage of regular functions to simplify the trace formula, are pursued to extend the results of chapter III to sufficiently many test functions to permit proving in chapter V the symmetric square lifting for all representations of $\mathrm{SL}(2, \mathbb{A})$ and self-contragredient representations of $\mathrm{PGL}(3, \mathbb{A})$.

Introduction

Put $\mathbf{H}_1 = \mathrm{PGL}(2)$. Let f_v (resp. f_{0v}, f_{1v}) denote a complex-valued, smooth (that is, locally-constant if F_v is nonarchimedean), compactly-supported function on G_v (resp. H_v, H_{1v}). If F_v is nonarchimedean put $K_{1v} = \mathbf{H}_1(R_v)$, and let f_v^0 (resp. f_{0v}^0, f_{1v}^0) be the measure of volume one which is supported on K_v (resp. K_{0v}, K_{1v}) and is constant on this group. Here we used the uniqueness of the Haar measure (up to a constant) to identify the space of locally-constant compactly-supported measures with the space of locally-constant compactly-supported functions on G_v (resp. H_v, H_{1v}) once a Haar measure is chosen.

At any place v, the functions f_v and f_{0v} (resp. f_v and f_{1v}) are called *matching* if they have matching orbital integrals. For a definition see section II.3. Briefly, they satisfy

$$\Delta(\delta\sigma)\Phi^{\mathrm{st}}(\delta, f_v dg) = \Delta_0(\gamma)\Phi^{\mathrm{st}}(\gamma, f_{0v} dh)$$

for every δ in G_v with regular norm $\gamma = N\delta$, and

$$\Delta(\delta\sigma)\Phi^{\mathrm{us}}(\delta, f_v dg) = \Delta_1(\gamma_1)\Phi_1(\gamma_1, f_{1v} dh_1)$$

for every δ in G_v with regular norm $\gamma_1 = N_1\delta$. Here $\Phi^{\mathrm{st}}(\delta, f_v dg)$ means "stable σ-orbital integral of $f_v dg$ at δ", and $\Phi^{\mathrm{us}}(\delta, f_v dg)$ is the "unstable σ-orbital integral of $f_v dg$ at δ". These are defined and studied in section II.3.

The Theorem of section II.1 asserts that $f_v^0 dg$ and $f_{0v}^0 dh$ are matching, and that $f_v^0 dg$ and $f_{1v}^0 dh_1$ are matching. This local proof relies on a twisted analogue of Kazhdan's decomposition of a compact element into its topologically unipotent and its absolutely semisimple parts. There are other proofs of these assertions (see, e.g., §4 of the paper [F2;II], for a proof of the first assertion), but they seem to be more complicated.

Let $fdg = \otimes_v f_v dg_v$ (resp. $f_0 dh = \otimes_v f_{0v} dh_v$, $f_1 dh_1 = \otimes_v f_{1v} dh_{1v}$) be measures on $\mathbf{G}(\mathbb{A})$ (resp. $\mathbf{H}(\mathbb{A})$, $\mathbf{H}_1(\mathbb{A})$) such that (1) $f_v dg_v = f_v^0 dg_v$, $f_{0v} dh_v = f_{0v}^0 dh_v$, $f_{1v} dh_{1v} = f_{1v}^0 dh_{1v}$ for almost all v, and such that (2) $f_v dg_v$ and $f_{0v} dh_v$, and $f_v dg_v$ and $f_{1v} dh_{1v}$, are matching for all v. The measures fdg, $f_0 dh$, $f_1 dh_1$ exist since the conditions (1) and (2) are compatible, namely $f_v^0 dg_v$ and $f_{0v}^0 dh_v$ as well as $f_v^0 dg_v$ and $f_{1v}^0 dh_{1v}$ are matching.

In section III.3, we defined various sums, denoted by I_i^*, of traces (such as $\operatorname{tr} \pi_0(f_0 dh)$, $\operatorname{tr} \pi_1(f_1 dh_1)$, $\operatorname{tr} \pi(fdg \times \sigma)$) of convolution operators ($\pi_0(f_0 dh)$, $\pi_1(f_1 dh_1)$ and $\pi(fdg \times \sigma)$). The sums I, I', I'', I_1' depend on fdg. The sums I_0, I_E, I_E', I_E'' depend on $f_0 dh$, and I_1 on $f_1 dh_1$. Put

$$\mathcal{I} = I + \frac{1}{2} I' + \frac{1}{4} I'' + \frac{1}{2} I_1' - I_0 - \frac{1}{4} \sum_E I_E'' + \frac{1}{2} \sum_E I_E' + \frac{1}{4} \sum_E I_E - \frac{1}{2} I_1.$$

We show in section V.2, that the global symmetric square lifting is a consequence of the following

THEOREM. *We have $\mathcal{I} = 0$ for any matching fdg, $f_0 dh$, $f_1 dh_1$ as above.*

It is also shown in section V.2, that when $\mathcal{I} = 0$ then I relates to I_0 and to the $\mu'(f_{T_E} dt)$, and $I_1 = I_1'$. Our proof is based on the usage of regular, or Iwahori type, functions.

It is clear from the proof given below that it applies to establish relatively effortlessly, and conceptually, the analytic part of the comparison of trace formulae for general test functions in any lifting situation where all groups involved have (split) rank bounded by one. In our case the ("twisted") rank of $G = \mathrm{PGL}(3)$ is one. In particular our technique establishes the comparison of trace formulae for any test functions in the cases of (1) basechange from U(3) to $\mathrm{GL}(3, E)$ which is studied in [F3] ([F3;IV], [F3;V], [F1;II] chapter IV, [F3;VI] and [F3;VIII]; [F3;VII] contains another proof of the trace formulae comparison for a general test function in the case of basechange from U(3) to $\mathrm{GL}(3, E)$; it relies on properties of quasispherical functions, but does not generalize to establish our Theorem); (2) cyclic basechange lifting for GL(2) (see [F1;IV] where our present technique is used to give a simple proof of this comparison); (3) basechange from U(2) to GL$(2,E)$ (see [F3;II]); (4) metaplectic correspondence for GL(2) (see [F1;I]).

The proof of the Theorem is based on the usage of regular functions in the sense of chapter III, [FK1], [FK2], and [F1;II], chapters III, IV. That such functions would be useful in this context was discovered by us while working on the joint paper [FK1] with D. Kazhdan, being inspired by the proof — see [FK1], sections 16, 17 — of the metaplectic correspondence for representations of GL(n) with a vector fixed by an Iwahori subgroup.

IV.1 The comparison

Although these functions can be introduced for any quasi-split group, to simplify the notations we discuss these functions here only in the case of the group GL(n) (and SL(n), PGL(n)).

Let F be a local nonarchimedean field, R its ring of integers, $\boldsymbol{\pi}$ a local uniformizer in $R, \mathbf{q} = \boldsymbol{\pi}^{-1}, q$ the cardinality of the residue field $R/(\boldsymbol{\pi}), |\cdot|$ the valuation on F normalized to have $|\boldsymbol{\pi}| = q^{-1}$ (thus $|\mathbf{q}| = q$), G the group GL(n,F), $K =$ GL(n,R) a maximal compact subgroup in G, B the Iwahori subgroup of G which consists of matrices in K which are upper triangular modulo $\boldsymbol{\pi}$, A the diagonal subgroup of G, $A(R) = A \cap K = A \cap B$, and U the upper triangular unipotent subgroup; AU is a minimal parabolic subgroup.

The vector $\mathbf{m} = (m_1, \dots, m_n)$ in $\mathbb{Z}^n$ is called *regular* if $m_i > m_{i+1}$ for all i $(1 \leq i < n)$. Let $\mathbf{q}^{\mathbf{m}}$ be the matrix diag$(\mathbf{q}^{m_1}, \dots, \mathbf{q}^{m_n})$ in A. The matrix $\mathbf{a} =$ diag$(a_1, \dots, a_n)$ in A is called *strongly regular* if $|a_i| > |a_{i+1}|$ for all i, and $\mathbf{m}$-*regular* if $\mathbf{a} = u\mathbf{q}^{\mathbf{m}}$ for a regular $\mathbf{m}$ and u in $\mathbf{A}(R)$. A conjugacy class in G is called *strongly* (resp. $\mathbf{m}$-)*regular* if it contains a strongly (resp. $\mathbf{m}$-) regular element. An element of G is called *regular* if its eigenvalues are distinct.

Denote by J the matrix whose (i,j) entry is $\delta_{i,n-j}$. Put $\sigma(g) = J^t g^{-1} J$. The elements g and g' of G are called σ-*conjugate* if there is x in G with $g' = xg\sigma(x)^{-1}$. For

$$\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n \qquad \text{put} \qquad \sigma\mathbf{m} = (-m_n, \dots, -m_2, -m_1),$$

and say that $\mathbf{m}$ is σ-*regular* if $\mathbf{m} + \sigma\mathbf{m}$ is regular. The element $\mathbf{a}$ of A is called $\mathbf{m}$-σ-*regular* if $\mathbf{m}$ is σ-regular and $\mathbf{a}\sigma(\mathbf{a})$ is $(\mathbf{m} + \sigma\mathbf{m})$-regular; $\mathbf{a}$

is called *strongly σ-regular* if it is $\mathbf{m}$-σ-regular for some $\mathbf{m}$. A σ-conjugacy class in G is called *strongly* (or $\mathbf{m}$-) *σ-regular* if it contains a strongly (or $\mathbf{m}$) σ-regular element in A. Note that if $\mathbf{a}$ is *$\mathbf{m}$-regular* then $\mathbf{a}$ is $\mathbf{m}$-σ-regular since $\mathbf{a}\sigma(\mathbf{a})$ is $(\mathbf{m}+\sigma(\mathbf{m}))$-regular. We have

1. Proposition. *If $\mathbf{a}$ is $\mathbf{m}$-regular then*
(1) *Each conjugacy class in G which intersects $B\mathbf{a}B$ is $\mathbf{m}$-regular.*
(2) *Each σ-conjugacy class in G which intersects $B\mathbf{a}B$ contains an $\mathbf{m}$-regular element in A; in particular it is $\mathbf{m}$-σ-regular.*

Proof. We shall prove (2); (1) follows by the same method on erasing σ throughout. Write $g' \sim g$ if g is σ-conjugate to g' in G. We have to show that any $b'\mathbf{a}b$ (b', b in B) is σ-conjugate to an $\mathbf{m}$-regular element. Since $\sigma B = B$, up to σ-conjugacy we may assume that $b' = 1$. Each element b in B can be written in a unique way as a product

$$b_0 b_- b_+, \qquad b_0 \in \mathbf{A}(R), \qquad b_- = 1 + n_-, \qquad b_+ = 1 + n_+,$$

where n_- (resp. n_+) is a lower (resp. upper) triangular nilpotent matrix. Put $\tilde{\mathbf{a}} = \mathbf{a}b_0$. Then

$$\begin{aligned}\mathbf{ab} &= \tilde{\mathbf{a}}b_- b_+ \sim \sigma(b_+)\tilde{\mathbf{a}}b_- = (\tilde{\mathbf{a}}b_-\tilde{\mathbf{a}}^{-1})\tilde{\mathbf{a}}(b_-^{-1}\tilde{\mathbf{a}}^{-1}\sigma(b_+)\tilde{\mathbf{a}}b_-)\\ &\sim \tilde{\mathbf{a}}(b_-^{-1}\tilde{\mathbf{a}}^{-1}\sigma(b_+)\tilde{\mathbf{a}}b_-)\sigma(\tilde{\mathbf{a}}b_-\tilde{\mathbf{a}}^{-1}).\end{aligned}$$

Denote by $|x|$ the maximum of the valuations of the entries of a matrix x in G. Put

$$b'_+ = \tilde{\mathbf{a}}^{-1}\sigma(b_+)\tilde{\mathbf{a}}, \qquad b'_- = \sigma(\tilde{\mathbf{a}}b_-\tilde{\mathbf{a}}^{-1}),$$

and also $n'_+ = b'_+ - 1$ and $n'_- = b'_- - 1$. Since σ stabilizes every congruence subgroup of G, and $\tilde{\mathbf{a}}$ is $\mathbf{m}$-regular, we have $|n'_+| < |n_+|$ and $|n'_-| < |n_-|$. Moreover, it is clear that

$$b_-^{-1}b'_+b_-b'_- = b''_0 b''_- b''_+ \qquad \text{with} \qquad \max(|n''_-|, |n''_+|) \le \max(|n'_-|, |n'_+|).$$

Repeating this process we obtain a matrix of the form $\mathbf{a}'(1+\varepsilon)$ with $\mathbf{m}$-regular $\mathbf{a}'$ and ε with $|\varepsilon|$ smaller than any given positive number. The proposition now follows. □

Let f be a locally constant compactly supported complex valued function on G, dx a Haar measure on G, and

$$\Phi^\sigma(\gamma, f dx) = \Phi(\gamma\sigma, f dx) = \int f(x^{-1}\gamma\sigma(x))dx/d_\gamma$$

the (*twisted* or) σ-*orbital integral* of fdg at the element γ of G (the integration is taken over $Z_G(\gamma\sigma)\backslash G$, where $Z_G(\gamma\sigma)$ is the σ-centralizer of γ in G, and d_γ is a Haar measure on $Z_G(\gamma\sigma)$). Denote by $\mathrm{Lie}(G)$ the Lie algebra of G. If $\mathbf{G} = \mathrm{GL}(n)$ then $\mathrm{Lie}(G) = M_n$ (the algebra of $n \times n$ matrices). Put $\sigma X = -J^t XJ$ for X in $\mathrm{Lie}(G)$. Denote by $\mathrm{Ad}(\gamma)$ the adjoint action of γ on $\mathrm{Lie}(G)$. We say that γ is σ-*regular* if $\gamma\sigma(\gamma)$ is regular (has distinct eigenvalues) in G. If γ is σ-regular, its σ-orbit is closed, and the convergence of $\Phi(\gamma\sigma, fdg)$ is clear; this is the only case to be used in this chapter, but the convergence of $\Phi(\gamma\sigma, fdg)$ is known in general. Put

$$\Delta(\gamma\sigma) = |\det(1 - \mathrm{Ad}(\gamma)\sigma)|\,\mathrm{Lie}(Z_G(\gamma\sigma)\backslash G)\,|^{1/2}.$$

This is well defined since $\mathrm{Ad}(\gamma)\sigma$ acts trivially on $Z_G(\gamma\sigma)$ and therefore trivially also on $\mathrm{Lie}(Z_G(\gamma\sigma))$. Put

$$F^\sigma(\gamma, fdg) = F(\gamma\sigma, fdg) = \Delta(\gamma\sigma)\Phi(\gamma\sigma, fdg).$$

Let U be the unipotent upper triangular subgroup in G, A the diagonal subgroup, and K the maximal compact subgroup $GL(n, R)$. Each of A, U, K is σ-invariant, and A normalizes U. Put $A^\sigma = \{a \in A; \sigma a = a\}$. For γ in A put

$$\boldsymbol{\delta}(\gamma) = |\det \mathrm{Ad}(\gamma)\sigma|\mathrm{Lie}(U)| = |\det \mathrm{Ad}(\gamma)|\mathrm{Lie}(U)|$$

($= |a/c|^2$ if $\gamma = \mathrm{diag}(a, b, c)$) and

$$f_U^\sigma(\gamma) = \boldsymbol{\delta}(\gamma)^{1/2} \int_{A^\sigma\backslash A} \int_U \int_K f(\sigma(k)^{-1}\sigma(a)^{-1}\gamma\, auk)\, dk\, du\, da.$$

A standard formula of change of variables (see, e.g., A1.3) asserts that for any σ-regular γ in A we have $F(\gamma\sigma, fdg) = f_U^\sigma(\gamma)$. Consequently it is clear from Proposition 1(2) that if f is (a multiple of) the characteristic function of $B\mathbf{a}B$, where $\mathbf{a}$ is an $\mathbf{m}$-regular element, then $F(\gamma\sigma, fdg)$ is a scalar multiple of the characteristic function of the union of the σ-conjugacy classes in G which contain an $\mathbf{m}$-regular element, namely of the set of the $\mathbf{m}$-σ-regular σ-conjugacy classes in G. Consequently we can introduce the following

Definition. For any regular $\mathbf{m}$ in Z^n let $\phi_{\mathbf{m},\sigma}$ denote the multiple of the characteristic function of $B\mathbf{q}^{\mathbf{m}}B$ such that $F(\gamma\sigma, \phi_{\mathbf{m},\sigma}dg)$ is zero unless γ lies in an $\mathbf{m}$-σ-regular σ-conjugacy class in G, where $F(\gamma\sigma, \phi_{\mathbf{m},\sigma}dg) = 1$.

Analogous definitions will now be introduced in the nontwisted case. We simply have to erase σ everywhere. Thus the orbital integral of a locally-constant compactly-supported complex-valued measure fdg on G at γ in G is denoted by $\Phi(\gamma, fdg) = \int f(x^{-1}\gamma x)dx/d_\gamma$. Here x ranges over $Z_G(\gamma)\backslash G$, where $Z_G(\gamma)$ is the centralizer of γ in G. If γ is regular, namely it has distinct eigenvalues $\gamma_1, \dots, \gamma_n$, the orbit of γ is closed and $\Phi(\gamma, fdg)$ is clearly convergent. Put

$$\Delta(\gamma) = |\det(1 - \mathrm{Ad}(\gamma))|\mathrm{Lie}(Z_G(\gamma)\backslash G)|^{1/2};$$

it is equal to

$$\left|\prod_{i<j}(\gamma_i - \gamma_j)^2\right|^{1/2} / |\det \gamma|^{(n-1)/2}.$$

Put $F(\gamma, fdg) = \Delta(\gamma)\Phi(\gamma, fdg)$. If γ lies in A put

$$\boldsymbol{\delta}(\gamma) = |\det \mathrm{Ad}(\gamma)|\mathrm{Lie}(U)|.$$

It is equal to $\prod_{i<j} |\gamma_i/\gamma_j|$. Put

$$f_U(\gamma) = \boldsymbol{\delta}(\gamma)^{1/2} \int_U \int_K f(k^{-1}\gamma nk)\, dk\, dn.$$

Since $F(\gamma, fdg) = f_U(\gamma)$ for all regular γ in A it is clear from Proposition 1(1) that if f is (a multiple of) the characteristic function of $B\mathbf{a}B$, where $\mathbf{a}$ is an $\mathbf{m}$-regular element, than $F(\gamma, fdg)$ is a scalar multiple of the characteristic function of the union of the $\mathbf{m}$-regular conjugacy classes in G. Consequently we can introduce the following

Definition. Denote by $\phi_{\mathbf{m}}$ the multiple of the characteristic function of $B\mathbf{q}^{\mathbf{m}}B$ such that $F(\gamma, \phi_{\mathbf{m}}dg)$ is 0 unless γ lies in an $\mathbf{m}$-regular conjugacy class, where $F(\gamma, \phi_{\mathbf{m}}) = 1$.

Let π be an admissible G-module. Let $\pi(fdg)$ be the convolution operator $\int f(g)\pi(g)dg$; it is of finite rank, hence has a trace, denoted by tr $\pi(fdg)$. It is easy to see that there exists a conjugacy invariant locally-constant complex-valued function χ on the regular set (distinct eigenvalues) of G, with tr $\pi(fdg) = \int_G \chi(g)f(g)dg$ for any fdg supported on the regular set of G. The function $\chi = \chi_\pi$ is called the *character* of π; it is clearly independent of the choice of the measure dg.

If V is the space of π, then $V_U = \{\pi(u)v - v; v \text{ in } V, u \text{ in } U\}$ is stabilized by A since A normalizes U, and V/V_U is an admissible (namely it has finite length) A-module denoted by π'_U. The A-module $\pi_U = \boldsymbol{\delta}^{-1/2}\pi'_U$ is called the *A-module of U-coinvariants of* π. The composition series of the admissible A-module π_U consists of finitely many irreducible A-modules, namely characters on A (since A is abelian). These characters are called here the *exponents* of π. The character $\chi(\pi_U)$ of π_U is the sum of the exponents of π.

If $\pi_U \neq \{0\}$ then by Frobenius reciprocity π is a subquotient of the G-module $I(\mu) = \text{ind}(\boldsymbol{\delta}^{1/2}\mu; AU, G)$ normalizedly induced from the character μ of A extended to AU by one on U; here μ is any exponent of π. Let $W = N(A)/A$ be the Weyl group of A in G; $N(A)$ is the normalizer of A in G. Put $w\mu$ for the character $a \mapsto \mu(w(a))$ of A. Define $J = (\delta_{i,n+1-i})$. The Theorem of [C1] asserts that $(\Delta\chi_\pi)(\mathbf{a}) = (\chi(\pi_U))(J\mathbf{a}J)$ for every strongly regular $\mathbf{a}$ in A. Hence $\chi(I(\mu)_U) = \Sigma\, w\mu$ (sum over w in W), and each exponent of π is of the form w in W. Since $\phi_{\mathbf{m}}$ is supported on the $\mathbf{m}$-regular set, the Weyl integration formula implies that

$$\text{tr } \pi(\phi_{\mathbf{m}}dg) = [W]^{-1}\int_A (\Delta\chi_\pi)(\mathbf{a})F(\mathbf{a}, \phi_{\mathbf{m}}dg)d\mathbf{a}$$

$$= (\chi(\pi_U))(\mathbf{q}^{\mathbf{m}})\int_{A(R)} \mu(\mathbf{a})d\mathbf{a}.$$

Namely the trace tr $\pi(\phi_{\mathbf{m}}dg)$ is zero unless the composition series of π_U consists of unramified characters, in which case (for a suitable choice of measures) tr $\pi(\phi_{\mathbf{m}}dg)$ is the sum of $\mu(\mathbf{q}^{\mathbf{m}})$ over the exponents (with multiplicities) of π. We conclude:

2. Proposition. *If μ is an unramified character of A then*

$$\text{tr}\, I(\mu; \phi_{\mathbf{m}}dg) = \sum_w (w\mu)(\mathbf{q}^{\mathbf{m}}) \qquad (w \text{ in } W).$$

Let V denote the space of π, $V_B(\pi)$ the subspace of B-fixed vectors in V, and $V_B(\mu)$ the space $V_B(\pi)$ when $\pi = I(\mu)$. Then $\pi(\phi_{\mathbf{m}}dg)$ acts on $V_B(\pi)$, and we have

3. PROPOSITION. *If μ in an unramified character of A then the dimension of $V_B(\mu)$ is the cardinality $[W]$ of W. The set $\{\psi_w; w \text{ in } W\}$ of functions on G such that ψ_w is supported on $AUwB$ and satisfies*

$$\psi_w(auwb) = (\mu\boldsymbol{\delta}^{1/2})(a) \quad (a \in A, \quad u \in U, \quad b \in B),$$

is a basis of the space $V_B(\mu)$.

PROOF. This is clear from the decomposition

$$AU\backslash G = (AU) \cap K\backslash(AU) \cap K \cdot W \cdot B. \qquad \square$$

For each i ($1 \leq i \leq n$) let $\mathbf{e}_i$ be the vector $(0,\ldots,0,1,0,\ldots,0)$ in $\mathbb{Z}^n$; the nonzero entry is at the i-th place. A vector $\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_j$ ($i \neq j$) is called here a *root* of A. It is called *positive* if $i < j$, negative if $i > j$, and *simple* if $j = i + 1$ ($1 < i < n$). Put

$$\rho = \sum_{\alpha > 0} \alpha \qquad (= (n-1, n-3, \ldots, 1-n)).$$

Then

$$\boldsymbol{\delta}(\mathbf{q}^{\mathbf{m}}) = q^{\langle \rho, \mathbf{m} \rangle}.$$

Denote by $\overline{U}$ the unipotent lower triangular subgroup. We have

4. PROPOSITION. (1) *If $\mathbf{m} = (m_1, \ldots, m_n) = \sum_{i=1}^n m_i \mathbf{e}_i$ satisfies $m_1 \geq \cdots \geq m_n$, and $h = \mathbf{q}^{\mathbf{m}}$, then the cardinality of the set BhB/B is $\boldsymbol{\delta}(h)$.*
(2) *Put $B_- = B \cap \overline{U}$. Then for every w in W, the cardinality of the set*

$$w[h^{-1}B_-h/B_- \cap h^{-1}B_-h]w^{-1}/\overline{U} \cap wh^{-1}B_-hw^{-1}$$

is $\boldsymbol{\delta}^{1/2}(h)/\boldsymbol{\delta}^{1/2}(whw^{-1})$.

PROOF. If $B_+ = B \cap U, B_0 = B \cap A$, then

$$B = B_-B_0B_+, \qquad h^{-1}B_-h \supset B_-, \qquad h^{-1}B_+h \subset B_+$$

and

$$BhB/B \simeq h^{-1}Bh \cdot B/B = h^{-1}B_-h \cdot B/B \simeq h^{-1}B_-h/h^{-1}B_-h \cap B_-;$$

(1) follows; the proof of (2) is similar. $\square$

The Weyl group W is isomorphic to the symmetric group S_n on n letters. It is generated by the simple transpositions $s_i = (i, i+1)$ ($1 \leq i \leq n$). The length function ℓ on W associates to each w in W the least nonnegative integer $\ell(w)$ such that w can be expressed as a product of $\ell(w)$ simple transpositions. It is easy to verify that $(\pi(\phi_{\mathbf{m}}dg)\psi_w)(u)$ is zero for every $u \neq w$ in W with $\ell(u) \geq \ell(w)$.

5. Proposition. *For every w in W we have*

$$(\pi(\phi_{\mathbf{m}} dg)\psi_w)(w) = \mu(whw^{-1})$$

(where $h = \mathbf{q}^{\mathbf{m}}$), and $\phi_{\mathbf{m}} dg$ is equal to

$$|BhB|^{-1}\boldsymbol{\delta}^{1/2}(h)\,\mathrm{ch}(BhB)dg.$$

Proof. Compute:

$$\begin{aligned}(\pi(\mathrm{ch}(BhB)dg)\psi_w)(w) &= \int_{BhB} \psi_w(wx)dx = |B| \sum_{x\in BhB/B} \psi_w(wh\cdot h^{-1}x)\\ &= |B|(\mu\boldsymbol{\delta}^{1/2})(whw^{-1}) \sum_{x\in h^{-1}B_-h/B_-\cap h^{-1}B_-h} \psi_w(wxw^{-1}\cdot w)\\ &= |B|(w\mu)(h)\cdot\boldsymbol{\delta}^{1/2}(whw^{-1})\cdot(\boldsymbol{\delta}^{1/2}(h)/\boldsymbol{\delta}^{1/2}(whw^{-1}))\psi_w(w)\\ &= |B|(w\mu)(h)\boldsymbol{\delta}^{1/2}(h)\psi_w(w) = |BhB|\cdot\boldsymbol{\delta}^{-1/2}(h)\cdot(w\mu)(h).\end{aligned}$$

Conclude:

$$\mathrm{tr}\ \pi[|BhB|^{-1}\boldsymbol{\delta}^{1/2}(h)\ \mathrm{ch}(BhB)dg] = \sum_w (w\mu)(h) = \mathrm{tr}\ \pi(\phi_{\mathbf{m}} dg).$$

Since $\phi_{\mathbf{m}}$ is by definition a multiple of $\mathrm{ch}(BhB)$, the proposition follows.□

We conclude that the matrix of $\pi(\phi_{\mathbf{m}} dg)$ with respect to the basis $\{\psi_w; w$ in $W\}$ of $V_B(\mu)$ (this basis is partially ordered by the length function ℓ on W) is of the form $Z+N$, where Z is a diagonal matrix with diagonal entries $\mu(whw^{-1})$ (w in W), and N is a strictly upper triangular nilpotent matrix of size $[W]\times[W]$. Thus we have $N^{[W]}=0$.

6. Proposition. *If $\mathbf{m} = (m_i)$ and $\mathbf{m}' = (m_i')$ satisfy*

$$m_i \geq m_{i+1}, \qquad m_i' \geq m_{i+1}' \qquad (1 \leq i < n)$$

then

$$\pi(\phi_{\mathbf{m}} dg)\pi(\phi_{\mathbf{m}'} dg) = \pi(\phi_{\mathbf{m}+\mathbf{m}'} dg).$$

Proof. Since $hB_-h^{-1} \subset B_-$ and $h^{-1}B_+h \subset B_+$, we have

$$B\mathbf{q}^{\mathbf{m}}B\mathbf{q}^{\mathbf{m}'}B = B\mathbf{q}^{\mathbf{m}}\mathbf{q}^{\mathbf{m}'}B = B\mathbf{q}^{\mathbf{m}+\mathbf{m}'}B. \qquad \square$$

We shall consider only operators $\pi(\phi_{\mathbf{m}} dg)$ with regular $\mathbf{m}$. Since the semigroup of $\mathbf{m}$ in $\mathbf{Z}^n$ with $m_i \geq m_{i+1} \geq 0$ $(1 \leq i < n)$ is generated by

$$\sum_{i=1}^{j} \mathbf{e}_i = (1, \dots, 1, 0, \dots, 0) \qquad (1 \leq j < n),$$

we need only consider (products of finitely many commuting) matrices of the form $(Z+N)^m$, $m \geq 0$.

7. Proposition. *Let Z be a diagonal matrix with entries z_α along the diagonal. Let $N = (n_{\alpha,\beta})$ be a strictly upper triangular matrix with $N^s = 0$. Then $(Z+N)^m$ is the matrix whose (α_1, α_r) entry is the sum over $r = 1, \dots, s$ of*

$$\sum_{\{\alpha_1 < \alpha_2 < \cdots < \alpha_r\}} n_{\alpha_1,\alpha_2} \cdots n_{\alpha_{r-1},\alpha_r} \sum_{1 \leq k \leq r} (-1)^{k-1} z_{\alpha_k}^m$$

$$\prod_{\substack{1 \leq i < j < r \\ i,j \neq k}} (z_{\alpha_i} - z_{\alpha_j}) / \prod_{1 \leq i < j \leq r} (z_{\alpha_i} - z_{\alpha_j}).$$

Proof. This is easily proven by induction. To obtain this formula, we argue as follows. The noncommutative binomial expansion, easily verified by induction, asserts

$$(Z+N)^m = \sum_{r=1}^{s} \left(\sum_{\{(i_j); \sum_{j=1}^{r} i_j = m+1-r\}} Z^{i_1} N Z^{i_2} \cdots N Z^{i_r} \right).$$

Here

$$Z^{i_1} N \cdots N Z^{i_r} = (z_{\alpha_1}^{i_1})(n_{\alpha_1,\alpha_2})(z_{\alpha_2}^{i_2}) \cdots (n_{\alpha_{r-1},\alpha_r})(z_{\alpha_r}^{i_r})$$

$$= \left(\sum_{\alpha_2, \alpha_3, \dots, \alpha_{r-1}} n_{\alpha_1,\alpha_2} n_{\alpha_2,\alpha_3} \cdots n_{\alpha_{r-1},\alpha_r} \cdot z_{\alpha_1}^{i_1} \cdots z_{\alpha_r}^{i_r} \right).$$

To take the sum over (i_j) we note that by induction we have

$$\sum_{\sum_{j=1}^{r} i_j = m+1-r} z_1^{i_1} \cdots z_r^{i_r} = \sum_{k=1}^{r} (-1)^{k+1} z_k^m \prod_{\substack{1 \leq i < j < r \\ i,j \neq k}} (z_i - z_j) / \prod_{1 \leq i < j \leq r} (z_i - z_j).$$

The proposition follows. □

As usual, let μ be an unramified character on A. Let $\psi_{K,\mu}$ be the function on G defined by

$$\psi_{K,\mu}(pk) = (\mu\boldsymbol{\delta}^{1/2})(p) \qquad (p \in P = AN, \quad k \in K).$$

It lies in the space of $I(\mu)$. Put $\mu_i = \mu(\mathbf{q}^{\mathbf{e}_i})$. Suppose that $\mu_i \neq q\mu_j$ for all $i \neq j$. Put

$$c_\alpha(\mu) = \frac{1-\mu_i/\mu_j}{1-\mu_i/q\mu_j} \quad \text{if} \quad \alpha = \alpha_{ij}, \tag{7.1}$$

and

$$c_w(\mu) = \prod_\alpha c_\alpha(\mu) \qquad (\alpha > 0, \quad w\alpha < 0).$$

The Weyl group W acts on the set of roots. Suppose that $\mu_i \neq \mu_j$ for all $i \neq j$. Then for each w in W there exists a unique G-morphism $R_{w,\mu}$ from $I(\mu)$ to $I(w\mu)$ which maps $\psi_{K,\mu}$ to $\psi_{K,w\mu}$; this is the content of [C2], Theorem 3.1, where our μ is denoted by χ, our $c_w(\mu)$ is denoted by $c_w(\chi)^{-1}$ in [C2], and it is shown in [C2], (3.1), that our $R_{w,\mu}$ has the form $c_w(\chi)^{-1}T_w$ (in the notations of [C2]). The uniqueness of $R_{w,\mu}$ implies that if $w = w_t \cdots w_2 w_1$ in W, then

$$R_{w,\mu} = R_{w_t, w_{t-1}\cdots w_2 w_1 \mu} \cdots R_{w_2, w_1\mu} R_{w_1,\mu}. \tag{7.2}$$

Put $c_i(\mu)$ for $c_{s_i}(\mu)$. The action of $R_{w,\mu}$ on $V_B(\mu)$ is described in [C2], Theorem 3.4, which asserts the following

8. PROPOSITION. *For each i $(1 \leq i < n)$, put $R_i = R_{s_i,\mu}$. If $\ell(s_i w) > \ell(w)$, then*

$$R_i(\psi_w) = (1 - c_i(\mu))\psi_w + q^{-1}c_i(\mu)\psi_{s_i w}$$

and

$$R_i(\psi_{s_i w}) = c_i(\mu)\psi_w + (1 - q^{-1}c_i(\mu))\psi_{s_i w}.$$

Next we analyze in greater detail the case when G is $H = \mathrm{SL}(2,F)$. Here we put $\mathbf{m} = (m, -m)$ where m is a positive integer, $h = \mathbf{q}^{\mathbf{m}} = \begin{pmatrix} \mathbf{q}^m & 0 \\ 0 & \mathbf{q}^{-m} \end{pmatrix}$.

Note that $\boldsymbol{\delta}(h) = q^{2m}$. Let z be a nonzero complex number, and μ the unramified character of

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \qquad \text{with} \qquad \mu\left(\begin{pmatrix} \mathbf{q} & 0 \\ 0 & 1/\mathbf{q} \end{pmatrix} \right) = z.$$

Thus, if $\tilde{\mu}$ is an extension of μ to the diagonal subgroup in GL(2), then $z = \tilde{\mu}_1/\tilde{\mu}_2$ in our previous notations. The Weyl group W consists of two elements. If s denotes the nontrivial one, put c for $c_s(\mu)$; then $c = (1 - z)/(1 - z/q)$. With respect to the basis $\{\psi_1, \psi_s\}$, the matrix of

$$R = R_{s,\mu} \qquad \text{is} \qquad \begin{pmatrix} 1-c & c \\ c/q & 1-c/q \end{pmatrix}.$$

Then

$$\frac{dc}{dz} = q(1-q)/(q-z)^2 \qquad \text{and} \qquad \det R = (1-qz)/(z-q).$$

Hence

$$R^{-1} = \frac{z-q}{1-qz} \begin{pmatrix} 1-c/q & -c \\ -c/q & 1-c \end{pmatrix}, \quad R' = \frac{d}{dz} R = \frac{1-q}{(z-q)^2} \begin{pmatrix} -q & q \\ 1 & -1 \end{pmatrix},$$

and

$$R' R^{-1} = \frac{q-1}{(z-q)(qz-1)} \begin{pmatrix} -q & q \\ 1 & -1 \end{pmatrix}.$$

9. Proposition. *The matrix of the operator $\pi(\phi_{\mathbf{m}} dg)$, where $\pi = I(\mu)$ and*

$$\phi_{\mathbf{m}} = |BhB|^{-1} \boldsymbol{\delta}^{1/2}(h) \operatorname{ch}(BhB),$$

with respect to the basis $\{\psi_1, \psi_s\}$, is

$$\begin{pmatrix} z^m & (q-1)z(1-z)^{-1}(z^{-m} - z^m) \\ 0 & z^{-m} \end{pmatrix}.$$

Proof. For w, u in $W = \{1, s\}$, we are to compute

$$|B|^{-1}(\pi(\operatorname{ch}(BhB)dg)\psi_w)(u) = \sum_{x \in h^{-1}B_-h/h^{-1}B_-h \cap B_-} \psi_w(uhx).$$

If $u = s$ we obtain $|BhB|\psi_w(sh)$, which is zero if $w = 1$ and

$$|BhB|(\mu \boldsymbol{\delta}^{1/2})(shs^{-1}) \qquad \text{if} \qquad w = s.$$

If $u = 1$ we obtain

$$(\mu\boldsymbol{\delta}^{1/2})(h)\sum_x \psi_w\left(\begin{pmatrix} 1 & 0 \\ \mathbf{q}^{2m-1}x & 1\end{pmatrix}\right) \qquad (x \in R/\boldsymbol{\pi}^{2m}R).$$

Using the relation

$$\begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix} = \begin{pmatrix} 1 & 1/t \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1/t & 0 \\ 0 & t\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & 1/t \\ 0 & 1\end{pmatrix}$$

it is clear that when $w = 1$ only the term of $x = 0$ in $R/\boldsymbol{\pi}^{2m}R$ is nonzero, and we obtain $(\mu\boldsymbol{\delta}^{1/2})(h)$. When $w = s$ only the terms of $x \neq 0$ are nonzero; there are $(q-1)q^{2m-i-1}$ elements x in $R/\boldsymbol{\pi}^{2m}R$ with absolute value q^{-i} $(0 \leq i < 2m)$, and our sum becomes

$$\begin{aligned}(q-1)\sum_{i=0}^{2m-1} q^{2m-i-1}(\mu\boldsymbol{\delta}^{1/2})&\begin{pmatrix} \mathbf{q}^{1-2m}\mathbf{q}^{i} & 0 \\ 0 & \mathbf{q}^{-i}\mathbf{q}^{2m-1}\end{pmatrix} \\ &= (q-1)\sum_{i=0}^{2m-1} q^{2m-i-1}(qz)^{i+1-2m} \\ &= (q-1)z^{1-m}(1-z)^{-1}(z^{-m}-z^{m}).\end{aligned}$$

Since $(\mu\boldsymbol{\delta}^{1/2})(h) = (qz)^m$ and $|BhB|^{-1}\boldsymbol{\delta}^{1/2}(h) = q^{-m}$, the proposition follows. □

10. Corollary. *For any $m \geq 0$ we have*

$$\begin{aligned}&\operatorname{tr}[R' \cdot R^{-1} \cdot I(\mu, \phi_{\mathbf{m}}dg)] \\ &= \frac{(q-1)/z}{(z-q)(z^{-1}-q)}[z^{-m} + qz^{m} - (q-1)z(z-1)^{-1}(z^{m} - z^{-m})].\end{aligned} \tag{10.1}$$

We shall now use these computations to express the trace formula for $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2, \mathbb{A})$ in a convenient form. Thus let F be a global field, fix a nonarchimedean place u of F, fix a function f_{0v} for all $v \neq u$ such that $f_{0v} = f_{0v}^0$ for almost all v.

11. PROPOSITION. *There exists a positive integer m_0, depending on $\{f_{0v}; v \neq u\}$, with the following property. Suppose that $m \geq m_0$; f_{0u} is the function $\phi_{\mathbf{m}}$ on H_u; f_0 is $\otimes_v f_{0v}$; and x is an element of H with eigenvalues in $F^\times$. Then $f_0(x) = 0$.*

PROOF. Denote the eigenvalues of x by a and a^{-1}. If $f_0(x) \neq 0$ then $f_{0v}(x) \neq 0$ for all v, and there are $C_{0v} \geq 1$ with $C_{0v} = 1$ for almost all v such that

$$C_{0v}^{-1} \leq |a|_v \leq C_{0v} \qquad (*)_v$$

holds for all $v \neq u$. Since a lies in $F^\times$ we have $\prod_v |a|_v = 1$. Hence $(*)_u$ holds with $C_{0u} = \prod_{v \neq u} C_{0v}$. But if $f_{0u} = \phi_{\mathbf{m}}$ and $f_{0u}(x) \neq 0$ then $|a|_u = q_u^m$ or q_u^{-m}. The choice of m_0 with $q_u^{m_0} > C_{0u}$ establishes the proposition. □

We conclude that for $f_0 = \otimes_v f_{0v}$ as in Proposition 11, the group theoretic side of the trace formula consists only of orbital integrals of elliptic regular elements; weighted orbital integrals and orbital integrals of singular classes do not appear.

In the representation theoretic side of the trace formula there appears a sum of traces $\operatorname{tr} \pi_0(f_0 dh)$, described as I_0, $\operatorname{tr} \eta(f_{T_E} dt)$ in Proposition III.3.3(1), and chapter V, (1.3). There are two additional terms, denoted by S_0, S_0' in Proposition III.3.3(1). They involve integrals over the analytic manifold of unitary characters $\mu(a) = \mu_0(a)|a|^s$ (s in $i\mathbb{R}$) of $\mathbb{A}^\times/F^\times$; each connected component of this manifold is isomorphic to $\mathbb{R}$. The first term, denoted by $S_0/2$ in Proposition III.3.3(1), is

$$\frac{1}{2} \sum_{\mu_0} \int_{i\mathbb{R}} \frac{m'(\mu)}{m(\mu)} \prod_v \operatorname{tr} I_0(\mu_v; f_{0v} dh_v)|ds|. \qquad (11.1)$$

The sum ranges over a set of representatives for the connected components, $m(\mu)$ is the quotient $L(1,\mu)/L(1,\mu^{-1})$ of values of L-functions (see section III.3). Since all sums and products in the trace formula are absolutely convergent we obtain

$$\int_{|z|=1} d(z)(z^m + z^{-m})|d^\times z|. \qquad (11.1)'$$

Here $d(z)$ is an integrable functions on the unit circle $|z| = 1$ in $\mathbb{C}$. We used the fact that $\operatorname{tr} I_0(\mu_u; \phi_{\mathbf{m}} dh) = z^m + z^{-m}$, where $z = \mu_u\left(\begin{pmatrix} \mathbf{q} & 0 \\ 0 & \mathbf{q}^{-1} \end{pmatrix}\right)$.

The second term, denoted by $S'_0/2$ in Proposition III3.3(1), is the sum over all places w of the terms

$$\frac{1}{2}\sum_{\mu_0}\int_{i\mathbb{R}} \operatorname{tr}[R_w^{-1}R'_w I_0(\mu_w)](f_{0w}dh_w)\cdot\prod_{v\neq w}\operatorname{tr}[I_0(\mu_v)](f_{0v}dh_v)|ds|. \tag{11.2$_w$}$$

The summands $(11.2)_w$ which are indexed by $w \neq u$ depend on f_{0u} via $\operatorname{tr}[I_0(\mu_u)](f_{0u}dh_u) = z^m + z^{-m}$; they can be included in the expression $(11.1)'$ on changing $d(z)$ to another function with the same properties. Left is only $(11.2)_u$, in which $\operatorname{tr}[R_w^{-1}R'_w I_0(\mu_w, f_{0w}dh_{0w})]$ is given by Corollary 10.

This completes our discussion of the trace formula for $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2,\mathbb{A})$. Clearly this discussion applies also in the case of $\mathbf{H}_1(\mathbb{A}) = \mathrm{PGL}(2,\mathbb{A})$. Again we take a global measure $f_1dh_1 = \otimes_v f_{1v}dh_{1v}$ (matching, as in the statement of the Theorem), whose component $f_{1u}dh_{1u}$ at u is sufficiently regular with respect to the other components, so that the analogue of Proposition 11 holds. The group theoretic part of the trace formula for $\mathbf{H}_1(\mathbb{A})$ then consists of orbital integrals of elliptic regular elements. There appears a sum of traces $\operatorname{tr}\pi_1(f_1dh_1)$, described as $\widetilde{I}_1$ in Proposition III.3.4(1) and in chapter V, (1.3), and a term analogous to (11.1) (or $(11.1')$, denoted by $S_1/2$ in Proposition III.3.4(1), and a sum of terms of the form $(11.2)_w$ over all places w of F, which comes from the term $S'_1/2$ of Proposition III.3.4(1). Note that the contribution of $\widetilde{I}_1$ to $\mathcal{I}$ is multiplied by 1/2.

We need consider only the analogue for $\mathbf{H}_1(\mathbb{A})$ of $(11.2)_u$, since $(11.2)_w$ for $w \neq u$ can be included in $(11.1)'$. Here write z for $\mu(\mathbf{q})$, when the induced representation $I_1(\mu)$ of $\mathbf{H}_1(F_u)$ from the character $\left(\begin{smallmatrix} a & * \\ 0 & b\end{smallmatrix}\right) \mapsto \mu(a/b)$ is considered. Then

$$\mu_1 = z, \qquad \mu_2 = z^{-1} \qquad \text{and} \qquad c = (1-z^2)/(1-z^2/q)$$

in the notations of (6.1). Hence

$$\frac{dc}{dz} = 2zq(1-q)/(q-z^2)^2, \qquad \det R = (1-qz^2)/(z^2-q),$$

$$R = \begin{pmatrix} 1-c & c \\ c/q & 1-c/q\end{pmatrix}, \qquad R^{-1}R' = \frac{2z(q-1)}{(z^2-q)(1-qz^2)}\begin{pmatrix} q & -q \\ -1 & 1\end{pmatrix}$$

and

$$I_1(\phi_{1,\mathbf{m}}) = \begin{pmatrix} z^m & (q-1)z(z^m - z^{-m})/(z - z^{-1}) \\ 0 & z^{-m} \end{pmatrix}$$

where $I_1 = I_1(\mu)$ $(= I_1(z))$ and

$$\phi_{1,\mathbf{m}} = |BhB|^{-1}\boldsymbol{\delta}^{1/2}(h)\,\mathrm{ch}(BhB)$$

is the function associated with $h = \begin{pmatrix} \mathbf{q}^m & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathbf{H}_1(F_u)$. Namely we have

12. Proposition. *For every $m \geq 0$ we have*

$$\mathrm{tr}[R^{-1}R'I_1(\mu, \phi_{1,\mathbf{m}}dh_1)] = \frac{2(q-1)/z}{(z^2-q)(z^{-2}-q)}$$
$$\cdot\,[qz^m + z^{-m} - (q-1)z(z^m - z^{-m})/(z - z^{-1})]. \tag{12.1}$$

This completes our discussion of the trace formula for $\mathbf{H}_1 = \mathrm{PGL}(2)$.

Remark. The above discussion applies for any group of rank one. For example it applies also in the case of the unitary group U(3) in three variables, defined by means of a quadratic extension E/F (see [F3;IV], [F3;V] and [F3;VI]). Here we take a place u which stays prime in E, and note that the definition of $c_w(\mu)$ in the quasi-split case is different from the split case discussed here; see [C2], p. 397.

It remains to carry out analogous discussion of the twisted trace formula of $\mathbf{G}(\mathbb{A}) = \mathrm{PGL}(3, \mathbb{A})$ for a function $f = \otimes_v f_v$ as in the Theorem whose component f_u at u is sufficiently regular with respect to the other components. Again the trace formula consists of:

(1) twisted orbital integrals of σ-elliptic regular elements only, by virtue of the immediate twisted analogue of Proposition 11;

(2) discrete sum described as I in chapter III, Remark 2.1, and I', I'' in chapter III, (2.3.2) and (2.4.1), and chapter V, (1.3);

(3) an integral as in $(11.1)'$, see S of chapter III, (2.2.4);

(4) a sum over w of terms analogous to $(11.2)_w$, see S' of chapter III, (2.2.5). Note that the contribution to our formulae is $(S + S')/4$, see the line prior to (2.2.4), chapter III. Only the term at $w = u$ has to be explicitly evaluated, and we proceed to establish the suitable analogue of Corollary 10 and Proposition 12 for PGL(3), twisted by σ.

Recall that if π is a G-module we define ${}^\sigma\pi$ to be the G-module ${}^\sigma\pi(g) = \pi(\sigma g)$. A G-module π is called σ-invariant if $\pi \simeq {}^\sigma\pi$. If μ' is a character of A, put $\sigma\mu'$ for the character $\mu' \circ \sigma$ of A. Then ${}^\sigma I(\mu')$ is $I(\sigma\mu')$. We denote by $\pi(\sigma)$ the operator from $I(\mu')$ to $I(\sigma\mu')$ which maps ψ in the space of $I(\mu')$ to $\psi \circ \sigma$. In particular, when μ' is unramified, $\pi(\sigma)$ maps $\psi_{w,\mu'}$ in $V_B(\mu')$ to $\psi_{\sigma w,\sigma\mu'}$ in $V_B(\sigma\mu')$. If $I(\mu')$ is σ-invariant then the classes $[I(\mu')]$ and $[I(\sigma\mu')]$ are equal as elements of the Grothendieck group $K(G,\sigma)$, and there exists w in W with $\sigma\mu' = w\mu'$.

If $G = \mathrm{PGL}(3,F)$ and $\mu' = \sigma\mu'$ then there is a character μ of $F^\times$ such that $\mu'(\mathrm{diag}(a,b,c)) = \mu(a/c)$. Suppose in addition that μ' is unramified, and fix as a basis of $V_B(\mu') = V_B(\sigma\mu')$ the set $\psi_1 = \psi_{id}$, $\psi_2 = \psi_{(12)}$,

$$\psi_3 = \psi_{(23)}, \quad \psi_4 = \psi_{(23)(12)}, \quad \psi_5 = \psi_{(12)(23)}, \quad \psi_6 = \psi_{(13)},$$

where

$$W = \{id, (12), (23), (12)(23), (23)(12), (13)\}.$$

Then the matrix of $\pi(\sigma)$ with respect to this basis is the 6×6 matrix whose nonzero entries are equal to one and located at $(1,1)$, $(2,3)$, $(3,2)$, $(4,5)$, $(5,4)$, $(6,6)$. Here $\pi = I(\mu')$. Denote by A the matrix of $\pi(\phi_{\mathbf{m}}dg)$, with $\mathbf{m} = (1,0,0)$, with respect to our basis, and by B the matrix of $\pi(\phi_{\mathbf{m}}dg)$ with $\mathbf{m} = (1,1,0)$. Then A^n (resp. B^m) is the matrix of $\pi(\phi_{\mathbf{m}}dg)$ with $\mathbf{m} = (n,0,0)$ (resp. $\mathbf{m} = (m,m,0)$), and $A^nB^m = B^mA^n$ by Proposition 6. A direct computation, as in Proposition 9, shows that

$$A = \begin{pmatrix} z & (q-1)z & 0 & 0 & 0 & q(q-1)z \\ 0 & 1 & 0 & q-1 & 0 & 0 \\ 0 & 0 & z & (q-1)z & (q-1)z & (q-1)^2z \\ 0 & 0 & 0 & z^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & q-1 \\ 0 & 0 & 0 & 0 & 0 & z^{-1} \end{pmatrix}$$

and

$$B = \begin{pmatrix} z & 0 & (q-1)z & 0 & 0 & q(q-1)z \\ 0 & z & 0 & (q-1)z & (q-1)z & (q-1)^2z \\ 0 & 0 & 1 & 0 & q-1 & 0 \\ 0 & 0 & 0 & 1 & 0 & q-1 \\ 0 & 0 & 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & z^{-1} \end{pmatrix}.$$

Here $z = \mu(\mathbf{q})$. Proposition 7 implies that

$$A^n = \begin{pmatrix} z^n & (q-1)z\alpha(n) & 0 & (q-1)^2z\beta(n) & 0 & q(q-1)z\gamma(n) \\ 0 & 1 & 0 & (q-1)\delta(n) & 0 & 0 \\ 0 & 0 & z^n & (q-1)z\gamma(n) & (q-1)z\alpha(n) & (q-1)^2z(\gamma(n)+\beta(n)) \\ 0 & 0 & 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (q-1)\delta(n) \\ 0 & 0 & 0 & 0 & 0 & z^{-n} \end{pmatrix},$$

where $\alpha(n) = (z^n - 1)/(z-1)$; $\beta(n)$

$$= [z^n(1-z^{-1}) - (z-z^{-1}) + z^{-n}(z-1)]/(z-1)(1-z^{-1})(z-z^{-1});$$
$$\gamma(n) = (z^n - z^{-n})/(z-z^{-1}); \qquad \delta(n) = (1-z^{-n})/(1-z^{-1});$$

and

$$B^m = \begin{pmatrix} z^m & 0 & (q-1)z\alpha(m) & 0 & (q-1)^2 z\beta(m) & q(q-1)z\gamma(m) \\ 0 & z^m & 0 & (q-1)z\alpha(m) & (q-1)z\gamma(m) & (q-1)^2 z(\beta(m)+\gamma(m)) \\ 0 & 0 & 1 & 0 & (q-1)\delta(m) & 0 \\ 0 & 0 & 0 & 1 & 0 & (q-1)\delta(m) \\ 0 & 0 & 0 & 0 & z^{-m} & 0 \\ 0 & 0 & 0 & 0 & 0 & z^{-m} \end{pmatrix}.$$

In particular we conclude the following

13. Proposition. *For any* $\mathbf{m} = (m_1, m_2, m_3)$ *with* $m_1 \geq m_2 \geq m_3$ *we have*

$$\mathrm{tr}[\pi(\phi_{\mathbf{m}} dg)\pi(\sigma)] = \mu'(h_{\mathbf{m}}) + \mu'(Jh_{\mathbf{m}}J) = \mu(h_{\mathbf{m}}\sigma(h_{\mathbf{m}})) + \mu(Jh_{\mathbf{m}}\sigma(h_{\mathbf{m}})J),$$

where $h_{\mathbf{m}} = \mathbf{q}^{\mathbf{m}}$, *that is, the trace is* $= z^{m_1-m_3} + z^{m_3-m_1}$.

On the other hand it is easy to compute the twisted character $\chi = \chi_\pi$ of $\pi = I(\mu')$; see I.1.6. Recall that χ is a locally constant function on the σ-regular set of G with $\mathrm{tr}\ \pi(fdg \times \sigma) = \int f(g)\chi(g)dg$ for every locally-constant function on the σ-regular set of G. Now the twisted character χ of $\pi = I(\mu')$ is supported on the set of g in G such that $g\sigma(g)$ is conjugate to a diagonal element, where

$$\Delta(h)\chi(h) = z^{m_1-m_3} + z^{m_3-m_1} \qquad \text{at} \qquad h = h_{\mathbf{m}}.$$

Using the Weyl integration formula we conclude that

$$\mathrm{tr}[\pi(\phi_{\mathbf{m},\sigma} dg)\pi(\sigma)] = z^{m_1-m_3} + z^{m_3-m_1},$$

where $\phi_{\mathbf{m},\sigma}$ is the unique multiple of $\mathrm{ch}(Bh_{\mathbf{m}}B)$ with $F^\sigma(h_{\mathbf{m}}, \phi_{\mathbf{m},\sigma} dg) = 1$. It follows from Proposition 13 that we have

14. Proposition. *We have*

$$\phi_{\mathbf{m},\sigma} = \phi_{\mathbf{m}}\ (= \boldsymbol{\delta}^{1/2}(h_{\mathbf{m}})|Bh_{\mathbf{m}}B|^{-1}\,\mathrm{ch}(Bh_{\mathbf{m}}B)).$$

The operator $R = R((13))$ from $V_B(\mu')$ to $V_B(J\mu')$ is the product of three operators, according to (7.2). Write

$$V_B(\mu_1, \mu_2, \mu_3) \qquad \text{for} \qquad V_B(\mu') \qquad \text{if} \qquad \mu_i \qquad (i = 1, 2, 3)$$

are the parameters associated to μ' in (7.1). Then R is the product of $R_1 = R((12))$ from $V_B(z, 1, z^{-1})$ to $V_B(1, z, z^{-1})$, then $R_2 = R((23))$ to $V_B(1, z^{-1}, z)$, and then $R_3 = R((12))$ to $V_B(z^{-1}, 1, z)$. Put

$$c_1 = (1 - z)/(1 - z/q), \qquad c_2 = (1 - z^2)/(1 - z^2/q),$$

and

$$A_1 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1/q & -1/q & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1/q & 0 & -1/q & 0 \\ 0 & 0 & 0 & 1/q & 0 & -1/q \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1/q & 0 & -1/q & 0 & 0 & 0 \\ 0 & 1/q & 0 & -1/q & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1/q & -1/q \end{pmatrix}.$$

Then $R_1 = R_3 = I + c_1 A_1$ and $R_2 = I + c_2 A_2$; further, $R = R_3 R_2 R_1$. Now denote (the right side of) (10.1) by $X(z; m)$, that of (12.1) by $Y(z; m)$, and $\mathrm{tr}[R^{-1} R' A^n B^m \pi(\sigma)]$ by $Z(z; n, m)$. Then we have

15. PROPOSITION. *For every $m, n \geq 0$ we have*

$$2X(z; n + m) + Y(z; n + m) = Z(z; n, m).$$

PROOF. We proved this using the symbolic manipulation language Mathematica. The difference of the two sides of the Proposition is denoted by DIFF in the file given in the Appendix below. It takes a computer a moment to arrive at the conclusion that DIFF=0. In this Appendix we denote A_1 by A, A_2 by B, c_1 by c, c_2 by d, R_i by Ri, R^{-1} by S, $\pi(\sigma)$ by s, $\alpha(n)$, etc., by an, etc., A^n, B^m by An, Bm, $Z(z; n, m)$ by Z, $X(z; n + m)$ by X, $Y(z; n + m)$ by Y. $\square$

REMARK. The fact that $Z(z; n, m)$ depends only on $n + m$ is remarkable.

16. Corollary. *The sum of twice* $(11.2)_u$ *for* $H = \mathrm{SL}(2,F)$ *with* $(11.2)_u$ *for* $H_1 = \mathrm{PGL}(2,F)$ *is equal to the term* $(11.2)_u$ *for* $G = \mathrm{PGL}(3,F)$.

Proof. It follows from Proposition 14 that the measure $\phi_{\mathbf{m},\sigma}dg$ with $\mathbf{m} = (m+n,n,0)$ matches the measure $\phi_{(m+n,-m-n)}dh$ on $H = \mathrm{SL}(2,F)$ and the measure $\phi_{(m+n,0)}dh_1$ on $H_1 = \mathrm{PGL}(2,F)$. Using Proposition III.3.1, Proposition III.3.3(1), and Proposition III.3.4(1), we obtain that $\mathcal{I}$ of chapter III, 3.5, is equal to

$$(S+S')/4 - (S_0+S_0')/2 - (S_1+S_1')/4$$

in the notations of chapter III. The S_i' are those leading to the $(11.2)_u$ here. The corollary then follows from Proposition 15. □

The Theorem can now be proven by a standard argument, see chapter V, (1.6.2). On the one hand $\mathcal{I}$ of the Theorem is a discrete sum of the form

$$\sum_i c_i(z_i^m + z_i^{-m}) + \sum_j a_j z_j^m,$$

where z_j lies in the finite set

$$\{q, \quad q^{-1}, \quad q^{1/2}, \quad q^{-1/2}, \quad -q^{1/2}, \quad -q^{-1/2}\},$$

and z_i in $|z_i| = 1$ or $q^{-1/2} < z_i < q^{1/2}$ or $-q^{1/2} < z_i < -q^{-1/2}$. On the other hand $\mathcal{I}$ is equal to an integral of the form $(11.1)'$. Here m is a sufficiently large positive integer. The argument of chapter V, (1.6.2), implies that the coefficients c_i and a_j are zero. In particular $\mathcal{I} = 0$, and the Theorem follows. □

IV.2 Appendix: Mathematica program

Here is a Mathematica program to compute DIFF:

```
A={{−1,1,0,0,0,0},{1/q,−1/q,0,0,0,0},{0,0,−1,0,1,0},
   {0,0,0,−1,0,1},{0,0,1/q,0,−1/q,0},{0,0,0,1/q,0,−1/q}};
B={{−1,0,1,0,0,0},{0,−1,0,1,0,0},{1/q,0,−1/q,0,0,0},
   {0,1/q,0,−1/q,0,0},{0,0,0,0,−1,1},{0,0,0,0,1/q,−1/q}};
c=(1−z)/(1−z/q);
```

```
d=(1−z ∧ 2)/(1−z ∧ 2/q);
h=IdentityMatrix[6];
R1=Together[h+c˜ A];
R2=Together[h+d˜ B];
R=Together[R1.(R2.R1)];
R′=Together[D[R,z]];
S1=Together[Inverse[R1]];
S2=Together[Inverse[R2]];
S=Together[S1.(S2.S1)];
s={{1,0,0,0,0,0},{0,0,1,0,0,0},{0,1,0,0,0,0},
   {0,0,0,0,1,0},{0,0,0,1,0,0},{0,0,0,0,0,1}};
T1=Together[(s.S).R′];
an=(z ∧ n−1)/(z−1);
am=(z ∧ m−1)/(z−1);
bn=(z ∧ n (1−1/z)−(z−1/z)+(1/z ∧n)(z−1))/((z−1)(1−1/z)(z−1/z));
bm=(z ∧ m (1−1/z)−(z−1/z)+(1/z ∧m)(z−1))/((z−1)(1−1/z)(z−1/z));
cn=(z ∧ n−1/z ∧ n)/(z−1/z);
cm=(z ∧ m−1/z ∧ m)/(z−1/z);
dn=(1−1/z ∧ n)/(1−1/z);
dm=(1−1/z ∧ m)/(1−1/z);
An={{z ∧ n,(q−1) z an,0,(q−1) ∧ 2 z bn,0,q (q−1) z cn},
   {0,1,0,(q−1) dn,0,0},
   {0,0,z ∧ n,(q−1) z cn, (q−1) z an, (q−1) ∧2 z (cn+bn)},
   {0,0,0,1/z ∧ n,0,0},{0,0,0,0,1,(q−1) dn}, {0,0,0,0,0,1/z ∧ n}};
Bm={{z ∧ m,0,(q−1) z am,0,(q−1) ∧ 2 z bm,q (q−1) z cm},
   {0,z ∧ m,0,(q−1) z am,(q−1) z cm,(q−1) ∧ 2 z (bm+cm)},
   {0,0,1,0,(q−1) dm,0},{0,0,0,1,0,(q−1) dm},
   {0,0,0,0,1/z ∧ m,0},{0,0,0,0,0,1/z ∧ m}};
T=Together[T1.(An.Bm)];
Z=Simplify[Sum[T[[i,i]],{i,6}]];
X=(1−q)(1/z ∧ (n+m)+q z ∧ (n+m)−(q−1) z (z ∧ (n+m)
   −1/z ∧ (m+n))/(z−1))/((q−z)(1−z q));
Y=2(1−q)z(q z ∧ (m+n)+1/z ∧ (m+n)−(q−1) z (z ∧ (m+n)
   −1/z ∧ (m+n))/(z−1/z))/((q−z ∧ 2)(1−q z ∧ 2));
DIFF=Factor[PowerExpand[Simplify[Z−(2 X+Y)]]]
```

V. APPLICATIONS OF A TRACE FORMULA

Summary. In this chapter the existence of the symmetric-square lifting of admissible and of automorphic representations from the group SL(2) to the group PGL(3) is proven. Complete local results are obtained, relating the character of an SL(2)-packet with the twisted character of a self-contragredient PGL(3)-module. The global results include introducing a definition of packets of cuspidal representations of $\mathrm{SL}(2,\mathbb{A})$ and relating them to self-contragredient automorphic $\mathrm{PGL}(3,\mathbb{A})$-modules which are not induced $I(\pi_1)$ from a discrete-spectrum representation π_1 of the maximal parabolic subgroup with trivial central character. The sharp results, which concern SL(2) rather than GL(2), are afforded by the usage of the trace formula. The surjectivity and injectivity of the correspondence implies that any self-contragredient automorphic $\mathrm{PGL}(3,\mathbb{A})$-module as above is a lift, and that the space of cuspidal $\mathrm{SL}(2,\mathbb{A})$-modules admits multiplicity one theorem and rigidity ("strong multiplicity one") theorem for packets (and not for individual representations).

V.1 Approximation

1.1 Discrete spectrum. Let $\mathbf{G}$ be a reductive group over a number field F with an anisotropic center. Let dg be a Haar measure on $\mathbf{G}(\mathbb{A})$. Let $L = L^2(G\backslash\mathbf{G}(\mathbb{A}))$ denote the space of square-integrable complex valued functions φ on $G\backslash\mathbf{G}(\mathbb{A})$ which are right smooth. The group $\mathbf{G}(\mathbb{A})$ acts on L by $(r(g)\varphi)(h) = \varphi(hg)$. An *automorphic* representation is an irreducible $\mathbf{G}(\mathbb{A})$-invariant subquotient, of the $\mathbf{G}(\mathbb{A})$-module L. The theory of Eisenstein series decomposes L as a direct sum of the discrete spectrum L_d, which is the sum of all irreducible submodules in L, and the continuous spectrum L_c. The continuous spectrum L_c is a direct integral of induced representations.

The space L_d decomposes as a direct sum with finite multiplicities of irreducible inequivalent representations, called *discrete spectrum*. Denote by L_0 the subspace of all cuspidal functions φ in L. Then L_0 is a $\mathbf{G}(\mathbb{A})$-submodule of L_d. Its irreducible constituents are called *cuspidal*.

Every irreducible admissible representation of $\mathbf{G}(\mathbb{A})$ factors as a restricted product $\pi = \otimes_v \pi_v$ over all primes v of local admissible irreducible representations π_v. This means that for almost all places π_v is unramified, namely has a nonzero $K_v = \mathbf{G}(R_v)$-fixed vector ξ_v^0, necessarily unique up to scalar. For all v the component π_v is admissible. The space of π is spanned by the products $\otimes_v \xi_v$, $\xi_v \in \pi_v$ for all v, $\xi_v = \xi_v^0$ for almost all v.

Put $\mathbf{G} = \mathrm{PGL}(3)$, $\mathbf{H} = \mathbf{H}_0 = \mathrm{SL}(2)$, $\mathbf{H}_1 = \mathrm{PGL}(2)$. The discrete-spectrum representations of any of these groups are cuspidal or one-dimensional automorphic representations. The notion of local lifting for unramified representations with respect to the dual groups homomorphisms $\lambda_0 \colon \widehat{H} \to \widehat{G}$, $\lambda_1 \colon \widehat{H}_1 \to \widehat{G}$ is defined in section I.1. We shall generalize this definition to deal with any local representation on formulating it in terms of characters. We shall write $\pi_v = \lambda_i(\pi_{iv})$ when π_{iv} lifts to π_v with respect to λ_i, once the notion is defined.

1.1.1 Normalization. Let π be a σ-*invariant* representation of $\mathbf{G}(\mathbb{A})$. Namely π is equivalent to the representation ${}^\sigma\pi(g) = \pi(\sigma g)$ of $\mathbf{G}(\mathbb{A})$. Then there exists an intertwining operator A on the space of π with $A\pi(g)A^{-1} = \pi(\sigma g)$ for all g in $\mathbf{G}(\mathbb{A})$. Assume that π is irreducible. Then by Schur's lemma the operator A^2, which intertwines π with itself, is a scalar which we normalize to be equal to 1. This specifies A up to a sign.

Fix a nontrivial additive character $\boldsymbol{\psi}$ of $\mathbb{A} \bmod F$. Denote by ψ the character of the upper triangular unipotent subgroup $\mathbf{N}(\mathbb{A})$, defined by $\psi(n) = \boldsymbol{\psi}(x+z)$, where

$$n = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $\psi(\sigma n) = \psi(n)$. Assume that π is *generic*, or realizable in the space of Whittaker functions. Namely there is a $\mathbf{G}(\mathbb{A})$-equivariant map $Y : \{W\} \to \pi$ onto π from the space of (Whittaker) functions W on $\mathbf{G}(\mathbb{A})$. These W satisfy $W(ngk) = \psi(n)W(g)$ for all g in $\mathbf{G}(\mathbb{A})$, n in $\mathbf{N}(\mathbb{A})$, and k in a compact open subgroup of $G(\mathbb{A})$, depending on W. $G(\mathbb{A})$ acts by $(\omega(g)W)(h) = W(hg)$. Then ${}^\sigma\pi$ is generic since $Y_\sigma : \{W\}^\sigma \to \pi$ by $Y_\sigma(W) = Y({}^\sigma W)$ is onto and $\mathbf{G}(\mathbb{A})$-equivariant:

$$Y_\sigma(\omega(g)W) = Y({}^\sigma(\omega(g)W)) = Y(\omega(\sigma g){}^\sigma W) = {}^\sigma\pi(g)Y({}^\sigma W).$$

We take A to be the operator on the space of π which maps $Y(W)$ to $Y({}^\sigma W)$.

This gives a normalization of the intertwining operator A on the generic representations, which is also local in the following sense. Each component π_v of $\pi = \otimes_v \pi_v$ is generic, thus there is a G_v-equivariant map Y_v onto π_v from the space of Whittaker functions W_v (which satisfy

$$W_v(n_v g_v k_v) = \psi_v(n_v) W_v(g_v),$$

where ψ_v is the restriction of ψ to $N_v = \mathbf{N}(F_v)$). Moreover, each W is a finite linear combination of products $\otimes_v W_v$; where for almost all v the component W_v is the (unique up to a scalar multiple) unramified (i.e., right $K_v = \mathbf{G}(R_v)$-invariant) Whittaker function W_v^0. In fact Y_v is $W_v \mapsto Y(W_v \otimes \otimes_{u \neq w} W_u)$ where W_u $(u \neq v)$ are fixed, $W_u = W_u^0$ at almost all u, such that $Y_v \neq 0$.

Now we can write A as a product $\otimes_v A_v$ over all places, where A_v is the operator intertwining π_v with ${}^\sigma\pi_v$, which maps $Y(W_v)$ to $Y({}^\sigma W_v)$. This is the normalization of the local operators used below. We put $\pi_v(\sigma) = A_v$, and $\pi_v(f_v dg_v \times \sigma)$ for the operator $\pi_v(f_v dg_v)A_v$ when π_v is a generic representation. Moreover, if π is normalizedly induced $I(\tau)$ from a generic representation of a parabolic subgroup and τ is σ-invariant, then the induction functor I defines $A_\pi = I(A_\tau)$.

In the special case when π_v is unramified, there exists a unique Whittaker function W_v^0 in the space of π_v with respect to ψ_v (provided ψ_v is unramified), with $W_v^0(k_v) = 1$ for k_v in $K_v = \mathbf{G}(R_v)$. It is mapped by $\pi_v(\sigma) = A_v$ to ${}^\sigma W_v^0$, which satisfies ${}^\sigma W_v^0(k_v) = 1$ for all k_v in K_v since K_v is σ-invariant. Namely A_v maps the unique K_v-fixed vector W_v^0 in the space of π_v to the unique K_v-fixed vector ${}^\sigma W_v^0$ in the space of ${}^\sigma\pi_v$, and we have ${}^\sigma W_v^0 = W_v^0$.

Hence A_v acts as the identity on the K_v-fixed vectors, and our local normalization coincides (for generic unramified representations) with the one used in the study of spherical functions in section I.1.2.

We take $\pi(\sigma)$ to be the identity if π is the (nongeneric) trivial representation of $\mathbf{G}(\mathbb{A})$. If $\pi = I(\mathbf{1}; \mathbf{P}(\mathbb{A}), \mathbf{G}(\mathbb{A}))$

$$= \{\phi\colon \mathbf{G}(\mathbb{A}) \to \mathbb{C}; \phi(pg) = \boldsymbol{\delta}_P^{1/2}(p)\phi(g), g \in \mathbf{G}(\mathbb{A}), p \in \mathbf{P}(\mathbb{A})\}$$

is the $\mathbf{G}(\mathbb{A})$-module normalizedly induced from the trivial representation $\mathbf{1}$ of the maximal parabolic subgroup $\mathbf{P}(\mathbb{A})$ of $\mathbf{G}(\mathbb{A})$ of type $(2,1)$ ($\boldsymbol{\delta}_P$ is the modular function of P), then the conjugate representation ${}^{\sigma}\pi$ is the induced $I(\mathbf{1}; {}^{\sigma}\mathbf{P}(\mathbb{A}), \mathbf{G}(\mathbb{A}))$ from the trivial representation of the parabolic ${}^{\sigma}\mathbf{P}(\mathbb{A})$ of type $(1,2)$. In this case we define $\pi(\sigma)$ by $(\pi(\sigma)\phi)(g) = \phi(\sigma g)$.

1.2 (Quasi) lifting. The automorphic representation $\pi_i = \otimes_v \pi_{iv}$ of $\mathbf{H}_i(\mathbb{A})$ (*quasi-*)*lifts* to the automorphic representation $\pi = \otimes_v \pi_v$ of $\mathbf{G}(\mathbb{A})$ if $\pi_v = \lambda_i(\pi_{iv})$ for (almost) all v.

1.2.1 Case of $\lambda_1(\pi_1) = I(\pi_1, 1)$. Let $\pi_1 = \otimes_v \pi_{1v}$ be an automorphic representation of $\mathbf{H}_1(\mathbb{A})$. Let $\pi = \otimes_v \pi_v$ be the representation $I(\pi_1, 1)$ of $\mathbf{G}(\mathbb{A})$ normalizedly induced from the representation $\pi_1 \times 1$ of its maximal parabolic subgroup $\mathbf{P}(\mathbb{A}) = \mathbf{M}(\mathbb{A})\mathbf{N}(\mathbb{A})$. Note that the Levi factor $\mathbf{M}(\mathbb{A})$ of $\mathbf{P}(\mathbb{A})$ is isomorphic to $\mathrm{GL}(2, \mathbb{A})$ and π_1 defines a representation of $\mathbf{M}(\mathbb{A})$ which is trivial on the center. Then π is irreducible, and also σ-invariant, since (1) ${}^{\sigma}\pi$ is the representation $I(\check{\pi}_1)$ induced from the contragredient $\check{\pi}_1$ of π_1, (2) $\check{\pi}_1$ is equivalent to π_1, being a representation of $\mathbf{H}_1(\mathbb{A}) = \mathrm{PGL}(2, \mathbb{A})$. We have that π_1 quasilifts to π by virtue of I.1.8 and I.3.10.

1.2.2 Case of $\lambda_0(\{\pi_0(\mu')\}) = I(\pi(\mu''), \chi_E)$, $\mu''(z) = \mu'(z/\overline{z})$. Let F be a local or global field. Let E be a quadratic extension of F. Put C_E for the Weil group $W_{E/E}$ (it is isomorphic to $E^\times$ if E is local, and to $\mathbb{A}_E^\times/E^\times$ if E is global). Put C_E^1 for the kernel of the norm map from C_E to C_F. Similarly we have E^1 and $\mathbb{A}_E^1$. Note that $\mathbb{A}_E^1/E^1 \simeq C_E^1$. The Weil group $W_{E/F}$ is an extension of $\mathrm{Gal}(E/F)$ by C_E. The sequence $1 \to W_{E/E} \to W_{E/F} \to \mathrm{Gal}(E/F) \to 1$ is exact. This $W_{E/F}$ can be described as the group generated by the z in C_E and τ with τ^2 in $C_F - N_{E/F}C_E$, under the relation $\tau z = \overline{z}\tau$; the bar indicates the action of the nontrivial element of $\mathrm{Gal}(E/F)$.

Let μ^* be a character of C_E. The two-dimensional induced representation $\mathrm{Ind}_E^F(\mu^*) = \mathrm{Ind}(\mu^*; W_{E/E}, W_{E/F})$ of $W_{E/F}$ in $\mathrm{GL}(2, \mathbb{C})$ can be realized as

$$W_{E/E} \ni z \mapsto \begin{pmatrix} \mu^*(z) & 0 \\ 0 & \mu^*(\overline{z}) \end{pmatrix} \times z, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ \mu^*(\tau^2) & 0 \end{pmatrix} \times \tau.$$

The image $\mathrm{Ind}_E^F(\mu')_0$ of $\mathrm{Ind}_E^F(\mu^*)$ in the dual group $\widehat{H} = \mathrm{PGL}(2, \mathbb{C})$ of $\mathbf{H} = \mathrm{SL}(2)$ is a projective two-dimensional representation. It depends only on the restriction μ' of μ^* of C_E^1.

Denote by χ_E the nontrivial (quadratic) character of C_F whose kernel is $N_{E/F}C_E$.

If F is local and $\mu^* = \overline{\mu}^*$ ($\overline{\mu}^*$ is the character defined by $\overline{\mu}^*(z) = \mu^*(\overline{z})$ for all $z \in C_E$), then there is a character μ of C_F with $\mu^*(z) = \mu(Nz)$ ($Nz = z\overline{z}$). We define the representation $\pi(\mu^*)$ of $\mathrm{GL}(2, F)$ associated with μ^* — or rather with $\mathrm{Ind}_E^F(\mu^*)$ — to be the induced representation $I(\mu, \mu\chi_E)$. In this case, where $\mu'(z/\overline{z}) = (\mu^*(z/\overline{z}) =)1$, we define the packet $\{\pi_0\} = \{\pi_0(\mu')\}$ of representations of $H = \mathrm{SL}(2, F)$ associated with $\mathrm{Ind}_E^F(\mu')_0$ to be the set of irreducible subquotients of the representation $I_0(\chi_E)$ normalizedly induced from the character $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \chi_E(a)$ of the Borel subgroup. This is the restriction of $I(\mu, \mu\chi_E)$ to H. It consists of two elements. In this case $\{\pi_0(\mu')\}$ is independent of μ^* since μ^* is trivial on C_E^1. The dependence of $\{\pi_0(\mu')\}$ on $\mu' = 1$ on C_E^1 is via E, that is χ_E.

If F is global, for almost all places v of F the character μ' is unramified, and then at an inert v we have $\mu'_v = 1$ on E_v^1. At v which splits in E/F the restriction of $\mathrm{Ind}_E^F(\mu^*)$ to W_{E_v/F_v} is a direct sum of two characters: μ_{1v}, μ_{2v}. This defines a representation $\pi(\mu_v^*) = I(\mu_{1v}, \mu_{2v})$ of $\mathrm{GL}(2, F_v)$ induced from the Borel subgroup. We denote by $\{\pi_0(\mu'_v)\}$ the set of constituents in the restriction of $I(\mu_{1v}, \mu_{2v})$ to $H_v = \mathrm{SL}(2, F_v)$. We shall denote by $\pi_0(\mu')$ (resp. $\pi(\mu')$) any discrete-spectrum automorphic representation of $\mathrm{SL}(2, \mathbb{A})$ (resp. $\mathrm{GL}(2, \mathbb{A})$) whose components for almost all v are in the above $\{\pi_0(\mu'_v)\}$ (resp. $\pi(\mu_v^*)$).

Applying the map $\lambda_0 = \mathrm{Sym}^2$ to $\mathrm{Ind}_E^F(\mu')_0$, we get the representation

$$z \mapsto \mathrm{diag}(\mu'(z/\overline{z}), 1, \mu'(\overline{z}/z)) \times z, \qquad \tau \mapsto \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix} \times \tau,$$

of $W_{E/F}$ in $\widehat{G} = \mathrm{SL}(3, \mathbb{C})$. It is the direct sum of the two-dimensional representation

$$\mathrm{Ind}_E^F(\mu'') = \mathrm{Ind}(\mu''; W_{E/E}, W_{E/F})$$

and the one-dimensional representation $x \mapsto \chi_E(x)$ of $W_{F/F}$, where we put $\mu''(z) = \mu'(z/\overline{z})$ ($z \in C_E$) and again χ_E is the quadratic character of $W_{F/F}$ associated with the quadratic extension E/F by class field theory.

This direct sum parametrizes the representation π of $\mathbf{G}(\mathbb{A})$ induced from the representation $\pi^* \times \chi$ of a maximal parabolic P, if there exists a $\mathrm{GL}(2, \mathbb{A})$-module $\pi^* = \pi^*(\mu'')$. The representation π is σ-invariant, since ${}^\sigma\pi$ is the representation induced from $\check{\pi}^* \times \chi^{-1}$. But χ is of order two, and for our π^* of the form $\pi^*(\mu'')$, the contragredient $\check{\pi}^*$ is $\pi^*\chi \simeq \pi^*$. It follows from I.1.8 that π_0 quasilifts to π.

Note that $\mathrm{Ind}_E^F(\mu'')$ is reducible precisely when $\mu'' = \overline{\mu}''$ $(= \mu''^{-1})$, equivalently: $\mu''^2 = 1$. In this case there is μ on C_F, $\mu^2 = 1$, with $\mu''(z) = \mu(z\overline{z})$, and $\mathrm{Ind}_E^F(\mu'') = \mu \oplus \mu\chi_E$, and $\pi^*(\mu'') = I(\mu, \mu\chi_E)$.

More generally, if π_0 is an automorphic representation (or rather its "packet", to be defined below) which conjecturally corresponds to a map $\rho\colon W_F \to \widehat{H}$, and π is one parametrized by the composition $\lambda_0 \circ \rho$ of ρ and $\lambda_0\colon \widehat{H} \to \widehat{G}$, then it is clear that π_0 quasilifts to π upon restricting ρ to the local Weil groups W_{F_v}. But it is not clear that given π_0, there exists such π which is the quasilift of π_0. For this we need to use the trace formula, which yields also local lifting at all places and global lifting.

1.3 Trace formula. To formulate the identity of traces of σ-invariant representations in $L^2(G\backslash \mathbf{G}(\mathbb{A}))$, and traces of representations in the spaces $L^2(H\backslash \mathbf{H}(\mathbb{A}))$ and $L^2(H_1\backslash \mathbf{H}_1(\mathbb{A}))$, with which we study the lifting, we now describe the terms which appear in it.

$$I = \sum_\pi m(\pi) \prod_v \operatorname{tr} \pi_v(f_v dg_v \times \sigma).$$

This sum is taken over a set of representatives for the equivalence classes of discrete-spectrum representations $\pi = \otimes_v \pi_v$ of $\mathbf{G}(\mathbb{A})$, and $m(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{G}(\mathbb{A})}(\pi, L_d)$ is the multiplicity of π in the discrete spectrum L_d. Multiplicity one theorem for $\mathrm{GL}(3, \mathbb{A})$ asserts that $m(\pi) = 1$ for all π. For almost all v the component π_v is unramified.

$$I' = \sum_E \sum_\tau \prod_v \operatorname{tr} I_v((\tau_v, \chi_{E_v}); f_v dg_v \times \sigma).$$

Here the first sum is over all quadratic extensions E of F, and χ_E denotes the quadratic character of $F^\times\backslash \mathbb{A}^\times$ whose kernel is $N_{E/F}(\mathbb{A}_E^\times)$. The second sum is over all cuspidal representations τ of $\mathrm{GL}(2, \mathbb{A})$ with $\tau \simeq \check{\tau} (= \chi_E \tau)$.

$$I'' = \sum_\eta \prod_v \operatorname{tr} I_v(\eta; f_v dg_v \times \sigma).$$

The sum is over the unordered triples $\eta = \{\chi, \xi\chi, \xi\}$, where χ, ξ are characters of $W_{F/F} = \mathbb{A}^\times/F^\times$ of order 2 (not 1), and $\chi \neq \xi$.

$$I_1 = \sum_{\pi_1} \prod_v \operatorname{tr} \pi_1(f_{1v} dh_{1v}),$$

and

$$I_1' = \frac{1}{2}\sum_{\pi_1}\prod_v \operatorname{tr} I_v((\pi_{1v},1); f_v dg_v \times \sigma).$$

Both sums extend over a set of representatives for the equivalence classes of the discrete-spectrum representations π_1 of $\mathbf{H}_1(\mathbb{A}) = \mathrm{PGL}(2,\mathbb{A})$. Multiplicity one implies that $m(\pi_1) = 1$, namely that each equivalence class consists of a single representation.

$$I_0 = \sum_{\pi_0} m(\pi_0)\prod_v \operatorname{tr}\pi_{0v}(f_{0v}dh_v).$$

The sum ranges over a set of representatives for the equivalence classes of the discrete-spectrum representations π_0 of $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2,\mathbb{A})$. They occur with finite multiplicities $m(\pi_0)$.

$$\frac{1}{4}\sum_E I_E'' - \frac{1}{2}\sum_E I_E' - \frac{1}{4}\sum_E I_E.$$

Here $I_E'' = \operatorname{tr} M(\chi_E)I_0(\chi_E, f_0 dh)$,

$$I_E' = \sum_{\mu'\neq\overline{\mu}'}{}' \ \mu'(f_{T_E}dt), \qquad I_E = \sum_{\mu'=\overline{\mu}'}{}' \ \mu'(f_{T_E}dt),$$

where $\sum'$ means here a sum over a set of representatives of equivalence classes $\mu' \sim \overline{\mu}'$.

Fix a representation π_v of G_v for almost all v. The rigidity theorem for $\mathrm{GL}(3,\mathbb{A})$ of [JS] implies that each of I, I_1', I' and I'' consists of at most one entry π with the above components for almost all v, and, moreover, at most one of the four terms has such a nonzero entry.

1.4 Lemma. *Let F be a local field. Suppose $\pi = I(\pi',\chi)$ is a σ-invariant representation of $\mathrm{PGL}(3,F)$ induced from a maximal parabolic subgroup, where π' is a square-integrable representation of the 2×2 factor and χ is a character. Then either $\chi = 1$ and π' is a representation π_1 of $H_1 = \mathrm{PGL}(2,F)$, or χ is a character of order 2, π' has central character χ, and $\pi' \simeq \check{\pi}'$ $(= \chi\pi')$.*

Remark. The lemma and its proof are valid also in the case where F is global and π is an automorphic representation of $\mathbf{G}(\mathbb{A}) = \mathrm{PGL}(3,\mathbb{A})$.

Proof. By definition of induction, ${}^{\sigma}\pi$ is $I(\check{\pi}', \chi^{-1})$, where $\check{\pi}'$ is the contragredient of π'. Since $I(\pi', \chi)$ is tempered, the square-integrable data (π', χ) is uniquely determined. Hence, as $I(\pi', \chi)$ is equivalent to $I(\check{\pi}', \chi^{-1})$, our π' is equivalent to $\check{\pi}'$ and $\chi = \chi^{-1}$. The central character of π' is $\chi = \chi^{-1}$ since π is a representation of $\mathrm{PGL}(3, F)$. If $\chi = 1$ then π' is a representation π_1 of $\mathrm{GL}(2, F)$ with trivial central character. If $\chi \neq 1$, since $\check{\pi}' = \chi\pi'$ we have $\pi' = \chi\pi'$. □

1.5 Regularity. Let F be a nonarchimedean local field, n a positive integer, μ a unitary character of $R^{\times}$, hence of $\mathbf{A}_0(R) = \{\mathrm{diag}(a, a^{-1}); |a| = 1\}$. We write H, G for $\mathbf{H}(F)$, $\mathbf{G}(F)$, etc. Recall that we write $\Phi(\gamma, f_0 dh_0)$ for the orbital integral of $f_0 dh_0$ at γ, and $F(\gamma, f_0 dh_0)$ for $\Delta_0(\gamma)\Phi(\gamma, f_0 dh_0)$. Let $\boldsymbol{\pi}$ be a generator of the maximal ideal in the ring R of integers in F.

Definition. Let S be the open closed set of γ in H which are conjugate to $\begin{pmatrix} a\boldsymbol{\pi}^n & 0 \\ 0 & a^{-1}\boldsymbol{\pi}^{-n} \end{pmatrix}$ in H, where a lies in $R^{\times}$. The function f_0 is called *regular* of type (n, μ) if f_0 is supported on S and

$$F(\mathrm{diag}(a\boldsymbol{\pi}^n, a^{-1}\boldsymbol{\pi}^{-n}), f_0 dh) = \mu(a)^{-1}$$

for every a in $R^{\times}$. When $\mu = 1$ we say that f_0 is regular of type n.

Analogous definition applies to f_1 and f. For example, we say that f is regular of type (n, μ) if the value of f at δ in G is zero unless δ is σ-conjugate to $\mathrm{diag}(a\boldsymbol{\pi}^n, 1, 1)$, and then

$$F^{\sigma}(\mathrm{diag}(a\boldsymbol{\pi}^n, 1, 1), f dg) = \mu(a)^{-1}.$$

1.5.1 Modules of coinvariants [BZ2]. Let (π, V) be an admissible G-module, N the upper triangular subgroup, V_N the quotient of V by the span of $n \cdot v - v$ (n in N, v in V). It is an A-module, as A normalizes N. The associated representation of A is denoted by ${}'\pi_N$, and we put

$$\pi_N = \boldsymbol{\delta}^{-1/2}\, {}'\pi_N, \qquad \text{where} \qquad \boldsymbol{\delta}(\mathrm{diag}(a, b, c)) = |a/c|^2.$$

It is an admissible representation, studied in [BZ2]. The function $\boldsymbol{\delta}$ is introduced to preserve unitarity ([BZ2], p. 444, last line). Since π is σ-invariant and N is σ-invariant, V_N is an $A \rtimes \langle\sigma\rangle$-module, and π_N is a σ-invariant representation of A. Its character on $A \times \sigma$ is denoted by $\chi^{\sigma}(\pi_N)$ (or $\chi^{\sigma}_{\pi_N}$), so that

$$\mathrm{tr}\,\pi_N(f da \times \sigma) = \int_A f(a)(\chi^{\sigma}(\pi_N))(a)\, da$$

for any smooth compactly supported function f on A. If π_i are all of the irreducible subquotients of π_N (repeated with multiplicities) which are equivalent to their σ-conjugates, then $\chi^\sigma(\pi_N) = \sum_i \chi^\sigma(\pi_i)$. The Deligne-Casselman theorem [C1] generalizes to our twisted case, and asserts that $\chi^\sigma_\pi(\delta) = \chi^\sigma({}'\pi_N)(\delta)$ (these are the unnormalized characters). Hence

$$(\Delta\chi^\sigma_\pi)(\delta) = (\chi^\sigma(\pi_N))(\delta) \qquad \text{for} \qquad \delta = \operatorname{diag}(ab,1,b) \qquad \text{with} \qquad |a|<1.$$

Similar definitions hold for representations π_0 of H. Again N is the upper triangular subgroup (of H), ${}'\pi_{0N}$ is defined as above and so is π_{0N}, where $\boldsymbol{\delta}(\operatorname{diag}(a,a^{-1})) = |a|^2$. The Theorem of [C1], which is stated for the unnormalized characters, implies that

$$(\Delta_0\chi_{\pi_0})(\gamma) = (\chi(\pi_{0N}))(\gamma) \qquad \text{at} \quad \gamma = \operatorname{diag}(a,a^{-1}) \quad \text{with} \quad |a|<1.$$

For any measure f_0dh on H, where $dh = \boldsymbol{\delta}^{-1}(a)dndadk = dadndk$, put

$$f_{0N}(\gamma) = \boldsymbol{\delta}^{1/2}(\gamma)\int_{\mathbf{H}(R)}\int_N f_0(k^{-1}\gamma nk)\,dn\,dk.$$

1.5.2 Computation. Let μ be a character of $F^\times$. The space of an induced representation $I_0(\mu)$ of $H = \mathrm{SL}(2,F)$ consists of all smooth $\varphi : H \to \mathbb{C}$ with $\varphi(n\operatorname{diag}(a,a^{-1})k) = |a|\mu(a)\phi(k)$ (here $\boldsymbol{\delta}(\operatorname{diag}(a,a^{-1})) = |a/a^{-1}| = |a|^2$). It is reducible when $\mu = \nu^{-1}$ ($\nu(a) = |a|$), where the composition series is described by the exact sequence $0 \to \mathbf{1} \to I_0(\nu^{-1}) \to \mathrm{sp} \to 0$, where $\mathbf{1}$ denotes the trivial representation of H and sp the Steinberg (or special) representation of H; or $\mu = \nu$, where $0 \to \mathrm{sp} \to I_0(\nu) \to \mathbf{1} \to 0$ is exact; or μ has order precisely two, where $I_0(\mu)$ is tempered, equal to the direct sum of the irreducible representations $I_0^+(\mu)$ and $I_0^-(\mu)$ of H.

Let f_0 be a regular function of type (n,μ), and π_0 an irreducible representation of H. Then, using the Weyl integration formula (see I.3.5), we have

$$\begin{aligned}\operatorname{tr}\pi_0(f_0dh) = \operatorname{tr}\pi_{0N}(f_{0N}da) &= \frac{1}{2}\int_{A_0}\chi(\pi_{0N})(a)F(a,f_0dh)\,da \\ &= \int_{\mathbf{A}_0(R)}\chi(\pi_{0N})(\operatorname{diag}(a\boldsymbol{\pi}^n, a^{-1}\boldsymbol{\pi}^{-n}))\mu^{-1}(a)\,da.\end{aligned}$$

If μ is ramified, that is, $\mu \neq 1$, then $\operatorname{tr}\pi_0(f_0dh)$ vanishes unless π_0 is a subquotient of the induced representation $I_0(\mu_1)$ of H, in the notations of

I.3.10, where μ_1 is a character of $A_0 \simeq F^\times$ with $\mu_1 = \mu$ on $\mathbf{A}_0(R) \simeq R^\times$. Then

$$(\chi(\pi_{0N}))(\mathrm{diag}(a, a^{-1})) = \mu_1(a) + \mu_1(a^{-1}),$$

and $\operatorname{tr} \pi_0(f_0 dh)$ is equal to $\mu_1(\boldsymbol{\pi}^n)$ if $\mu^2 \neq 1$ on $\mathbf{A}_0(R)$. If $\mu^2 = 1$ but $\mu_1^2 \neq 1$ then $I_0(\mu_1)$ is irreducible and $\operatorname{tr} \pi_0(f_0 dh)$ is equal to $z^n + z^{-n}$, where $z = \mu_1(\boldsymbol{\pi}^n)$. If $\mu_1^2 = 1$ but $\mu_1 \neq 1$ then $I_0(\mu_1)$ is reducible and $\operatorname{tr} \pi_0(f_0 dh) = \mu_1(\boldsymbol{\pi}^n)$ for any of the two constituents π_0 of $I_0(\mu_1)$.

Suppose that $\mu = 1$. In this case, if $\operatorname{tr} \pi_0(f_0 dh) \neq 0$ then π_0 is a constituent of $I_0(\mu_1)$ where μ_1 is unramified. Hence π_0 has a nonzero vector fixed under the action of an Iwahori subgroup, by [Bo3], Lemma 4.7. We have

$$\operatorname{tr} I_0(\mu_1; f_0 dh) = \mu_1(\boldsymbol{\pi}^n) + \mu_1(\boldsymbol{\pi}^n)^{-1},$$

and this is the value of $\operatorname{tr} \pi_0(f_0 dh)$ when $I_0(\mu_1)$ is irreducible. Reducibility occurs when $z = \mu_1(\boldsymbol{\pi})$ is equal to $q = |\boldsymbol{\pi}|^{-1}$, q^{-1} or -1. If $z = q$ or q^{-1}, then the composition series of $I_0(\mu_1)$ consists of the trivial representation $\mathbf{1}$ and the special representation sp. Then $\operatorname{tr} \mathbf{1}(f_0 dh) = q^n$ and $\operatorname{tr} \mathrm{sp}(f_0 dh) = q^{-n}$. If $z = -1$ then $I_0(\mu_1)$ is the direct sum of two irreducibles π_0, and $\operatorname{tr} \pi_0(f_0 dh) = (-1)^n$ for each of them.

1.5.3 Twisted computation. Let f be a regular function of type (n, μ), and π a σ-invariant irreducible representation of G. The twisted Weyl integration formula (see I.3.5) implies that

$$\operatorname{tr} \pi(f dg \times \sigma) = \int_{R^\times} (\chi(\pi_N))(\mathrm{diag}(a\boldsymbol{\pi}^n, 1, 1) \times \sigma) \mu^{-1}(a)\, da.$$

This vanishes unless π is a subquotient of a representation $I(\eta)$ of G induced from a character $\eta = (\mu_1, \mu_2, \mu_3)$ of A, such that $\mu_2 = 1$ and $\mu_1\mu_3 = 1$ (by σ-invariance) and $\mu_1 = \mu$ on $R^\times$. As explained in (1.5.1), we have

$$\chi(\pi_N)(\mathrm{diag}(a, b, c) \times \sigma) = \mu_1(a/c) + \mu_1(c/a).$$

Put $z = \mu_1(\boldsymbol{\pi}^n)$. Then $\operatorname{tr} I(\eta)(f dg \times \sigma)$ is equal to z^n, unless $\mu^2 = 1$ when it is equal to $z^n + z^{-n}$. These are the values of $\operatorname{tr} \pi(f dg \times \sigma)$ if π is an irreducible $I(\eta)$.

The reducibility results of [BZ2] imply that if $I(\eta)$ is reducible, and its twisted character $\chi^\sigma_{I(\eta)}$ is nonzero, then its twisted character is equal to that of

$$I(\nu^{-1}, 1, \nu) \qquad \text{or} \qquad I(\chi\nu^{-1/2}, 1, \chi\nu^{1/2}),$$

where χ is a character of $F^\times$ with $\chi^2 = 1$, and ν denotes the character $\nu(x) = |x|$. Then $\mu = 1$ or $\mu = \chi$ (respectively), and $\operatorname{tr} I(\eta)(fdg \times \sigma) = z^n + z^{-n}$. Here z equals q or $q^{1/2}\chi(\boldsymbol{\pi})$, and $\chi(\boldsymbol{\pi})$ equals 1 or -1.

In the first case, where $z = q$, the composition series of $I(\eta)$ consists of (1) the trivial representation $\mathbf{1}$, and $\operatorname{tr} \mathbf{1}(fdg \times \sigma) = q^n$; (2) the Steinberg representation st, and $\operatorname{tr} \operatorname{st}(fdg \times \sigma) = q^{-n}$; and some other non-$\sigma$-invariant irreducibles.

In the second case, where $z = \chi(\boldsymbol{\pi})q^{1/2}$, the composition series of $I(\eta)$ consists of two σ-invariant irreducibles. Let $\operatorname{sp}(\chi)$ and $\mathbf{1}(\chi)$ denote the special and one-dimensional subquotients of the induced representation $I(\nu^{1/2}, \nu^{-1/2})\chi$ of GL(2,F). Let P denote a maximal proper parabolic subgroup of G; its Levi factor is isomorphic to GL(2). Then the composition series of $I(\eta)$ consists of the irreducibles $I_P(\operatorname{sp}(\chi), 1)$ and $I_P(\mathbf{1}(\chi), 1)$ normalizedly induced from P, and

$$\operatorname{tr}[I_P(\operatorname{sp}(\chi), 1)](fdg \times \sigma) = z^{-n}, \qquad \operatorname{tr}[I_P(\mathbf{1}(\chi), 1)](fdg \times \sigma) = z^n.$$

It is clear that when $\mu = 1$ and $\operatorname{tr} \pi(fdg \times \sigma) \neq 0$, then the irreducible π has a vector fixed by the action of an Iwahori subgroup, again by [Bo3], Lemma 4.7.

1.6 Comparison. Let F be a global field. Suppose that $fdg = \otimes_v f_v dg_v$ and $f_i dh_i = \otimes_v f_{iv} dh_{iv}$ are products of smooth compactly supported measures $f_v dg_v$ and $f_{iv} dh_{iv}$ on G_v and H_{iv}. Suppose that $f_v dg_v$ and $f_{iv} dh_{iv}$ are the unit elements $f_v^0 dg_v$ and $f_{iv}^0 dh_{iv}$ in the Hecke algebras $\mathbb{H}$ and $\mathbb{H}_i$ (see I.1.2) of G_v and H_{iv} for almost all v. Suppose that $f_{iv} dh_{iv} = \lambda_i^*(f_v dg_v)$ for all v in the notations of section II.3, namely $\Phi^{\mathrm{st}}(\delta\sigma, f_v dg_v) = \Phi^{\mathrm{st}}(\gamma, f_{0v} dh_v)$ whenever $\gamma = N\delta$ (see II.3.1), and a similar statement of matching orbital integrals for $f_{1v} dh_{1v}$, relating $\Phi^{\mathrm{us}}(\delta\sigma, f_v dg_v)$ with $\Phi(N_1\delta, f_{1v} dh_{1v})$ (see II.3.4). It is shown in section II.3 that for each $f_v dg_v$ there exists $f_{iv} dh_{iv}$ and for each $f_{iv} dh_{iv}$ there exists $f_v dg_v$ with $f_{iv} dh_{iv} = \lambda_i^*(f_v dg_v)$, and in section II.1 that $f_{0v}^0 dh_v = \lambda_0^*(f_v^0 dg_v)$ and that $f_{1v}^0 dh_{1v} = \lambda_1^*(f_v^0 dg_v)$.

Had we not proved that $f_{1v}^0 dh_{1v} = \lambda_1^*(f_v^0 dg_v)$ we could have worked with fdg which has the property that there is a place u' of F such that $\Phi^{\mathrm{us}}(\delta\sigma, f_{u'} dg_{u'}) = 0$ for all σ-regular δ in $G_{u'}$.

Recall that δ is called σ-regular in G_v (resp. σ-elliptic, σ-split) if $\gamma = N\delta$ is regular (resp. elliptic, split) in H_v, where N is the norm map defined in I.2.

Working with such fdg we could choose $f_{1u}dh_{1u}$ to be 0, hence $f_1dh_1 = \otimes_v f_{1v}dh_{1v}$ to be 0, and $I_1 = 0$. Consequently, we would not need to know that $f_{1v}^0 dh_{1v} = \lambda_1^*(f_v^0 dg_v)$ for almost all v. But then we could derive only partial results, on cuspidal representations π with a discrete-series component.

Fix a finite place u of F. Fix $f_v dg_v$, $f_{0v}dh_v$, $f_{1v}dh_{1v}$, for all $v \neq u$ to be matching. Put $f^u dg^u = \otimes_v f_v dg_v$, $f_0^u dh^u = \otimes_v f_{0v}dh_v$, $f_1^u dh^u = \otimes_v f_{1v}dh_{1v}$ (product over $v \neq u$). Proposition III.3.5 and the last paragraph of section IV show that we have

1.6.1 Lemma. *There exists an absolutely integrable function $d(z)$ on the unit circle in $\mathbb{C}^\times$, and a positive integer n' depending on $f^u dg^u$, $f_0^u dh^u$, $f_1^u dh_1^u$, such that if $f_u dg_u$, $f_{0u}dh_u$, $f_{1u}dh_{1u}$ are regular of type n, $n \geq n'$, then*

$$I_n = I + \tfrac{1}{2}I' + \tfrac{1}{4}I'' + \tfrac{1}{2}I_1' - \left[I_0 + \tfrac{1}{4}\sum_E I_E'' - \tfrac{1}{2}\sum_E I_E' - \tfrac{1}{4}\sum_E I_E\right] - \tfrac{1}{2}I_1$$

is equal to

$$J_n = \int_{|z|=1} d(z)(z^n + z^{-n})\, d^\times z.$$

Indeed, $\operatorname{tr} I_0(\mu, f_u dg_u) = z^n + z^{-n}$, where $z = \mu(\boldsymbol{\pi})$.

Remark. As the one-dimensional representation which appears in I_0 lifts to the one-dimensional representation in I, we may assume that I and I_0 consist of cuspidal representations only.

1.6.2 Proposition. *The function $d(z)$ in the integral J_n is equal to* 0.

Proof. The sum of the I's in I_n can be written as

$$\sum_i c_i(z_i^n + z_i^{-n}) + a_0 q^n + a_1 q^{-n} + a_2 q^{n/2}$$

$$+ a_3 q^{-n/2} + a_4(-q^{1/2})^n + a_5(-q^{-1/2})^n,$$

where a_i and c_i are complex numbers, the sum is absolutely convergent, and c_i is a sum of $\operatorname{tr}\pi^u(f^u dg^u \times \sigma)$, $\operatorname{tr}\pi_0^u(f_0^u dh^u)$ etc. with coefficient $1, \frac{1}{2}$ or $\frac{1}{4}$, over the $\pi^u, \ldots$ such that $\pi = \pi^u \otimes \pi_u, \ldots$ appears in the sum of

$I, \dots,$ where $\pi_u = I(\eta)$ determines z_i as in (1.5.2), (1.5.3) (with $\mu = 1$). Here $z_i \neq q, q^{-1}, q^{1/2}, q^{-1/2}, -q^{1/2}, -q^{-1/2}$, $q = q_u$.

We shall use the following comments. All representations in the trace formula have unitarizable components. Hence each z_i lies in the compact subset $X' = X'(q)$ in $\mathbb{C}$ which is the union of the unit circle $|z| = 1$ and the real segments $q^{-1} \leq z \leq q$ and $q^{-1} \leq -z \leq q$. Let $X = X(q)$ be the quotient of X' by the equivalence relation $z^{-1} \sim z$. Then X is a compact Hausdorff space. Let $B = B(q)$ be the space spanned over $\mathbb{C}$ by the functions $f_n(z) = z^n + z^{-n}$ on X, where $n \geq 0$. It is closed under multiplication, contains the scalars, and separates points of X. Moreover, if f lies in B then its complex conjugate $\overline{f}$ does too. Hence the Stone-Weierstrass theorem implies the following

LEMMA. *B is dense in the sup norm in the space of complex-valued continuous functions on X.*

Our argument is based on the observation that the terms in the identity $I_n = J_n$ with coefficients a_i are finite in number. We shall first prove that $J_n = 0$ and $d(z) = 0$ and $c_i = 0$ for all i. It will then follow from a standard linear independence argument for finitely many characters that each a_i is zero. Since we do not know *apriori* that $a_{2i} = a_{2i+1}$, we cannot express I_n in terms of values of f_n. The first step of the proof is then to eliminate the a_i. This would let us express I_n in terms of values of f_n, but we need to observe that only sufficiently large n are known to us now to satisfy $I_n = J_n$.

To eliminate the terms a_i we construct a rational function $r(x)$ whose zeroes are precisely $q^{\pm 1}$, $q^{\pm 1/2}$, $-q^{\pm 1/2}$, and whose pole is only at 0. Namely

$$r(x) = (qx^2 - 1)(qx^{-2} - 1)(qx - 1)(qx^{-1} - 1)$$

$$= q^2x^3 - q(q^2+1)x^2 - q(q^2 - q + 1)x + (q^2+1)^2$$

$$- q(q^2 - q + 1)x^{-1} - q(q^2+1)x^{-2} + q^2x^{-3}.$$

Note that $r(x^{-1}) = r(x)$.

Correspondingly we define

$$G_n = q^2 f_{n+3} - q(q^2+1)f_{n+2} - q(q^2 - q + 1)f_{n+1}$$

$$+ (q^2+1)^2 f_n - q(q^2 - q + 1)f_{n-1} - q(q^2+1)f_{n-2} + q^2 f_{n-3},$$

and we take the linear combination of I_n's:

$$q^2 I_{n+3} - q(q^2+1)I_{n+2} - q(q^2-q+1)I_{n+1} + (q^2+1)^2 I_n \\ - q(q^2-q+1)I_{n-1} - q(q^2+1)I_{n-2} + q^2 I_{n-3}.$$

The terms with coefficients a_i become zero, and we obtain

$$\sum_i c_i G_n(z_i) = \int_{|z|=1} d(z) G_n(z) d^\times z.$$

Note that

$$G_{n+3}(z) = (z^{n+3} + z^{-n-3}) r(z) = f_{n+3}(z) r(z).$$

Hence for $n \geq n' + 3$ we have

$$\sum_i c_i r(z_i) f_n(z_i) = \int_{|z|=1} d(z) r(z) f_n(z) d^\times z. \tag{1.6.3}$$

The z_i are all on the unit circle S^1. Let S be the quotient of S^1 by the relation $z \sim z^{-1}$. Suppose that the sum is nonempty, that is, there is some $z_i \in S$ with $c_i \neq 0$. Rearranging indices we may assume that $i = 0$. The absolute convergence of the sum and integral implies that there is $c > 0$ with

$$\int_{|z|=1} |d(z) r(z)| d^\times z \leq c,$$

and for a given $\varepsilon > 0$, an $m > 0$ with

$$\sum_{i \geq m} |c_i r(z_i)| < \varepsilon.$$

The Lemma implies that there is a function f in B, which is a linear combination of f_n's over $\mathbb{C}$, with $f(z_0) = 1$, which is bounded by 2 on S and whose value outside a small neighborhood of z_0 is small. The only problem is that (1.6.3) holds only for n bigger than some n'. To overcome this, take k larger than the sum of n' and the degree of f ($\deg f_n = n$), such that z_0^k is close to one. Then $|z^k + z^{-k}| \leq 2$ on S, and we may apply (1.6.3) with $f_n(z)$ replaced by

$$g(z) = f(z)(z^k + z^{-k})$$

to obtain a contradiction to $c_0 \neq 0$. Of course $r(z_0) \neq 0$ as $r \neq 0$ on S. The same proof shows that $d(z)$ is zero on S^1. Indeed, as $c_i = 0$ for all i, if $d(z_0) \neq 0$, we apply (1.6.3) with f_n replaced by f which is small outside a small neighborhood of z_0, and with $f(z_0) = 1$. The proposition follows. □

1.6.4 Correction. In the proof of Proposition 5 in [F1;IV] we should work with

$$r(x) = -(q^{1/2}x - 1)(q^{1/2}x^{-1} - 1) = q^{1/2}x - (q+1) + q^{1/2}x^{-1}$$

and

$$G_n = q^{1/2}f_{n+1} - (q+1)f_n + q^{1/2}f_{n-1}$$

which satisfy $G_{n+1}(z) = f_{n+1}(z)r(z)$, instead of with F_n of page 756, line 2 from the bottom, of [F1;IV].

1.7 Density. For a global function f whose components at u', u'' are supported on the σ-elliptic regular set, the twisted trace formula takes the form (see chapter III, (3.2.5)).

$$I + \tfrac{1}{2}I' + \tfrac{1}{4}I'' + \tfrac{1}{2}I'_1 = \sum_{\{\delta\}} c_\gamma \Phi(\delta\sigma, f dg). \tag{1.7.1}$$

The sum is over all conjugacy classes of elements δ in G whose norm $\gamma = N\delta$ in H is elliptic regular. The c_γ are volume factors, see chapter III, (1.2.1). The sum is finite. With analogous conditions on $f_0 dh$, the stable trace formula for H takes the form

$$I_0 + \tfrac{1}{4}\sum_E I''_E - \tfrac{1}{2}\sum_E I'_E - \tfrac{1}{4}\sum_E I_E = \sum_{\{\gamma\}} c_\gamma \Phi^{\mathrm{st}}(\gamma, f_0 dh).$$

The sum over $\{\gamma\}$ is over all stable conjugacy classes of elliptic regular elements in H. The c_γ are as above and the sum is again finite. The following is a twisted analogue of Kazhdan [K2].

Proposition. *Let F_u be a local field. Suppose that* $\operatorname{tr}\pi_u(f_u dg_u \times \sigma) = 0$ *for all admissible π_u. Then the twisted orbital integral $\Phi(\delta, f_u dg_u)$ of $f_u dg_u$ is* 0 *for all δ in G_u.*

Remark. It suffices to make the assumption of the proposition only for the π_u which are the component at u of the π which make a contribution (1.7.1).

Proof. By virtue of II.3 it suffices to consider only σ-regular δ. Choose a global field F whose completion at a place u is our F_u. Choose places u', u''. Since G is dense in G_u and $\Phi(\delta\sigma, f_u dg_u)$ is smooth on the σ-regular set, it suffices to show that in each neighborhood of δ in G_u there exists a σ-regular δ_0 in G with $\Phi(\delta_0\sigma, f_u dg_u) = 0$. We choose such δ_0 which is σ-elliptic at the places u', u''. We choose fdg whose components at u', u'' are supported on the σ-regular elliptic set, so that (1.7.1) holds, such that the component of fdg at u is our $f_u dg_u$, and $\Phi(\delta_0\sigma, f_v dg_v) \neq 0$ for all $v \neq u$. The assumption of the proposition implies that

$$\sum_{\{\delta\}} c_\gamma \Phi(\delta\sigma, fdg) = 0.$$

The sum ranges over all σ-conjugacy classes of σ-elliptic regular δ in G. Since fdg is compactly supported it is clear that the eigenvalues of $N\delta$ lie in a finite set (depending on the support of fdg). These eigenvalues determine the stable σ-conjugacy class of δ. By Corollary I.2.3.1, given a place u and stably σ-conjugate δ, δ' which are not σ-conjugate, there is a place $v \neq u$ where δ, δ' are not σ-conjugate. Hence we may restrict the support of $f^u dg^u = \otimes_{v \neq u} f_v dg_v$ to have $\Phi(\delta\sigma, f^u dg^u) = 0$ for all δ in the sum unless δ is σ-conjugate to δ_0. Since

$$\Phi(\delta_0\sigma, f^u dg^u) \neq 0 \qquad \text{and} \qquad \Phi(\delta_0\sigma, fdg) = 0,$$

and $c_\gamma \neq 0$, it follows that $\Phi(\delta_0\sigma, f_u dg_u) = 0$, as asserted. □

We shall now adapt the above techniques to show that corresponding spherical functions have matching stable orbital integrals, using the Fundamental Lemma of section II.1, that the unit elements of the Hecke algebras are matching. Our method is new. It is based on the usage of regular functions. The method was extended in [FK1] and [F1;V] to deal with groups of general rank. As noted in [F1;VI], page 3, there is a gap in [F1;V]. It is filled in an appendix of the paper [F2;V], and by Labesse, Duke Math. J. 61 (1990), 519-530, Proposition 8, p. 525. We checked — but did not write up — that this result can also be proven by a method of Clozel, which is also global (both Clozel's and our technique are motivated by the global technique of Kazhdan [K2], Appendix), but relies instead on properties of spherical, not Iwahori, functions. In fact Clozel writes in [Cl2], p. 151, line 3, that his method is the one used in this work. But his assertion is not true. Langlands wrote an unpublished long set of notes, using combinatorics on buildings, to prove the matching statement. In any case we believe that our method is the simplest available.

As in I.3.4, I.3.8 and II.3.1, we write $\lambda_0^*(fdg) = f_0dh$ if fdg and f_0dh are *matching* (have matching stable orbital integrals), and $\tilde{\lambda}_0(fdg) = f_0dh$ if fdg and f_0dh are *corresponding* spherical functions (see I.1; they satisfy $\operatorname{tr}\pi(fdg \times \sigma) = \operatorname{tr}\pi_0(f_0dh)$ for all unramified π_0 and π with $\pi = \lambda_0(\pi_0)$).

1.7.2 PROPOSITION. *For each fdg in $\mathbb{H}$ we have $\lambda_0^*(fdg) = f_0dh$ if $\tilde{\lambda}_0(fdg) = f_0dh$.*

PROOF. As in (1.7) it suffices to consider a σ-regular δ_0 in G which is σ-elliptic at u', u''. We choose $f^u dg^u = \otimes_v f_v dg_v$ $(v \neq u)$ whose components at u', u'' are supported on the σ-regular set, with $\Phi^{\mathrm{us}}(f_{u'}dg_{u'})$ identically zero and $\Phi^{\mathrm{st}}(\delta_0\sigma, f^u dg^u) \neq 0$. The component at u is taken to be a regular measure of any type n. The measure $f_0dh = f_0^u dh^u \otimes f_{0u}dh_u$ is taken in a parallel fashion, so that fdg, f_0dh have matching orbital integrals. Hence

$$\sum c_\gamma \Phi^{\mathrm{st}}(\gamma, f_0dh) = \sum c_\gamma \Phi^{\mathrm{st}}(\delta\sigma, fdg), \tag{1.7.3}$$

where the sums, which range over stable conjugacy classes, are finite. Recall from I.2.3 that the norm map is a bijection from the set of stable σ-conjugacy classes in G, to the set of stable conjugacy classes in H. By (1.7.1) we obtain the identity $I_n = 0$, where I_n is defined in Lemma 1.6.1. We write $I_n = 0$ as in the proof of (1.6.2) in the form

$$\sum c(\pi_{0u}) \operatorname{tr}\pi_{0u}(f_{0u}dh_u) = 0 \quad \text{or} \quad \sum c_i(z_i^n + z_i^{-n}) = 0. \tag{1.7.4}$$

As in (1.6.2) we conclude that each coefficient c_i, or $c(\pi_{0u})$, is zero. In particular we can take the subsum in (1.7.4) over spherical π_{0u} only, and it is equal to zero also when $f_{0u}dh_u$, f_udg_u are replaced by corresponding spherical functions as in our proposition. Hence we obtain (1.7.3) where $f_{0u}dh_u$, f_udg_u are now corresponding spherical functions. As the sums are finite we can reduce the support of the component $f_{0u'}dh_{u'}$, so that the only entry to the sums in (1.7.3) is δ_0. Indeed, a stable σ-conjugacy class δ is determined by the eigenvalues of $\delta\sigma(\delta)$. Since $\Phi^{\mathrm{st}}(\delta_0, f^u dg^u)$ is nonzero by construction, we have

$$\Phi^{\mathrm{st}}(\delta_0\sigma, f_udg_u) = \Phi^{\mathrm{st}}(N\delta_0, f_{0u}dh_u)$$

for all σ-regular δ_0 (in G, hence in G_u), as asserted. □

1.8 Proposition. *Let V be a finite set of places of F including the archimedean places. Fix a conjugacy class t_v in $\widehat{H}$ for all v outside V. For any choice of matching $f_v dg_v$, $f_{0v} dh_v$ $(= \lambda_0^*(f_v dg_v))$, and $f_{1v} dh_{1v}$ $(= \lambda_1^*(f_v dg_v))$ for v in V, we have*

$$I + \tfrac{1}{2} I' + \tfrac{1}{4} I'' + \tfrac{1}{2} I_1' = I_0 + \tfrac{1}{4} \sum_E I_E'' - \tfrac{1}{2} \sum_E I_E' - \tfrac{1}{4} \sum_E I_E + \tfrac{1}{2} I_1, \tag{1.8.1}$$

where I, I_0, I_1, I_E, ... are defined by products $\prod_{v \in V} \operatorname{tr} \pi_v(f_v dg_v \times \sigma)$, ..., over v in V only, the sums in I, I_0, I_1, I_E, ... are taken only over those π, π_0, π_1, μ' on $\mathbb{A}_E^1/E^1$, ... whose component at v outside V is unramified and parametrized by the conjugacy class $\lambda_0(t_v)$ in $\widehat{G}$ or t_v in $\widehat{H}$ or $\lambda_1(t_v)$ in $\widehat{H}_1$ or $\eta_v'(\boldsymbol{\pi})$ with image t_v under λ_E.

Proof. The proof of (1.6.2) applied inductively to the elements in a set U of places outside V, implies that

$$\sum_i c_i \prod_{v \notin V \cup U} f_{0v}^{\vee}(t_{iv}) = 0.$$

Here the product ranges over v outside $V \cup U$, the sum is over all sequences $\{t_{iv};\ v \text{ outside } V\}$ in $\widehat{H}$ with $t_{iv} = t_v$ for v in U, and c_i is defined by the difference of the left and right sides of (1.8.1) (corresponding to the sequence $\{t_{iv}\}$). We have to show that $c_i = 0$ for all i. Suppose $c_0 \neq 0$. Choose a positive m with

$$\sum_{i \geq m} |c_i| < \tfrac{1}{2} |c_0|,$$

and a set U disjoint from V so that for each $1 \leq i < m$ there is u in U with $t_{iu} \neq t_u$. Applying our identity with this U and with $f_{0v}^{\vee} = 1$ (thus $f_{0v} = f_{0v}^0$) for all v outside $V \cup U$, we obtain a contradiction which proves the proposition. □

1.8.2 Theorem. *Under the conditions of* (1.8) *at most one of the sums I, I', I'', I_1' is nonempty, and consists of a single summand.*

Proof. This follows from the rigidity theorem of [JS]. □

1.8.3 COROLLARY. *Fix a nonarchimedean u in V and a character μ_{1u} of $F_u^\times$. Then the trace identity* (1.8.1) *holds where the products in I, I_0, I_E, ... are taken to range only over the places $v \neq u$ in V, and the sums in I, I_0, I_E, ... are the subsums of those specified in* (1.8) *where π_0 has the component $I_0(\mu_{1u})$ at u, π has the component $\lambda(I_0(\mu_{1u})) = I(\mu_{1u}, 1, 1/\mu_{1u})$ at u, π_1 has the component $\lambda_1^{-1}(I(\mu_{1u}, 1, 1/\mu_{1u})) = I_1(\mu_{1u}, 1/\mu_{1u})$, and μ' on $\mathbb{A}_E^1/E^1$ has $\lambda_E(\mu'_E) = I_0(\mu_{1u})$.*

PROOF. Denote by μ the restriction of μ_{1u} to $R_u^\times$. The case of $\mu = 1$ is dealt with in Proposition 1.6.2 (or (1.8)). That of $\mu^2 = 1$ is the same. If $\mu^2 \neq 1$ let f'_{0u} be a regular function of type (n, μ), and consider $f_{0u} = f'_{0u} + \overline{f}'_{0u}$; note that the complex conjugate $\overline{f}'_{0u}$ is of type (n, μ^{-1}). Then $\operatorname{tr} \pi_{0u}(f_{0u} dh_u)$ vanishes unless π_{0u} is a constituent of an induced $I_0(\mu_{1u})$ from a character μ_{1u} of $F_u^\times = A_{0u}$ whose restriction to $R_u^\times$ is μ, in which case $\operatorname{tr} \pi_{0u}(f_{0u} dh_u)$ equals $z^n + z^{-n}$ for a suitable z. As the same observations apply on the twisted side, and for H_{1u}, applying the Stone-Weierstrass theorem as in (1.6.2) the corollary follows. □

It would simplify matters to remove the terms associated with H_1 in our trace identity (1.8.1).

1.9 PROPOSITION. *Let F_u be a local field. Every irreducible admissible representation π_{1u} of H_{1u} λ_1-lifts to the σ-invariant representation $\pi_u = I(\pi_{1u})$ of G_u.*

PROOF. If π_{1u} is fully induced, the result is proven in I.3.10. Suppose that u is nonarchimedean and π_{1u} is square integrable. Choose a totally imaginary number field whose completion at a place u is our F_u. Choose two nonarchimedean places u_1, $u_2 \neq u$ of F. Choose cuspidal representations π'_{1u_i} of H_{1u_i}. Construct cuspidal representations $\tilde{\pi}_1$ and $\tilde{\pi}'_1$ of $\mathbf{H}_1(\mathbb{A})$ whose components at u_1, u_2 are our cuspidal π'_{1u_i}; outside u, u_1, u_2 and the archimedean places the components are unramified; and at u we take $\tilde{\pi}_1$ to have our component π'_{1u}, while $\tilde{\pi}'_1$ is taken to have (at u) an unramified component. Such $\tilde{\pi}_1$ and $\tilde{\pi}'_1$ are constructed using the simple trace formula on $\mathbf{H}_1(\mathbb{A})$. Note that if π_{1u} is special, the fact that $\tilde{\pi}'_{1u_1}$ is cuspidal would guarantee that the component of $\tilde{\pi}_1$ at u is the special π_{1u} and not the one-dimensional complement in the induced representation.

We now apply the trace identity (1.8.1) fixing the conjugacy classes $\{t_v;\ v \notin V\}$ so that the sum I'_1 has the contribution $I(\tilde{\pi}_1)$. Consequently the sums I, I', I'' are empty. We evaluate at a test measure such that $f_{1u_1}dh_{1u_1}$ is supported on the elliptic set of H_{1u_1} and $\operatorname{tr}\pi'_{1u_1}(f_{1u_1}dh_{1u_1}) \neq 0$. We can then choose $f_{0u_1}dh_{u_1}$ to be identically zero, and $f_{u_1}dg_{u_1}$ to be a matching function on G_{u_1}. Then the terms I_0, I'_E, I_E are zero. The trace identity (1.8.1) reduces to $I'_1 = I_1$, and there is only one entry in each sum, thus

$$\prod_v \operatorname{tr} I(\tilde{\pi}_{1v}; f_v dg_v \times \sigma) = \prod_v \operatorname{tr} \tilde{\pi}_{1v}(f_{1v}dh_{1v}). \tag{1.9.0}$$

Now the product can be taken only over the set $\{u, u_1, u_2\}$, as all other components of $\tilde{\pi}_1$ are induced. Working with the cuspidal representation $\tilde{\pi}'_1$ instead of with $\tilde{\pi}_1$, we obtain the same identity (1.9.0), but with product ranging only over the set of v in $\{u_1, u_2\}$. The quotient of the two identities is

$$\operatorname{tr} I(\pi_{1u}; f_u dg_u \times \sigma) = \operatorname{tr} \pi_{1u}(f_{1u}dh_{1u}).$$

This holds for all matching measures $f_{1u}dh_{1u}$ and $f_u dg_u$. Hence π_{1u} λ_1-lifts to $I(\pi_{1u})$.

If π_{1u} is one dimensional it is contained in an induced I_{1u} whose composition series consists of π_{1u} and a special representation sp_{1u}. The result (character relation) for I_{1u} and for sp_{1u} implies the result for π_{1u}. This comment applies also when $F_u = \mathbb{C}$, the field of complex numbers, where the trivial representation is the difference between two fully induced representations of $\mathbf{H}_1(\mathbb{C})$. This comment would apply also when $F_u = \mathbb{R}$ is the field of real numbers once we prove the proposition for square-integrable representations of $\mathbf{H}_1(\mathbb{R})$.

To deal with the real case we take $F = \mathbb{Q}$. Then $F_u = \mathbb{R}$. We construct a cuspidal representation π_1 of $\mathbf{H}_1(\mathbb{A})$ whose component at the real place is the π_{1u} of the proposition, and whose component at some nonarchimedean place w is cuspidal. Once again we apply (1.8.1) to get $I'_1 = I_1$ with a single term π_1, and the products in (1.9.0) reduce to $v = u$ by virtue of the result for the nonarchimedean places. □

1.9.1 COROLLARY. *In the trace identity* (1.8.1) *we have* $I'_1 = I_1$. *Every discrete-spectrum representation* π_1 *of* $\mathbf{H}_1(\mathbb{A})$ λ_1*-lifts to the* σ*-invariant representation* $I(\pi_1, 1)$ *of* $\mathbf{G}(\mathbb{A})$. *Every* σ*-invariant representation of* $\mathbf{G}(\mathbb{A})$ *of the form* $I(\pi_1, 1)$ *is the* λ_1*-lift of* π_1 *on* $\mathbf{H}_1(\mathbb{A})$. *A* σ*-invariant representation of* $\mathbf{G}(\mathbb{A})$ *which has a component* $I(\pi_{1u})$ *where* π_{1u} *is not fully induced, is of the form* $I(\pi_1, 1)$ *for a discrete-spectrum* π_1. *The* σ*-twisted character* *of a σ-invariant representation $\pi_u = I(\pi_{1u})$ is σ-unstable ($\chi_\pi^\sigma(\delta) = -\chi_\pi^\sigma(\delta')$ if δ, δ' are stably σ-conjugate but not σ-conjugate). If a σ-invariant representation of $\mathbf{G}(\mathbb{A})$ has a σ-elliptic σ-unstable component, then it is of the form $I(\pi_1, 1)$. Any σ-invariant σ-elliptic σ-unstable representation π_u of G_u is of the form $I(\pi_{1u})$. Any σ-invariant σ-elliptic representation π_u of G_u is either σ-stable or σ-unstable.*

PROOF. The σ-twisted character of $I(\pi_{1u})$ is σ-unstable by the character relation

$$\operatorname{tr} \pi_{1u}(f_{1u} dh_{1u}) = \operatorname{tr} I(\pi_{1u}; f_u dg_u \times \sigma).$$

Since $I'_1 = I_1$, every component of a contribution to the sums I, I', I'' in (1.8.1) are stable (depend only on $f_{0v} dh_v$, or on the stable σ-orbital integrals of $f_v dg_v$, for every v). Using a pseudo-coefficient of π_u and the twisted trace formula we can construct a σ-invariant representation π of $\mathbf{G}(\mathbb{A})$ which occurs in I, I', I'' or I_1 whose component at u is our π_u. If π_u is σ-unstable, π must occur in I_1 and $\pi_u = I(\pi_{1u})$. If not, π will occur in I, I' or I''. □

Now that we eliminated the terms $I'_1 = I_1$ in (1.8.1), and we know that no factors $\operatorname{tr} I(\pi_{1u}; f_u dg_u \times \sigma)$ may appear in I, I', I'', we may rewrite (1.8.1) in the form

$$\begin{aligned}
&\sum_\pi \prod_v \operatorname{tr} \pi_v(f_v dg_v \times \sigma) + \tfrac{1}{2} \sum_E \sum_\tau \prod_v \operatorname{tr} I((\tau_v, \chi_{E_v}); f_v dg_v \times \sigma) \\
&\quad + \tfrac{1}{4} \sum_\eta \prod_v \operatorname{tr} I(\eta_v; f_v dg_v \times \sigma) \\
&= \sum_{\pi_0} m(\pi_0) \prod_v \operatorname{tr} \pi_{0v}(f_{0v} dh_v) - \tfrac{1}{2} \sum_E \sideset{}{'}\sum_{\mu' \neq \overline{\mu}'} \prod_v \mu'_v(f_{T_{E_v}} dt) \\
&\quad + \tfrac{1}{4} \sum_E \prod_v \operatorname{tr} R(\chi_{E_v}) I_0(\chi_{E_v}, f_{0v} dh_v) - \tfrac{1}{4} \sum_E \sum_{\mu' = \overline{\mu}'} \prod_v \mu'_v(f_{T_{E_v}} dt).
\end{aligned} \tag{1.9.2}$$

On the left the first sum ranges over the set of discrete-spectrum σ-invariant automorphic representations of $\mathbf{G}(\mathbb{A})$.

The second sum is over all quadratic extensions E of F, and χ_E denotes the quadratic character of $\mathbb{A}^\times/F^\times$ whose kernel is $N_{E/F}(\mathbb{A}_E^\times)$. The second sum is over all cuspidal representations τ of $\mathrm{GL}(2,\mathbb{A})$ with $\tau \simeq \check{\tau}(= \chi_E\tau)$.

The third sum is over the unordered triples $\eta = \{\chi, \mu\chi, \mu\}$, where χ, μ are characters of $W_{F/F} = \mathbb{A}^\times/F^\times$ of order 2 (not 1), and $\chi \neq \mu$.

The first sum on the right is over the equivalence classes of discrete-spectrum automorphic representations π_0 of $\mathbf{H}(\mathbb{A})$. The coefficients $m(\pi_0)$ are the multiplicities. The last two sums range over the quadratic extensions E of F, and all characters μ' of $\mathbb{A}_E^1/E^1$, up to the equivalence relation $\mu' \sim \overline{\mu}'$. This is indicated by the prime in $\sum'$.

The products are taken over v in V, as specified in (1.8) and (1.8.3). Namely we fix classes t_v in $\widehat{H}$ for all $v \notin V$, or $I_0(\mu_{1v})$, and only those π, π_0, μ' on $\mathbb{A}_E^1/E^1$ that have components at $v \notin V$ specified by t_v or $I_0(\mu_{1v})$ via our liftings ($t(\pi_v) = \lambda(t_v)$, $t(\pi_{0v}) = t_v$, $\lambda_E(\mu'_v) = t_v$) occur in our sums.

1.9.3 Lemma. (1) *The conclusion of* (1.8.3) *holds at a complex place.*

(2) *If F is totally imaginary then* (1.8.1) *holds where all archimedean places are omitted* (*in the sense of* (1.8.3)) *from V; then the sums in* (1.9.2) *are finite for a fixed choice of $f_{0v}dh_v$, f_vdg_v (v in V, $v \neq \infty$).*

Proof. (1) Let π be an irreducible admissible σ-invariant representation of $\mathbf{G}(\mathbb{C})$ which appears as a component at a complex place of an automorphic representation on the left of (1.9.2). Since the trivial representation of $\mathbf{H}(\mathbb{A})$ lifts to the trivial representation of $\mathbf{G}(\mathbb{A})$, we may assume that π is generic, in which case it is induced from a character of a Borel subgroup, hence it is the lift of an induced π_0; here we use the description [Vo], Theorem 6.2(f), of generic (= large) representations of $\mathbf{G}(\mathbb{C})$.

For (2), the sums are finite by a classical theorem of Harish-Chandra (see [BJ], 4.3(i), p. 195), which asserts that there are only finitely many automorphic representations π of $\mathbf{G}(\mathbb{A})$ with a fixed infinitesimal character and a C-fixed vector; C is an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$, and $\mathbb{A}_f$ denotes the finite adèles. The conditions of this theorem are satisfied in our case since we fixed the archimedean components of the π and the π_0, and we choose f_vdg_v ($v \neq \infty$) to be invariant under such fixed C. □

1.10 Lemma. *Let E_i be quadratic extensions of F and μ'_i ($i = 1,2$) characters of $\mathbb{A}_{E_i}^1/E_i^1$ such that the two-dimensional projective* (*image in* $\mathrm{PGL}(2,\mathbb{C})$) *representations* $(\mathrm{Ind}_{E_1}^F \mu'_1)_{0v}$ *and* $(\mathrm{Ind}_{E_2}^F \mu'_2)_{0v}$ *are equivalent for all v outside a finite set V. Then* (1) $E_1 = E_2$ *and* $\mu'_1 = \mu'_2$ *or* ${\mu'_2}^{-1}$, *or* (2)

$\mu_i'^2 = 1 \neq \mu_i'$ and $\{\mu_1, \chi_{E_1}\mu_1, \chi_{E_1}\} = \{\mu_2, \chi_{E_2}\mu_2, \chi_{E_2}\}$ where μ_i is defined by $\mu_i''(z) = \mu_i(N_{E_i/F}z)$ (it is unique up to multiplication by χ_{E_i}), and μ_i'' is defined by $\mu_i''(z) = \mu_i'(z/\overline{z})$, $z \in \mathbb{A}_{E_i}^\times/E_i^\times$.

PROOF. By Chebotarev's density theorem we may assume $(\mathrm{Ind}_{E_1}^F \mu_1')_0 \simeq (\mathrm{Ind}_{E_2}^F \mu_2')_0$. Applying λ_0 we then get $\mathrm{Ind}_{E_1}^F \mu_1'' \oplus \chi_{E_1} \simeq \mathrm{Ind}_{E_2}^F \mu_2'' \oplus \chi_{E_2}$. If one of the Ind is irreducible we obtain that both are irreducible, $\chi_{E_1} = \chi_{E_2}$ so $E_1 = E_2$, and $\mu_1' = \mu_2'$ or $\mu_2'^{-1}$.

If the Ind are reducible, $\mu_i'^2 = 1$. If $\mu_i' = 1$, $\mathrm{Ind}_{E_i}^F \mu_i'' = \chi_{E_i} \oplus 1$. If $\mu_i' \neq 1$ $(= \mu_i'^2)$ then $\mathrm{Ind}_{E_i}^F \mu_i'' = \mu_i \oplus \mu_i\chi_{E_i}$, so the lemma follows. □

REMARK. Analogous proof — based on applying λ_0 — establishes the local analogue, namely that if $(\mathrm{Ind}_{E_1}^F \mu_1')_0$ and $(\mathrm{Ind}_{E_2}^F \mu_2')_0$ are equivalent then (1) or (2).

1.10.1 COROLLARY. *Let E be a quadratic extension of F. Let $\mu' \neq 1$ be a character of $\mathbb{A}_E^1/E^1$ with $\mu_u' \neq 1$ at a place u of F where E_u is a field. Then there exists a cuspidal representation $\pi_0 = \pi_0(\mu')$ of $\mathrm{SL}(2,\mathbb{A})$ with $\lambda_E(t(\mu_v')) = t(\pi_{0v})$ for almost all v. If $\mu' = 1$ the conclusion holds with $\pi_0 = I_0(\chi_E)$.*

PROOF. Set up (1.9.2) with V such that μ_v' is unramified outside V, such that our E and μ' make the only contribution on the right. At $u \in V$ choose f_{0u} with $\Phi^{\mathrm{st}}(\gamma, f_{0u}dh_u) \equiv 0$, and $\Phi^{\mathrm{us}}(\gamma, f_{0u}dh_u) \equiv 0$ unless $\gamma \in E_u^1$, $\gamma \neq \overline{\gamma}$, and $\mu_u'(f_{T_{E_u}}dt) \neq 0$. For fdg matching f_0dh the sums I, I', I'' are zero, and (1.9.2) becomes $\sum_{\pi_0} m(\pi_0) \prod_v \mathrm{tr}\, \pi_{0v}(f_{0v}dh_v) = \frac{1}{2}\prod_v \mu_v'(f_{T_{E_v}}) \neq 0$. Hence there is π_0 with $\lambda_E(\mu_v') = \pi_{0v}$ for all $v \notin V$. □

REMARK. The assumption that there is a place u where E_u is a field and $\mu_u' \neq 1$ will be removed once we complete the local theory.

1.10.2 Construction. Given a quadratic extension E_1 of the global field F, and a character $\mu_1' \neq 1 = \mu_1'^2$ of $\mathbb{A}_{E_1}^1/E_1^1$, let us find the E_2 and μ_2' with $(\mathrm{Ind}_{E_2}^F \mu_2')_0 = (\mathrm{Ind}_{E_1}^F \mu_1')_0$. For this, note that there is a quadratic character μ_1 of $\mathbb{A}_F^\times/F^\times\mathbb{A}_F^{\times 2}$, nontrivial on $F^\times N_{E_1/F}\mathbb{A}_{E_1}^\times/F^\times\mathbb{A}_F^{\times 2}$, such that $\mu_1''(z) = \mu_1'(z/\tau_1 z)$ is $\mu_1(z\tau_1 z)$ for all $z \in \mathbb{A}_{E_1}^\times$. Here τ_1 generates $\mathrm{Gal}(E_1/F)$. Indeed, we have $\mu_1'' = \overline{\mu}_1''$ where $\overline{\mu}_1''(z) = \mu_1''(\overline{z})$, and the sequence $1 \to E_1^1 \to E_1^\times \to N_{E_1/F}E_1^\times \to 1$ defined by the norm $N_{E_1/F}$ is exact. This μ_1 is determined uniquely up to multiplication by $\chi_1 = \chi_{E_1}$, the nontrivial character of $\mathbb{A}_F^\times/F^\times N_{E_1/F}\mathbb{A}_{E_1}^\times$. Now the characters

$\chi_2 = \mu_1$ and $\chi_3 = \mu_1\chi_1$ determine the quadratic extensions E_2 and E_3 of F, and the biquadratic extensions E_iE_j of F for any $i \neq j$ are all equal to $E_1E_2E_3$. Define characters μ_i'' on $\mathbb{A}_{E_i}^\times/E_i^\times\mathbb{A}_F^\times$ and μ_i' on $\mathbb{A}_{E_i}^1/E_i^1$ by $\mu_i''(z) = \mu_i'(z/\tau_i z) = \mu_i(z\tau_i z)$, where τ_i generates $\mathrm{Gal}(E_i/F)$ and $\mu_i = \chi_1$ (or $= \chi_1\chi_i$).

Analogous construction applies in the local case.

V.2 Main theorems

Let F be a global field. Fix a place u to be nonarchimedean, unless otherwise specified. Put $\mathbf{H} = \mathrm{SL}(2)$, $\mathbf{H}_1 = \mathrm{PGL}(2)$, $\mathbf{G} = \mathrm{PGL}(3)$. An irreducible σ-invariant G_u-module π_u is called σ-*elliptic* if its twisted character is not identically zero on the σ-elliptic regular set.

2.1 Proposition. *Given a cuspidal representation π'_{0u} of H_u there exists (i) a σ-invariant σ-stable σ-elliptic generic tempered representation π_u of G_u which is not Steinberg, and (ii) for each π_{0u} a nonnegative integer $m(\pi_{0u})$ with $m(\pi'_{0u}) \neq 0$ which is equal to 0 if π_{0u} is one dimensional or special, such that for all matching $f_u dg_u$, $f_{0u} dh_u$ we have*

$$\operatorname{tr}\pi_u(f_u dg_u \times \sigma) = \sum m(\pi_{0u}) \operatorname{tr}\pi_{0u}(f_{0u} dh_u). \qquad (2.1.1)$$

Given an open compact subgroup C_u of $H_u = \mathbf{H}(F_u)$, there are only finitely many terms π_{0u} in the sum which have nonzero C_u-fixed vector.

For each σ-invariant σ-stable σ-elliptic representation π_u of G_u there are $m(\pi_{0u})$ for which (2.1.1) holds.

If u is real and π'_{0u} is square integrable, (2.1.1) holds with an absolutely convergent sum.

Remark. (2.1.1) holds of course when π'_{0u} is special. Then π_u is Steinberg, and the sum consists of π'_{0u} alone.

Proof. Choose a totally imaginary field F whose completion at a place u is our local field F_u. Let π'_0 be a cuspidal representation of $\mathbf{H}(\mathbb{A})$ which has the component π'_{0u} at u, its component at another finite place w is special, and it is unramified at any other finite place. It is easy to construct such π'_0 using the trace formula for $\mathbf{H}(\mathbb{A})$, and a function $f_0 dh = \otimes_v f_{0v} dh_v$ whose component at u is a matrix coefficient of π_{0u}, at w it is a pseudo-coefficient of the special representation, at the other finite places it is the unit element of the Hecke algebra, and at the infinite places the component has small compact support near the identity.

Apply Proposition 1.8 with π_0' and the set $V = \{u, w\}$. By 1.9.1 $I_1' = I_1$ is removed from (1.8.1). Take $f_{0w}dh_w$ to be a pseudo-coefficient of the special representation. Its orbital integrals are stable, namely $f_{T_{E_w}} \equiv 0$ for all E_w, hence all terms on the right of (1.8.1) belong to I_0. We obtain the right side of (2.1.1). If we take $f_{0u}dh_u$ to be a matrix-coefficient of π_{0u}' we obtain a positive integer (the multiplicity of π_0' in the cuspidal spectrum of $\mathbf{H}(\mathbb{A})$) on the right of (1.8.1). Hence there exists a (necessarily unique under the conditions of (1.8)) term π on the left of (1.8.1). If $f_w dg_w$ is a measure which matches a pseudo-coefficient of the special representation, then

$$\langle \chi_{\pi_w}, \chi_{\mathrm{St}_w} \rangle_e = \operatorname{tr} \pi_w(f_w dg_w \times \sigma) \neq 0$$

by the orthogonality relations I.4.7. Hence the component of π at w is the Steinberg St_w. Then π is a σ-invariant cuspidal representation in I of (1.8.1), and (2.1.1) follows. Note that π_u is σ-stable since the right side depends only on $f_{0u}dh_u$. Moreover, π_u is generic since π is cuspidal. Consequently π_u is tempered, since it is σ-elliptic and generic.

Further, π_u is not Steinberg. Indeed, if it were, then it would be the lift of the special π_{0u}'', and (2.1.1) would become

$$\operatorname{tr} \pi_{0u}''(f_{0u}dh_u) = \sum m(\pi_{0u}) \operatorname{tr} \pi_{0u}(f_{0u}dh_u).$$

Taking $f_{0u}dh_u$ to be a matrix-coefficient of π_{0u}' we would conclude that $m(\pi_{0u}')$ is 0.

No π_{0u} is special. Indeed, taking $f_{0u}dh_u$ to be a pseudo-coefficient of a special π_{0u}, we obtain $m(\pi_{0u})$ on the right of (2.1.1), and on the left 0, by the twisted orthogonality relations of I.4.7.

Harish-Chandra's theorem quoted in (1.9.3) implies the finiteness claim.

The final claim was already observed in 1.9.1: using a pseudo-coefficient of π_u and the twisted trace formula we may construct π in I with the component π_u (and a Steinberg component). □

2.1.2 Proposition. *Only square-integrable* π_{0u} *appear in the sum of* (2.1.1). *This holds also when* u *is real.*

Proof. In the nonarchimedean case the π_{0u} on the right are cuspidal H_{0u}-modules, or irreducible constituents in the composition series of an induced H_{0u}-module. Fix a character μ_1 of $\mathbf{A}_0(R_u) \simeq R_u^\times$, and let $f_{0u}dh_u$ be an (n,μ_1)-regular function with $n \geq 1$. Then

$$\operatorname{tr} \pi_{0u}(f_{0u}dh_u + \overline{f}_{0u}dh_u)$$

vanishes unless π_{0u} is a constituent of $I_0(\mu)$ with $\mu = \mu_1$ on $R_u^\times$, where its value is $z^n + z^{-n}$, where $z = \mu(\boldsymbol{\pi})$. Hence the right side takes the form $\sum_i c_i(z_i^n + z_i^{-n})$. The sum is absolutely convergent, and $|z_i| = 1$, or $z_i = \overline{z}_i$, and $q_u^{-1} < |z_i| < q_u$ (by unitarity). It is also clear from the last assertion of Proposition 2.1 that this sum is finite. On the left, since π_u is σ-elliptic and generic, if the value of $\operatorname{tr} \pi_u(f_u dg_u \times \sigma)$ is not zero then π_u is induced from the special representation of a maximal parabolic subgroup of G_u, and $\operatorname{tr} \pi_u(f_u dg_u \times \sigma)$ is equal to $q_u^{-n/2}$. Applying the Stone-Weierstrass theorem as in (1.6.2) we conclude that $c_i = 0$ for all i. In particular the π_{0u} on the right are cuspidals, and π_u on the left is not induced from the special representation of the maximal parabolic.

When F_u is real, the sum is again absolutely convergent. The representation π_{0u} is either square integrable, and then $\operatorname{tr} \pi_{0u}(f_{0u}dh_u) = z^n$ for a suitable $f_{0u} = f_{0u}(n)$ and $z = z(\pi_{0u})$ with $|z| < 1$, or

$$\operatorname{tr} \pi_{0u}(f_{0u}dh_u) = \operatorname{tr}[I_0(\mu)](f_{0u}dh_u)$$

has the form $z^n + z^{-n}$. The argument of (1.6.2) implies the proposition. □

2.1.3 Proposition. *The sum of* (2.1.1) *is finite.*

Proof. For simplicity, omit u from the notations. The equality (2.1.1) shows that fdg depends only on its stable σ-orbital integrals. Hence the σ-character χ_π^σ of π is a σ-stable function. Then we can define $\chi_H(N\delta) = \chi_\pi^\sigma(\delta)$ on the σ-regular σ-elliptic set. List the π_0 with $m(\pi_0) \geq 1$ on the right of (2.1.1) as π_{0i} $(i = 1, 2, \dots)$. Choose matrix coefficients f_{0i} of π_{0i}. Put $f_0 dh = \sum_{1 \leq i \leq a} f_{0i} dh$. Put $'\Phi(\gamma, f_0 dh) = |Z_H(\gamma)|^{-1}\Phi(\gamma, f_0 dh)$. For our $f_0 dh$ it is equal to $\sum_{1 \leq i \leq a} \chi_{\pi_{0i}}(\gamma)$. Then the left side of (2.1.1) is

$$\operatorname{tr} \pi(fdg \times \sigma) = \langle \chi_H, {'\Phi}(f_0 dh) \rangle_e$$

$$\leq \langle \chi_H, \chi_H \rangle_e^{1/2} \cdot \left\langle \sum_{1 \leq i \leq a} \chi_{\pi_{0i}}, {'\Phi}(f_0 dh) \right\rangle_e^{1/2}.$$

Note that $\langle\sum_{1\le i\le a}\chi_{\pi_{0i}}, {}'\Phi(f_0dh)\rangle_e = a$. The right side of (2.1.1) is

$$\sum_{1\le i\le a} m(\pi_{0i}) \ge a.$$

But $a \le \langle\chi_H,\chi_H\rangle_e^{1/2}\sqrt{a}$ implies $a \le \langle\chi_H,\chi_H\rangle_e$. Hence the sum of (2.1.1) is finite. □

2.1.4 PROPOSITION. *In* (2.1.1), *the square-integrable* π_{0u} *determines uniquely the tempered* π_u.

PROOF. For simplicity, omit u. Set up (2.1.1) for π and π'. Thus $\chi_{\pi,H}(N\delta) = \chi_\pi^\sigma(\delta) = \sum_{\pi_0} m(\pi,\pi_0)\chi_{\pi_0}(N\delta)$ and $\chi_{\pi',H}(N\delta) = \chi_{\pi'}^\sigma(\delta) = \sum_{\pi_0} m(\pi',\pi_0)\chi_{\pi_0}(N\delta)$. The sums are finite and $m(\pi,\pi_0)\ge 0$, $m(\pi',\pi_0)\ge 0$. Orthogonality relations for characters on H give

$$\langle\chi_{\pi,H},\chi_{\pi',H}\rangle_e = \sum_{\pi_0} m(\pi,\pi_0)m(\pi',\pi_0) \ge 0.$$

This is nonzero iff there is a π_0 with $m(\pi,\pi_0) > 0$ and $m(\pi',\pi_0) > 0$, in which case $\pi \simeq \pi'$ by the orthogonality relations for twisted characters. □

It follows that the relations (2.1.1) define a partition of the set of square-integrable representations of H_u into finite sets.

DEFINITION. The set of irreducible representations π_u which occur in the sum of (2.1.1) is called a *packet*.

The packets then partition the set of equivalence classes of square-integrable representations of H_u. The packet of a Steinberg (= special) representation $\mathrm{sp}(\chi)$ of H_u consists only of $\mathrm{sp}(\chi)$. Here $\chi : F_u^\times/F_u^{\times 2} \to \{\pm 1\}$, and $\mathrm{sp}(\chi) = \chi\,\mathrm{sp}$ is defined by the exact sequence $0 \to \mathrm{sp}(\chi) \to I_0(\chi\nu_u^{1/2}) \to \chi\mathbf{1}_u \to 0$. We define the packet of a one-dimensional representation $\chi\mathbf{1}_u$ to consist only of $\chi\mathbf{1}_u$. The same applies to any nontempered representation and to any irreducible induced representation $I_0(\mu_u)$, thus $\mu_u \ne \chi\nu_u^{1/2}$, $\chi\nu_u^{-1/2}$, $\chi \ne 1 = \chi^2$. In these cases (2.1.1) holds:

$$\mathrm{tr}\,\mathrm{St}(\chi)(f_u dg_u \times \sigma) = \mathrm{tr}\,\mathrm{sp}(\chi)(f_{0u}dh_u),$$

$$\mathrm{tr}\,\chi\mathbf{1}_{\mathrm{PGL}(3,F_u)}(f_u dg_u \times \sigma) = \mathrm{tr}\,\chi\mathbf{1}(f_{0u}dh_u),$$

$$\mathrm{tr}\,I(\mu_u, 1, \mu_u^{-1}; f_u dg_u \times \sigma) = \mathrm{tr}\,I_0(\mu_u; f_{0u}dh_u).$$

When $\mu_u \neq 1 = \mu_u^2$ the induced $I_0(\mu_u)$ is the direct sum of two irreducible representations $I_0^+(\mu_u)$ and $I_0^-(\mu_u)$, and we define them to be in the same packet. In this case (2.1.1) holds with $\pi_u = I(\mu_u, 1, \mu_u)$. The superscript + or − is determined by:

2.1.5 PROPOSITION. *Let μ'_u be the trivial character on E_u^1, where E_u is the quadratic extension determined by $\chi_{E,u} \neq 1 = \chi_{E,u}^2$. For matching $f_{0u}dh_u$, $f_{T_{E_u}}dt$ we have*

$$\mu'_u(f_{T_{E_u}}dt) = \operatorname{tr} I_0^+(\chi_{E,u})(f_{0u}dh_u) - \operatorname{tr} I_0^-(\chi_{E,u})(f_{0u}dh_u).$$

PROOF. Several proofs of this are known. See [LL], Lemma 3.6, or [K1]. We shall use (1.9.2). For that we choose a global quadratic extension E/F whose completion at a place u is our E_u/F_u, which is unramified at all other places, and write (1.9.2) such that (only) the terms associated with E and $\mu' = 1$ on $\mathbb{A}_E^1/E^1$ contribute. The intertwining operator $M(\chi_E)$ is the product of the scalar $m(\chi_E) = L(1, \chi_E^{-1})/L(1, \chi_E) = 1$ and $\otimes_v R(\chi_{E,v})$, where the normalized intertwining operator $R(\chi_{E,v})$ acts on $I_0^+(\chi_{E,u})$ as 1 and on $I_0^-(\chi_{E,u})$ as -1 (defining the superscript). Applying "generalized linear independence" of characters at the places other than u, (1.9.2) takes the form

$$\operatorname{tr} R(\chi_{E,u}) I_0(\chi_{E,u})(f_{0u}dh_u) = \mu'_u(f_{T_{E_u}}dt). \qquad \square$$

Let E_u be a quadratic field extension of F_u; denote by E_u^1 the group of elements in E_u whose norm in F_u is one, as usual.

2.2 PROPOSITION. *Given a character μ'_u of $C_{E_u}^1 = E_u^1$ there are non-negative integers $m'(\pi_{0u})$ and a cuspidal (if $\mu'_u \neq 1$) representation $\pi(\mu''_u)$ of* $\mathrm{GL}(2, F_u)$ *such that*

$$\mu'_u(f_{T_{E_u}}dt) + \operatorname{tr} I(\pi(\mu''_u), \chi_{Eu}; f_u dg_u \times \sigma) = 2 \sum m'(\pi_{0u}) \operatorname{tr} \pi_{0u}(f_{0u}dh_u) \tag{2.2.1}$$

for all matching $f_{0u}dh_u$, $f_u dg_u$, $f_{T_{E_u}}dt$, where $\mu''_u(z) = \mu'_u(z/\overline{z})$ $(z \in E_u^\times)$. The sum is absolutely convergent and includes neither the trivial nor the special representation.

REMARK. Here u may be a real place.

PROOF. If u is nonarchimedean we work with a totally imaginary F. If u is real take $F=\mathbb{Q}$ and imaginary quadratic E. The claim is clear if (1) u splits E/F or, by 2.1.5, if (2) $\mu'_u = 1$, where $\pi_0(\mu'_u)$ is the induced representation $I_0(\chi_{E_u})$, $\pi(\mu''_u)$ is $I(\chi_{E_u},1)$ and

$$\mu'_u(f_{T_{E_u}}\,dt) = \operatorname{tr} I_0^+(\chi_{E_u})(f_{0u}dh_u) - \operatorname{tr} I_0^-(\chi_{E_u})(f_{0u}dh_u),$$

$$\operatorname{tr} I(\chi_{E_u},1,\chi_{E_u}; f_u dg_u \times \sigma) = \operatorname{tr} I_0(\chi_{E_u}, f_{0u}dh_u).$$

If $\mu'_u \neq 1$ on E^1_u we fix a finite split place $w \neq u$ and a character μ'_w of E^1_w with ${\mu'_w}^2 \neq 1$. Let μ' be a character of C^1_E which has the specified components at u and w, and all its components at the finite $v \neq u, w$ are unramified, except perhaps at a place $v' \neq u, w$ which splits in E if u is real. It is easy to construct such μ' using the trace (or Poisson summation) formula for the pair $\mathbb{A}^1_E$ and E^1, and a function $f = \otimes_v f_v$ with $f(1) \neq 0$; with $f_u = \overline{\mu}'_u$; $f_w = \overline{\mu}'_w$; f_v is the characteristic function of the maximal compact subgroup of E^1_v for all finite $v \neq u, w, v'$; and f_v is supported on a small compact neighborhood of 1 if v is complex (when u is finite) or if v is v' (if u is real).

Since ${\mu'_w}^2 \neq 1$ we have $\mu'^2 \neq 1$. We apply Proposition 1.8 with μ' on the right of (1.9.2). Then $\pi_0(\mu')$ appears on the right, in I_0.

We claim that there is a nonzero term on the left of (1.9.2), namely in I, I' or I''. If not, using the usual argument of linear independence of characters of (1.8.3), and Lemma 1.10, we conclude from (1.9.2) that $\sum_{\pi_{0u}} m'(\pi_{0u}) \operatorname{tr} \pi_{0u}(f_{0u}dh_u) = \frac{1}{2}\mu'_u(f_{T_{E_u}}dt)$. As $m'(\pi_{0u}) \geq 0$, the argument of 2.1.2 shows that only square-integrable π_{0u} would occur here. As $m(\pi_{0u}) \geq 0$, we may use the orthogonality relations on H_u with (2.1.1):

$$\sum_{\pi_{0u}} m(\pi_{0u}) \operatorname{tr} \pi_{0u}(f_{0u}dh_u) = \operatorname{tr} \pi_u(f_u dg_u \times \sigma),$$

to conclude that since μ'_u defines a σ-unstable function $\chi_{\mu'_u}$ on the elliptic set H_{ue} of H_u, and $\chi^\sigma_{\pi_u}$ a σ-stable function $\chi_{\pi_u,H}$ on H_{ue}, they are orthogonal to each other, so $0 = \sum_{\pi_{0u}} m(\pi_{0u})m'(\pi_{0u}) \geq 0$ and all $m'(\pi_{0u})$ are zero. Here we used the finiteness of (2.1.1), and that each π_{0u} occurs in (2.1.1) for some π_u. We conclude that there *is* a (unique) contribution π to one of I, I' or I''. Clearly its local components are the same as those of what $I(\pi(\mu''), \chi_E)$ should be at all split and unramified places. So we have a term π in I', which we name $I(\pi(\mu''), \chi_E)$. In particular its component at u is denoted $I(\pi(\mu''_u), \chi_{Eu})$.

To obtain (2.2.1) we apply the argument of (1.8.3) at all places (including w, v' or the complex places). □

COROLLARY. (2.1.1) *holds with* $\pi_u = I(\pi(\mu''_u), \chi_{E_u})$.

PROOF. In (2.2.1), $f_u dg_u$ depends only on its stable σ-orbital integrals. Hence the stable σ-orbital integrals of a pseudo-coefficient $f_{\pi_u} dg_u$ of $I(\pi(\mu''_u), \chi_{E_u})$ are nonzero on the σ-regular σ-elliptic set. Use the twisted trace formula with a test measure fdg with the component $f_{\pi_u} dg_u$ at a place u, and a pseudo-coefficient of a Steinberg representation (which is σ-invariant) at a place w, to create a global σ-invariant cuspidal π on $\mathbf{G}(\mathbb{A})$ with component π_u at u, Steinberg at w, unramified elsewhere. Apply (1.9.2) as in (2.1) to get (2.1.1) with $\pi_u = I(\pi(\mu''_u), \chi_{E_u})$. □

In particular we conclude that $\pi(\mu''_u)$ is uniquely determined by μ'_u, by 2.1.4.

2.2.2 PROPOSITION. *If* $\mu'_u \neq 1$, *only square-integrable* π_{0u} *appear in the sum of* (2.2.1). *The same holds also when u is real.*

PROOF. The proof of 2.1.2 applies here too. □

2.2.3 PROPOSITION. *The sum of* (2.2.1) *is finite.*

PROOF. For simplicity, omit u from the notations. The sum of (2.1.1) is finite. We substitute it for $\operatorname{tr} I(\cdots)$ in (2.2.1), to get

$$\mu'(f_{T_E} dt) = \sum_{i \geq 1} m''_i \operatorname{tr} \pi_{0i}(f_0 dh).$$

Here we labeled the π_0 with $2m'(\pi_0) - m(\pi_0) \neq 0$ by $i \geq 1$, m''_i are the integers $2m'(\pi_{0i}) - m(\pi_{0i})$, $2m'(\pi_{0i})$ is from (2.2.1) and $m(\pi_{0i})$ are the (finitely many nonzero) coefficients from (2.1.1).

Recall that we have $f_{T_E}(t)dt = \kappa(b)\Delta_0(t)\Phi^{\mathrm{us}}(t, f_0 dh)$ on $t \in T_E$. In Proposition II.1.8 we defined a function $\chi(t) = \chi_{\mu'}(t)$ on t in the regular set of H to be the unstable function which is zero unless $t \in T_0$ (up to stable conjugacy), in which case it is $\kappa(b)\Delta_0(t)^{-1}\mu'(t)$. By Proposition II.1.8 $\mu'(f_{T_E} dt) = \langle \chi, {}'\Phi(f_0 dh) \rangle_e$. This is

$$\leq \langle \chi, \chi \rangle_e^{1/2} \cdot \left\langle {}'\Phi(f_0 dh), \sum_{1 \leq i \leq a} \frac{|m''_i|}{m''_i} \chi_{\pi_{0i}} \right\rangle_e^{1/2}$$

for $f_0dh = \sum_{1\le i\le a} \frac{|m_i''|}{m_i''} f_{\pi_{0i}} dh$. Here $f_{\pi_{0i}} dh$ is a pseudo-coefficient of π_{0i}. Hence $\mu'(f_{T_E}dt) \le \langle\chi,\chi\rangle_e^{1/2}\sqrt{a}$. But for our f_0dh, $\sum_{i\ge1} m_i'' \operatorname{tr}\pi_{0i}(f_0dh) = \sum_{1\le i\le a} |m_i''| \ge a$. Hence $a \le \langle\chi,\chi\rangle_e$, our sum is finite and so is the sum of (2.2.1). □

2.2.4 Corollary. *Let F be a local field. If $\mu'^2 \ne 1$ there exist irreducible inequivalent cuspidal representations $\pi_0^+(\mu')$ and $\pi_0^-(\mu')$ such that for all matching measures f_0dh and $f_{T_E}dt$ we have*

$$\mu'(f_{T_E}dt) = \operatorname{tr}\pi_0^+(\mu')(f_0dh) - \operatorname{tr}\pi_0^-(\mu')(f_0dh).$$

If $\mu' \ne 1 = \mu'^2$ the same holds except that $\pi_0^+(\mu')$ and $\pi_0^-(\mu')$ are sums with multiplicity one of irreducibles, have no irreducible in common, and contain together 4 irreducibles.

Proof. Since μ_1' defines a σ-unstable function $\chi_{\mu_1'}$ on the elliptic set H_e of H for all μ_1', and χ_π^σ a σ-stable function $\chi_{\pi,H}$ on H_e, they are orthogonal to each other. Therefore $0 = \sum_{\pi_0} m(\pi_0)m''(\pi_0)$, and $m(\pi_0) \ge 0$ for all π_0, imply that $m''(\pi_0)$ takes both positive and negative values for each μ'.

Proposition II.1.8 asserts that $\langle\chi_{\mu'},\chi_{\mu'}\rangle_e$ of the proof of 2.2.3 is 2 if $\mu'^2 \ne 1$ and 4 if $\mu'^2 = 1$. The (end of the) proof of 2.2.3 shows that $(\sum_{i\ge1} |m_i''|)^2 \le a\langle\chi_{\mu'},\chi_{\mu'}\rangle_e$. If $\langle\chi_{\mu'},\chi_{\mu'}\rangle_e = 2$, $a = 2$ and $|m_i''| = 1$. If $\langle\chi_{\mu'},\chi_{\mu'}\rangle_e = 4$, a might be 2, 3 or 4 (but not 1, as m_i'' takes both positive and negative values). Had there been an m_i'' with absolute value at least 2, $(\sum_{i\ge1} |m_i''|)^2$ would be at least $(2+a-1)^2 > 4a$. Hence all (nonzero) $|m_i''|$ are 1. Then $\chi_{\mu'}$ is the difference of the characters of two disjoint (have no irreducible in common) cuspidal representations of H, which we name $\pi_0^+(\mu')$ and $\pi_0^-(\mu')$. From $\langle\chi_{\mu'},\chi_{\mu'}\rangle_e = 4$ we conclude that $\pi_0^+(\mu') \oplus \pi_0^-(\mu')$ is the direct sum of 4 irreducibles. □

If $\mu'^2 \ne 1$, substituting the identity displayed in (2.2.4) back in (2.2.1) we get

$$\operatorname{tr} I(\pi(\mu''),\chi_E; fdg\times\sigma) = (2m(\pi_0^+(\mu'))+1)\operatorname{tr}\pi_0^+(\mu')(f_0dh)$$

$$+ (2m(\pi_0^-(\mu'))+1)\operatorname{tr}\pi_0^-(\mu')(f_0dh) + 2\sum_{\pi_0} m(\pi_0)\operatorname{tr}\pi_0(f_0dh). \tag{2.2.5}$$

The sum over π_0 is finite and the m are nonnegative integers. Applying orthogonality (stable character against an unstable character) with the identity of (2.2.4) we conclude that $m(\pi_0^+(\mu')) = m(\pi_0^-(\mu'))$.

Denote by E_i $(1 \le i \le 3)$ distinct quadratic extensions of the local field F, and by μ_i' a quadratic character of E_i^1. Thus $\mu_i''(z) = \mu_i'(z/\overline{z}) = \mu_i''(\overline{z}) = \mu_i(N_{E_i/F} z)$, where μ_i is a quadratic character of $F^\times$ nontrivial on $N_{E_i/F} E_i^\times$. We choose μ_i to be trivial on $N_{E_j/F} E_j^\times$, $j \ne i$.

2.2.6 PROPOSITION. *There are cuspidal irreducible representations π_{0j}, $1 \le j \le 4$, such that*

$$\mu_i'(f_{T_{E_i}} dt) = \operatorname{tr} \pi_{01}(f_0 dh) + \operatorname{tr} \pi_{0i+1}(f_0 dh) - \operatorname{tr} \pi_{0j}(f_0 dh) - \operatorname{tr} \pi_{0j'}(f_0 dh)$$

where $\{i+1, j, j'\} = \{2, 3, 4\}$.

PROOF. The character relation $\mu_i'(f_{T_{E_i}} dt) = \sum_{1 \le j \le 4} \varepsilon_{ij} \operatorname{tr} \pi_{0j}(f_0 dh)$, where π_{0j} are irreducible cuspidal and $\{\varepsilon_{ij}; 1 \le j \le 4\} = \{1, -1\}$, implies the character relation, with nonnegative coefficients, where $\mu_i = \chi_{E_i}$ is associated with E_i $(1 \le i \le 3)$,

$$\operatorname{tr} I(\mu_1, \mu_2, \mu_3; f dg \times \sigma) = \sum_{1 \le j \le 4} (2m_j + 1) \operatorname{tr} \pi_{0j}(f_0 dh) \qquad (2.2.7)$$

$$+ 2 \sum_{\pi_0 \ne \pi_{0j}} m(\pi_0) \operatorname{tr} \pi_0(f_0 dh).$$

Namely the π_{0j} are those with odd coefficients. Hence the set $\{\pi_{0j}\}$ is independent of i (that is, of μ_i').

Our claim is that for each i, $1 \le i \le 3$, precisely two out of the four ε_{ij}, $1 \le j \le 4$, are 1. If not, we may assume that $\varepsilon_{1j} = (1, -1, -1, -1)$. Using the orthogonality relations

$$0 = \langle \chi_{\mu_i'}, \chi_{\mu_j'} \rangle_e = \sum_{1 \le k \le 4} \varepsilon_{ik} \varepsilon_{jk}$$

we may assume that $\varepsilon_{2j} = (-1, 1, -1, -1)$. Using orthogonality of the stable σ-character of $I(\mu_1, \mu_2, \mu_3)$ against the unstable characters $\chi_{\mu_i'}$, $i = 1, 2$, we conclude that

$$2m_1 + 1 = 2m_2 + 1 + 2m_3 + 1 + 2m_4 + 1,$$

$$2m_2 + 1 = 2m_1 + 1 + 2m_3 + 1 + 2m_4 + 1.$$

Hence $m_3 + m_4 + 1 = 0$, contradicting $m_j \geq 0$. Hence precisely two out of ε_{ij}, $1 \leq j \leq 4$, are 1, for each i. Using the orthogonality of $\chi_{\mu'_i}$ and $\chi_{\mu'_j}$ we conclude that up to reordering, $\varepsilon_{1j} = (1, 1, -1, -1)$, $\varepsilon_{2j} = (1, -1, 1, -1)$, $\varepsilon_{3j} = (1, -1, -1, 1)$. □

Put $\pi_0^+(\mu'_i) = \pi_{01} \oplus \pi_{0i+1}$ and $\pi_0^-(\mu'_i) = \pi_{0j} \oplus \pi_{0j'}$ (when $\mu'_i \neq 1 = \mu_i'^2$). Note that the superscript + or − depends on i in μ'_i. Recall that packets were defined after 2.1.4.

The next result holds for all $\operatorname{tr} I(\pi(\mu''), \chi_E; fdg \times \sigma)$. It asserts that all $m(\pi_0)$ in (2.2.5) and (2.2.7) are 0.

2.2.8 PROPOSITION. (1) *The (finite) sum over* π_0 *in* (2.2.5) *and in* (2.2.7) *is empty.*
(2) *The* m_j *in* (2.2.7) *are independent of* j.

PROOF. (1) Introduce the class functions on the elliptic regular set of H:

$$\chi^1 = (2m+1) \sum_{1 \leq j \leq 4} \chi_{\pi_{0j}} \qquad \text{if} \qquad \mu' \neq 1 = \mu'^2 \neq 1$$

$(= (2m+1)(\chi_{\pi^+} + \chi_{\pi^-})$ if $\mu'^2 \neq 1)$ and $\chi^0 = 2\sum_{\pi_0} m(\pi_0)\chi_{\pi_0}$. Also write χ_I^σ for the class function on the regular set of H whose value at the stable conjugacy class Ng is $\chi_{I(\pi(\mu''),\chi_E)}(g \times \sigma)$.

Our first claim is that χ^1 (and χ^0) is stable. It suffices to show that $\langle \chi^1, \chi_{\mu'_1} \rangle_e$ is 0 for every quadratic extension E of F and every character μ'_1 of E. But this follows on applying orthogonality relations with the identities of 2.2.4 and 2.2.6, and on using 2.1.4.

Next we claim that χ^0 is zero. If not,

$$\chi = \langle \chi^1 + \chi^0, \chi^1 \rangle_0 \cdot \chi^0 - \langle \chi^1 + \chi^0, \chi^0 \rangle_0 \cdot \chi^1$$

is a nonzero stable function on the elliptic regular set of H. (Note that $\langle \chi^0, \chi^1 \rangle_0 = 0$). Choose $f'_{v_0} dg_{v_0}$ on G_{v_0} such that $'\Phi(t, f'_{v_0} dg_{v_0} \times \sigma) = \chi(Nt)$ on the σ-elliptic σ-regular set of G_{v_0} and it is zero outside the σ-elliptic set. As usual fix a totally imaginary field F and create a cuspidal σ-invariant representation π which is unramified outside v_0, v_1, has the component St_{v_1} at v_1 and $\operatorname{tr} \pi_{v_0}(f'_{v_0} dg_{v_0} \times \sigma) \neq 0$. Since π is cuspidal as usual by generalized linear independence of characters we get the local identity

$$\operatorname{tr} \pi_{v_0}(f_{v_0} dg_{v_0} \times \sigma) = \sum_{\pi_{0,v_0}} m^1(\pi_{0,v_0}) \operatorname{tr} \pi_{0,v_0}(f_{0,v_0} dh_{v_0})$$

for all matching $f_{v_0}dg_{v_0}$, $f_{0,v_0}dh_{v_0}$. The local representation $\pi = \pi_{v_0}$ is perpendicular to $I(\pi(\mu''),\chi_E)$ since $\langle\chi,\chi^0+\chi^1\rangle_0 = 0$, and $\chi^0+\chi^1 = \chi^\sigma_{I(\pi(\mu''),\chi_E)}$. Since $\chi^1+\chi^0$ is perpendicular to the σ-twisted character χ^σ_Π of any σ-invariant representation Π inequivalent to $I(\pi(\mu''),\chi_E)$, χ is also perpendicular to all χ^σ_Π, hence $\operatorname{tr}\Pi(f'_{v_0}dg_{v_0}\times\sigma) = 0$ for all σ-invariant representations Π, contradicting the construction of π_{v_0} with $\operatorname{tr}\pi_{v_0}(f'_{v_0}dg_{v_0}\times\sigma)\neq 0$. Hence $\chi = 0$, which implies that $\chi^0 = 0$, as required.

(2) follows on using orthogonality of the σ-character of the stable, induced $I(\mu_1,\mu_2,\mu_3)$, against the unstable characters $\chi_{\mu'_i}$, $i = 1, 2$. □

An irreducible representation of $\mathrm{SL}(2,F_u)$ (resp. $\mathrm{GL}(2,F_u)$) is called *monomial* if it is of the form $\pi_0(\mu'_u)$ (resp. $\pi(\mu^*_u)$) for a character μ'_u of E^1_u (resp. μ^*_u of $E^\times_u$) where E_u is a quadratic extension of F_u. A cuspidal representation is called nonmonomial if it is not monomial. A packet is defined to be the set of π_0 which appear on the right of (2.1.1).

2.2.9 PROPOSITION. (1) *If π_u on the left of* (2.1.1) *is cuspidal then π'_{0u} is nonmonomial, it is the only term on the right of* (2.1.1)*, and $m(\pi'_{0u}) = 1$. The residual characteristic is* 2.
(2) *The packet $\{\pi_0(\mu')\}$ is the set of irreducibles in $\pi_0^+(\mu')$ and in $\pi_0^-(\mu')$. It consists of four irreducibles if $\mu'\neq 1 = \mu'^2$, in which case there are three pairs (E_i,μ'_i) with $\mu'_1 = \mu'$ and $\{\pi_0(\mu'_j)\} = \{\pi_0(\mu')\}$, $1\leq j\leq 3$, and of* 2 *irreducibles otherwise. If $\mu' = 1$ on E^1 then $\pi_0^\pm(\mu') = I_0^\pm(\chi_E)$. In all other cases a packet consists of a single irreducible.*

PROOF. (1) For a cuspidal π_u we have twisted orthonormality relations for its character (II.4.3.1), namely $\langle\chi_{\pi_u},\chi_{\pi_u}\rangle_e = 1$ in the notations of II.4.4. On the right the orthogonality relations for characters (II.4.2) imply that

$$\left\langle \sum m(\pi_{0u})\chi_{\pi_{0u}}, \sum m(\pi_{0u})\chi_{\pi_{0u}} \right\rangle$$

is equal to $\sum m(\pi_{0u})^2$. It follows that the sum consists of a single π_{0u} with coefficient $m(\pi_{0u}) = 1$. It is nonmonomial since the cuspidal π_u is orthogonal to any $I(\pi(\mu''),\chi_E)$.

Nonmonomial representations exist only in even residual characteristic $p = 2$. See Deligne [D5], Proposition 3.1.4, and Tunnell [Tu].

(2) follows on applying 2.1.4 to 2.2.7, using 2.2.8. The right side of (2.1.1) defines a packet. □

REMARK. A packet $\{\pi_0\}$ contains an unramified π_0^0 and has cardinality $[\{\pi_0\}] \neq 1$ only if it is $I_0(\chi_E)$ where E is the unramified extension of F.

2.3 PROPOSITION. *For $g \in \mathrm{GL}(2, F)$ put $\pi_0^g(h) = \pi_0(g^{-1}hg)$. Put*

$$G(\pi_0) = \{g \in \mathrm{GL}(2, F); \pi_0^g \simeq \pi_0\},$$

$$G_E = \{g \in \mathrm{GL}(2, F); \det g \in N_{E/F}E^\times\}.$$

Then:
(0) *The packet $\{\pi_0\}$ of π_0 consists of the distinct irreducibles π_0^g, $g \in \mathrm{GL}(2, F)$.*
(1) *If $[\{\pi_0\}] = 1$ then $G(\pi_0) = \mathrm{GL}(2, F)$.*
(2) *If $[\{\pi_0\}] = 2$, thus $\{\pi_0\} = \pi_0(\mu')$, $\mu'^2 \neq 1$, μ' on E^1, then $G(\pi_0) = G_E$.*
(3) *If $[\{\pi_0\}] = 4$, thus $\{\pi_0\} = \pi_0(\mu_i')$, $\mu_i' \neq 1 = \mu'^2$, μ_i' on E_i^1 $(1 \leq i \leq 3)$, then $G(\pi_0) = \cap_{1 \leq i \leq 3} G_{E_i}$. Moreover, each $\pi_0^\pm(\mu_i')$ consists of two irreducibles $\pi_{0ij}^\pm (j = 1, 2)$ with $\pi_{0i2}^\pm = \pi_{0i1}^{\pm g}$, $g \in G_{E_i} - G(\pi_0)$.*

PROOF. We use the identity

$$\pi_0^g(f_0 dh) = \int \pi_0(g^{-1}hg) f_0(h) dh = \int \pi_0(h) f_0(ghg^{-1}) dh = \pi_0({}^g f_0 dh),$$

and the fact that $f_0 dh$ and ${}^g f_0 dh$ have the same stable orbital integrals. The distribution $f_0 dh \mapsto \sum_{\{\pi_0\}} \operatorname{tr} \pi_0(f_0 dh)$ is stably invariant (it depends only on the stable orbital integrals of $f_0 dh$), since

$$\operatorname{tr} \pi(f dg \times \sigma) = (m + 1) \sum_{\{\pi_0\}} \operatorname{tr} \pi_0(f_0 dh)$$

for the lift π of $\{\pi_0\}$. Hence we have

$$\sum_{\{\pi_0\}} \operatorname{tr} \pi_0(f_0 dh) = \sum_{\{\pi_0\}} \operatorname{tr} \pi_0({}^g f_0 dh) = \sum_{\{\pi_0\}} \operatorname{tr} \pi_0^g(f_0 dh).$$

Hence $\{\pi_0^g\} = \{\pi_0\}$ (the packet of π_0^g is the same as that of π_0; $g \mapsto \pi_0^g$ permutes the irreducibles in the packet $\{\pi_0\}$). Then $[\mathrm{GL}(2, F) : G(\pi_0)] = [\{\pi_0\}]$ and in particular (0) and (1) follow.

For a quadratic extension E of F and a torus $T_E \simeq E^1$ in $\mathrm{SL}(2, F)$, $f_{T_E}(t)dt$ depends on $\Phi(t, f_0 dh) - \Phi(t^g, f_0 dh)$ with any $g \in \mathrm{GL}(2, F) - G_E$. The centralizer $Z_{\mathrm{GL}(2,F)}(t)$ of t in $\mathrm{GL}(2, F)$ is the torus T_E^* in $\mathrm{GL}(2, F)$

centralizing T_E. It has $\det T_E^* = NE^\times$, hence $\Phi(t, f_0 dh) = \Phi(t, {}^h f_0 dh)$ for all $h \in \mathrm{SL}(2,F)T_E^*$, thus for all $h \in G_E$ (same holds with t replaced by t^g). Then the character relation

$$\mu'(f_{T_E} dt) = \operatorname{tr} \pi_0^+(\mu')(f_0 dh) - \operatorname{tr} \pi_0^-(\mu')(f_0 dh)$$

does not change on replacing $f_0 dh$ by ${}^h f_0 dh$ if $\det h \in NE^\times$. Hence for such h, if $\pi_0 \in \pi_0^\pm(\mu')$ then $\pi_0^h \in \pi_0^\pm(\mu')$. (2) and (3) follow. □

2.3.1 Lemma. *Let H'' be a subgroup of index 2 in H'.*
(1) *The restriction $\pi|H''$ to H'' of an admissible irreducible representation π of H' is irreducible or the direct sum of two irreducibles π_1, π_2 with $\pi_2 = \pi_1^g$ for any $g \in H' - H''$.*
(2) *Any irreducible admissible representation π_1 of H'' is contained in the restriction to H'' of an irreducible admissible representation of H'.*

Proof. (1) If the restriction of π to H'' is reducible, its space, V, contains a nontrivial irreducible H''-invariant subspace W. If $g \in H' - H''$ then $V = W + \pi(g)W$ and $W \cap \pi(g)W$ is H'-invariant, hence zero. Thus $V = W \oplus \pi(g)W$ and $\pi|H'' = \pi_1 \oplus \pi_2$ with $\pi_2 = \pi_1^g$.

(2) Define $\pi = \operatorname{Ind}_{H''}^{H'} \pi_1$. If $\pi_1^g \neq \pi_1$ for some, hence any, $g \in H' - H''$, then π is irreducible, $\omega\pi = \pi$ if $\omega|H'' = 1$, and the restriction of π to H'' contains π_1. Otherwise let $A : W \to W$ be an operator intertwining π_1^g with π_1 (W denotes the space of π_1): $A\pi_1(g^{-1}hg) = \pi_1(h)A$ ($h \in H''$). Schur's lemma permits us to choose $A^2 = \pi_1(g^2)$. Extend π_1 to a representation π' of H' by $\pi'(g) = A$. Then π is $\pi' \oplus \omega\pi'$ where ω is the nontrivial character of H'/H''. The restriction of π' to H'' is π_1. □

2.3.2 Proposition. *For every packet $\{\pi_0\}$ of $\mathrm{SL}(2,F)$ and character ω of $F^\times = \mathbf{Z}(F)$ (= center of $\mathrm{GL}(2,F)$) with $\omega(-I) = \pi_0(-I)$ there exists a unique irreducible representation π^* of $\mathrm{GL}(2,F)$ with central character ω whose restriction to $\mathrm{SL}(2,F)$ contains $\pi_0 \in \{\pi_0\}$. We have that $\pi^*|\,\mathrm{SL}(2,F)$ is the direct sum of the π_0 in $\{\pi_0\}$, and $\mu\pi^* \simeq \pi^*$ iff μ is 1 on $G(\pi_0) = \{g \in \mathrm{GL}(2,F); \pi_0^g = \pi_0\}$.*

Proof. Extend π_0 to $\mathrm{SL}(2,F)\mathbf{Z}(F)$ by ω on $\mathbf{Z}(F)$. Extend π_0 from $\mathrm{SL}(2,F)\mathbf{Z}(F)$ to $G(\pi_0)$. If $[\{\pi_0\}] = 1$, $G(\pi_0) = \mathrm{GL}(2,F)$ and we obtain an irreducible π^* of $\mathrm{GL}(2,F)$ whose restriction to $\mathrm{SL}(2,F)$ is π_0. Moreover, $\mu\pi^* = \pi^*$ for a character μ of $\mathrm{GL}(2,F)$ only if $\mu = 1$.

If $[\{\pi_0\}] = 2$, define $\pi^* = \mathrm{Ind}_{G_E}^{\mathrm{GL}(2,F)}(\pi_0)$. It is irreducible, $\chi_E\pi^* = \pi^*$ where χ_E is the nontrivial character on $\mathrm{GL}(2,F)$ with kernel G_E, and $\pi^*|G_E = \pi_0 \oplus \pi_0^g$ with $g \in \mathrm{GL}(2,E) - G_E$, thus $\pi|\,\mathrm{SL}(2,F) = \{\pi_0\}$.

If $[\{\pi_0\}] = 4$, $\pi_0 \in \pi_0^{\pm}(\mu_i')$, put $\tilde{\pi}_0^{\pm}(\mu_i') = \mathrm{Ind}_{G(\pi_0)}^{G_{E_i}}(\pi_0)$. It is irreducible, $\chi_{E_j} \cdot \tilde{\pi}_0^{\pm}(\mu_i') = \tilde{\pi}_0^{\pm}(\mu_i')$ for the character χ_{E_j} of $G_{E_i}/G(\pi_0)$ (which is the restriction to G_{E_i} of the character of $\mathrm{GL}(2,F)/G_{E_j}$ where $\{\pi_0\} = \{\pi_0(\mu_j')\}$, μ_j' on E_j^1, $j \neq i$), and $\tilde{\pi}_0^{\pm}(\mu_i')|G_{E_i} = \pi_0^{\pm}(\mu_i')$, a direct sum of two irreducibles. Further we put $\pi^* = \mathrm{Ind}_{G_{E_i}}^{\mathrm{GL}(2,F)}(\tilde{\pi}_0^{+}(\mu_i'))$. It is irreducible, $\chi_{E_j} \cdot \pi^* = \pi^*$ for all $j = 1,2,3$, and $\pi^*|G_{E_i} = \tilde{\pi}_0^{+}(\mu_i') \oplus \tilde{\pi}_0^{-}(\mu_i')$, and $\pi^*|G(\pi_0)$ is the direct sum of the irreducibles in $\{\pi_0\}$ (as is $\pi^*|\,\mathrm{SL}(2,F)$). Moreover, π^* is independent of j $(= 1,2,3)$, and $\omega\pi^* = \pi^*$ only for $\omega = \chi_{E_j}$ $(j = 1,2,3)$ or $\omega = 1$. □

The packet $\{\pi_0(\mu')\}$ depends on the projective induced representation $\mathrm{Ind}_{W_E}^{W_F}(\mu')_0$, hence $\{\pi_0(\mu')\} = \{\pi_0(\overline{\mu}')\}$ where $\overline{\mu}'(x) = \mu'(\overline{x})$, conjugation of E over F. Thus a better notation is $\{\pi_0(\mathrm{Ind}_E^F(\mu')_0)\}$. Extending the character μ' of C_E^1 to μ^* on C_E we lift the projective representation $\mathrm{Ind}_E^F(\mu')_0$ to the two-dimensional representation $\mathrm{Ind}_E^F(\mu^*)$ of $W_{E/F}(= W_F/W_E^c)$:

$$C_E \ni z \mapsto \begin{pmatrix} \mu^*(z) & 0 \\ 0 & \mu^*(\overline{z}) \end{pmatrix}, \quad \sigma \mapsto \begin{pmatrix} 0 & 1 \\ \mu^*(\sigma^2) & 0 \end{pmatrix}.$$

Thus $\mathrm{Ind}_E^F(\mu')_0$ is the composition of $\mathrm{Ind}_E^F(\mu^*)$ and $\mathrm{GL}(2,\mathbb{C}) \to \mathrm{PGL}(2,\mathbb{C})$; it depends only on the restriction μ' of μ^* from C_E to C_E^1. The determinant of $\mathrm{Ind}_E^F(\mu^*)$ is

$$C_E \ni z \mapsto \mu^*(z\overline{z}), \quad \sigma \mapsto \chi_E(\sigma^2)\mu^*(\sigma^2).$$

It factorizes as the composition of the norm $N : W_{E/F} \to C_F$, $C_E \ni z \mapsto z\overline{z}$, $\sigma \mapsto \sigma^2 \in C_F - N_{E/F}C_E$, and the character $\omega(x) = \chi_E(x)\mu^*(x)$ on C_E.

Definition. The representation $\pi(\mu^*)$, or more precisely $\pi(\mathrm{Ind}_E^F \mu^*)$, of $\mathrm{GL}(2,F)$, is the π^* of 2.3.2 associated with $\omega = \chi_E \cdot \mu^*|F^\times$ and $\{\pi_0(\mu')\}$, $\mu' = \mu^*|E^1$, if $\mu^* \neq \overline{\mu}^*$ (or $\mu' \neq 1$).

If $\mu^* = \overline{\mu}^*$, thus $\mu' = 1$, then $\mathrm{Ind}_E^F(\mu^*) = \mu \oplus \chi_E\mu$ is reducible, where $\mu^*(z) = \mu(z\overline{z})(z \in E^\times)$ defines μ and $\chi_E\mu$ on $F^\times$. Define $\pi(\mu^*)$, or $\pi(\mathrm{Ind}_E^F \mu^*)$, to be the induced representation $I(\mu, \chi_E\mu)$ of $\mathrm{GL}(2,F)$. Its restriction to $\mathrm{SL}(2,F)$ is $I_0(\chi_E)$, a tempered reducible representation, $= \pi_0^+(\chi_E) \oplus \pi_0^-(\chi_E)$.

Note that given μ' on E^1 and ω on $F^\times$ with $\omega(-1) = \chi_E(-1)\mu'(-1)$ there is μ^* on $E^\times$ extending μ' and ω.

We have $\pi(\mu^*)|\operatorname{SL}(2,F) = \{\pi_0(\mu')\}$ and $\chi_E \cdot \pi(\mu^*) = \pi(\mu^*)$. If $\mu^{*2} \neq \overline{\mu}^{*2}$, thus $\mu'^2 \neq 1$, then $\eta \cdot \pi(\mu^*) = \pi(\mu^*)$ implies $\eta = \chi_E$ or 1.

If $\mu^* \neq \overline{\mu}^*$ but $\mu^{*2} = \overline{\mu}^{*2}$, thus $\mu' \neq 1 = \mu'^2$, then $\eta \cdot \pi(\mu^*) = \pi(\mu^*)$ implies that $\eta = \chi_{E_i}$ (or 1), where E_1, E_2, E_3 are the quadratic extensions of F with $\{\pi_0(\mu')\} = \{\pi_0(\mu'_i)\}$. If $E_1 = E$ and $\mu'_1 = \mu'$, recall that E_i, μ'_i are defined by $\mu'_1(z/\overline{z}) = \mu'_1(\overline{z}/z) = \mu_1(z\overline{z})$ ($z \in E_1^\times$), μ_1 extends to $F^\times$ from $N_{E_1/F}E_1^\times$ as χ_{E_2} or $\chi_{E_3} = \chi_{E_2}\chi_{E_1}$ (these are the only characters whose restriction to $NE_1^\times$ is the quadratic character μ_1, thus μ'_1, namely μ_1 defines E_2, E_3), and we define $\mu'_i(z/\overline{z}) = \mu'_i(\overline{z}/z) = \mu_i(z\overline{z})$ on $z \in E_i^\times$ where now bar indicates $\operatorname{Gal}(E_i/F)$-action, where $\mu_i = \chi_{E_j}|NE_i^\times (j \neq i)$. The signs $\omega(-1) = \chi_{E_i}(-1)\mu'_i(-1)$ are independent of i since $\{\pi_0(\mu'_i)\}$ share central character, being independent of i. We extend μ'_i and ω to μ_i^* on $E_i^\times$ to get $\pi(\mu^*) = \pi(\mu_1^*) = \pi(\mu_2^*) = \pi(\mu_3^*)$ with $\eta \cdot \pi(\mu^*) = \pi(\mu^*)$ iff $\eta = \chi_{E_i}$ $(1 \leq i \leq 3)$ or $\eta = 1$. Note that on $F^\times$ we have $\chi_{E_i}(x)\mu_i^*(x) = \omega(x) = \chi_{E_j}(x)\mu_j^*(x)$, thus $\mu_j^*(x) = \chi_{E_i}(x)\chi_{E_j}(x)\mu_i^*(x)$ on $F^\times$.

The groups $\operatorname{SL}(2)$ and $\operatorname{GL}(2) = \operatorname{SL}(2) \rtimes \mathbb{G}_m$ are closely related. It is useful to compare their representation theories. Generalizing the question a little, put — in the rest of this subsection 2.3 —

$$\mathbf{G} = \operatorname{GSp}(n) = \{g \in \operatorname{GL}(2n); {}^t gJg = \lambda J\}, \quad J = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix},$$

$w = \operatorname{antidiag}(1, \dots, 1)$, for the group of symplectic similitudes of (semisimple) rank n and $\mathbf{H} = \operatorname{Sp}(n) = \{g \in \operatorname{GL}(2n); {}^t gJg = J\}$ for the symplectic group of rank n. Note that $\mathbf{H} \subset \operatorname{SL}(2n)$, $\operatorname{GSp}(1) = \operatorname{GL}(2)$, $\operatorname{Sp}(1) = \operatorname{SL}(2)$, and $\operatorname{GSp}(n) = \operatorname{Sp}(n) \rtimes \mathbb{G}_m$ by $h = g\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \operatorname{Sp}(n)$ if λ is the factor of similitude of $g \in \operatorname{GSp}(n)$. For $g \in \operatorname{GL}(2)$, $\lambda(g) = \det g$.

Let F be a local field of characteristic 0, put $G = \operatorname{GSp}(n,F)$, $H = \operatorname{Sp}(n,F)$, Z for the center of G (it consists of the scalar matrices zI, $z \in F^\times$, and $\lambda(zI) = z^2$, $I = I_{2n}$), and $H^+ = ZH$. Let Λ be a subgroup of $F^\times$ containing $F^{\times 2}$. Define $H_\Lambda = \{g \in G; \lambda(g) \in \Lambda\}$. Then $H_{F^{\times 2}} = H^+$, $H_{F^\times} = G$, and $G/H_\Lambda = F^\times/\Lambda$, $H_\Lambda/H = \Lambda/F^{\times 2}$. Since the product HZ is direct, and $Z \cap H$ is the center $\{\pm I\}$ of H, an irreducible admissible representation π_0 of H extends to H^+ on extending its central character from $\{\pm I\}$ to $F^\times$.

2.3.3 PROPOSITION. (1) *The restriction* $\pi|H_\Lambda$ *to* H_Λ *of an admissible irreducible representation* π *of* G *is the direct sum of* $\leq [F^\times/\Lambda]$ *irreducible representations* π_Λ. *If* $\pi|H_\Lambda$ *contains* π_Λ *and* π'_Λ *then* $\pi'_\Lambda = \pi^g_\Lambda$ *for some* $g \in G$.
(2) *Any irreducible admissible representation of* H_Λ *is contained in the restriction to* H_Λ *of an irreducible admissible representation of* G.

PROOF. Since $F^\times/F^{\times 2}$ is a finite product of copies of $\mathbb{Z}/2$, it suffices to prove the claims with G, H_Λ replaced by $H'' = H_{\Lambda''} \supset H' = H_{\Lambda'}$. This is done in Lemma 2.3.1. □

Let N denote the unipotent upper triangular subgroup of G. Then $N \subset H$. Let ψ be a generic character of N, thus

$$\psi = \psi_\alpha : (u_{ij}) \mapsto \psi_0\Big(\sum \alpha_i u_{i,i+1}\Big), \qquad \alpha_i \in F^\times, \qquad 1 \leq i \leq n$$

and $\psi_0 : F \to \mathbb{C}^1$ is a nontrivial character. There is a single orbit of generic characters under the action of the diagonal subgroup of $G : a \cdot \psi_1(u) = \psi_1(\mathrm{Int}(a)u) = \psi_\alpha(u)$, where ψ_1 is ψ_α with all $\alpha_i = 1$, $\mathrm{Int}(a)u = aua^{-1}$, and

$$a = \mathrm{diag}(a_1, \ldots, a_n, \lambda/a_n, \ldots, \lambda/a_1)$$

with $a_i/a_{i+1} = \alpha_i (1 \leq i < n)$ and $a_n/(\lambda/a_n) = \alpha_n$. The orbits of the generic ψ under the action of diagonal subgroup of H_Λ are parametrized by $F^\times/\Lambda$, as $\lambda \in \Lambda$.

If $A \subset B$, denote by Ind_A^B the functor of induction from A to B, and by Res_A^B the functor of restriction from B to A ([BZ1]). An irreducible representation π_Λ of H_Λ is called *ψ-generic* if $\pi_\Lambda \hookrightarrow \mathrm{Ind}_N^{H_\Lambda} \psi$. Clearly π_Λ is ψ-generic iff it is $a \cdot \psi$-generic, for any a in H_Λ. Thus we can talk about generic representations of G without specifying ψ. We say that π_Λ is generic if it is ψ-generic for some ψ. Every infinite-dimensional representation of $\mathrm{GL}(2, F)$ is generic.

2.3.4 PROPOSITION. (1) *Suppose* π *is a generic irreducible representation of* G. *Any constituent* π_Λ *of* $\mathrm{Res}_{H_\Lambda}^G \pi$ *occurs with multiplicity one and is* ψ*-generic for some* ψ.
(2) *Any* ψ*-generic* π_Λ *is contained in* $\mathrm{Res}_{H_\Lambda}^G \pi$ *where* π *is generic.*

PROOF. For (2), if $\pi_\Lambda \subset \mathrm{Ind}_N^{H_\Lambda} \psi$ then $\pi = \mathrm{Ind}_{H_\Lambda}^G \pi_\Lambda \subset \mathrm{Ind}_N^G \psi$ and $\pi_\Lambda \subset \mathrm{Res}_{H_\Lambda}^G \pi$.

For (1), if $\pi \subset \operatorname{Ind}_N^G \psi$ then

$$\pi_\Lambda \subset \operatorname{Res}_{H_\Lambda}^G \pi \subset \operatorname{Res}_{H_\Lambda}^G \operatorname{Ind}_N^G \psi = \sum_\lambda \operatorname{Ind}_N^{H_\Lambda}(\lambda \cdot \psi),$$

where the sum ranges over $\lambda \in G/H_\Lambda = F^\times/\Lambda$. Since π_Λ is irreducible there is a λ with $\pi_\Lambda \subset \operatorname{Ind}_N^{H_\Lambda}(\lambda \cdot \psi)$, hence $\operatorname{Ind}_{H_\Lambda}^G \pi_\Lambda \subset \operatorname{Ind}_N^G \psi$. By Frobenius reciprocity

$$\operatorname{Hom}_{H_\Lambda}(\operatorname{Res}_{H_\Lambda}^G \pi, \pi_\Lambda) = \operatorname{Hom}_G(\pi, \operatorname{Ind}_{H_\Lambda}^G \pi_\Lambda).$$

Composing $\pi \hookrightarrow \operatorname{Ind}_{H_\Lambda}^G \pi_\Lambda \to \operatorname{Ind}_N^G \psi$, since $\dim_{\mathbb{C}} \operatorname{Hom}_G(\pi, \operatorname{Ind}_N^G \psi) \leq 1$ (by the uniqueness of the Whittaker model) the proposition follows, namely the multiplicity of π_Λ in $\operatorname{Res}_{H_\lambda}^G \pi$ is at most one. □

Definition. Given π_Λ of H_Λ let $\Gamma(\pi_\Lambda) \subset F^\times$ be the group of factors of similitudes $\lambda = \lambda(g)$ of the $g \in G$ with $\pi_\Lambda^g \simeq \pi_\Lambda$, where $\pi_\Lambda^g(h) = \pi_\Lambda(g^{-1}hg)(h \in H_\Lambda)$. Since $\lambda(H_\Lambda) = \Lambda$, $\Gamma(\pi_\Lambda) \supset \Lambda$. Given π of G, put $X(\pi) = \{\omega \in \operatorname{Hom}(G, \mathbb{C}^\times);\ \omega\pi \simeq \pi\}$. Note that a character $\omega : G \to \mathbb{C}^\times$ factorizes via λ, thus $\omega(g) = \omega_0(\lambda(g))$ for a character $\omega_0 : F^\times \to \mathbb{C}^\times$. For such ω_0 we also write $\omega_0\pi : g \mapsto \omega_0(\lambda(g))\pi(g)$. As usual $\omega\pi : g \mapsto \omega(g)\pi(g)$.

2.3.5 Proposition. *π_Λ is ψ-generic and ψ'-generic iff $\psi' = a \cdot \psi$ for a diagonal a with $\lambda(a) \in \Gamma(\pi_\Lambda)$.*

Proof. If π_Λ lies in $\operatorname{Ind}_N^{H_\Lambda} \psi = \{\varphi : H_\Lambda \to \mathbb{C};\ \varphi(uh) = \psi(u)\varphi(h),\ \varphi \text{ smooth}\}$ and $\lambda(a) \in \Gamma(\pi_\Lambda)$, then $\pi_\Lambda \subset \{\varphi'; \varphi'(h) = \varphi(\operatorname{Int}(a)h)\}$, and $\varphi'(uh) = \psi(\operatorname{Int}(a)u)\varphi'(h)$.

If $\pi_\Lambda \subset \operatorname{Ind}_N^{H_\Lambda}(\psi)$ and $\pi_\Lambda \subset \operatorname{Ind}_N^{H_\Lambda}(a \cdot \psi)$ then $\pi_\Lambda^a \subset \operatorname{Ind}_N^{H_\Lambda}(\psi)$ where $\pi_\Lambda^a(h) = \pi_\Lambda(a^{-1}ha)$. The uniqueness of the Whittaker model for H_Λ implies $\pi_\Lambda \simeq \pi_\Lambda^a$, hence $\lambda(a) \in \Gamma(\pi_\Lambda)$. □

2.3.6 Proposition. *Suppose (π, V) is generic and*

$$(\pi_\Lambda, V_1) \subset \operatorname{Res}_{H_\Lambda}^G(\pi, V).$$

(1) *$\omega \in X(\pi)$ iff ω is trivial on $G(\pi_\Lambda) = \{g \in G; \lambda(g) \in \Gamma(\pi_\Lambda)\}$.*
(2) *The number of irreducibles in $\operatorname{Res}_{H_\Lambda}^G \pi$ is*
$\#X(\pi) = [G : G(\pi_\Lambda)] = [F^\times : \Gamma(\pi_\Lambda)]$.
(3) *If π_Λ lies also in (σ, W) then $\sigma \simeq \omega\pi$, $\omega|H_\Lambda = 1$. If π and $\omega\pi$ contain the same π_Λ then $\omega = \omega_1\omega_2$, $\omega_2|H_\Lambda = 1$, $\omega_2\pi \simeq \pi$.*

PROOF. Suppose $\operatorname{Res}^G_{H_\Lambda}(\pi, V) = \oplus_{1\le i\le r}(\pi_{\Lambda i}, V_i)$, (π_{Vi}, V_i) irreducible, and $\pi_{\Lambda 1} = \pi_\Lambda$. Then V_1 is invariant under $G(\pi_\Lambda)$. Denote the representation of $G(\pi_\Lambda)$ on V_1 by π_1. Then $\pi = \operatorname{Ind}^G_{G(\pi_\Lambda)} \pi_1$. Hence $r = [G : G(\pi_\Lambda)]$ and every character of $G/G(\pi_\Lambda)$ lies in $X(\pi)$. If $\omega \in X(\pi)$ and A intertwines π and $\omega\pi$, then $A : V_i \to V_i$ (since V_i, V_j are inequivalent for $i \ne j$, as π is generic) acts as a scalar on the irreducible V_i. Hence π_Λ and $\omega\pi_\Lambda$ are equal, not only equivalent. Hence ω is trivial on $G(\pi_\Lambda)$.

For (3), if π_Λ lies also in (σ, W) then σ is generic and $\operatorname{Res}^G_{H_\Lambda}(\sigma, W) = \oplus_{1\le i\le r}(\sigma_{\Lambda i}, W_i)$, $\sigma_{\Lambda i}$ irreducible, inequivalent, with $\sigma_{\Lambda 1} = \pi_\Lambda$. Again $r = [G : G(\pi_\Lambda)]$, and $G(\pi_\Lambda)$ acts on W_1. Then $\sigma_1 = \omega_1\pi_1$ with a character ω_1 of $G(\pi_\Lambda)/H_\Lambda$. Then $\sigma = \operatorname{Ind}^G_{G(\pi_\Lambda)} \sigma_1$ is equivalent to $\omega\pi$, where ω is any extension of ω_1 from $G(\pi_\Lambda)$ to G. Thus, if π and $\omega\pi$ have the same restriction to H_Λ then there is ω_1 on G/H_Λ with $\omega\pi = \omega_1\pi$, so $\omega = \omega_1\omega_2$ where $\omega_2 = \omega\omega_1^{-1}$ satisfies $\omega_2\pi \simeq \pi$. □

2.4 DEFINITION. Let F be a number field. For each place v of F, let $\{\pi_{0v}\}$ be a packet of representations of $H_v = \mathrm{SL}(2, F_v)$. Suppose $\{\pi_{0v}\}$ contains an unramified π^0_{0v} for almost all v. An irreducible π^0_{0v} is called *unramified* if it has a nonzero $K_{0v} = \mathrm{SL}(2, R_v)$-fixed vector. The global *packet* $\{\pi_0\}$ associated with this local data is the set of all products $\otimes_v\pi^0_{0v}$ with $\pi_{0v} \in \{\pi_{0v}\}$ for all v and with $\pi_{0v} = \pi^0_{0v}$ for almost all v.

Let E/F be a quadratic extension, and μ' a character of $C^1_E = \mathbb{A}^1_E/E^1$. Then the local packets $\{\pi_0(\mu'_v)\}$ define a global packet, denoted $\{\pi_0(\mu')\}$. If $\mu' = 1$ it is the set of constituents of the representation $I_0(\chi_E)$ normalizedly induced from the character $\chi_E : \mathbb{A}^\times/F^\times N_{E/F}\mathbb{A}^\times_E \xrightarrow{\sim} \{\pm 1\}$. If $\mu' \ne 1$ the packet $\{\pi_0(\mu')\}$ contains a cuspidal representation. If $\mu' \ne 1 = \mu'^2$ there are 3 quadratic extensions $E_1 = E$, E_2, E_3 of F and characters $\mu'_1 = \mu$, μ'_2, μ'_3 of $C^1_{E_1}$, $C^1_{E_2}$, $C^1_{E_3}$ with $\{\pi_0(\mu'_1)\} = \{\pi_0(\mu'_2)\} = \{\pi_0(\mu'_3)\}$.

All irreducibles in a packet have the same central character, which is trivial at almost all places since the center of $\mathrm{SL}(2, F_v)$ is $\pm I$. If the packet contains an automorphic representation, its central character is trivial on the rational element $-I$.

Let ω be a character of $C_F = \mathbb{A}^\times/F^\times$ whose restriction to the center $\mathbf{Z}_H(\mathbb{A})$ of $\mathbf{H}(\mathbb{A})$ coincides with the central character of $\{\pi_0\}$. Then $\{\pi_{0v}\}$ and ω_v define a unique representation π^*_v of $\mathrm{GL}(2, F_v)$ with central character ω_v as in 2.3.2. It is unramified wherever $\{\pi_{0v}\}$ and ω_v are. Define $\pi^* = \otimes_v\pi^*_v$ to be the representation of $\mathrm{GL}(2, \mathbb{A})$ associated with $\{\pi_0\}$ and ω.

In particular, the extension μ^* to C_F of the character μ' of C_F^1 defines a representation $\pi^*(\mu^*)$, or $\pi^*(\mathrm{Ind}_E^F \mu^*)$, on using $\{\pi_0(\mu')\}$ and the (central) character $\omega = \chi_E \cdot \mu^*|C_F$ on C_F. If $\mu^* = \overline{\mu}^*$ then there is $\mu : C_F \to \mathbb{C}^\times$ with $\mu^*(z) = \mu(z\overline{z})$ $(z \in C_E)$, $\mathrm{Ind}_E^F \mu^* = \mu \oplus \mu\chi_E$ and $\pi^*(\mu^*) = I(\mu, \mu\chi_E)$. Moreover, $\pi^*(\mu^* \cdot \mu \circ N_{E/F}) = \mu \cdot \pi^*(\mu^*)$ for any characters μ of C_F and μ^* of C_E.

Our aim is to show that the integer $m (= m(\pi_0^+) = m(\pi_0^-)$ in (2.2.5), $= m_j$ in (2.2.7)) is zero. Our purely local proof is given in Proposition 2.5. We begin with a global proof, patterned on [LL], which shows that there is at most one cuspidal representation in any packet $\{\pi_0(\mu')\}$, $\mu' \neq 1$, and its multiplicity is one. Using the trace identity (1.9.2) and the local character relations 2.2.5 and 2.2.7, it follows at once that $m = 0$ in (2.2.5) and (2.2.7).

2.4.1 Lemma. *Let π_0 be an irreducible representation of* $\mathrm{SL}(2, \mathbb{A})$ *such that $m(\pi_0^g) \neq m(\pi_0)$ for some $g \in \mathrm{GL}(2, \mathbb{A})$. Then there is a quadratic extension E of F and a character $\mu' \neq 1$ of $\mathbb{A}_E^1/E^1$ such that $\pi_0 \in \{\pi_0(\mu')\}$.*

Proof. This follows at once from the identity (1.9.2) and the local character relations 2.2.4-7, and Proposition 2.2.8. □

2.4.2 Lemma. *Let E be a quadratic extension of F and μ^* a character of $C_E = \mathbb{A}_E^\times/E^\times$. Then $\pi^*(\mu^*)$ is automorphic, cuspidal if $\mu^* \neq \overline{\mu}^*$.*

Proof. If $\mu^* \neq \overline{\mu}^*$ then $\mu' = \mu^*|C_E^1$ is $\neq 1$, and the claim follows from each of the following propositions. □

In the following proposition we take $\mathbf{H} = \mathrm{Sp}(n)$, $\mathbf{G} = \mathrm{GSp}(n)$.

2.4.3 Proposition. (1) *Every automorphic cuspidal representation π_0 of $\mathbf{H}(\mathbb{A})$ is contained in an automorphic cuspidal representation π of $\mathbf{G}(\mathbb{A})$.*
(2) *If π contains π_0 and π_0' then $\pi_0' = \pi_0^h$ for some h in $\mathbf{G}(\mathbb{A})$, where $\pi_0^h(g) = \pi_0(h^{-1}gh)$.*
(3) *If π and π' are generic and contain π_0 then $\pi' = \omega\pi$ for a character ω of $\mathbb{A}^\times$.*

Proof. (2) follows from 4.1(1), and (3) from 4.4(3). For (1), extend π_0 to an automorphic representation of $\mathbf{H}^+(\mathbb{A})$, $\mathbf{H}^+ = \mathbf{ZH}$, by extending the central character of π_0 to $Z\backslash\mathbf{Z}(\mathbb{A})$; $\mathbf{Z}$ denotes here the center of $\mathbf{G}$. Put

$$(\pi, V_\pi) = \mathrm{Ind}((\pi_0, V_{\pi_0}); \mathbf{H}^+(\mathbb{A}), \mathbf{G}(\mathbb{A})).$$

Here the space V_{π_0} of π_0 is a subspace of the space $L^2(H^+\backslash\mathbf{H}^+(\mathbb{A}))$ of cusp forms on $\mathbf{H}^+(\mathbb{A})$. Define a linear functional $l\colon V_{\pi_0} \to \mathbb{C}$ by $l(\varphi) = \varphi(1)$. Note that

$$l(\pi_0(\gamma)\pi_0(g)\varphi) = \varphi(\gamma g) = \varphi(g) = l(\pi_0(g)\varphi)$$

for all γ in H^+ since φ is automorphic. It suffices to construct an embedding of the space V_π of π into $L_0(G\backslash\mathbf{G}(\mathbb{A}))$. The induced representation π operates by right translation in the space V_π of functions $f\colon \mathbf{G}(\mathbb{A}) \to V_{\pi_0}$ which are compactly supported modulo $\mathbf{H}^+(\mathbb{A})$ and satisfy

$$f(sg) = \pi_0(s)f(g) \quad (s \in \mathbf{H}^+(\mathbb{A}), \quad g \in \mathbf{G}(\mathbb{A})).$$

Define a functional L on the space V_π of π by

$$L(f) = \sum_{u \in F^\times/F^{\times 2}} l\left(f\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).$$

The sum converges since f is compactly supported modulo $\mathbf{H}^+(\mathbb{A})$. Since

$$L(\pi(g)f) = \sum l\left(f\left(\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right)\right)$$

and l is H^+-invariant, it follows that L is G-invariant. The map intertwining V_π and $L^2(G\backslash\mathbf{G}(\mathbb{A}))$ is defined by

$$f \mapsto \phi_f, \qquad \phi_f(g) = L(\pi(g)f).$$

It is clearly nonzero. Since the unipotent radical of any parabolic subgroup of $\mathbf{G}$ lies in $\mathbf{H}$, ϕ_f is a cusp form. The induced representation π is reducible, and we deduce that one of its irreducible components is automorphic and cuspidal. □

Let $\pi^* = \otimes\pi_v^*$ be an irreducible representation of $\mathrm{GL}(2,\mathbb{A})$. Let π_0 be an irreducible constituent of the restriction of π^* to $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2,\mathbb{A})$. Any other constituent has the form $\pi_0^g : h \mapsto \pi_0(g^{-1}hg)$ for a g in $\mathrm{GL}(2,\mathbb{A})$. The group $G(\pi_0) = \{g \in \mathrm{GL}(2,\mathbb{A});\ \pi_0^g \simeq \pi_0\}$ contains $\mathrm{SL}(2,\mathbb{A})$, hence it is normal in $\mathrm{GL}(2,\mathbb{A})$ and depends only on π^*; denote it also by $G(\pi^*)$. Let $X(\pi^*)$ be the group of characters $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ with $\omega\pi^* = \pi^*$. It consists of the characters trivial on $G(\pi^*) = G(\pi_0)$, hence depends only on π_0 and can be denoted by $X(\pi_0)$. Let $Y(\pi^*)$ be the set of characters ω of $\mathbb{A}^\times$ for which $\omega\pi^*$ is automorphic and cuspidal. Put $Y = \mathrm{Hom}(\mathbb{A}^\times/F^\times, \mathbb{C}^\times)$. Then $X(\pi^*)$ and Y act on $Y(\pi^*)$ by multiplication.

2.4.4 PROPOSITION. *Let π_0 be an irreducible infinite dimensional representation of* $\mathrm{SL}(2,\mathbb{A})$ *with* $\pi_0(-I)=1$. *Then* $Y(\pi^*)/YX(\pi^*)$ *has cardinality* $\Sigma_g m(\pi_0^g)$, *where* $m(\pi_0)$ *denotes the multiplicity of* π_0 *in*

$$L^2(\mathrm{SL}(2,F)/\,\mathrm{SL}(2,\mathbb{A}))$$

and g *ranges over* $\mathrm{GL}(2,\mathbb{A})/G(\pi_0)\,\mathrm{GL}(2,F)$.

PROOF. Extend π_0 to $\mathbf{G}(\mathbb{A})$ by the central character χ of π^*, where $\mathbf{G}=\mathbf{Z}\,\mathrm{SL}(2)$ and $\mathbf{Z}$ is the center of $\mathrm{GL}(2)$. Since $\mathrm{SL}(2,F)\cdot\mathbf{Z}_S(\mathbb{A})\backslash\,\mathrm{SL}(2,\mathbb{A})=\mathbf{G}(F)\mathbf{Z}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$, where $\mathbf{Z}_S=\mathbf{Z}\cap\mathrm{SL}(2)$ and $\mathbf{Z}$ is the center of $\mathbf{G}$, it suffices to prove the proposition for a π_0 of $\mathbf{G}(\mathbb{A})$ with $\pi_0|\mathbf{Z}(\mathbb{A})=\pi^*|\mathbf{Z}(\mathbb{A})$, where $m(\pi_0)$ is the multiplicity of π_0 in $L_0^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))_\chi$, the space of cusp forms on $\mathbf{G}(\mathbb{A})$ transforming under $\mathbf{Z}(\mathbb{A})$ by χ. We have

$$L_0=L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))_\chi,\quad L_1=L^2(\mathrm{GL}(2,F)\backslash\,\mathrm{GL}(2,F)\mathbf{G}(\mathbb{A}))_\chi,$$

$$L^*=L^2(\mathrm{GL}(2,F)\backslash\,\mathrm{GL}(2,\mathbb{A}))_\chi.$$

Let s_0, s_1, s^* be the representations of $\mathbf{G}(\mathbb{A})$, $\mathrm{GL}(2,F)\mathbf{G}(\mathbb{A})$, $\mathrm{GL}(2,\mathbb{A})$, on L_0, L_1, L^*. As spaces, $L_0=L_1$. As representations,

$$s^*=\mathrm{Ind}(\mathrm{GL}(2,\mathbb{A}),\mathrm{GL}(2,F)\mathbf{G}(\mathbb{A}),s_1).$$

Let π_0 be an irreducible occurring in s_0 with multiplicity $m(\pi_0)$. Put $G_1(\pi_0)=G(\pi_0)\cap\mathrm{GL}(2,F)\mathbf{G}(\mathbb{A})$. Then π_0 extends to a representation σ of $G(\pi_0)$ on the same space. Put $\sigma_1=\sigma|G_1(\pi_0)$.

Let V_0 be the subspace of L_0 transforming according to π_0. Under $G_1(\pi_0)$ it transforms according to

$$\oplus_{i=1}^{m(\pi_0)}\omega_i\sigma_1,$$

where ω_i are characters of $\mathbf{G}(\mathbb{A})/G_1(\pi_0)$. The smallest invariant subspace V_1 of L_1 containing V_0 transforms according to

$$\oplus_i\,\mathrm{Ind}(\mathrm{GL}(2,F)\mathbf{G}(\mathbb{A}),G_1(\pi_0),\omega_i\sigma_1).$$

Each summand here is irreducible. From $s^*=\mathrm{Ind}(s_1)$ we obtain

$$s^*=\bigoplus_{\pi_0/\sim}\oplus_{i=1}^{m(\pi_0)}\mathrm{Ind}(\mathrm{GL}(2,\mathbb{A}),G_1(\pi_0),\omega_i\sigma_1),$$

where $\pi_0 \sim \pi_0'$ if $\pi_0' = \pi_0^g$, $g \in \mathrm{GL}(2, F)$, as such π_0^g defines the same σ_1 as π_0 does.

The induction can be performed in two steps, the first being

$$\mathrm{Ind}(G(\pi_0), G_1(\pi_0), \omega_i \sigma_1) = \bigoplus_{\omega} \omega\sigma,$$

where the sum ranges over all characters ω of $G(\pi)$ which equal ω_i on $G_1(\pi_0)$; note that $G(\pi_0)/G_1(\pi_0)$ is a subquotient of $\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}$, hence compact. Then $s^* = \bigoplus_{\pi_0/\approx} s^*_{\pi_0}$, where

$$s^*_{\pi_0} = \bigoplus_{g \in \mathrm{GL}(2,\mathbb{A})/G(\pi_0)\,\mathrm{GL}(2,F)} \bigoplus_{i=1}^{m(\pi_0^g)} \bigoplus_{\omega} \mathrm{Ind}(\mathrm{GL}(2,\mathbb{A}), G(\pi_0), \omega\sigma^g),$$

and $\pi_0 \approx \pi_0'$ if $\pi_0' = \pi_0^g$ for some $g \in \mathrm{GL}(2, \mathbb{A})$. Each summand in $s^*_{\pi_0}$ is irreducible and its restriction to $\mathbf{G}(\mathbb{A})$ contains π_0, hence consists of π_0^g, $g \in \mathrm{GL}(2, \mathbb{A})$. Since $\mathrm{Ind}(\mathrm{GL}(2, \mathbb{A}),\ G(\pi_0),\ \omega\sigma^g)$, where σ^g is the extension of π_0^g to $G(\pi_0)$, is independent of g, by multiplicity one for $\mathrm{GL}(2, \mathbb{A})$ there is at most one g in $\mathrm{GL}(2, \mathbb{A})/G(\pi_0)\,\mathrm{GL}(2, F)$ with $m(\pi_0^g) \neq 0$.

Since $\pi^* = \omega \cdot \mathrm{Ind}(\mathrm{GL}(2, \mathbb{A}),\ G(\pi_0),\ \sigma)$ for some character ω of $\mathrm{GL}(2, \mathbb{A})$, $Y(\pi^*)$ is empty precisely when $m(\pi_0^g) = 0$ for all g. If $Y(\pi^*)$ is not empty, we may assume that π^* is automorphic cuspidal. We claim that $Y(\pi^*) = YY'(\pi^*)$, where $Y'(\pi^*)$ consists of $\omega_1 \in Y(\pi^*)$ with $\omega_1^2 = 1$. Indeed, identifying characters of $\mathrm{GL}(2, \mathbb{A})$ and $\mathbb{A}^\times$ via det (thus $\omega(g) = \omega(\det g)$), a character ω in $Y(\pi^*)$ is a character on $\mathbb{A}^\times/F^{\times 2}$. Define a character $\eta : F^\times\mathbb{A}^{\times 2} \to \mathbb{C}^\times$ by $\eta|F^\times = 1$ and $\eta(x^2) = \omega(x^2)$, $x \in \mathbb{A}^{\times 2}$. It is well defined as $F^\times \cap \mathbb{A}^{\times 2} = F^{\times 2}$ and $\omega|F^{\times 2} = 1$. Extend η to $\mathbb{A}^\times/F^\times$, and define ω_1 by $\omega = \eta\omega_1$. Then $\omega_1\pi^* \subset s^*$ and $\omega_1^2 = 1$. Thus $\omega_1 \in Y'(\pi^*)$. Each element of $X(\tilde{\pi})$ is of order 2, and $Y' = Y \cap Y'(\pi^*)$ is the group of characters $\omega : \mathbb{A}^\times/F^\times\mathbb{A}^{\times 2} \to \mathbb{C}^\times$. We then wish to compute the cardinality of

$$Y(\pi^*)/YX(\pi^*) = Y'(\pi^*)Y/YX(\pi^*)$$
$$= Y'(\pi^*)/X(\pi^*) \cdot Y \cap Y'(\pi^*) = Y'(\pi^*)/X(\pi^*)Y'.$$

Then multiplying $\mathrm{Ind}(\mathrm{GL}(2, \mathbb{A}),\ G(\pi_0),\ \omega\sigma)$, where $\omega|G_1(\pi_0) = \omega_i$, by a character ω^* of $\mathrm{GL}(2, \mathbb{A})$ whose restriction to $G_1(\pi_0)$ is ω_j/ω_i, we shall get $\mathrm{Ind}(\mathrm{GL}(2, \mathbb{A}),\ G(\pi_0),\ \omega'\sigma)$ where $\sigma'|G_1(\pi_0) = \omega_j$. Multiplying

$$\mathrm{Ind}(\mathrm{GL}(2, \mathbb{A}), G(\pi_0), \omega\sigma)$$

by ω^* on $\mathrm{GL}(2,\mathbb{A})$ whose restriction to $G_1(\pi_0)$ is 1 simply permutes the summands in the sum over ω such that $\omega|G_1(\pi_0)=\omega_i$. The characters ω_i are all different, by multiplicity one theorem for $\mathrm{GL}(2,\mathbb{A})$. A character lies in $X(\pi^*)$ iff it is trivial on $G(\pi_0)$. It is in Y' iff it is 1 on $\mathbf{G}(\mathbb{A})\,\mathrm{GL}(2,F)$. Hence it lies in $Y'X(\pi^*)$ iff it is trivial on $G_1(\pi_0)$. It follows that $Y'(\pi^*)/X(\pi^*)Y'$ acts simply transitively on the set of irreducibles in $s^*_{\pi_0}$, a set with cardinality $\sum_g m(\pi_0^g)$, g ranges over the finite set $\mathrm{GL}(2,\mathbb{A})/G(\pi_0)\,\mathrm{GL}(2,F)$, and all multiplicities $m(\pi_0^g)$ but one are zero. □

2.4.5 PROPOSITION. *Let $\omega\neq 1$ be a character of C_F with $\omega\pi^*=\pi^*$; π^* is a cuspidal representation of* $\mathrm{GL}(2,\mathbb{A})$. *Then $\omega=\chi_E$ for some quadratic extension E of F, and $\pi^*=\pi^*(\mu^*)$ for a character $\mu^*\neq\overline{\mu}^*$ of C_E.*

PROOF. As $\omega\pi^*=\pi^*$, $\omega^2=1$ and $\omega=\chi_E$ for some E. Put $G_E(\mathbb{A})=\{g\in\mathrm{GL}(2,\mathbb{A});\det g\in N_{E/F}\mathbb{A}_E^\times\}$. The restriction of π^* to $\ker\chi_E=G_E(\mathbb{A})\,\mathrm{GL}(2,F)$ is $\pi_1\oplus\pi_2$, π_i irreducible, $\pi_2=\pi_1^g$, $\chi_E(g)=-1$. The restriction map from $L_0^2(\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A}))$ to

$$L_0^2(\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,F)G_E(\mathbb{A}))\oplus L_0^2(\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,F)G_E(\mathbb{A})g),$$

restricted to the space V_{π^*} of π^*, is nonzero, hence one of π_1, π_2 is cuspidal automorphic.

Put $G_E=\{g\in\mathrm{GL}(2,F);\ \det g\in N_{E/F}E^\times\}$. If both π_1 and π_2 are cuspidal, namely contained in

$$L_0^2(\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,F)G_E(\mathbb{A}))=L_0^2(G_E\backslash G_E(\mathbb{A})),$$

we view π_1, π_2 as cuspidal representations of $G_E(\mathbb{A})$.

Taking the Fourier expansion with respect to $\mathbf{N}(\mathbb{A})/\mathbf{N}(F)$ we conclude that there are characters ψ_1, ψ_2 of $\mathbb{A}/F$ such that

$$\pi_i\subset\mathrm{Ind}(G_E(\mathbb{A}),\mathbf{N}(\mathbb{A}),\psi_i).$$

As $\pi_2=\pi_1^g$, we have

$$\pi_2\subset\mathrm{Ind}(G_E(\mathbb{A}),\mathbf{N}(\mathbb{A}),\psi_1^g),\qquad \psi_1^g(x)=\psi_1(x\det g).$$

But $\psi_2(x)=\psi_1(\beta x)$ for some $\beta\in F^\times$. As $G_E(F_v)=G(\pi_{2v})$ for all v, by Proposition 4.3 we have $1=\chi_E(\beta)=\chi_E(\det g)=-1$. The contradiction implies that only one of π_1, π_2 is cuspidal. Hence the multiplicity $m(\pi_0^g)$, where π_0 is an irreducible in the restriction of π_1 to $\mathrm{SL}(2,\mathbb{A})$, is not constant in $g\in\mathrm{GL}(2,\mathbb{A})$. The proposition follows by Lemma 2.4.1. □

2.4.6 PROPOSITION. *Let E/F be a quadratic extension, μ^* a character of C_E with $\mu^* \neq \overline{\mu}^*$, $\pi^* = \pi^*(\mu^*)$, and ω a character of $\mathbb{A}^\times$ such that $\pi^*_\omega = \omega\pi^*$ is automorphic. Then there is a character β of $\mathbb{A}^\times$ with $\beta\pi^* = \pi^*$, and a character α of $\mathbb{A}^\times/F^\times$, with $\omega = \alpha\beta$.*

PROOF. We have $\chi_E \cdot \pi^* = \pi^*$, hence $\chi_E \pi^*_\omega = \pi^*_\omega$. By 2.4.5 there is a character $\mu^*_\omega \neq \overline{\mu}^*_\omega$ of C_E with $\pi^*_\omega = \pi^*(\mu^*_\omega)$. Since $\pi^*(\mu^*_\omega) = \omega\pi^*(\mu^*)$, the projective representations $\mathrm{Ind}_E^F(\mu^*_\omega)_0$ and $\mathrm{Ind}_E^F(\mu^*)_0$ have equivalent restrictions to the local Weil groups W_{E_v/F_v} at every place v. Hence their symmetric squares are equivalent locally, whence globally by Chebotarev's density theorem. As

$$\mathrm{Sym}^2(\mathrm{Ind}_E^F(\mu^*_\omega)_0) = \mathrm{Ind}_E^F(\mu^*_\omega/\overline{\mu}^*_\omega) \oplus \chi_E,$$

we conclude that $\mu^*_\omega/\overline{\mu}^*_\omega$ is equal to $\mu^*/\overline{\mu}^*$ or to $\overline{\mu}^*/\mu^*$. Hence μ^*_ω/μ^*, or $\mu^*_\omega/\overline{\mu}^*$, takes the same value at z and $\overline{z}$ in C_E. Then there exists a character α of $\mathbb{A}^\times/F^\times$ with $\mu^*_\omega(z) = \mu^*(z)\alpha(N_{E/F}z)$ or $\mu^*_\omega(z) = \mu^*(\overline{z})\alpha(N_{E/F}z)$. In both cases $\pi^*_\omega = \alpha\pi^*$, and $\beta = \omega/\alpha$ satisfies $\beta\pi^* = \pi^*$. □

It follows from Propositions 2.4.4 and 2.4.6 that (1) in each packet $\{\pi_0(\mu')\}$, $\mu' \neq 1$ a character of C_E^1, there is a cuspidal representation π_0; (2) any other cuspidal representation has the form π_0^g, $g \in \mathrm{GL}(2,F)$; (3) all other representations in the packet, which are of the form π_0^g, $g \in \mathrm{GL}(2,\mathbb{A}) - \mathrm{GL}(2,F)G(\pi)$, do not occur in the cuspidal spectrum.

The cuspidal representations occur with multiplicity one.

Indeed, applying the trace identity (1.9.2) in the form $\frac{1}{2}I' = I_0 - \frac{1}{2}I'_E$ (see (1.8.1)) where $\mu'^2 \neq 1$ makes the only contribution to I'_E, and using the character relations 2.2.4-7 (recall that the $m(\pi_0)$ are 0 by Proposition 2.2.8) to replace the representations in I', I'_E by π_0 on $\mathrm{SL}(2,\mathbb{A})$, we conclude that the $m(\pi_0^+) = m(\pi_0^-)$ of 2.2.5 and $m = m_j$ of 2.2.7 are zero for each component of our global character μ'. The identity (1.9.2) then takes the form

$$\prod_{v\in V}(2m_v+1)\,\mathrm{tr}\{\pi_{0v}\}(f_{0v}dh_v) + \prod_{v\in V}[\mathrm{tr}\,\pi^+_{0v}(f_{0v}dh_v) - \mathrm{tr}\,\pi^-_{0v}(f_{0v}dh_v)]$$

$$= 2\sum_{\pi_0} m(\pi_0) \prod_{v\in V} \mathrm{tr}\,\pi_{0v}(f_{0v}dh_v).$$

The set V is finite, and the sum ranges over the cuspidal π_0 with unramified component in $\{\pi_{0v}(\mu'_v)\}$ for all $v \notin V$. Since there is a π_0 in the sum with $m(\pi_0) = 1$, we cannot have $2m_v + 1 > 1$ for any v (in V).

Since each local character $\mu'_v \neq 1$ of E^1_v can be embedded as a local component of a global character μ', $\mu'^2 \neq 1$, of $\mathbb{A}^1_E/E^1$, we proved the following.

2.5 Proposition. *The integer* m $(= m(\pi_0^+) = m(\pi_0^-)$ *in* (2.2.5), $= m_j$ *in* (2.2.7)) *is* 0. *For every* $a \in F^\times$ *there is just one* $\boldsymbol{\psi}^a_H$*-generic* π_H *in the sum* ($\dim_{\mathbb{C}} \neq 0$, *necessarily* $= 1$).

We now give a purely local proof of this proposition, which is independent of the subsections 2.3 and 2.4 above. It is based on the following theorem of Rodier [Rd], p. 161, (for any split group H) which computes the number of $\boldsymbol{\psi}_H$-Whittaker models of the admissible irreducible representation π_H of H in terms of values of the character $\operatorname{tr}\pi_H$ or χ_{π_H} of π_H at the measures $\psi_{H,n}dh$ which are supported near the origin.

2.5.1 Proposition. *The multiplicity* $\dim_{\mathbb{C}} \operatorname{Hom}_H(\operatorname{ind}_{U_H}^H \boldsymbol{\psi}^a_H, \pi_H)$ *is*

$$= \lim_n |H_n|^{-1} \operatorname{tr}\pi_H(\psi^a_{H,n}dh) \quad \left(= \lim_n |H_n|^{-1} \int_{H_n} \chi_{\pi_H}(h)\psi^a_{H,n}(h)dh \right).$$

The limit here and below stabilizes for large n. We proceed to explain the notations. Thus $\boldsymbol{\psi}_H : U_H \to \mathbb{C}^1 (= \{z \in \mathbb{C}; |z| = 1\})$ is a generic (nontrivial on each simple root subgroup) character on the unipotent radical U_H of a Borel subgroup B_H of H.

A $\boldsymbol{\psi}_H$-*Whittaker vector* is a vector in the space of the compactly induced representation $\operatorname{ind}_{U_H}^H(\boldsymbol{\psi}_H)$. This space consists of all functions $\varphi : H \to \mathbb{C}$ with $\varphi(uhk) = \boldsymbol{\psi}_H(u)\varphi(h)$, $u \in U_H$, $h \in H$, $k \in K_\varphi$, where K_φ is a compact open subgroup depending on φ, which are compactly supported on $U_H\backslash H$. The group H acts by right translation. The multiplicity $\dim_{\mathbb{C}} \operatorname{Hom}_H(\operatorname{ind}_{U_H}^H \boldsymbol{\psi}_H, \pi_H)$ of any irreducible admissible representation π_H of H in the space of $\boldsymbol{\psi}_H$-Whittaker vectors is known to be 0 or 1. In the latter case we say that π_H has a $\boldsymbol{\psi}_H$-Whittaker model or that it is $\boldsymbol{\psi}_H$-*generic.*

The maximal torus A_H in B_H normalizes U_H and so acts on the set of generic characters by $a \cdot \boldsymbol{\psi}_H(u) = \boldsymbol{\psi}_H(\operatorname{Int}(a)u)$. We need this only for our $H = \operatorname{SL}(2, F)$. We may take $U_H = \{u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\}$, and define $\boldsymbol{\psi}^a_H : U_H \to \mathbb{C}^1$

by $\boldsymbol{\psi}_H^a(u) = \psi(ax)$, where $a \in F^\times$ and $\psi : F \to \mathbb{C}^1$ is a fixed additive character which is 1 on the ring R of integers of F, but $\neq 1$ on $\boldsymbol{\pi}^{-1}R$. Since $\operatorname{diag}(a, a^{-1}) \cdot \boldsymbol{\psi}_H^b = \boldsymbol{\psi}_H^{ba^2}$, the A_H-orbits of generic characters are parametrized by $F^\times/F^{\times 2}$.

Let $\mathcal{H}_0$ be the ring of 2×2 matrices with entries in R and trace zero. Write $\mathcal{H}_n = \boldsymbol{\pi}^n \mathcal{H}_0$ and $H_n = \exp(\mathcal{H}_n)$. For $n \geq 1$ we have $H_n = {}^tU_{H,n}A_{H,n}U_{H,n}$, where $U_{H,n} = U_H \cap H_n$, and $A_{H,n}$ is the group of diagonal matrices in H_n. Define a character $\psi_{H,n}^a : H \to \mathbb{C}^1$, supported on H_n, by

$$\psi_{H,n}^a({}^tbu) = \psi(ax\boldsymbol{\pi}^{-2n}) \quad \text{at} \quad {}^tb \in {}^tU_{H,n}A_{H,n}, \quad u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in U_{H,n}.$$

Alternatively, by

$$\psi_{H,n}^a(\exp X) = \operatorname{ch}_{\mathcal{H}_n}(X)\psi(\operatorname{tr}[X\boldsymbol{\pi}^{-2n}\beta_{H,a}]),$$

where $\operatorname{ch}_{\mathcal{H}_n}$ indicates the characteristic function of $\mathcal{H}_n = \boldsymbol{\pi}^n\mathcal{H}_0$ in $\mathcal{H}$, and $\beta_{H,a} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$.

We need a twisted analogue of Rodier's theorem. It can be described as follows.

Let π be an admissible irreducible representation of G which is also σ-invariant: $\pi \simeq {}^\sigma\pi$, where ${}^\sigma\pi(\sigma(g)) = \pi(g)$. Then there exists an intertwining operator $A : \pi \to {}^\sigma\pi$, with $A\pi(g) = \pi(\sigma(g))A$ for all $g \in G$. Since π is irreducible, by Schur's lemma A^2 is a scalar which we may normalize by $A^2 = 1$. Thus A is unique up to a sign. Denote by G' the semidirect product $G \rtimes \langle\sigma\rangle$. Then π extends to G' by $\pi(\sigma) = A$. If π is generic, namely $\operatorname{Hom}_G(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi) \neq 0$ where $\operatorname{ind}_U^G \boldsymbol{\psi}$ is the space of Whittaker functions ($\varphi : G \to \mathbb{C}$ with $\varphi(ugk) = \boldsymbol{\psi}(u)\varphi(g)$, $u \in U$, $g \in G$, k in a compact open subgroup K_φ of G depending on φ, φ compactly supported on $U\backslash G$), then A is normalized by $A\varphi = {}^\sigma\varphi$ where ${}^\sigma\varphi(g) = \varphi(\sigma(g))$.

Let $G = \operatorname{GL}(3, F)$ and $a, b \in F^\times$. Define a character $\boldsymbol{\psi}^{a,b} : U \to \mathbb{C}^1$ on the unipotent subgroup $U = \left\{ u = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}$ of G by $\boldsymbol{\psi}^{a,b}(u) = \psi(ax + by)$. This one-dimensional representation has the property that $\boldsymbol{\psi}^{a,b}(\sigma(u)) = \boldsymbol{\psi}^{a,b}(u)$ for all u in U precisely when $a = b$. Put $\boldsymbol{\psi}^a = \boldsymbol{\psi}^{a,a}$. The group $\{\operatorname{diag}(a, 1, 1/a)\}$ of σ-invariant diagonal matrices in G acts simply transitively on the set of σ-invariant characters $\boldsymbol{\psi}^a$.

Let $\mathfrak{g}_0$ be the ring of 3×3 matrices with entries R. Write $\mathfrak{g}_n = \boldsymbol{\pi}^n\mathfrak{g}_0$ and $G_n = \exp(\mathfrak{g}_n)$. For $n \geq 1$ we have $G_n = {}^tU_nA_nU_n$, where $U_n = U\cap G_n$, and A_n is the group of diagonal matrices in G_n. Define a character $\psi_n^a : G \to \mathbb{C}^1$ supported on G_n by $\psi_n^a({}^tbu) = \psi(a(x+y)\boldsymbol{\pi}^{-2n})$ where ${}^tb \in {}^tU_nA_n$, $u \in U_n$. Alternatively, $\psi_n^a : G \to \mathbb{C}^1$ is defined by

$$\psi_n^a(\exp X) = \mathrm{ch}_{\mathfrak{g}_n}(X)\psi(\mathrm{tr}[X\boldsymbol{\pi}^{-2n}\beta_a]) \quad \text{where} \quad \beta_a = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix}.$$

This ψ_n^a is σ-invariant, and multiplicative on G_n.

The σ-twisted analogue of Rodier's theorem of interest to us (see E3 below) is as follows. Let $\mathrm{ch}_{G_n^\sigma}$ denote the characteristic function of $G_n^\sigma = \{g = \sigma g; g \in G_n\}$ in G_n.

Proposition 2.5.2. *The multiplicity*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{ind}_U^G \boldsymbol{\psi}^a, \pi) = \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_U^G \boldsymbol{\psi}^a, \pi)$$

is (*independent of* a *and*) *equal, for all sufficiently large* n, *to*

$$|G_n^\sigma|^{-1} \int_{G_n^\sigma} \chi_\pi^\sigma(g)\psi_n^a(g)dg.$$

Proof of Proposition 2.5. We are given the identity

$$\mathrm{tr}\,\pi(fdg \times \sigma) = (2m+1)\sum_{\pi_H} \mathrm{tr}\,\pi_H(f_Hdh).$$

The sum ranges over finitely many (in fact, two times the cardinality of the packet of $\pi_0(\mu')$) inequivalent square-integrable irreducible admissible representations π_H of $\mathrm{SL}(2,F)$, and π is generic with trivial central character. The number m is a nonnegative integer, independent of π_H. Note that in this proof we use the index H for what is usually indexed by 0 in this part, to be consistent with the notations of 2.5.1 and 2.5.2.

The identity for all matching test measures fdg and f_Hdh implies an identity of characters:

$$\chi_\pi^\sigma(\delta) = (2m+1)\sum_{\pi_H} \chi_{\pi_H}(\gamma)$$

for all $\delta \in G = \mathrm{GL}(3,F)$ with regular norm $\gamma \in H = \mathrm{SL}(2,F)$. The norm map $\delta \mapsto N\delta$ sends the stable σ-conjugacy class of δ to the stable conjugacy class of $N\delta$, which is determined by the two non-1 eigenvalues of $\delta\sigma(\delta)$. If $\delta \in G_n^\sigma$ then $\sigma\delta = \delta$, $\delta\sigma\delta = \delta^2$, and we are interested in the eigenvalues of δ^2. Now $\delta = \exp X$, $X \in \mathfrak{g}_n^\sigma$, $\sigma\delta = \exp(d\sigma X)$, where $d\sigma X = -J^t X J$, and $X = d\sigma X$ has the form $\begin{pmatrix} x & y & 0 \\ z & 0 & y \\ 0 & z & -x \end{pmatrix}$. Its eigenvalues are $0, \pm\sqrt{x^2+yz}$. Thus the norm $N\delta$ is the stable conjugacy class in $\mathrm{SL}(2,F)$ of $\exp Y$, $Y = 2\left(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right)$, as the eigenvalues of Y are $\pm 2\sqrt{x^2+yz}$. The norm map is compatible then with the isomorphism $G_n^\sigma \xrightarrow{\sim} H_n$, $e^X \mapsto e^Y$, when $p \neq 2$.

For $X \in \mathfrak{g}_n^\sigma$ the value

$$\psi_n^a(\exp X) = \mathrm{ch}_{\mathfrak{g}_n}(X)\psi(\mathrm{tr}[X\boldsymbol{\pi}^{-2n}\beta_a]) = \mathrm{ch}_{\mathfrak{g}_n^\sigma}(X)\psi(2ay\boldsymbol{\pi}^{-2n})$$

is equal to $\psi_{H,n}^a(\exp Y) = \mathrm{ch}_{\mathcal{H}_n}(Y)\psi(2ay\boldsymbol{\pi}^{-2n})$, namely for $\delta \in G_n^\sigma$ we have $\psi_n^a(\delta) = \psi_{H,n}^a(N\delta)$. Then

$$|G_n^\sigma|^{-1}\int_{G_n^\sigma}\chi_\pi^\sigma(\delta)\psi_n^a(\delta)d\delta = |H_n|^{-1}\int_{H_n}(2m+1)\sum_{\pi_H}\chi_{\pi_H}(\gamma)\psi_{H,n}^a(\gamma)d\gamma.$$

It follows that for any a in $F^\times$ we have

$$1 = \dim_{\mathbb{C}}\mathrm{Hom}_G(\mathrm{ind}_U^G\boldsymbol{\psi}^a,\pi) = (2m+1)\sum_{\pi_H}\dim_{\mathbb{C}}\mathrm{Hom}_H(\mathrm{ind}_{U_H}^H\boldsymbol{\psi}_H^a,\pi_H).$$

Hence $m=0$ and there is precisely one $\boldsymbol{\psi}_H^a$-generic π_H in the sum ($\dim_{\mathbb{C}} \neq 0$, necessarily $=1$), for every a. □

We say that π_0 λ_0-*lifts* to the (necessarily σ-invariant) representation π of G (and we write $\pi = \lambda_0(\pi_0)$) if π_0 and π satisfy (2.1.1). In terms of characters this can be rephrased as follows (cf. (I.3.3)).

Definition. An irreducible admissible representation π_0 of H_0 λ_0-*lifts* to the representation π of G (and we write $\pi = \lambda_0(\pi_0)$) if

$$\chi_\pi(\delta\sigma) = \chi_{\{\pi_0\}}(N\delta)$$

for all σ-regular elements δ of G, where $\{\pi_0\}$ denotes the packet of π_0.

2.6 Theorem. *Let F be a local field.*
(1) A one-dimensional, special, nonmonomial, type $\pi_0(\mu')$, representation of H, lifts to a one-dimensional, Steinberg, cuspidal, $I(\pi(\mu''),\chi_E)$, representation of G (respectively).
(2) A σ-invariant admissible irreducible representation π of G is a λ_0-lift of a packet $\{\pi_0\}$ of H precisely when it is σ-stable ($\chi_\pi^\sigma(\delta)$ depends only on the stable σ-conjugacy class of δ in G). Thus a σ-invariant π is a λ_0-lift unless it is of the form $I(\pi_1,1)$, where π_1 is an elliptic representation of H_1. In particular, a σ-invariant irreducible cuspidal representation π of G is σ-stable and is the λ_0-lift of a nonmonomial representation π_0 of H. This case may occur only if the residual characteristic of F is 2.

Proof. This follows from I.3.9 (case of special and trivial representations), 2.2.9(1) (nonmonomial case), (1.9) (case of $I(\pi_1,1)$), as well as (1.4) (list of σ-invariant representations), and 2.2.9(2), which asserts that $\pi_0(\mu'_u)$ lifts to $I(\pi(\mu''_u),\chi_u)$.

If π is a σ-invariant cuspidal representation of a local G, using the twisted trace formula we can construct a global cuspidal σ-invariant cuspidal representation of $\mathbf{G}(\mathbb{A})$ whose component at some place is our π. The global representation cannot be of the form $I(\pi_1,1)$, hence our local π is σ-stable, as asserted. □

Remark. It will be interesting to give a direct local proof (not using the trace formulae) that every σ-invariant cuspidal G-module π is σ-stable.

Definition. Let F be a number field. For each place v of F, let $\{\pi_{0v}\}$ be a packet of representations of $H_v = \mathrm{SL}(2,F_v)$, containing an unramified π_{0v}^0 for almost all v. We say that π_{0v}^0 is unramified if it has a nonzero K_{0v}-fixed vector where $K_{0v} = \mathrm{SL}(2,R_v)$.

The associated global *packet* is the set of products $\otimes_v \pi_{0v}$ with $\pi_{0v} \in \{\pi_{0v}\}$ for all v and with $\pi_{0v} = \pi_{0v}^0$ for almost all v.

If E is a quadratic extension of F and μ' a character of $\mathbb{A}_E^1/E^1$, define $\{\pi_0(\mu')\}$ by $\{\pi_0(\mu'_v)\}$ for all v.

Write $\varepsilon(\mu'_v,\pi_{0v}) = \pm 1$ if $\pi_{0v} \in \pi_0^{\pm}(\mu'_v)$. Note that $\varepsilon(\mu'_v,\pi_{0v}^0) = 1$.

For $\pi_0 = \otimes_v \pi_{0v} \in \{\pi_0(\mu')\}$ put $\varepsilon(\mu',\pi_0) = \prod_v \varepsilon(\mu'_v,\pi_{0v})$.

If $\mu'^2 \neq 1$ put $m(\pi_0) = \frac{1}{2}(1+\varepsilon(\mu',\pi_0))$.

If $\mu' \neq 1 = \mu'^2$ there are three pairs (E_i,μ'_i) such that $\mu'_1 = \mu'$ and $\{\pi(\mu'_i)\} = \{\pi(\mu')\}$, $i = 1,2,3$. For $\pi_0 = \otimes_v \pi_{0v} \in \{\pi_0(\mu')\}$ put $m(\pi_0) = \frac{1}{4}[1+\sum_{1\le i\le 3}\varepsilon(\mu'_i,\pi_0)]$.

The *unstable* discrete spectrum of $L(\mathrm{SL}(2,F)\backslash\mathrm{SL}(2,\mathbb{A}))$ is defined to consist of all packets of the form $\{\pi(\mu')\}$. The *stable* spectrum is its complement. A packet is named (un)stable if it lies in the (un)stable spectrum.

Our main goal is to describe all automorphic discrete-spectrum representations of $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2,\mathbb{A})$, namely the decomposition of the discrete spectrum of $L(\mathrm{SL}(2,F)\backslash\mathrm{SL}(2,\mathbb{A}))$.

2.7 Theorem. *Let F be a number field.*
(1) *The packets partition the discrete spectrum of* $\mathrm{SL}(2,\mathbb{A})$. *Thus if π_0' and π_0 are cuspidal, and $\pi_{0v}' \simeq \pi_{0v}$ for almost all v, then $\{\pi_0\} = \{\pi_0'\}$.*
(2) *Every packet $\{\pi_0\}$ of representations of* $\mathrm{SL}(2,\mathbb{A})$ *λ_0-lifts to a unique automorphic representation π of* $\mathrm{PGL}(3,\mathbb{A})$. *The λ_0-lift π is one dimensional if π_0 is one dimensional. It is cuspidal if $\{\pi_0\}$ is cuspidal but not of the form $\{\pi_0(\mu')\}$. It is of the form $I(\pi(\mu''),\chi_E)$, $\mu''(z) = \mu'(z/\overline{z})$ on $z \in \mathbb{A}_E^\times$, if π_0 is in a packet $\{\pi_0(\mu')\}$ associated with a character μ' of $\mathbb{A}_E^1/E^1$. If $\mu' \neq 1 = \mu'^2$ then $I(\pi(\mu''),\chi_E) = I(\mu,\mu\chi_E,\chi_E)$ where $\mu'(z) = \mu(z\overline{z})$ ($z \in \mathbb{A}_E^\times$) defines μ on $\mathbb{A}^\times/F^\times$ up to multiplication by $\chi_E : \mathbb{A}^\times/F^\times N_{E/F}\mathbb{A}_E^\times \xrightarrow{\sim} \{\pm 1\}$.*
(3) *Each cuspidal π_0 occurs only once in the cuspidal spectrum of*

$$L(\mathrm{SL}(2,F)\backslash\mathrm{SL}(2,\mathbb{A})).$$

Every π_0 in a stable packet (not of the form $\pi(\mu')$) occurs with multiplicity one in the cuspidal spectrum. A $\pi_0 \in \pi(\mu')$, $\mu'^2 \neq 1$, occurs with multiplicity $m(\pi_0) = \frac{1}{2}(1+\varepsilon(\mu',\pi_0))$.

A $\pi_0 \in \pi(\mu')$, $\mu' \neq 1 = \mu'^2$, occurs with multiplicity $m(\pi_0)$ equal to $\frac{1}{4}[1+\sum_{1\le i\le 3}\varepsilon(\mu_i',\pi_0)]$.
(4) *Every σ-invariant automorphic representation π of* $\mathrm{PGL}(3,\mathbb{A})$ *which is neither of the form $I(\mu,1,\mu^{-1})$, where μ is a character of $\mathbb{A}^\times/F^\times$, nor of the form $I(\pi_1,1)$, where π_1 is a discrete-spectrum representation of* $\mathrm{PGL}(2,\mathbb{A})$, *is the λ_0-lift of a unique cuspidal packet $\{\pi_0\}$ of* $\mathrm{SL}(2,\mathbb{A})$. *Such a π has no component of the form $I(\pi_{1v},1)$ where π_{1v} is elliptic.*

Remark. (1) is the *rigidity theorem* for packets of the cuspidal representations of $\mathrm{SL}(2,\mathbb{A})$. (3) is the *multiplicity one theorem* for the cuspidal representations of $\mathbf{H}(\mathbb{A}) = \mathrm{SL}(2,\mathbb{A})$.

Proof. This follows from the trace formulae identity (1.8.1), noting as in 1.9.1 that $I_1' = I_1$ can be removed, and from our local lifting results, on applying our usual arguments of "generalized linear independence of characters". Indeed, fixing E and μ', using 1.10 we see that (1.8.1) takes the form $\frac{1}{2}I' + \frac{1}{2}I'_E = I_0$ namely

$$\frac{1}{2}\prod_{v\notin V}\operatorname{tr} I(\pi(\mu''_v),\chi_{E,v};f_v dg_v \times \sigma) + \frac{1}{2}\prod_{v\notin V}\mu'_v(f_{E_v}dt_{E,v})$$

$$= \sum_{\pi_0} m(\pi_0)\prod_{v\notin V}\operatorname{tr}\pi_{0v}(f_{0v}dh_v)$$

if $\mu'^2 \neq 1$, or $\frac{1}{4}I' + \frac{1}{4}\sum_E I_E = I_0$, namely

$$\frac{1}{4}\prod_{v\notin V}\operatorname{tr} I(\mu_{1v},\mu_{2v},\mu_{3v};f_v dg_v \times \sigma) + \frac{1}{4}\sum_{1\le i\le 3}\prod_{v\notin V}\mu'_{iv}(f_{E_i v}dt_{E_i v})$$

$$= \sum_{\pi_0} m(\pi_0)\prod_{v\notin V}\operatorname{tr}\pi_{0v}(f_{0v}dh_v)$$

with $\{\mu_1,\mu_2,\mu_3\} = \{\mu,\mu\chi_E,\chi_E\}$. The local lifting results and linear independence of characters show that there are π_0 on the right which λ_0-lift to $I(\pi(\mu''),\chi_E)$ if $\mu'^2 \neq 1$ or to $I(\mu,\mu\chi_E,\chi_E)$ if $\mu' \neq 1 = \mu'^2$, and all the π_0 that occur are in the packet $\{\pi_0(\mu')\}$, with multiplicities as stated in the last two sentences of (3).

At this stage (1.8.1) is reduced to $I = I_0$. Then (4) is clear, as by 1.4 we need to consider only a σ-invariant cuspidal π. It contributes the only term in I, hence π is the λ_0-lift of some $\{\pi_0\}$, again by the local character relations and linear independence of characters. Each member of $\{\pi_0\}$ occurs with multiplicity $m(\pi_0) = 1$, by the local character relations.

It remains to show that each cuspidal π_0 lifts to some π, namely that if π_0 contributes to I_0 in the equality $I = I_0$, then I is not empty. But this follows from linear independence of characters, or alternatively on using I.4.3.1. □

2.7.1 Corollary [GJ]. *If a unitary cuspidal representation* $\tilde{\pi}_0$ *of* $\mathrm{GL}(2,\mathbb{A})$ *has a local component* $\tilde{\pi}_{0v}$ *of the form* $I_{0v}(\mu_1\nu_v^t,\mu_2\nu_v^{-t})$, $|\mu_i| = 1$, $\nu_v(x) = |x|_v$, $t \ge 0$, *then* $t < \frac{1}{4}$.

Proof. This follows at once from the existence of the lifting λ_0. The restriction $\{\pi_0\}$ of $\tilde{\pi}_0$ to $\mathbf{H}(\mathbb{A})$ is a discrete-spectrum packet, which lifts to an automorphic representation π of $\mathbf{G}(\mathbb{A})$. In particular, the induced $\tilde{\pi}_0(\mu_1\nu_v^t, \mu_2\nu_v^{-t})$ lifts to $I(\mu\nu_v^{2t}, 1, \mu\nu_v^{-2t})$, $\mu = \mu_1/\mu_2$, which is unitarizable only if $2t < \frac{1}{2}$. □

For any representation $\tilde{\pi}_{0v}$ of $\mathrm{GL}(2, F_v)$ and character χ_v of $F_v^\times$ put

$$L_2(s, \tilde{\pi}_{0v}, \chi_v) = L(s, \tilde{\pi}_{0v}\chi_v \times \check{\tilde{\pi}}_{0v})/L(s, \chi_v),$$

and

$$\varepsilon_2(s, \tilde{\pi}_{0v}, \chi_v; \psi_v) = \varepsilon(s, \tilde{\pi}_{0v}\chi_v \times \check{\tilde{\pi}}_{0v}; \psi_v)/\varepsilon(s, \chi_v; \psi_v).$$

Here $\check{\tilde{\pi}}_{0v}$ is the contragredient of $\tilde{\pi}_{0v}$ and ψ_v is a nontrivial additive character of F_v. The L-functions depend only on the packet $\{\pi_{0v}\}$ defined by π_{0v}. As in [GJ], we say that $\tilde{\pi}_{0v}$ *L-lifts* to a representation π_v of G_v if π_v is σ-invariant and

$$L(s, \pi_v\chi_v) = L_2(s, \tilde{\pi}_{0v}, \chi_v), \qquad \varepsilon(s, \pi_v\chi_v; \psi_v) = \varepsilon_2(s, \tilde{\pi}_{0v}, \chi_v; \psi_v),$$

for any character χ_v of $F_v^\times$. Here π_v is viewed as a representation of $\mathrm{GL}(3, F_v)$ with a trivial central character. Gelbart and Jacquet [GJ], Propositions 3.2, 3.3, showed for nonmonomial $\tilde{\pi}_{0v}$ that $\{\pi_{0v}\}$ L-lifts to the lift π_v of $\{\pi_{0v}\}$. If $\tilde{\pi}_0$ is an automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ and χ is a character of $F^\times\backslash\mathbb{A}^\times$, then the function $L_2(s, \tilde{\pi}_0, \chi)$ is defined to be the product over all v of the $L_2(s, \tilde{\pi}_{0v}, \chi_v)$.

2.7.2 COROLLARY [GJ]. *If π_0 is cuspidal and not monomial (of the form $\pi_0(\mu')$), then*

$$L_2(s, \pi_0, \chi) = L(s, \pi\chi)$$

for any character χ of $F^\times\backslash\mathbb{A}^\times$, where π is the lift of π_0. Hence $L_2(s, \pi_0, \chi)$ is entire for any χ.

PROOF. The local factors of the two global products are equal unless π_v is cuspidal, but then both local factors are equal to 1.

It is easy to deduce from this [GJ], p. 535, that π_{0v} L-lifts to its lift π_v in the remaining case, where π_v is cuspidal. □

Corollary 2.7.2 was proved directly using the Rankin-Shimura method in [GJ], where it was used as the key tool to prove that each π_{0v} and π_0 L-lift to their lifts. The advantage of the trace formula is in characterizing the image of the lifting, establishing character relations and proving the multiplicity one theorem and the rigidity theorem for $\mathrm{SL}(2,\mathbb{A})$, in addition to proving the existence of the lifting.

2.7.3 Multiplicities. Following [LL], the packets can be described in duality with the dual group. Namely, if F is local, the character relations define a duality $\langle .,. \rangle : \mathcal{C}_\varphi \times \{\pi_0\} \to \{\pm 1\}$ between the packet $\{\pi_0\}$ which is parametrized by $\varphi : W_F \to {}^L H = \widehat{H} \times W_F$, and $\mathcal{C}_\varphi = C_\varphi / C_\varphi^0$. Here C_φ is the centralizer $Z(\varphi(W_F), {}^L H)$ of $\varphi(W_F)$ in ${}^L H$; as usual, superscript 0 means connected component of the identity. Indeed, suppose φ is

$$(\mathrm{Ind}_{W_E}^{W_F} \mu')_0 : W_F \to \widehat{H} = \mathrm{PGL}(2,\mathbb{C}).$$

When $\mu' = 1$ on E^1, $\varphi = (\chi_E \oplus 1)_0$ on W_F factorizes via $F^\times$, and $C_\varphi = \langle w_0, A \rangle$, where A is the diagonal subgroup, w is the antidiagonal matrix, and index 0 means image in $\mathrm{PGL}(2,\mathbb{C})$. Hence $\mathcal{C}_\varphi$ is $\mathbb{Z}/2$. If $\mu'^2 \neq 1$ then $C_\varphi = \langle w_0 \rangle$. If $\mu'^2 = 1 \neq \mu'$ then $C_\varphi = \langle w_0, \mathrm{diag}(-1,1)_0 \rangle$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$.

If $\{\pi_0\}$ is a global packet containing a cuspidal representation, which is associated with a homomorphism $\varphi : W_F \to {}^L H$, then the local packet $\{\pi_{0v}\}$ is associated with the restriction φ_v of φ to the decomposition group W_{F_v}, $C_\varphi = Z(\varphi(W_F), {}^L H) \subset C_{\varphi_v} = Z(\varphi(W_{F_v}), {}^L H)$, $C_\varphi^0 \subset C_{\varphi_v}^0$ induce $\mathcal{C}_\varphi \to \mathcal{C}_{\varphi_v}$. For π_0 in $\{\pi_0\}$ let $\langle s, \pi_0 \rangle$ be $\prod_v \langle s, \pi_{0v} \rangle$. Then the multiplicity $m(\pi_0)$ of π_0 in the discrete spectrum is $|\mathcal{C}_\varphi|^{-1} \sum_{s \in \mathcal{C}_\varphi} \langle s, \pi_0 \rangle$, at least where we know to associate $\{\pi_0\}$ with φ, namely in the monomial case.

Unipotent, nontempered representations, their quasipackets and multiplicities in the discrete spectrum, are described by conjectures of Arthur [A2]. However, for our group SL(2) these are only the trivial representation.

V.3 Characters and genericity

In this section we reduce Proposition 2.5.2 to Proposition 2.5.1 for G (not for H), so we begin by recalling the main lines in Rodier's proof in the context of G. Fix $d = \mathrm{diag}(\boldsymbol{\pi}^{-r+1}, \boldsymbol{\pi}^{-r+3}, \dots, \boldsymbol{\pi}^{r-1})$. Put $V_n = d^n G_n d^{-n}$ and $\boldsymbol{\psi}_n(u) = \psi_n(d^{-n} u d^n)$ $(u \in V_n)$. Note that $\theta(d) = d$, $\theta(G_n) = G_n$, $\theta(U_n) = U_n$, $\theta\psi_n = \psi_n$, and that the entries in the jth line $(j \neq 0)$ above or below the diagonal of $v = (v_{ij})$ in V_n lie in $\boldsymbol{\pi}^{(1-2j)n} R$ (thus $v_{i,i+j} \in \boldsymbol{\pi}^{(1-2j)n} R$ if $j > 0$, and also when $j < 0$). Thus $V_n \cap U$ is a θ-invariant strictly increasing sequence of compact and open subgroups of U whose union is U, while $V_n \cap (\overline{U}H)$ — where $\overline{U}H$ is the lower triangular subgroup of G — is a strictly decreasing sequence of compact open subgroups of G whose intersection is the element I of G.

Note that $\boldsymbol{\psi}_n = \boldsymbol{\psi}$ on $V_n \cap U$.

Consider the induced representations $\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n$, and the intertwining operators

$$A_n^m : \operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n \to \operatorname{ind}_{V_m}^G \boldsymbol{\psi}_m,$$

$$(A_n^m \varphi)(g) = ((|V_m|^{-1} 1_{V_m} \boldsymbol{\psi}_m) * \varphi)(g) = |V_m|^{-1} \int_{V_m} \boldsymbol{\psi}_m(u)\varphi(u^{-1}g)du$$

(g in G, φ in $\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n$, $|V_m|$ denotes the volume of V_m, 1_{V_m} denotes the characteristic function of V_m). Note that for $m \geq n \geq 1$

$$(A_n^m \varphi)(g) = ((|V_m \cap U|^{-1} 1_{V_m \cap U} \boldsymbol{\psi}) * \varphi)(g)$$

$$= |V_m \cap U|^{-1} \int_{V_m \cap U} \boldsymbol{\psi}(u)\varphi(u^{-1}g)du.$$

Hence $A_m^\ell \circ A_n^m = A_n^\ell$ for $\ell \geq m \geq n \geq 1$. So $(\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n, A_n^m \ (m \geq n \geq 1))$ is an inductive system of representations of G. Denote by $(I, A_n : \operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n \to I)$ $(n \geq 1)$ its limit.

The intertwining operators $\phi_n : \operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n \to \operatorname{ind}_U^G \boldsymbol{\psi}$,

$$(\phi_n(\varphi))(g) = (\boldsymbol{\psi} 1_U * \varphi)(g) = \int_U \boldsymbol{\psi}(u)\varphi(u^{-1}g)du,$$

satisfy $\phi_n \circ A_n^m = \phi_n$ if $m \geq n \geq 1$. Hence there exists a unique intertwining operator $\phi : I \to \operatorname{ind}_U^G \boldsymbol{\psi}$ with $\phi \circ A_n = \phi_n$ for all $n \geq 1$. Proposition 3 of [Rd] asserts that

Lemma 3.1. *The map ϕ is an isomorphism of G-modules.*

Lemma 3.2. *There exists $n_0 \geq 1$ such that*

$$\boldsymbol{\psi}_n * \boldsymbol{\psi}_m * \boldsymbol{\psi}_n = |V_n||V_m \cap V_n|\boldsymbol{\psi}_n$$

for all $m \geq n \geq n_0$.

Proof. This is Lemma 5 of [Rd]. We review its proof (the first displayed formula in the proof of this Lemma 5, [Rd], p. 159, line -8, should be erased).

There are finitely many representatives u_i in $U \cap V_m$ for the cosets of V_m modulo $V_n \cap V_m$. Denote by $\varepsilon(g)$ the Dirac measure in a point g of G. Consider

$$(\varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m \cap V_n})(g) = \int_G \varepsilon(u_i)(gh^{-1})(\boldsymbol{\psi}_n 1_{V_m \cap V_n})(h)dh$$

$$= \boldsymbol{\psi}_n(u_i^{-1}g) = \boldsymbol{\psi}_m(u_i)^{-1}\boldsymbol{\psi}_m(g).$$

Note here that if the left side is nonzero, then $g \in u_i(V_m \cap V_n) \subset V_m$. Conversely, if $g \in V_m$, then $g \in u_i(V_m \cap V_n)$ for some i. Hence $\boldsymbol{\psi}_m = \sum_i \boldsymbol{\psi}_m(u_i)\varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m \cap V_n}$, thus

$$\boldsymbol{\psi}_n * \boldsymbol{\psi}_m * \boldsymbol{\psi}_n = \sum_i \boldsymbol{\psi}_m(u_i)\boldsymbol{\psi}_n * \varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m \cap V_n} * \boldsymbol{\psi}_n.$$

Since $\boldsymbol{\psi}_n 1_{V_m \cap V_n} * \boldsymbol{\psi}_n = |V_m \cap V_n|\boldsymbol{\psi}_n$, this is $= \sum_i \boldsymbol{\psi}_m(u_i)|V_m \cap V_n|\boldsymbol{\psi}_n * \varepsilon(u_i) * \boldsymbol{\psi}_n$. But the key Lemma 4 of [Rd] asserts that $\boldsymbol{\psi}_n * \varepsilon(u) * \boldsymbol{\psi}_n \neq 0$ implies that $u \in V_n$. Hence the last sum reduces to a single term, with $u_i = 1$, and we obtain

$$= |V_m \cap V_n|\boldsymbol{\psi}_n * \boldsymbol{\psi}_n = |V_m \cap V_n||V_m|\boldsymbol{\psi}_n.$$

This completes the proof of the lemma. □

LEMMA 3.3. *For an inductive system $\{I_n\}$ of G-modules we have*

$$\mathrm{Hom}_G(\varinjlim I_n, \pi) = \varprojlim \mathrm{Hom}_G(I_n, \pi).$$

PROOF. See, e.g., Rotman [Rt], Theorem 2.27. □

COROLLARY. *We have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n |G_n|^{-1} \operatorname{tr} \pi(\psi_n dg).$$

PROOF. As the $\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$ are increasing with n, if they are bounded we get that they are independent of n for sufficiently large n. Hence the left side of the corollary is equal to $\lim_n \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$. This is equal to $\lim_n \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_{G_n}^G \psi_n, \pi)$ since $\boldsymbol{\psi}_n(v) = \psi_n(d^{-n}vd^n)$. This is equal to $\lim_n \dim_{\mathbb{C}} \mathrm{Hom}_{G_n}(\psi_n, \pi|G_n)$ by Frobenius reciprocity. This is equal to the right side of the corollary since $|G_n|^{-1}\pi(\psi_n dg)$ is a projection from π to the space of ξ in π with $\pi(g)\xi = \psi_n(g)\xi$ $(g \in G_n)$, a space whose dimension is then $|G_n|^{-1}\operatorname{tr}\pi(\psi_n dg)$. □

We can now discuss the twisted case. Note that since $\theta\boldsymbol{\psi}_n = \boldsymbol{\psi}_n$, the representations $\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n$ are θ-invariant, where θ acts on $\varphi \in \operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n$ by $\varphi \mapsto \theta\varphi$, $(\theta\varphi)(g) = \varphi(\theta g)$. Similarly $\theta\boldsymbol{\psi} = \boldsymbol{\psi}$ and $\operatorname{ind}_U^G \boldsymbol{\psi}$ is θ-invariant. We then extend these representations ind of G to the semidirect product $G' = G \rtimes \langle\theta\rangle$ by putting $(I(\theta)\varphi)(g) = \varphi(\theta(g))$.

Let π be an irreducible admissible representation of G. Suppose it is θ-invariant. Thus there exists an intertwining operator $A : \pi \to {}^\theta\pi$, where ${}^\theta\pi(g) = \pi(\theta(g))$, with $A\pi(g) = \pi(\theta(g))A$. Then A^2 commutes with every $\pi(g)$ $(g \in G)$, hence A^2 is a scalar by Schur's lemma, and can be normalized to be 1. We extend π from G to $G' = G \rtimes \langle\theta\rangle$ by putting $\pi(\theta) = A$ once A is chosen.

If $\operatorname{Hom}_G(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi) \neq 0$, its dimension is 1. Choose a generator ℓ : $\operatorname{ind}_U^G \boldsymbol{\psi} \to \pi$. Define $A : \pi \to \pi$ by $A\ell(\varphi) = \ell(I(\theta)\varphi)$. Then

$$\operatorname{Hom}_G(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi) = \operatorname{Hom}_{G'}(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi).$$

Similarly we have $\quad \operatorname{Hom}_G(\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n, \pi) = \operatorname{Hom}_{G'}(\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$.
The right side in the last equality can be expressed as

$$\operatorname{Hom}_{G'}(\operatorname{ind}_{G_n}^G \psi_n, \pi) = \operatorname{Hom}_{G'_n}(\psi'_n, \pi|G'_n) \qquad (G'_n = G_n \rtimes \langle\theta\rangle).$$

The last equality follows from Frobenius reciprocity, where we extended ψ_n to a homomorphism ψ'_n on G'_n whose value at $1\times\theta$ is 1. Thus $\psi'_n = \psi_n^1 + \psi_n^\theta$ with $\psi_n^\alpha(g \times \beta) = \delta_{\alpha\beta}\psi_n(g)$, $\alpha, \beta \in \{1, \theta\}$.

Now $\operatorname{Hom}_{G'_n}(\psi'_n, \pi|G'_n)$ is isomorphic to the space π_1 of vectors ξ in π with $\pi(g)\xi = \psi_n(g)\xi$ for all g in G'_n. In particular $\pi(g)\xi = \psi_n(g)\xi$ for all g in G_n, and $\pi(\theta)\xi = \xi$. Clearly $|G'_n|^{-1}\pi(\psi'_n dg')$ is a projection from the space of π to π_1. It is independent of the choice of the measure dg'. Its trace is then the dimension of the space Hom. We conclude a twisted analogue of the theorem of [Rd]:

Proposition 3.1. *We have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n |G'_n|^{-1}\operatorname{tr}\pi(\psi'_n dg')$$

where the limit stabilizes for a large n.

Note that G'_n is the semidirect product of G_n and the two-element group $\langle\theta\rangle$. With the natural measure assigning 1 to each element of the discrete group $\langle\theta\rangle$, we have $|G'_n| = 2|G_n|$. The right side is then

$$\frac{1}{2}\lim_n |G_n|^{-1}\operatorname{tr}\pi(\psi_n dg) + \frac{1}{2}\lim_n |G_n|^{-1}\operatorname{tr}\pi(\psi_n dg\times\theta)$$

(as $\psi'_n = \psi^1_n + \psi^\theta_n$, $\psi^1_n = \psi_n$ and $\operatorname{tr}\pi(\psi^\theta_n dg) = \operatorname{tr}\pi(\psi_n dg\times\theta)$). By the nontwisted version of Rodier's theorem,

$$\dim_{\mathbb{C}}\operatorname{Hom}_G(\operatorname{ind}_U^G\boldsymbol{\psi},\pi) = \lim_n |G_n|^{-1}\operatorname{tr}\pi(\psi_n dg),$$

we conclude that for θ-invariant π

PROPOSITION 3.2. *We have*

$$\dim_{\mathbb{C}}\operatorname{Hom}_{G'}(\operatorname{ind}_U^G\boldsymbol{\psi},\pi) = \lim_n |G_n|^{-1}\operatorname{tr}\pi(\psi_n dg\times\theta). \qquad\square$$

PROPOSITION 3.3. *The terms in the limit on the right of Proposition* 2 *are equal to*

$$|G^\theta_n|^{-1}\int_{G^\theta_n}\chi^\theta_\pi(g)\psi_n(g)dg.$$

PROOF. Consider the map $G^\theta_n\times G^\theta_n\backslash G_n \to G_n$, $(u,k)\mapsto k^{-1}u\theta(k)$. It is a closed immersion. More generally, given a semisimple element s in a group G, we can consider the map $Z_{G^0}(s)\times Z_{G^0}(s)\backslash G^0 \to G^0$ by $(u,k)\mapsto k^{-1}usks^{-1}$. Our example is: $(s,G) = (\theta, G_n\times\langle\theta\rangle)$.

Our map is in fact an analytic isomorphism since G_n is a small neighborhood of the origin, where the exponential $e:\mathfrak{g}_n\to G_n$ is an isomorphism. Indeed, we can transport the situation to the Lie algebra $\mathfrak{g}_n$. Thus we write $k = e^Y$, $u = e^X$, $\theta(k) = e^{(d\theta)(Y)}$, $k^{-1}u\theta(k) = e^{X-Y+(d\theta)(Y)}$, up to smaller terms. Here $(d\theta)(Y) = -J^{-1}{}^tYJ$. So we just need to show that $(X,Y)\mapsto X-Y+(d\theta)(Y)$, $Z_{\mathfrak{g}_n}(\theta)+\mathfrak{g}_n(\operatorname{mod} Z_{\mathfrak{g}_n}(\theta))\to\mathfrak{g}_n$, is bijective. But this is obvious since the kernel of $(1-d\theta)$ on $\mathfrak{g}_n$ is precisely $Z_{\mathfrak{g}_n}(\theta) = \{X\in\mathfrak{g}_n; (d\theta)(X) = X\}$.

Changing variables on the terms on the right of Proposition 2 we get the equality:

$$|G_n|^{-1}\int_{G_n}\chi^\theta_\pi(g)\psi_n(g)dg$$

$$= |G_n|^{-1} \int_{G_n^\theta} \int_{G_n^\theta \backslash G_n} \chi_\pi^\theta(k^{-1} u\theta(k))\psi_n(k^{-1}u\theta(k))dkdu.$$

But $\theta\psi_n = \psi_n$, ψ_n is multiplicative on G_n, G_n is compact, and χ_π^θ is a θ-conjugacy class function, so we end up with

$$= |G_n^\theta|^{-1} \int_{G_n^\theta} \chi_\pi^\theta(u)\psi_n(u)du.$$

Our proposition, and Proposition 2.5.2, follow. □

V.3.1 Germs of twisted characters

Harish-Chandra [HC2] showed that χ_π is locally integrable (Thm 1, p. 1) and has a germ expansion near each semisimple element γ (Thm 5, p. 3), of the form:

$$\chi_\pi(\gamma \exp X) = \sum_{\mathcal{O}} c_\gamma(\mathcal{O}, \pi)\widehat{\mu}_{\mathcal{O}}(X).$$

Here $\mathcal{O}$ ranges over the nilpotent orbits in the Lie algebra $\mathfrak{m}$ of the centralizer M of γ in G, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$, $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform with respect to a symmetric nondegenerate G-invariant bilinear form B on $\mathfrak{m}$ and a selfdual measure, and $c_\gamma(\mathcal{O}, \pi)$ are complex numbers. Both $\mu_{\mathcal{O}}$ and $c_\gamma(\mathcal{O}, \pi)$ depend on a choice of a Haar measure $d_{\mathcal{O}}$ on the centralizer $Z_G(X_0)$ of $X_0 \in \mathcal{O}$, but their product does not. The X ranges over a small neighborhood of the origin in $\mathfrak{m}$. We shall be interested only in the case of $\gamma = 1$, and thus omit γ from the notations.

Suppose that G is quasi-split over F, and U is the unipotent radical of a Borel subgroup B. Let $\boldsymbol{\psi} : U \to \mathbb{C}^1$ be the nondegenerate character of U (its restriction to each simple root subgroup is nontrivial) specified in Rodier [Rd], p. 153. The number $\dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi)$ of $\boldsymbol{\psi}$-Whittaker functionals on π is known to be zero or one. Let $\mathfrak{g}_0$ be a selfdual lattice in the Lie algebra $\mathfrak{g}$ of G. Denote by ch_0 the characteristic function of $\mathfrak{g}_0$ in $\mathfrak{g}$. Rodier [Rd], p. 163, showed that there is a regular nilpotent orbit $\mathcal{O} = \mathcal{O}_{\boldsymbol{\psi}}$ such that $c(\mathcal{O}, \pi)$ is not zero iff $\dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi)$ is one, in fact $\widehat{\mu}_{\mathcal{O}}(\mathrm{ch}_0)c(\mathcal{O}, \pi)$ is one in this case. Alternatively put, normalizing $\mu_{\mathcal{O}}$ by $\widehat{\mu}_{\mathcal{O}}(\mathrm{ch}_0) = 1$, we have $c(\mathcal{O}, \pi) = \dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi)$. This is shown in [Rd] for all p if $G = \mathrm{GL}(n, F)$, and for general quasi-split G for all $p \geq 1 + 2\sum_{\alpha \in S} n_\alpha$, if the longest root is $\sum_{\alpha \in S} n_\alpha \alpha$ in a basis S of the root system. A generalization of Rodier's theorem to degenerate Whittaker models and nonregular nilpotent orbits is given in Moeglin-Waldspurger [MW]. See [MW], I.8, for the normalization of measures. In particular they show that $c(\mathcal{O},\pi)>0$ for the nilpotent orbits $\mathcal{O}$ of maximal dimension with $c(\mathcal{O},\pi)\neq 0$.

Harish-Chandra's results extend to the twisted case. The twisted character is locally integrable (Clozel [Cl2], Thm 1, p. 153), and there exist unique complex numbers $c^\theta(\mathcal{O},\pi)$ ([Cl2], Thm 3, p. 154) with $\chi_\pi^\theta(\exp X) = \sum_{\mathcal{O}} c^\theta(\mathcal{O},\pi)\widehat{\mu}_{\mathcal{O}}(X)$. Here $\mathcal{O}$ ranges over the nilpotent orbits in the Lie algebra $\mathfrak{g}^\theta$ of the group G^θ of the $g\in G$ with $g=\theta(g)$. Further, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$ (it is unique up to a constant, not unique as stated in [HC2], Thm 5, and [Cl2], Thm 3); $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform, and X ranges over a small neighborhood of the origin in $\mathfrak{g}^\theta$.

In this section we compute the expression displayed in Proposition 3 using the germ expansion $\chi_\pi^\sigma(\exp X) = \sum_{\mathcal{O}} c^\sigma(\mathcal{O},\pi)\widehat{\mu}_{\mathcal{O}}(X)$. This expansion means that for any test measure fdg supported on a small enough neighborhood of the identity in G we have

$$\int_{\mathfrak{g}^\sigma} f(\exp X)\chi_\pi^\sigma(\exp X)dX$$

$$=\sum_{\mathcal{O}} c^\sigma(\mathcal{O},\pi)\int_{\mathcal{O}}\left[\int_{\mathfrak{g}^\sigma} f(\exp X)\psi(\mathrm{tr}(XZ))dX\right]d\mu_{\mathcal{O}}(Z).$$

Here $\mathcal{O}$ ranges over the nilpotent orbits in $\mathfrak{g}^\sigma$, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$, $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform. The X range over a small neighborhood of the origin in $\mathfrak{g}^\sigma$. Since we are interested only in the case of the symmetric square, and to simplify the exposition, we take $G=\mathrm{GL}(n,F)$ and the involution σ, $\sigma(g)=J^{-1t}g^{-1}J$. In this case there is a unique regular nilpotent orbit $\mathcal{O}_0$.

We normalize the measure $\mu_{\mathcal{O}_0}$ on the orbit $\mathcal{O}_0$ of β in $\mathfrak{g}^\sigma$ by the requirement that $\widehat{\mu}_{\mathcal{O}_0}(\mathrm{ch}_0^\sigma)$ is 1, thus that $\int_{\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma} d\mu_{\mathcal{O}_0}(X) = q^{n\dim(\mathcal{O}_0)}$ for large n. Equivalently a measure on an orbit $\mathcal{O}\simeq G/Z_G(Y)$ $(Y\in\mathcal{O})$ is defined by a measure on its tangent space $m=\mathfrak{g}/Z_{\mathfrak{g}}(Y)$ ([MW], p. 430) at Y, taken to be the selfdual measure with respect to the symmetric bilinear nondegenerate F-valued form $B_Y(X,Z)=\mathrm{tr}(Y[X,Z])$ on m.

PROPOSITION 3.4. *If π is a σ-invariant admissible irreducible representation of G and $\mathcal{O}_0$ is the regular nilpotent orbit in $\mathfrak{g}^\sigma$, then the coefficient $c^\sigma(\mathcal{O}_0,\pi)$ in the germ expansion of the σ-twisted character χ_π^σ of π is equal to*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\operatorname{ind}_U^G \boldsymbol{\psi},\pi) = \dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{ind}_U^G \boldsymbol{\psi},\pi).$$

This number is one if π is generic, and zero otherwise.

PROOF. We compute the expression displayed in Proposition 3 as in [MW], I.12. It is a sum over the nilpotent orbits $\mathcal{O}$ in $\mathfrak{g}^\sigma$, of $c^\sigma(\mathcal{O},\pi)$ times

$$|G_n^\sigma|^{-1}\widehat{\mu}_{\mathcal{O}}(\psi_n\circ e) = |G_n^\sigma|^{-1}\mu_{\mathcal{O}}(\widehat{\psi_n\circ e}) = |G_n^\sigma|^{-1}\int_{\mathcal{O}} \widehat{\psi_n\circ e}(X)d\mu_{\mathcal{O}}(X).$$

The Fourier transform (with respect to the character ψ_E) of $\psi_n\circ e$,

$$\widehat{\psi_n\circ e}(Y) = \int_{\mathfrak{g}^\sigma} \psi_n(\exp Z)\overline{\psi}_E(\operatorname{tr} ZY)dZ = \int_{\mathfrak{g}_n^\sigma} \psi_E(\operatorname{tr} Z(\boldsymbol{\pi}^{-2n}\beta - Y))dZ,$$

is the characteristic function of $\boldsymbol{\pi}^{-2n}\beta + \boldsymbol{\pi}^{-n}\mathfrak{g}_0^\sigma = \boldsymbol{\pi}^{-2n}(\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma)$ multiplied by the volume $|\mathfrak{g}_n^\sigma| = |G_n^\sigma|$ of $\mathfrak{g}_n^\sigma$. Hence we get

$$= \int_{\mathcal{O}\cap(\boldsymbol{\pi}^{-2n}(\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma))} d\mu_{\mathcal{O}}(X) = q^{n\dim(\mathcal{O})}\int_{\mathcal{O}\cap(\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma)} d\mu_{\mathcal{O}}(X).$$

The last equality follows from the homogeneity result of [HC2], Lemma 3.2, p. 18. For sufficiently large n we have that $\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma$ is contained only in the orbit $\mathcal{O}_0$ of β. Then only the term indexed by $\mathcal{O}_0$ remains in the sum over $\mathcal{O}$, and

$$\int_{\mathcal{O}_0\cap(\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma)} d\mu_{\mathcal{O}_0}(X) = \int_{\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma} d\mu_{\mathcal{O}_0}(X)$$

equals $q^{-n\dim(\mathcal{O}_0)}$ (cf. [MW], end of proof of Lemme I.12). The proposition follows. □

VI. COMPUTATION OF A TWISTED CHARACTER

Summary. We provide a purely local computation of the (elliptic) twisted (by "transpose-inverse") character of the representation $\pi = I(\mathbf{1})$ of $\mathrm{PGL}(3, F)$ over a p-adic field F induced from the trivial representation of the maximal parabolic subgroup. This computation is purely local, and independent of our results on the theory of the symmetric square lifting of automorphic and admissible representations of SL(2) to PGL(3), derived using the trace formula. This independent purely local computation gives an alternative verification of a special case of our results on character relations. The material of this chapter is based on the works [FK4] with D. Kazhdan and [FZ1] with D. Zinoviev.

Introduction

Let F be a local field. Put $\mathbf{G} = \mathrm{PGL}(3)$, $\quad G = \mathbf{G}(F)$,

$$\mathbf{H}_1 = \mathrm{PGL}(2), \qquad H_1 = \mathbf{H}_1(F), \qquad J = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix},$$

and $\sigma\delta = J^t\delta^{-1}J$ for δ in G. Fix an algebraic closure $\overline{F}$ of F. The elements δ, δ' of G are called (*stably*) σ-*conjugate* if there is g in G (resp. $\mathbf{G}(\overline{F})$) with $\delta' = g^{-1}\delta\sigma(g)$. To state our result, we first recall the results of I.2 concerning these classes. For any δ in $\mathrm{GL}(3, F)$, $\delta\sigma(\delta)$ lies in $\mathrm{SL}(3, F)$ and depends only on the image of δ in G. The eigenvalues of $\delta\sigma(\delta)$ are λ, 1, λ^{-1} (see end of I.2.1), with $[F(\lambda) : F] \leq 2$; δ is called σ-*regular* if $\lambda \neq \pm 1$. In this case we write (as in I.2.2) $\gamma_1 = N_1\delta$ for the conjugacy class in H_1 which corresponds to the conjugacy class with eigenvalues λ, 1, λ^{-1} in $\mathrm{SO}(3, F)$ under the isomorphism $H_1 = \mathrm{SO}(3, F)$ (i.e., γ_1 is the image in H_1 of a conjugacy class in $\mathrm{GL}(2, F)$ with eigenvalues a, b with $a/b = \lambda$). It is shown in I.2.3 that the map N_1 is a bijection from the set of stable regular σ-conjugacy classes in G to the set of regular conjugacy classes in H_1 (clearly, we say that a conjugacy class γ_1 in H_1 is regular if $\lambda = a/b \neq \pm 1$). The set of σ-conjugacy classes in the stable σ-conjugacy class of a σ-regular δ is shown in I.2.3 to be parametrized by $F^\times/NE^\times$, where E is the field extension $F(\lambda)$ of F, and N is the norm from E to F. Explicitly, if the quotients of the eigenvalues of the regular element γ_1 are λ and λ^{-1}, choose α, β in E with $\lambda = -\alpha/\beta$ (for example with $\beta = 1$ if $E = F$, and with $\overline{\beta} = \alpha$ if $E \neq F$). Let a be an element of $\mathrm{GL}(2, F)$ with eigenvalues α, β. Put

$$e = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } h_1 = \begin{pmatrix} x & 0 & y \\ 0 & 1 & 0 \\ z & 0 & t \end{pmatrix} \text{ if } h = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

Then $\delta_u = (uae)_1$ is a complete set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of the δ with $N_1\delta$ equals γ_1, as u varies over $F^\times/NE^\times$ (a set of cardinality one or two). In addition we associate (in I.2.4) to δ a sign $\boldsymbol{\kappa}(\delta)$, as follows: $\boldsymbol{\kappa}(\delta)$ is 1 if the quadratic form

$$x \quad (\in F^3) \quad \mapsto \quad {}^t x\delta J x \qquad (\text{equivalently} \qquad x \mapsto \frac{1}{2}{}^t x[\delta J + {}^t(\delta J)]x)$$

represents zero, and $\boldsymbol{\kappa}(\delta) = -1$ if this quadratic form is anisotropic. It is clear that $\boldsymbol{\kappa}(\delta)$ depends only on the σ-conjugacy class of δ, but it is not constant on the stable σ-conjugacy class of δ.

Put

$$\Delta_1(\gamma_1) = |(a-b)^2/ab|^{1/2}$$

if a, b are the eigenvalues of a representative in $\mathrm{GL}(2, F)$ of γ_1, and

$$\Delta(\delta) = |(1-\lambda^2)(1-\lambda^{-2})|^{1/2}$$

if $\lambda = a/b$. Thus

$$\Delta_1(\gamma_1) = |(1-\lambda)(1-\lambda^{-1})|^{1/2}, \quad \text{and} \quad \Delta(\delta)/\Delta_1(\gamma_1) = |(1+\lambda)(1+\lambda^{-1})|^{1/2}.$$

Suppose that F is a nonarchimedean; denote by R its ring of integers. Put $K = G(R)$, $K_1 = H_1(R)$. By a G-module π (resp. H_1-module π_1) we mean an admissible representation of G (resp. H_1) in a complex space. An irreducible G-module π is called σ*-invariant* if it is equivalent to the G-module ${}^\sigma\pi$, defined by ${}^\sigma\pi(g) = \pi(\sigma g)$. In this case there is an intertwining operator A on the space of π with $\pi(g)A = A\pi(\sigma g)$ for all g. Since $\sigma^2 = 1$ we have $\pi(g)A^2 = A^2\pi(g)$ for all g, and since π is irreducible A^2 is a scalar by Schur's lemma. We choose A with $A^2 = 1$. This determines A up to a sign, and when π has a Whittaker model, V.1.1.1 specifies a normalization of A which is compatible with a global normalization. A G-module π is called *unramified* if the space of π contains a nonzero K-fixed vector. The dimension of the space of K-fixed vectors is bounded by one if π is irreducible. If π is σ-invariant and unramified, and $v_0 \neq 0$ is a K-fixed vector in the space of π, then Av_0 is a multiple of v_0 (since $\sigma K = K$), namely $Av_0 = cv_0$, with $c = \pm 1$. Replace A by cA to have $Av_0 = v_0$, and put $\pi(\sigma) = A$. As verified in V.1.1.1, when π is (irreducible) unramified and has a Whittaker model, both normalizations of the intertwining operator are equal.

For any π and locally constant compactly supported (test) function f on G the convolution operator

$$\pi(fdg) = \int_G f(g)\pi(g)dg$$

has finite rank. If π is σ-invariant put

$$\pi(fdg \times \sigma) = \int_G f(g)\pi(g)\pi(\sigma)dg.$$

Denote by $\mathrm{tr}\pi(fdg \times \sigma)$ the trace of the operator $\pi(fdg \times \sigma)$. It depends on the choice of the Haar measure dg, but the (*twisted*) *character* χ_π^σ of π does not; χ_π^σ is a locally-integrable complex-valued function on G (see [Cl2], [HC2]) which is σ-conjugacy invariant and locally-constant on the σ-regular set, with

$$\mathrm{tr}\,\pi(fdg \times \sigma) = \int_G f(g)\chi_\pi^\sigma(g)dg$$

for all test functions f on G.

A Levi subgroup of a maximal parabolic subgroup P of G is isomorphic to $\mathrm{GL}(2, F)$. Hence an H_1-module π_1 extends to a P-module trivial on the unipotent radical N of P. Let $\boldsymbol{\delta}$ denote the character of P which is trivial on N and whose value at $p = mn$ is $|\det h|$ if m corresponds to h

in $\mathrm{GL}(2,F)$. Explicitly, if P is the upper triangular parabolic subgroup of type (2,1), and m in M is represented in $\mathrm{GL}(3,F)$ by

$$m = \begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix}, \qquad \text{then} \qquad \boldsymbol{\delta}(m) = |(\det m')/m''^2|$$

(m' lies in $\mathrm{GL}(2,F)$, m'' in $\mathrm{GL}(1,F)$). Denote by $I(\pi_1)$ the G-module $\pi = \mathrm{ind}(\boldsymbol{\delta}^{1/2}\pi_1; P, G)$ normalizedly induced from π_1 on P to G. It is clear from [BZ1] that when $I(\pi_1)$ is irreducible then it is σ-invariant, and it is unramified if and only if π_1 is unramified.

We say that a σ-invariant irreducible representation π of G is σ-unstable if for any σ-regular stably σ-conjugate but not σ-conjugate elements δ, δ' of G we have $\chi_\pi^\sigma(\delta') = -\chi_\pi^\sigma(\delta)$.

Of course $\delta \neq \delta'$ as here exist only when $F(\lambda) \neq F$, namely when $N_1\delta$ is elliptic regular.

Let χ_{π_1} be the character of π_1; it is a locally-integrable complex-valued conjugacy-invariant function on H_1 which is smooth on the regular set and satisfies

$$\operatorname{tr} \pi_1(f_1) = \int_{H_1} f_1(g)\chi_{\pi_1}(g)dg$$

for all f_1 on H_1. We now assume that F has characteristic zero and odd residual characteristic.

In this chapter we prove, by direct, local computation, the following

THEOREM. *If* $\mathbf{1}$ *is the trivial* H_1*-module,* $\pi = I(\mathbf{1})$*, and* δ *a* σ*-regular element of* G *with elliptic regular norm* $\gamma_1 = N_1\delta$*, then*

$$(\Delta(\delta)/\Delta_1(\gamma_1))\chi_\pi^\sigma(\delta) = \boldsymbol{\kappa}(\delta).$$

VI.1 Proof of theorem, anisotropic case

To compute the character of π we shall express π as an integral operator in a convenient model, and integrate the kernel over the diagonal. Denote by $\mu = \mu_s$ the character $\mu(x) = |x|^{(s+1)/2}$ of $F^\times$. It defines a character $\mu_P = \mu_{s,P}$ of P, trivial on N, by

$$\mu_P(p) = \mu((\det m')/m''^2)$$

if $p = mn$ and $m = \begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix}$ with m' in $\mathrm{GL}(2, F)$, m'' in $\mathrm{GL}(1, F)$. If $s = 0$, the $\mu_P = \boldsymbol{\delta}^{1/2}$. Let W_s be the space of complex-valued smooth functions ψ on G with $\psi(pg) = \mu_P(p)\psi(g)$ for all p in P and g in G. The group G acts on W_s by right translation: $(\pi_s(g)\psi)(h) = \psi(hg)$. By definition, $I(\pi_1)$ is the G-module W_s with $s = 0$. The parameter s is introduced for purposes of analytic continuation.

We prefer to work in another model V_s of the G-module W_s. Let V denote the space of column 3-vectors over F. Let V_s be the space of smooth complex-valued functions ϕ on $V - \{0\}$ with $\phi(\lambda v) = \mu(\lambda)^{-3}\phi(v)$. The expression $\mu(\det g)\phi({}^t gv)$, which is initially defined for g in $\mathrm{GL}(3, F)$, depends only on the image of g in G. The group G acts on V_s by

$$(\tau_s(g)\phi)(v) = \mu(\det g)\phi({}^t gv).$$

Let $v_0 \neq 0$ be a vector of V such that the line $\{\lambda v_0; \lambda \text{ in } F\}$ is fixed under the action of ${}^tP$. Explicitly, we take $v_0 = {}^t(0, 0, 1)$. It is clear that the map

$$V_s \to W_s, \quad \phi \mapsto \psi = \psi_\phi,$$

where

$$\psi(g) = (\tau_s(g)\phi)(v_0) = \mu(\det g)\phi({}^t gv_0),$$

is a G-module isomorphism, with inverse

$$\psi \mapsto \phi = \phi_\psi, \qquad \phi(v) = \mu(\det g)^{-1}\psi(g)$$

if $v = {}^t gv_0$ (G acts transitively on $V - \{0\}$).

For $v = {}^t(x, y, z)$ in V put $|v| = \max(|x|, |y|, |z|)$. Let V^0 be the quotient of the set of v in V with $|v| = 1$ by the equivalence relation $v \sim \alpha v$ if α is a unit in R. Denote by $\mathbb{P}V$ the projective space of lines in $V - \{0\}$. If Φ is a function on $V - \{0\}$ with $\Phi(\lambda v) = |\lambda|^{-3}\Phi(v)$ and $dv = dx\, dy\, dz$, then $\Phi(v)dv$ is homogeneous of degree zero. Define

$$\int_{\mathbb{P}V} \Phi(v)dv \quad \text{to be} \quad \int_{V^0} \Phi(v)dv.$$

Clearly we have

$$\int_{\mathbb{P}V} \Phi(v)dv = \int_{\mathbb{P}V} \Phi(gv)d(gv) = |\det g| \int_{\mathbb{P}V} \Phi(gv)dv.$$

Put $\nu(x) = |x|$ and $m = 3(s-1)/2$. Note that $\nu/\mu_s = \mu_{-s}$. Put $\langle v, w\rangle = {}^t vJw$. Then

$$\langle gv, \sigma(g)w\rangle = \langle v, w\rangle.$$

1. Lemma. *The operator* $T_s : V_s \to V_{-s}$,

$$(T_s\phi)(v) = \int_{\mathbb{P}V} \phi(w)|\langle w, v\rangle|^m dw,$$

converges when $\mathrm{Re}(s) > 1/3$ *and satisfies*

$$T_s\tau_s(g) = \tau_{-s}(\sigma g)T_s$$

for all g *in* G *where it converges.*

Proof. We have

$$\begin{aligned}(T_s(\tau_s(g)\phi))(v) &= \textstyle\int (\tau_s(g)\phi)(w)|{}^t wJv|^m dw = \mu(\det g) \int \phi({}^t gw)|{}^t wJv|^m dw\\ &= |\det g|^{-1}\mu(\det g) \textstyle\int \phi(w)|{}^t({}^t g^{-1}w)Jv|^m dw\\ &= (\mu/\nu)(\det g) \textstyle\int \phi(w)|{}^t wJ \cdot Jg^{-1}Jv|^m dw\\ &= (\mu/\nu)(\det g) \textstyle\int \phi(w)|\langle w, \sigma({}^t g)v\rangle|^m dw = (\nu/\mu)(\det \sigma g) \cdot (T_s\phi)(\sigma({}^t g)v)\\ &= [(\tau_{-s}(\sigma g))(T_s\phi)](v),\end{aligned}$$

as required. □

The spaces V_s are isomorphic to the space W of locally-constant complex-valued functions on V^0, and T_s is equivalent to an operator T_s^0 on W. The proof of Lemma 1 implies also

1. Corollary. *The operator* $T_s^0 \circ \tau_s(g^{-1})$ *is an integral operator with kernel*

$$(\mu/\nu)(\det \sigma g)|\langle w, \sigma({}^t g^{-1})v\rangle|^m \qquad (v, w \text{ in } V^0)$$

and trace

$$\mathrm{tr}[T_s^0 \circ \tau_s(g^{-1})] = (\nu/\mu)(\det g) \int_{V^0} |{}^t vgJv|^m dv.$$

Remark. (1) In the domain where the integral converges, it is clear that $\mathrm{tr}[T_s^0 \circ \tau_s(g^{-1})]$ depends only on the σ-conjugacy class of g if (and only if) $s = 0$. (2) We evaluate below this integral at $s = 0$ in a case where it converges for all s, and no analytic difficulties occur. However, to compute the trace of the analytic continuation of $T_s^0 \circ \tau_s(g^{-1})$ it suffices to compute this trace for s in the domain of convergence, and then evaluate the resulting expression at the desired s. Indeed, for each compact open σ-invariant subgroup K of G the space W_K of K-biinvariant functions in W is finite dimensional. Denote by $p_K : W \to W_K$ the natural projection. Then $p_K \circ T_s^0 \circ \tau_s(g^{-1})$ acts on W_K, and the trace of the analytic continuation of $p_K \circ T_s^0 \circ \tau_s(g^{-1})$ is the analytic continuation of the trace of $p_K \circ T_s^0 \circ \tau_s(g^{-1})$. Since K can be taken to be arbitrarily small the claim follows.

Next we normalize the operator $T = T_s$ so that it acts trivially on the one-dimensional space of K-fixed vectors in V_s. This space is spanned by the function ϕ_0 in V_s with $\phi_0(v) = 1$ for all v in V^0. Fix a local uniformizer $\boldsymbol{\pi}$ in R. Let q be the cardinality of the quotient field of R. Normalize the valuation $|\cdot|$ by $|\boldsymbol{\pi}| = q^{-1}$. Normalize the measure dx by $\int_{|x|\le 1} dx = 1$, so that $\int_{|x|=1} dx = 1 - q^{-1}$. In particular, the volume of V^0 is

$$(1-q^{-3})/(1-q^{-1}) = 1 + q^{-1} + q^{-2}.$$

2. Lemma. *We have*

$$(T\phi_0)(v_0) = (1 - q^{-3(s+1)/2})(1 - q^{(1-3s/2})^{-1}\phi_0(v_0).$$

When $s = 0$ the constant is

$$-q^{-1/2}(1 + q^{-1/2} + q^{-1}).$$

Proof.

$$\int \phi_0(v)|{}^t vJv_0|^m dv = \int_{V^0} |x|^m dx\ dy\ dz$$

$$= (1 - q^{-3(s+1)/2}) \int_{|x|\le 1} |x|^m dx / \int_{|x|=1} dx,$$

as asserted. □

To complete the proof of the proposition we have to compute

$$\mathrm{tr}[T \circ \tau_s(\delta^{-1})], \qquad T = T_s^0.$$

Put $a = \begin{pmatrix} \alpha & 1 \\ \theta & \alpha \end{pmatrix}$ with $\alpha \ne 0$ in F and θ in $F - F^2$ with $|\theta| = 1$ or $|\theta| = q^{-1}$. Put

$$\delta = \delta_u = u(u^{-1}ae)_1 = \begin{pmatrix} -\alpha & 0 & 1 \\ 0 & u & 0 \\ -\theta & 0 & \alpha \end{pmatrix},$$

where u ranges over a set of representatives in $F^\times$ for $F^\times/NE^\times$, where $E = F(\theta^{1/2})$. Then $\det\delta = u(\theta - \alpha^2)$. The eigenvalues of

$$\delta\sigma(\delta) = (-(\det a)^{-1}a^2)_1$$

are λ, 1, λ^{-1} where

$$\lambda = -(\alpha + \theta^{1/2})/(\alpha - \theta^{1/2}).$$

We have

$$(1+\lambda)(1+\lambda^{-1}) = \left(1 - \frac{\alpha + \theta^{1/2}}{\alpha - \theta^{1/2}}\right)\left(1 - \frac{\alpha - \theta^{1/2}}{\alpha + \theta^{1/2}}\right) = \frac{-4\theta}{\alpha^2 - \theta},$$

hence $(\nu/\mu)(\det\delta)\Delta(\delta)/\Delta_1(\gamma_1)$ is equal to

$$|u(\alpha^2 - \theta)|^{(1-s)/2}|4\theta/(\alpha^2 - \theta)|^{1/2} = |4u\theta|^{1/2}|u(\alpha^2 - \theta)|^{-s/2}.$$

Further,

$$\delta J = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & u & 0 \\ \alpha & 0 & -\theta \end{pmatrix},$$

hence ${}^t v\delta J v = x^2 + uy^2 - \theta z^2$. Consequently

$$\frac{\Delta(\delta)}{\Delta_1(\gamma_1)} \operatorname{tr}[T \circ \tau_s(\delta^{-1})]$$

$$= |4u\theta|^{1/2}|u(\alpha^2 - \theta)|^{-s/2} \int_{V^0} |uy^2 + x^2 - \theta z^2|^{3(s-1)/2} dx dy dz.$$

We are interested in the value of this expression at $s = 0$. When $\boldsymbol{\kappa}(\delta) = 1$ the quadratic form $uy^2 + x^2 - \theta z^2$ represents zero. Then the integral converges only for s with $\operatorname{Re}(s) > 2/3$, but not at $s = 0$. At $s = 0$ the integral can be evaluated by analytic continuation. However when $\boldsymbol{\kappa}(\delta) = -1$ the quadratic form $uy^2 + x^2 - \theta z^2$ is anisotropic, hence reaches a nonzero minimum (in valuation) on the compact set $|v| = 1$. Consequently the integral converges for all values of s, and we may restrict our attention to the case of $s = 0$. Here the character depends only on the σ-conjugacy class of δ, and we may take $|u| = 1$ if $|\theta| = q^{-1}$, and $|u| = q^{-1}$ if $|\theta| = 1$. Then $|u\theta|^{1/2} = q^{-1/2}$ and

$$\int_{|v|=1} |uy^2 + x^2 - \theta z^2|^{-3/2} dx dy dz = (1 + q^{-1/2} + q^{-1}) \int_{|x|=1} dx.$$

We conclude that

$$\frac{\Delta(\delta)}{\Delta_1(\gamma_1)}\operatorname{tr}[\tau_s(\delta)\circ T]=\boldsymbol{\kappa}(\delta)(T\phi_0)(v_0)$$

when $\boldsymbol{\kappa}(\delta)=-1$. Since

$$\chi_\pi^\sigma(\delta)=\operatorname{tr}[\tau_s(\delta)\circ T]/(T\phi_0)(v_0),$$

the theorem follows for δ with $\boldsymbol{\kappa}(\delta)=-1$.

VI.2 Proof of theorem, isotropic case

When $\boldsymbol{\kappa}(\delta)=1$ we prove the theorem on computing $\operatorname{tr}[T_s^0\circ\tau_s(\delta^{-1})]$ by analytic continuation, namely first for large $\operatorname{Re}(s)$ and then on evaluating the resulting expression at $s=0$.

The theorem asserts that the value at $s=0$ of

$$|4u\theta|^{1/2}|u(\alpha^2-\theta)|^{-s/2}\int_{V^0}|x^2+uy^2-\theta z^2|^{3(s-1)/2}\,dxdydz$$

is

$$-\boldsymbol{\kappa}(\delta)q^{-1/2}(1+q^{-1/2}+q^{-1}).$$

This equality is verified in the last section when the quadratic form $x^2+uy^2-\theta z^2$ is anisotropic, in which case $\boldsymbol{\kappa}(\delta)=-1$ and the integral converges for all s.

Here we deal with the case where the quadratic form is **isotropic**, in which case $\boldsymbol{\kappa}(\delta)=1$, the integral converges only in some half plane of s, and the value at $s=0$ is obtained by analytic continuation.

Recall that F is a local nonarchimedean field of odd residual characteristic; R denotes the (local) ring of integers of F; $\boldsymbol{\pi}$ signifies a generator of the maximal ideal of R. Denote by q the number of elements of the residue field $R/\boldsymbol{\pi}R$ of R. By $\mathbb{F}$ we mean a set of representatives in R for the finite field $R/\boldsymbol{\pi}$. The absolute value on F is normalized by $|\boldsymbol{\pi}|=q^{-1}$.

The case of interest is that where $E=F(\sqrt{\theta})$ is a quadratic extension of F, thus $\theta\in F^\times-F^{\times 2}$. Since the twisted character depends only on the twisted conjugacy class, we may assume that $|\theta|$ and $|u|$ lie in $\{1,q^{-1}\}$.

0. LEMMA. *We may assume that the quadratic form $x^2 + uy^2 - \theta z^2$ takes one of three avatars:*

$$x^2 - \theta z^2 - y^2, \quad \theta \in R - R^2; \qquad x^2 - \boldsymbol{\pi} z^2 + \boldsymbol{\pi} y^2; \qquad \textit{or} \qquad x^2 - \boldsymbol{\pi} z^2 - y^2.$$

PROOF. (1) If E/F is unramified, then $|\theta| = 1$, thus $\theta \in R^\times - R^{\times 2}$. The norm group $N_{E/F}E^\times$ is $\pi^{2\mathbb{Z}}R^\times$. If $x^2 - \theta z^2 + uy^2$ represents 0 then $-u \in R^\times$. If -1 is not a square, thus $\theta = -1$, then u is -1 (get $x^2 - z^2 - y^2$) or $u = 1$ (get $x^2 - z^2 + y^2$, equivalent case). If $-1 \in R^{\times 2}$, the case of

$$u = \theta \qquad (x^2 - \theta z^2 + \theta y^2 = \theta(y^2 + \theta^{-1}x^2 - z^2))$$

is equivalent to the case of $u = -1$. So wlog $u = -1$ and the form is $x^2 - \theta z^2 - y^2$, $|u\theta| = 1$.
(2) If E/F is ramified, $|\theta| = q^{-1}$ and $N_{E/F}E^\times = (-\theta)^{\mathbb{Z}}R^{\times 2}$. The form $x^2 - \theta z^2 + uy^2$ represents zero when $-u \in R^{\times 2}$ or $-u \in -\theta R^{\times 2}$. Then the form looks like $x^2 - \theta z^2 + \theta y^2$ with $u = \theta$ and $|\theta u| = q^{-2}$, or $x^2 - \theta z^2 - y^2$ with $u = -1$ and $|\theta u| = q^{-1}$. The Lemma follows. □

We are interested in the value at $s = -3/2$ of the integral $I_s(u, \theta)$ of $|x^2 + uy^2 - \theta z^2|^s$ over the set $V^0 = V/\sim$, where

$$V = \{\mathbf{v} = (x, y, z) \in R^3; \max\{|x|, |y|, |z|\} = 1\}$$

and $\sim$ is the equivalence relation $\mathbf{v} \sim \alpha\mathbf{v}$ for $\alpha \in R^\times$.

The set V^0 is the disjoint union of the subsets

$$V_n^0 = V_n^0(u, \theta) = V_n(u, \theta)/\sim,$$

where

$$V_n = V_n(u, \theta) = \{\mathbf{v}; \max\{|x|, |y|, |z|\} = 1, |x^2 + uy^2 - \theta z^2| = 1/q^n\},$$

over $n \geq 0$, and of the set $\{\mathbf{v}; x^2 + uy^2 - \theta z^2 = 0\}/\sim$, whose volume is zero.

Thus we have

$$I_s(u, \theta) = \sum_{n=0}^{\infty} q^{-ns} \operatorname{Vol}(V_n^0(u, \theta)).$$

PROPOSITION. *The value of* $|u\theta|^{1/2} I_s(u,\theta)$ *at* $s = -3/2$ *is*

$$-q^{-1/2}(1 + q^{-1/2} + q^{-1}).$$

The problem is simply to compute the volumes

$$\mathrm{Vol}(V_n^0(u,\theta)) = \mathrm{Vol}(V_n(u,\theta))/(1-1/q) \qquad (n \geq 0).$$

1. LEMMA. *When* $\theta = \boldsymbol{\pi}$ *and* $u = -1$, *thus* $|u\theta| = 1/q$, *we have*

$$\mathrm{Vol}(V_n^0) = \begin{cases} (1-1/q), & \text{if } n = 0, \\ 2q^{-1}(1-1/q) + 1/q^2, & \text{if } n = 1, \\ 2q^{-n}(1-1/q), & \text{if } n \geq 2. \end{cases}$$

PROOF. Recall that

$$V_0 = V_0(-1,\boldsymbol{\pi}) = \{(x,y,z);\ \max\{|x|,|y|,|z|\} = 1,\ |x^2 - y^2 - \boldsymbol{\pi} z^2| = 1\}.$$

Since $|z| \leq 1$, we have $|\boldsymbol{\pi} z^2| < 1$, and

$$1 = |x^2 - y^2 - \boldsymbol{\pi} z^2| = |x^2 - y^2| = |x-y||x+y|.$$

Thus $|x-y| = |x+y| = 1$. If $|x| \neq |y|$, $|x \pm y| = \max\{|x|,|y|\}$. We split V_0 into three distinct subsets, corresponding to the cases $|x| = |y| = 1$; $|x| = 1$, $|y| < 1$; and $|x| < 1$, $|y| = 1$. The volume is then

$$\mathrm{Vol}(V_0) = \int_{|z|\leq 1} \int_{|x|=1} \left[\int_{|y|=1, |x-y|=|x+y|=1} \right] dydxdz$$

$$+ \int_{|z|\leq 1} \left[\int_{|x|=1} \int_{|y|<1} + \int_{|x|<1} \int_{|y|=1} \right] dydxdz$$

$$= \int_{|x|=1} \left[\int_{|y|=1, |x-y|=|x+y|=1} \right] dydx + \frac{2}{q}\left(1 - \frac{1}{q}\right) = \left(1 - \frac{1}{q}\right)^2.$$

To consider the V_n with $n \geq 1$, where $|x^2 - y^2 - \boldsymbol{\pi} z^2| = 1/q^n$, recall that any p-adic number a such that $|a| \leq 1$ can be written as a power series in $\boldsymbol{\pi}$:

$$a = \sum_{i=0}^{\infty} a_i \boldsymbol{\pi}^i = a_0 + a_1 \boldsymbol{\pi} + a_2 \boldsymbol{\pi}^2 + \cdots \qquad (a_i \in \mathbb{F}).$$

In particular $|a| = 1/q^n$ implies that $a_0 = a_1 = \cdots = a_{n-1} = 0$, and $a_n \neq 0$. If

$$x = \sum_{i=0}^{\infty} x_i \boldsymbol{\pi}^i, \qquad y = \sum_{i=0}^{\infty} y_i \boldsymbol{\pi}^i, \qquad z = \sum_{i=0}^{\infty} z_i \boldsymbol{\pi}^i \qquad (x_i, y_i, z_i \in \mathbb{F}),$$

then

$$x^2 = \sum_{i=0}^{\infty} a_i \boldsymbol{\pi}^i, \qquad y^2 = \sum_{i=0}^{\infty} b_i \boldsymbol{\pi}^i, \qquad z^2 = \sum_{i=0}^{\infty} c_i \boldsymbol{\pi}^i,$$

where

$$a_i = \sum_{j=0}^{i} x_j x_{i-j}, \quad b_i = \sum_{j=0}^{i} y_j y_{i-j}, \quad c_i = \sum_{j=0}^{i} z_j z_{i-j} \quad (a_i, b_i, c_i \in \mathbb{F}).$$

We have

$$x^2 - y^2 - \boldsymbol{\pi} z^2 = \sum_{i=0}^{\infty} f_i \boldsymbol{\pi}^i \qquad (f_i \in \mathbb{F}),$$

where $f_0 = a_0 - b_0$, $f_i = a_i - b_i - c_{i-1}$ $(i \geq 1)$. Since $|x^2 - y^2 - \boldsymbol{\pi} z^2| = 1/q^n$, we have that $f_0 = f_1 = \cdots = f_{n-1} = 0$, and $f_n \neq 0$. Thus we obtain the relations (for a, b, c in the set $\mathbb{F}$, which (modulo $\boldsymbol{\pi}$) is the field $R/\boldsymbol{\pi}$):

$$a_0 - b_0 = 0, \qquad a_i - b_i - c_{i-1} = 0 \qquad (i = 1, ..., n-1), \qquad a_n - b_n - c_{n-1} \neq 0.$$

Recall that together with $\max\{|x|, |y|, |z|\} = 1$, these relations define the set V_n.

To compute the volume of V_n we integrate in the order: $\cdots dydzdx$. From $a_0 - b_0 = 0$ it follows that $y_0 = \pm x_0$, and from $a_i - b_i - c_{i-1}$ $(i \geq 1)$ it follows that

$$2y_0 y_i = a_i - c_{i-1} - \sum_{j=1}^{i-1} y_j y_{i-j},$$

where in the case of $i = 1$ the sum over j is empty.

Let $n \geq 2$. When $i = 1$ we have $2x_0 x_1 - 2y_0 y_1 - z_0^2 = 0$. So if $x_0 = 0$ (in $R/\boldsymbol{\pi}$, i.e. $|x| < 1$), it follows that $y_0 = 0$ and $z_0 = 0$ (i.e. $|y| < 1$, $|z| < 1$). This contradicts the fact that $\max\{|x|, |y|, |z|\} = 1$. Thus $|x| = 1$. In this case $y_0 \neq 0$ and (for $n \geq 2$) we have:

$$\mathrm{Vol}(V_n) = \int_{|x|=1} \int_{|z| \leq 1} \left[\int dy \right] dzdx,$$

where the variable y is such that once written as $y = y_0 + y_1\boldsymbol{\pi} + y_2\boldsymbol{\pi}^2 + \cdots$, it has to satisfy: $y_0 = \pm x_0$, and y_i $(i = 1, \ldots, n-1)$ is defined uniquely from $a_i - b_i - c_{i-1} = 0$, and $y_n \neq$ some value defined by $a_n - b_n - c_{n-1} \neq 0$. Thus when $n \geq 2$,

$$\mathrm{Vol}(V_n) = \frac{2}{q}\left(\frac{1}{q}\right)^{n-1}\left(1 - \frac{1}{q}\right)^2 = \frac{2}{q^n}\left(1 - \frac{1}{q}\right)^2.$$

Let $n = 1$. When $i = 1$ we have $2x_0x_1 - 2y_0y_1 - z_0^2 \neq 0$. So if $x_0 = 0$ (i.e. $|x| < 1$), it follows that $y_0 = 0$ and $z_0 \neq 0$ (i.e. we have an additional contribution from $|x| < 1$, $|y| < 1$, $|z| = 1$). Thus,

$$\mathrm{Vol}(V_1) = \frac{2}{q}\left(1 - \frac{1}{q}\right)^2 + \frac{1}{q^2}\left(1 - \frac{1}{q}\right).$$

The lemma follows. □

2. Lemma. *When u and θ equal $\boldsymbol{\pi}$, thus $|u\theta| = 1/q^2$, we have*

$$\mathrm{Vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-1}(1 - 1/q), & \text{if } n = 1, \\ 2q^{-n}(1 - 1/q), & \text{if } n \geq 2. \end{cases}$$

Proof. To compute $\mathrm{Vol}(V_0)$, recall that

$$V_0 = \{(x, y, z); \max\{|x|, |y|, |z|\} = 1, |x^2 + \boldsymbol{\pi}(y^2 - z^2)| = 1\}.$$

Since $|y| \leq 1$, $|z| \leq 1$, we have $|x^2 + \boldsymbol{\pi}(y^2 - z^2)| = |x^2| = 1$, and so

$$\mathrm{Vol}(V_0) = \int_{|z| \leq 1}\int_{|y| \leq 1}\int_{|x| = 1} dx dy dz = 1 - \frac{1}{q}.$$

To compute $\mathrm{Vol}(V_n)$, $n \geq 1$, recall that

$$V_n = \{(x, y, z); \max\{|x|, |y|, |z|\} = 1, |x^2 + \boldsymbol{\pi}(y^2 - z^2)| = 1/q^n\}.$$

Following the notations of Lemma 1 we write

$$x^2 + \boldsymbol{\pi}(y^2 - z^2) = \sum_{i=0}^{\infty} f_i \boldsymbol{\pi}^i \qquad (f_i \in \mathbb{F}),$$

where $f_0 = a_0$ and $f_i = a_i + b_{i-1} - c_{i-1}$ $(i \geq 1)$. The condition which defines V_n is that $f_0 = f_1 = \cdots = f_{n-1} = 0$ and $f_n \neq 0$. The equation $f_0 = 0$ implies that $x_0 = 0$ (i.e. $|x| < 1$). We arrange the order of integration to be: $\cdots dydzdx$.

When $n \geq 2$, since $x_0 = 0$, $f_1 = 0$ implies that $y_0^2 - z_0^2 = 0$. Using $\max\{|x|, |y|, |z|\} = 1$ we conclude that $y_0 = \pm z_0 \neq 0$ (i.e. $|z| = 1$, $|z^2 - y^2| < 1$). Thus we have

$$\mathrm{Vol}(V_n) = \int_{|x|<1} \int_{|z|=1} \left[\int dy \right] dzdx$$

where the variable y is such that once written as $y = y_0 + y_1\boldsymbol{\pi} + y_2\boldsymbol{\pi}^2 + \cdots$, it has to satisfy: $y_0 = \pm z_0$, and y_i $(i = 1, \ldots, n-2)$ is defined uniquely from $a_i + b_{i-1} - c_{i-1} = 0$, and $y_{n-1} \neq$ some value defined by $a_n + b_{n-1} - c_{n-1} \neq 0$. Thus when $n \geq 2$,

$$\mathrm{Vol}(V_n) = \frac{1}{q}\frac{2}{q}\left(\frac{1}{q}\right)^{n-2}\left(1 - \frac{1}{q}\right)^2 = \frac{2}{q^n}\left(1 - \frac{1}{q}\right)^2.$$

When $n = 1$ we have $f_0 = 0$, $f_1 \neq 0$. These amount to $x_0 = 0$, $y_0 \neq \pm z_0$. Separating the two cases $z_0 = 0$, and $z_0 \neq 0$, we obtain

$$\mathrm{Vol}(V_1) = \int_{|x|<1} \int_{|z|<1} \int_{|y|=1} dydzdx + \int_{|x|<1} \int_{|z|=1} \int_{|y^2 - z^2|=1} dydzdx$$

$$= \frac{1}{q^2}\left(1 - \frac{1}{q}\right) + \frac{1}{q}\left(1 - \frac{1}{q}\right)\left(1 - \frac{2}{q}\right) = \frac{1}{q}\left(1 - \frac{1}{q}\right)^2.$$

The Lemma follows. □

3. Lemma. *When E/F is unramified, thus $|u\theta| = 1$, we have*

$$\mathrm{Vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-n}(1 - 1/q)(1 + 1/q), & \text{if } n \geq 1. \end{cases}$$

Proof. First we compute $\mathrm{Vol}(V_0)$. Recall that

$$V_0 = \{(x, y, z); \max\{|x|, |y|, |z|\} = 1, |x^2 - y^2 - \theta z^2| = 1\}.$$

Since $|x^2-y^2-\theta z^2|\le\max\{|x|,|y|,|z|\}$,

$$V_0=\{(x,y,z)\in R^3;|x^2-y^2-\theta z^2|=1\}.$$

Making the change of variables $u=x+y$, $v=x-y$, we obtain

$$V_0=\{(u,v,z)\in R^3;|uv-\theta z^2|=1\}.$$

Assume that $|uv|<1$. Since $|uv-\theta z^2|=1$, it follows that $|z|=1$. The contribution from the set $|uv|<1$ is

$$\int_{|z|=1}\left[\int_{|u|<1}\int_{|v|\le 1}+\int_{|u|=1}\int_{|v|<1}\right]dudvdz$$

$$=\left(1-\frac{1}{q}\right)\left(\frac{1}{q}+\left(1-\frac{1}{q}\right)\frac{1}{q}\right)=\frac{1}{q}\left(1-\frac{1}{q}\right)\left(2-\frac{1}{q}\right).$$

Assume that $|uv|=1$, i.e. $|u|=|v|=1$. We arrange the order of integration as: $dudvdz$. If $|z|<1$ then $|uv-\theta z^2|=|uv|=1$. If $|z|=1$ we introduce $U(v,z)=\{u;\|u|=1,\|uv-\theta z^2|=1\}$, a set of volume $1-2/q$, and note that the contribution from the set $|uv|=1$ is

$$\int_{|z|<1}\int_{|v|=1}\int_{|u|=1}dudvdz+\int_{|z|=1}\int_{|v|=1}\int_{U(v,z)}dudvdz.$$

The sum of the two integrals is

$$\frac{1}{q}\left(1-\frac{1}{q}\right)^2+\left(1-\frac{1}{q}\right)^2\left(1-\frac{2}{q}\right)=\left(1-\frac{1}{q}\right)^3.$$

Adding the contributions from $|uv|<1$ and $|uv|=1$ we then obtain

$$\mathrm{Vol}(V_0)=\frac{1}{q}\left(1-\frac{1}{q}\right)\left(2-\frac{1}{q}\right)+\left(1-\frac{1}{q}\right)^3=1-\frac{1}{q}.$$

Next we compute $\mathrm{Vol}(V_n)$, $n\ge 1$. Recall that

$$V_n=\{(x,y,z);\max\{|x|,|y|,|z|\}=1,|x^2-y^2-\theta z^2|=1/q^n\}.$$

Making the change of variables $u=x+y$, $v=x-y$, we obtain

$$V_n=\{(u,v,z);\max\{|u+v|,|u-v|,|z|\}=1,|uv-\theta z^2|=1/q^n\}.$$

Since the set $\{v = 0\}$ is of measure zero, we assume that $v \neq 0$. Then $|uv - \theta z^2| = 1/q^n$ implies that $u = \theta z^2 v^{-1} + t v^{-1} \boldsymbol{\pi}^n$, where $|t| = 1$. There are two cases.

Assume that $|v| = 1$. Note that if $|z| = 1$, then $\max\{|u+v|, |u-v|, |z|\} = 1$ is satisfied, and if $|z| < 1$, then (recall that $n \geq 1$)

$$|u| = |\theta z^2 v^{-1} + t v^{-1} \boldsymbol{\pi}^n| \leq \max\{|z^2|, q^{-n}\} < 1,$$

and $|u + v| = |v| = 1$. So $|v| = 1$ implies $\max\{|u+v|, |u-v|, |z|\} = 1$. Further, since $|v| = 1$, we have $du = q^{-n} dt$. Thus the contribution from the set with $|v| = 1$ is

$$\int_{|z|\leq 1} \int_{|v|=1} \int_{|uv-\theta z^2|=1/q^n} du dv dz$$

$$= \int_{|z|\leq 1} \int_{|v|=1} \int_{|t|=1} \frac{dt}{q^n} dv dz = \frac{1}{q^n} \left(1 - \frac{1}{q}\right)^2.$$

Assume that $|v| < 1$. Note that if $|z| = 1$, since $|u| \leq 1$ we have $q^{-n} = |uv - \theta z^2| = |\theta z^2| = 1$, a contradiction. Thus $|z| < 1$, and in order to satisfy $\max\{|u+v|, |u-v|, |z|\} = 1$, we should have $|u| = 1$. The contribution from the set with $|v| < 1$ is

$$\int_{|z|<1} \int_{|u|=1} \int_{|uv-\theta z^2|=1/q^n} dv du dz.$$

We write $v = \theta z^2 u^{-1} + t u^{-1} \boldsymbol{\pi}^n$, where $|t| = 1$, and $dv = q^{-n} dt$. The integral equals

$$\int_{|z|<1} \int_{|u|=1} \int_{|t|=1} \frac{dt}{q^n} du dz = \frac{1}{q} \frac{1}{q^n} \left(1 - \frac{1}{q}\right)^2.$$

Adding the contributions from $|v| = 1$ and $|v| < 1$ we obtain

$$\mathrm{Vol}(V_n) = \frac{1}{q^n} \left(1 - \frac{1}{q}\right)^2 + \frac{1}{q} \frac{1}{q^n} \left(1 - \frac{1}{q}\right)^2 = \frac{1}{q^n} \left(1 - \frac{1}{q}\right)^2 \left(1 + \frac{1}{q}\right).$$

The Lemma follows. □

This completes the proof of the proposition, so that we provided a purely local proof of (the character relation of) the theorem. We believe that analogous computations can be carried out in other lifting situations, to provide direct and local computations of twisted characters. A step in this direction is taken in [FZ2] and in [FZ3].

PART 2. AUTOMORPHIC REPRESENTATIONS OF THE UNITARY GROUP U(3,E/F)

INTRODUCTION

1. Functorial overview

Let E/F be a quadratic Galois extension of local or global fields. Let $\mathbf{G}$ denote the quasi-split unitary group $\mathbf{U}(3, E/F)$ in 3 variables over F which splits over E. Our aim is to *determine the admissible and automorphic representations of this group* by means of the trace formula and the theory of liftings.

To be definite, we define $\mathbf{G}$ by means of the form $J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$. Thus $\tau \in \mathrm{Gal}(\overline{F}/F)$ acts on $g = (g_{ij}) \in \mathbf{G}(\overline{F}) = \mathrm{GL}(3, \overline{F})$ by $\tau g = (\tau g_{ij})$ if $\tau|E = 1$, and $\tau g = \theta(\tau g_{ij})$ if $\tau|E \neq 1$ where $\theta(g) = J {}^t g^{-1} J$, and ${}^t g$ indicates the transpose (g_{ji}) of g. Denote by $x \mapsto \overline{x}$ the action of the nontrivial element of $\mathrm{Gal}(E/F)$ on $x \in E$ and componentwise $\overline{g} = (\overline{g}_{ij})$ on g in $\mathbf{G}(E) = \mathrm{GL}(3, E)$. Put $\sigma(g) = \theta(\overline{g})$. Thus the group $G = \mathbf{G}(F)$ of F-points on $\mathbf{G}$ is

$$\{g \in \mathbf{G}(E);\ gJ {}^t\overline{g} = J\} = \{g \in \mathrm{GL}(3, E);\ \sigma(g) = g\}.$$

Write $\mathrm{U}(n, E/F)$ for the group $\mathbf{U}(n, E/F)(F)$ of F-points on $\mathbf{U}(n, E/F)$.

When F is the field $\mathbb{R}$ of real numbers, the group $\mathbf{G}(\mathbb{R})$ of $\mathbb{R}$-points on $\mathbf{G}$ is usually denoted by $\mathrm{U}(2, 1; \mathbb{C}/\mathbb{R})$, and the notation $\mathrm{U}(3; \mathbb{C}/\mathbb{R})$ is reserved for its anisotropic inner form. We too shall use the $\mathbb{R}$-notations in the $\mathbb{R}$-case (but only in this case).

When $E = F \oplus F$ is not a field, $\mathbf{G}(F) = \mathrm{GL}(3, F)$.

Our work is based on basechange lifting to $\mathbf{U}(3, E/F)(E) = \mathrm{GL}(3, E)$. We define this last group as an algebraic group over F by $\mathbf{G}' = \mathrm{R}_{E/F}\,\mathbf{G}$. Thus $\mathbf{G}'(\overline{F}) = \mathrm{GL}(3, \overline{F}) \times \mathrm{GL}(3, \overline{F})$, and $\tau \in \mathrm{Gal}(\overline{F}/F)$ acts as $\tau(x, y) = (\tau x, \tau y)$ if $\tau|E = 1$, and $\tau(x, y) = \iota\theta(\tau x, \tau y)$ if $\tau|E \neq 1$. Here $\theta(x, y) = (\theta(x), \theta(y))$ and $\iota(x, y) = (y, x)$. In particular $\mathbf{G}'(E) = \mathrm{GL}(3, E) \times \mathrm{GL}(3, E)$ while $G' = \mathbf{G}'(F) = \{(x, \sigma x); x \in \mathrm{GL}(3, E)\}$. Another main aim of this part is to *determine the admissible representations* Π *of* $\mathrm{GL}(3, E)$ *and the automorphic representations* Π *of* $\mathrm{GL}(3, \mathbb{A}_E)$ *which are* σ*-invariant*: ${}^\sigma\Pi \simeq$

Π, where ${}^\sigma\pi(g)=\pi(\sigma(g))$, and again $\sigma(g)=\theta(\overline{g})$ and $\theta(g)=J{}^tg^{-1}J$. In other words, we are interested in the representations $\Pi'(x,\sigma x)=\Pi(x)$ of G' or $\mathbf{G}'(\mathbb{A})$ — admissible or automorphic — which are ι-invariant: ${}^\iota\Pi'\simeq\Pi'$, where ${}^\iota\Pi'(x,\sigma x)=\Pi'(\sigma x,x)$.

The lifting, part of the principle of functoriality, is defined by means of an L-group homomorphism $b:{}^LG\to{}^LG'$. We are interested in this and related L-group homomorphisms not in the abstract but since via the Satake transform they specify an explicit lifting relation of unramified representations, crucial for stating the global lifting, from which we deduce the local lifting. For our work it suffices to specify the lifting of unramified representations. For this reason we reduce the discussion of functoriality here to a minimum. Thus the L-group ${}^LG$ (see [Bo2]) is the semidirect product of the connected component, $\widehat{G}=\mathrm{GL}(3,\mathbb{C})$, with a group which we take here to be the relative Weil group $W_{E/F}$. We could have equally worked with the absolute Weil group W_F and its subgroup W_E. Note that $W_F/W_E\simeq W_{E/F}/W_{E/E}\simeq\mathrm{Gal}(E/F)$, $W_{E/F}=W_F/W_E^c$, and $W_{E/E}=W_E/W_E^c=W_E^{\mathrm{ab}}$ is the abelianized W_E. Here W_E^c is the commutator subgroup of W_E (see [D2], [Tt]). Now the relative Weil group $W_{E/F}$ is an extension of $\mathrm{Gal}(E/F)$ by $W_{E/E}=C_E$, $=E^\times$ (locally) or $\mathbb{A}_E^\times/E^\times$ (globally). Thus

$$W_{E/F}=\langle z\in C_E,\sigma;\sigma^2\in C_F-N_{E/F}C_E,\sigma z=\overline{z}\sigma\rangle$$

and we have an exact sequence

$$1\to W_{E/E}=C_E\to W_{E/F}\to\mathrm{Gal}(E/F)\to 1.$$

Here $W_{E/F}$ acts on $\widehat{G}$ via its quotient $\mathrm{Gal}(E/F)=\langle\sigma\rangle$, $\sigma:g\mapsto\theta(g)=J{}^tg^{-1}J$. Further, ${}^LG'$ is $\widehat{G}'\rtimes W_{E/F}$, $\widehat{G}'=\mathrm{GL}(3,\mathbb{C})\times\mathrm{GL}(3,\mathbb{C})$, where $W_{E/F}$ acts via its quotient $\mathrm{Gal}(E/F)$ by $\sigma=\iota\theta$, $\theta(x,y)=(\theta(x),\theta(y))$, $\iota(x,y)=(y,x)$.

The basechange map $b:{}^LG\to{}^LG'$ is $x\times w\mapsto(x,x)\times w$. In fact $\mathbf{G}$ is an ι-twisted endoscopic group of $\mathbf{G}'$ (see Kottwitz-Shelstad [KS]) with respect to the twisting ι. Namely $\widehat{G}$ is the centralizer $Z_{\widehat{G}'}(\iota)=\{g\in\widehat{G}';\iota(g)=g\}$ of the involution ι in $\widehat{G}'$. Note that $\mathbf{G}$ is an elliptic ι-endoscopic group, which means that $\widehat{G}$ is not contained in any parabolic subgroup of $\widehat{G}'$.

The F-group $\mathbf{G}'$ has another elliptic ι-endoscopic group $\mathbf{H}$, whose dual group ${}^LH$ has connected component $\widehat{H}=Z_{\widehat{G}'}((s,1)\iota)$, where

$s = \mathrm{diag}(-1, 1, -1)$. Then $\widehat{H}$ consists of the (x, y) with

$$(x, y) = (s, 1)\iota \cdot (x, y) \cdot [(s, 1)\iota]^{-1} = (s, 1)(y, x)(s, 1) = (sys, x),$$

thus $y = x$ and $x = sys = sxs$. In other words $\widehat{H}$ is $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$, embedded in $\widehat{G} = \mathrm{GL}(3, \mathbb{C})$ as (a_{ij}), $a_{ij} = 0$ if $i + j$ is odd, a_{22} is the $\mathrm{GL}(1, \mathbb{C})$-factor, while $[a_{11},\ a_{13};\ a_{31},\ a_{33}]$ is the $\mathrm{GL}(2, \mathbb{C})$-factor. Now ${}^L H$ is isomorphic to a subgroup ${}^L H_1$ of ${}^L G'$, and the factor $W_{E/F}$, acting on $\widehat{G}'$ by $\sigma = \iota\theta$, induces on $\widehat{H}_1$ the action $\sigma(x, x) = (\theta x, \theta x)$, namely $W_{E/F}$ acts on $\widehat{H}_1$ via its quotient $\mathrm{Gal}(E/F)$ and $\sigma(x)$ is $\theta(x)$. If we write $x = (a, b)$ with a in $\mathrm{GL}(2, \mathbb{C})$ and b in $\mathrm{GL}(1, \mathbb{C})$, $\sigma(a, b)$ is $(w^t a^{-1} w, b^{-1})$, where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We prefer to work with $\mathbf{H} = \mathbf{U}(2, E/F) \times \mathbf{U}(1, E/F)$, whose dual group ${}^L H$ is the semidirect product of $\widehat{H} = \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$ ($\subset \widehat{G}$) and $W_{E/F}$ which acts via its quotient $\mathrm{Gal}(E/F)$ by $\sigma : x \mapsto \varepsilon\theta(x)\varepsilon$, $\varepsilon = \mathrm{diag}(1, -1, -1)$. We denote by $e' : {}^L H \to {}^L G'$ the map $\widehat{H} \hookrightarrow \widehat{G}'$ by $x \mapsto (x, x)$, and $\sigma \mapsto (\theta(\varepsilon), \varepsilon)\sigma$, $z \mapsto z$ ($\in W_{E/F}$). Here $\mathbf{U}(1, E/F)$ is the unitary group in a single variable: its group of F-points is $E^1 = \{x \in E^\times; x\overline{x} = 1\} = \{z/\overline{z}; z \in E^\times\}$. The quasi-split unitary group $\mathbf{U}(2, E/F)$ in two variables has F-points consisting of the a in $\mathrm{GL}(2, E)$ with $a = \varepsilon w^t \overline{a}^{-1} w \varepsilon$.

The homomorphism $e' : {}^L H \to {}^L G'$ factorizes through the embedding $i : {}^L H' \to {}^L G'$, where $\mathbf{H}'$ is the endoscopic group (not elliptic and not ι-endoscopic) of $\mathbf{G}'$ with $\widehat{H}' = Z_{\widehat{G}'}((s, s))$. Thus $\widehat{H}' = \widehat{H} \times \widehat{H} \subset \widehat{G}'$, $\mathrm{Gal}(E/F)$ permutes the two factors, and $\mathbf{H}' = \mathrm{R}_{E/F}\,\mathbf{U}(2, E/F) \times \mathrm{R}_{E/F}\,\mathbf{U}(1, E/F)$, so that $H' = \mathbf{H}'(F) = \mathrm{GL}(2, E) \times \mathrm{GL}(1, E)$. The map $b'' : {}^L H \to {}^L H'$ is the basechange homomorphism, $b'' : x \mapsto (x, x)$ for $x \in \widehat{H}$, $z \mapsto z$, $\sigma \mapsto (\theta(\varepsilon), \varepsilon)\sigma$ on $W_{E/F}$. Thus $e' = i \circ b''$.

The lifting of representations implied by b is the basechange lifting, described in the text below. On the $\mathrm{U}(1, E/F)$ factor it is $\mu \mapsto \mu'$, where μ' is a character of $\mathrm{GL}(1, E)$ which is σ-invariant, thus $\mu' = {}^\sigma\mu'$ where ${}^\sigma\mu'(x) = \mu'(\overline{x}^{-1})$. Then $\mu'(x) = \mu(x/\overline{x})$, $x \in E^\times$, where μ is a character of $E^1 = \mathrm{U}(1, E/F)$. The lifting implied by the embedding $i : {}^L H' \to {}^L G'$ is simply normalized induction, taking a representation (ρ', μ') of $\mathrm{GL}(2, E) \times \mathrm{GL}(1, E)$ to the normalizedly induced representation $I(\rho', \mu')$ from the parabolic subgroup of type $(2, 1)$. In particular, if ρ' is irreducible with central character $\omega_{\rho'}$ and $\Pi = I(\rho', \mu')$ has central character ω', then $\omega' = \omega_{\rho'} \cdot \mu'$, and so $\mu' = \omega'/\omega_{\rho'}$ is uniquely determined by ω' and $\omega_{\rho'}$. Since we fix the central character ω' ($= {}^{\sigma}\omega'$), we shall talk about the lifting $i : \rho' \to \Pi$, meaning that $\Pi = I(\rho', \omega'/\omega_{\rho'})$.

Similarly if e' maps a representation (ρ, μ) of $H = \mathrm{U}(2, E/F) \times \mathrm{U}(1, E/F)$ to $\Pi = I(\rho', \mu')$ where $(\rho', \mu') = b((\rho, \mu))$, then $\omega_\Pi(x) = \omega_\rho(x/\overline{x})\mu(x/\overline{x})$, and so μ is uniquely determined by the central character $\omega' = \omega_\Pi$ of Π and ω_ρ of ρ. Hence we talk about the lifting $e' : \rho \mapsto \Pi$, meaning that $\Pi = I(b(\rho),\, \omega'/\omega'_\rho)$, where $\omega'_\rho(x) = \omega_\rho(x/\overline{x})$ and $b(\rho)$ is the basechange of ρ.

The (elliptic ι-endoscopic) F-group $\mathbf{G}$ (of $\mathbf{G}'$) has a single proper elliptic endoscopic group $\mathbf{H}$. Here $\widehat{H} = Z_{\widehat{G}}(s)$ and $W_{E/F}$ acts via its quotient $\mathrm{Gal}(E/F)$ by $\sigma(x) = \varepsilon\theta(x)\varepsilon^{-1}$, $x \in \widehat{H}$. Thus to define ${}^L H \to {}^L G$ to extend $\widehat{H} \hookrightarrow \widehat{G}$ and $\sigma \mapsto \varepsilon \times \sigma$ to include the factor $W_{E/F}$, we need to map $z \in C_E = W_{E/E} = \ker[W_{E/F} \to \mathrm{Gal}(E/F)] = E^\times$ or $\mathbb{A}_E^\times/E^\times$, to $\mathrm{diag}(\kappa(z), 1, \kappa(z)) \times z$, where $\kappa : C_E/N_{E/F}C_E \to \mathbb{C}^\times$ is a homomorphism whose restriction to C_F is nontrivial (namely of order two). Indeed, $\sigma^2 \in C_F - N_{E/F}C_E$, and $\sigma^2 \mapsto \varepsilon\theta(\varepsilon) \times \sigma^2$, where $\varepsilon\theta(\varepsilon) = \mathrm{diag}(-1, 1, -1) = s$. We denote this homomorphism by $e : {}^L H \to {}^L G$ and name it the "*endoscopic map*". The group $\mathbf{H}$ is $\mathbf{U}(2, E/F) \times \mathbf{U}(1, E/F)$. If a representation $\rho \times \mu$ of $H = \mathbf{H}(F)$ or $\mathbf{H}(\mathbb{A})$ e-lifts to a representation π of $G = \mathbf{G}(F)$ or $\mathbf{G}(\mathbb{A})$, then $\omega_\pi = \kappa\omega_\rho\mu$, where the central characters ω_π, ω_ρ, μ are all characters of E^1 (or $\mathbb{A}_E^1/E^1$ globally). Note that $\kappa(z/\overline{z}) = \kappa^2(z)$. We fix $\omega = \omega_\pi$, hence $\mu = \omega_\pi/\omega_\rho\kappa$ is determined by κ and by the central character ω_ρ of ρ, and so it suffices to talk on the endoscopic lifting $\rho \mapsto \pi$, meaning $(\rho, \omega/\omega_\rho\kappa) \mapsto \pi$.

The homomorphism e factorizes via $i : {}^L H' \to {}^L G'$ and the unstable basechange map $b' : {}^L H \to {}^L H'$, $x \mapsto (x, x)$ for $x \in \widehat{H}$, $\sigma \mapsto (\varepsilon\theta(\varepsilon), 1)\sigma$, $z \mapsto (\kappa(z)_1, \kappa(z)_1)z$ for $z \in C_E$. Here $\kappa(z)_1$ indicates $\mathrm{diag}(\kappa(z), 1, \kappa(z))$. The basechange map on the factors $\mathbf{U}(1, E/F)$ and $\mathrm{GL}(1, \mathbb{C})$ is $\mu \mapsto \mu'$, $\mu'(z) = \mu(z/\overline{z})$, and $b : {}^L U(1) \to {}^L U(1)'$ is $x \mapsto (x, x)$, $b|W_{E/F}$ is the identity.

Let us summarize our L-group homomorphisms:

$$
\begin{array}{ccccc}
{}^L G = \mathrm{GL}(3,\mathbb{C}) \rtimes W_{E/F} & \xrightarrow{b} & {}^L G' & & \\
e\uparrow & & i\uparrow & \nwarrow^{e'} & \\
{}^L H = \mathrm{GL}(2,\mathbb{C}) \rtimes W_{E/F} & \underset{b'}{\rightarrow} & {}^L H' & \underset{b''}{\leftarrow} & {}^L H \ = \mathrm{GL}(2,\mathbb{C}) \rtimes W_{E/F}
\end{array}
$$

where ${}^L G' = [\mathrm{GL}(3, \mathbb{C}) \times \mathrm{GL}(3, \mathbb{C})] \rtimes W_{E/F}$
and ${}^L H' = [\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})] \rtimes W_{E/F}$.

Implicit is a choice of a character ω' on C_E and ω on C_E^1 related by $\omega'(z) = \omega(z/\overline{z})$.

The definition of the endoscopic map e and the unstable basechange map b' depend on a choice of a character $\kappa : C_E/N_{E/F}C_E \to \mathbb{C}^1$ whose restriction to C_F is nontrivial.

An L-groups homomorphism $\lambda : {}^LG \to {}^LG'$ defines — via the Satake transform — a lifting of unramified representations. It leads to a definition of a norm map N relating stable (σ-) conjugacy classes in G' to stable conjugacy classes in G based on the map $\delta \mapsto \delta\sigma(\delta)$, $G' \to G'$. In the local case it also leads to a suitable definition of matching of compactly supported smooth (locally constant in the p-adic case) complex valued functions on G and G'. Functions f on G and ϕ on G' are matching if a suitable (determined by λ) linear combination of their (σ-) orbital integrals over a stable conjugacy class, is related to the analogous object for the other group, via the norm map. Symbolically: "$\Phi_\phi^\kappa(\delta\sigma) = \Phi_f^{\mathrm{st}}(N\delta)$". We postpone the precise definition to the text below (in brief, the stable orbital integrals of f match the σ-twisted stable orbital integrals of ϕ, the orbital integrals of ${}'\phi$ match the σ-twisted unstable orbital integrals of ϕ, and the unstable orbital integrals of f match the stable orbital integrals of ${}'f$), but state the names of the related functions according to the diagram of the L-groups above:

$$\begin{array}{ccccc} f & \xleftarrow{b} & \phi & & \\ e\downarrow & & & \searrow^{e'} & \\ {}'f & & & & {}'\phi \end{array}$$

In fact we fix characters ω', ω on the centers $Z' = E^\times$ of $G' = \mathrm{GL}(3,E)$, $Z = E^1$ of $G = \mathrm{U}(3,E/F)$, related by $\omega'(z) = \omega(z/\overline{z})$, $z \in Z' = E^\times$, and consider ϕ on G' with $\phi(zg) = \omega'(z)^{-1}\phi(g)$, $z \in Z' = E^\times$, smooth and compactly supported $\bmod Z'$, f on G with $f(zg) = \omega(z)^{-1}f(g)$, $z \in Z = E^1$, smooth and compactly supported $\bmod Z$, but according to our conventions ${}'f \in C_c^\infty(H)$ and ${}'\phi \in C_c^\infty(H)$ are compactly supported, where now $H = \mathrm{U}(2,E/F)$.

Our representation theoretic results can be schematically put in a diagram:

$$\begin{array}{ccccccccc} \pi & \overset{b}{\leftrightarrow} & \Pi & I(\rho'\otimes\kappa) & & I(\rho') & & & \\ e\uparrow & & & \uparrow i & & i\uparrow & & \nwarrow^{e'} & \\ \rho & \xrightarrow{b'} & & \rho'\otimes\kappa & & \rho' & \underset{b''}{\leftarrow} & & \rho \end{array}$$

Here we make use of our results in the case of basechange from $\mathrm{U}(2, E/F)$ to $\mathrm{GL}(2, E)$, namely that $b''(\rho) = \rho'$ iff $b'(\rho) = \rho' \otimes \kappa$, in the bottom row of the diagram. We describe these liftings in the next section, and in particular the structure of packets of representations on $G = \mathrm{U}(3, E/F)$. Both are defined in terms of character relations.

Nothing will be gained from working with the group of unitary similitudes

$$\mathrm{GU}(3, E/F) = \{(g, \boldsymbol{\lambda}) \in \mathrm{GL}(3, E) \times E^\times; gJ^t\overline{g} = \boldsymbol{\lambda}J\},$$

as it is the product $E^\times \cdot \mathrm{U}(3, E/F)$, where $E^\times$ indicates the diagonal scalar matrices, and $E^\times \cap \mathrm{U}(3, E/F)$ is E^1, the group of $x = z/\overline{z}$, $z \in E^\times$. Indeed, the transpose of $gJ^t\overline{g} = \boldsymbol{\lambda}J$ is $\overline{g}J^tg = \boldsymbol{\lambda}J$, hence $\boldsymbol{\lambda} = \boldsymbol{\lambda}(g) \in F^\times$, and taking determinants we get $x\overline{x} = \boldsymbol{\lambda}^3$ where $x = \det g$. Hence $\boldsymbol{\lambda} \in N_{E/F}E^\times \subset F^\times$, say $\boldsymbol{\lambda} = (u\overline{u})^{-1}$, $u \in E^\times$, then $ug \in \mathrm{U}(3, E/F)$. Since an irreducible representation has a central character, working with admissible or automorphic representations of $\mathrm{U}(3, E/F)$ is the same as working with such a representation of $\mathrm{GU}(3, E/F)$: just extend the central character from the center $Z = \mathbf{Z}(F) = E^1$ (locally, or $\mathbf{Z}(\mathbb{A}) = \mathbb{A}^1$ globally) of $G = \mathbf{G}(F)$ (or $\mathbf{G}(\mathbb{A})$), to the center $E^\times$ (or $\mathbb{A}_E^\times$) of the group of similitudes.

2. Statement of results

We begin with our *local results*. Let E/F be a quadratic extension of nonarchimedean local fields of characteristic 0, put $G' = \mathrm{GL}(3, E)$, and denote by G or $\mathrm{U}(3, E/F)$ the group of F-points on the quasi-split unitary group in three variables over F which splits over E. We realize G as the group of g in G' with $\sigma(g) = g$, where $\sigma(g) = \theta(\overline{g})$, $\theta(g) = J {}^t g^{-1} J$, $\overline{g} = (\overline{g_{ij}})$ and ${}^t g = (g_{ji})$ if $g = (g_{ij})$, and

$$J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}.$$

Similarly, we realize the group of F-points on the quasi-split unitary group H, or $\mathrm{U}(2, E/F)$, in two variables over E/F as the group of h in $H' = \mathrm{GL}(2, E)$ with $\sigma(h) = \varepsilon\theta(\overline{h})\varepsilon$, $\theta(h) = w {}^t h^{-1} w$, $\varepsilon = \mathrm{diag}(1, -1)$ and

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let N denote the norm map from E to F, and E^1 the unitary group $\mathrm{U}(1, E/F)$, consisting of $x \in E^\times$ with $Nx = 1$.

Let ϕ, f, ${}'f$ denote complex valued locally constant functions on G', G, H. The function ${}'f$ is compactly supported. The functions ϕ, f transform under the centers $Z' \simeq E^\times$, $Z \simeq E^1$ of G', G by characters ω'^{-1}, ω^{-1} which are matching ($\omega'(z) = \omega(z/\overline{z})$, $z \in E^\times$), and are compactly supported modulo the center. The spaces of such functions are denoted by $C_c^\infty(G', \omega'^{-1})$, $C_c^\infty(G, \omega^{-1})$, $C_c^\infty(H)$. Assume they are *matching*. Thus the "stable" orbital integrals "$\Phi^{\mathrm{st}}(N\delta, f dg)$" of $f dg$ match the twisted "stable" orbital integrals "$\Phi^{\sigma,\mathrm{st}}(\delta, \phi dg')$" of $\phi dg'$, and the unstable orbital integrals of $f dg$ match the stable orbital integrals of ${}'f dh$. These notions are defined in I.2 below; dg is a Haar measure on G, dg' on G', dh on H.

By a *G-module* π, or a representation π of G, we mean an admissible representation of G. If such a π is irreducible it has a central character by Schur's lemma. We work only with π which has the fixed central character ω, thus $\pi(zg) = \omega(z)\pi(g)$ for all $g \in G$, $z \in Z$. For $f dg$ as above the operator $\pi(f dg)$ has finite rank, hence it has trace $\mathrm{tr}\,\pi(f dg) \in \mathbb{C}$. We denote by χ_π the character [HC2] of π. It is a complex valued function on G which is conjugacy invariant and locally constant on the regular set, with central character ω. Moreover it is locally integrable with $\operatorname{tr}\pi(fdg) = \int \chi_\pi(g)f(g)dg$ (g in G) for all measures dg on G and f in $C_c^\infty(G,\omega^{-1})$.

Definition. A G'-module Π is called *σ-invariant* if ${}^\sigma\Pi \simeq \Pi$, where ${}^\sigma\Pi(g) = \Pi(\sigma(g))$.

For such Π there is an intertwining operator $A : \Pi \to {}^\sigma\Pi$, thus $A\Pi(g) = \Pi(\sigma g)A$ for all $g \in G$. Assume that Π is irreducible. Then Schur's lemma implies that A^2 is a (complex) scalar. We normalize it to be 1. This determines A up to a sign. Extend Π to $G' \rtimes \langle\sigma\rangle$ by $\Pi(\sigma) = A$.

The twisted character $g \mapsto \chi_\Pi^\sigma(g) = \chi_\Pi(g \times \sigma)$ of such Π is a function on G' which depends on the σ-conjugacy classes and is locally constant on the σ-regular set. Further it is locally integrable ([Cl2]) and satisfies, for all measures ϕdg,

$$\operatorname{tr}\Pi(\phi dg \times \sigma) = \int \chi_\Pi^\sigma(g)\phi(g)dg \qquad (g \text{ in } G').$$

Definition. A σ-invariant G'-module Π is called *σ-stable* if its twisted character χ_Π^σ depends only on the stable σ-conjugacy classes in G, namely $\operatorname{tr}\Pi(\phi dg' \times \sigma)$ depends only on fdg. It is called *σ-unstable* if

$$\chi_\Pi^\sigma(\delta) = -\chi_\Pi^\sigma(\delta')$$

whenever δ, δ' are σ-regular σ-stably conjugate but not σ-conjugate, equivalently, $\operatorname{tr}\Pi(\phi dg' \times \sigma)$ depends only on ${}'fdh$.

An element of G' is called σ-elliptic if its norm in G is elliptic, namely lies in an anisotropic torus. It is called σ-regular if its norm is regular, namely its centralizer is a torus.

A σ-invariant G'-module Π is called *σ-elliptic* if its σ-character χ_Π^σ is not identically zero on the σ-elliptic σ-regular set.

We first deal with the σ-unstable σ-invariant representations.

Unstable Representations. *Every σ-invariant irreducible representation Π is σ-stable or σ-unstable. All σ-unstable σ-elliptic Π are of the form $I(\rho')$, normalizedly induced from the maximal parabolic subgroup; on* *the 2×2 factor the H'-module ρ' is obtained by the stable basechange map b'' from an elliptic representation ρ of H. We have*

$$\operatorname{tr} I(\rho'; \phi dg' \times \sigma) = \operatorname{tr} \rho('f dh)$$

for all matching measures $'f dh$ and $\phi dg'$.

Our preliminary basechange result is

Local Basechange. *Let Π be a σ-stable tempered G'-module. For every tempered G-module π there exist nonnegative integers $m'(\pi) = m'(\pi, \Pi)$ which are zero except for finitely many π, so that for all matching $\phi dg'$, $f dg$ we have*

$$\operatorname{tr} \Pi(\phi dg' \times \sigma) = \sum_{\pi} m'(\pi) \operatorname{tr} \pi(f dg). \qquad (*)$$

This relation defines a partition of the set of (*equivalence classes of*) *tempered irreducible G-modules into disjoint finite sets*: *for each π there is a unique Π for which $m'(\pi) \neq 0$.*

Definition. (1) We call the finite set of π which appear in the sum on the right of $(*)$ a *packet*. Denote it by $\{\pi\}$, or $\{\pi(\Pi)\}$. It consists of tempered G-modules.
(2) Π is called the *basechange lift* of (each element π in) the packet $\{\pi(\Pi)\}$.

To refine the identity $(*)$ we prove that the multiplicities $m'(\pi)$ are equal to 1, and count the π which appear in the sum. The result depends on the σ-stable Π. First we note that:

List of the σ-stable Π. *The σ-stable Π are the σ-invariant Π which are square integrable, one dimensional, or induced $I(\rho' \otimes \kappa)$ from a maximal parabolic subgroup, where on the 2×2 factor the H'-module $\rho' \otimes \kappa$ is the tensor product of an H'-module ρ' obtained by the stable basechange map b'' in our diagram, and the fixed character κ of C_E/NC_E which is nontrivial on C_F.*

In the local case $C_E = E^\times$ and N is the norm from E to F. Namely $\rho' \otimes \kappa$ is obtained by the unstable map b' in our diagram, from a packet $\{\rho\}$ of H-modules (defined in [F3;II]). Our main local results are as follows:

Local Results. (1) *If Π is square integrable then it is σ-stable and the packet $\{\pi(\Pi)\}$ consists of a single square-integrable G-module π. If Π is of the form $I(\rho' \otimes \kappa)$, and ρ' is the stable basechange lift of a square-integrable H-packet $\{\rho\}$, then Π is σ-stable and the cardinality of $\{\pi(\Pi)\}$ is twice that of $\{\rho\}$.*

Remark. In the last case we denote $\{\pi(\Pi)\}$ also by $\{\pi(\rho)\}$, and say that $\{\rho\}$ *endo-lifts* to $\{\pi(\rho)\} = \{\pi(I(\rho \otimes \kappa))\}$.

Let $\{\rho\}$ be a square-integrable H-packet. It consists of one or two elements.

Local Results. (2) *If $\{\rho\}$ consists of a single element then $\{\pi\}$ consists of two elements, π^+ and π^-, and we have the character relation*

$$\operatorname{tr} \rho('f dh) = \operatorname{tr} \pi^+(f dg) - \operatorname{tr} \pi^-(f dg)$$

for all matching measures $'f dh$, $f dg$. If $\{\rho\}$ consists of two elements, then there are four members in $\{\pi(\rho)\}$, and three distinct square-integrable H-packets $\{\rho_i\}$ $(i = 1, 2, 3)$ with $\{\pi(\rho_i)\} = \{\pi(\rho)\}$. With this indexing, the four members of $\{\pi_i\}$ can be indexed so that we have the relations

$$\operatorname{tr}\{\rho_i\}('f dh) = \operatorname{tr} \pi_1(f dg) + \operatorname{tr} \pi_{i+1}(f dg) - \operatorname{tr} \pi_{i'}(f dg) - \operatorname{tr} \pi_{i''}(f dg) \tag{$**$}$$

for all matching $f dg$, $'f dh$. Here i', i'' are so that $\{i+1,\ i',\ i''\} = \{2, 3, 4\}$. A single element in the packet has a Whittaker model. It is π^+ if $[\{\rho\}] = 1$, and π_1 if $[\{\rho\}] = 2$.

Remark. The proof that a packet contains no more than one generic member is given only in the case of odd residual characteristic. It depends on a twisted analogue of Rodier [Rd].

In the case of the Steinberg (or "special") H-module $s(\mu)$, which is the complement of the one-dimensional representation $1(\mu) : g \mapsto \mu(\det g)$ in the suitable induced representation of H, we denote their stable basechange lifts by $s'(\mu')$ and $1'(\mu')$. Here μ is a character of $C_E^1 = E^1$ (norm-one subgroup in $E^\times$), and $\mu'(a) = \mu(a/\overline{a})$ is a character of $C_E = E^\times$.

Local Results. (3) *The packet $\{\pi(s(\mu))\}$ consists of a cuspidal $\pi^- = \pi_\mu^-$, and the square-integrable subrepresentation $\pi^+ = \pi_\mu^+$ of the induced*

G-module $I = I(\mu'\kappa\nu^{1/2})$. Here I is reducible of length two, and its nontempered quotient is denoted by $\pi^\times = \pi^\times_\mu$. The character relations are

$$\begin{aligned}
\operatorname{tr}(s(\mu))('f dh) &= \operatorname{tr}\pi^+(f dg) - \operatorname{tr}\pi^-(f dg),\\
\operatorname{tr}(1(\mu))('f dh) &= \operatorname{tr}\pi^\times(f dg) + \operatorname{tr}\pi^-(f dg),\\
\operatorname{tr} I(s'(\mu')\otimes\kappa;\phi dg'\times\sigma) &= \operatorname{tr}\pi^+(f dg) + \operatorname{tr}\pi^-(f dg),\\
\operatorname{tr} I(1'(\mu')\otimes\kappa;\phi dg'\times\sigma) &= \operatorname{tr}\pi^\times(f dg) - \operatorname{tr}\pi^-(f dg).
\end{aligned}$$

As the basechange character relations for induced modules are easy, we obtained the character relations for all (not necessarily tempered) σ-stable G'-modules.

If π is a nontempered irreducible G-module then its packet $\{\pi\}$ is defined to consist of π alone. For example, the packet of $\pi^\times$ consists only of $\pi^\times$. Also we make the following:

DEFINITION. Let μ be a character of $C^1_E = E^1$. The *quasi-packet* $\{\pi(\mu)\}$ of the nontempered subquotient $\pi^\times = \pi^\times_\mu$ of $I(\mu'\kappa\nu^{1/2})$ consists of $\pi^\times$ and the cuspidal $\pi^- = \pi^-_\mu$.

Note that $\pi^\times$ is unramified when E/F and μ are unramified.

Thus a packet consists of tempered G-modules, or of a single nontempered element. A quasi-packet consists of a nontempered $\pi^\times$ and a cuspidal π^-. The packet of π^- consists of π^- and π^+, where π^+ is the square-integrable constituent of $I(\mu'\kappa\nu^{1/2})$. These local definitions are made for global purposes.

We shall now state our *global results.*

Let E/F be a quadratic extension of number fields, $\mathbb{A}_E$ and $\mathbb{A} = \mathbb{A}_F$ their rings of adèles, $\mathbb{A}^\times_E$ and $\mathbb{A}^\times$ their groups of idèles, N the norm map from E to F, $\mathbb{A}^1_E$ the group of E-idèles with norm 1, $C_E = \mathbb{A}^\times_E/E^\times$ the idèle class group of E, ω a character of $C^1_E = \mathbb{A}^1_E/E^1$, ω' a character of C_E with $\omega'(z) = \omega(z/\overline{z})$. Denote by $\mathbf{H}$, or $\mathbf{U}(2, E/F)$, and by $\mathbf{G}$, or $\mathbf{U}(3, E/F)$, the quasi-split unitary groups associated to E/F and the forms εw and J as defined in the local case. These are reductive F-groups. We often write G for $\mathbf{G}(F)$, H for $\mathbf{H}(F)$, and $G' = \mathrm{GL}(3, E)$ for $\mathbf{G}'(F) = \mathbf{G}(E)$, where $\mathbf{G}' = \mathrm{R}_{E/F}\,\mathbf{G}$ is the F-group obtained from $\mathbf{G}$ by restriction of scalars from E to F. Note that the group of E-points $\mathbf{G}'(E)$ is $\mathrm{GL}(3, E)\times\mathrm{GL}(3, E)$.

Denote the places of F by v, and the completion of F at v by F_v. Put $G_v=\mathbf{G}(F_v)$, $G'_v=\mathbf{G}'(F_v)=\mathrm{GL}(3,E_v)$, $H_v=\mathbf{H}(F_v)$. Note that at a place v which splits in E we have that $\mathbf{U}(n,E/F)(F_v)$ is $\mathrm{GL}(n,F_v)$. When v is nonarchimedean denote by R_v the ring of integers of F_v. When v is also unramified in E put $K_v=\mathbf{G}(R_v)$. Also put $K_{Hv}=\mathbf{H}(R_v)$ and $K'_v=\mathbf{G}'(R_v)=\mathrm{GL}(3,R_{E,v})$, where $R_{E,v}$ is the ring of integers of $E_v=E\otimes_F F_v$. When v splits we have $E_v=F_v\oplus F_v$ and $R_{E,v}=R_v\oplus R_v$.

Write $L^2(G,\omega)$ for the space of right-smooth complex-valued functions ϕ on $G\backslash\mathbf{G}(\mathbb{A})$ with $\phi(zg)=\omega(z)\phi(g)$ ($g\in\mathbf{G}(\mathbb{A})$, $z\in\mathbf{Z}(\mathbb{A})$, $\mathbf{Z}$ being the center of $\mathbf{G}$). The group $\mathbf{G}(\mathbb{A})$ acts by right translation: $(r(g)\phi)(h)=\phi(hg)$. The $\mathbf{G}(\mathbb{A})$-module $L^2(G,\omega)$ decomposes as a direct sum of (1) the discrete spectrum $L^2_d(G,\omega)$, defined to be the direct sum of all subrepresentations, and (2) the continuous spectrum $L^2_c(G,\omega)$, which is described by Langlands theory of Eisenstein series as a continuous sum.

The $\mathbf{G}(\mathbb{A})$-module $L^2_d(G,\omega)$ further decomposes as a direct sum of the cuspidal spectrum $L^2_0(G,\omega)$, consisting of cusp forms ϕ, and the residual spectrum $L^2_r(G,\omega)$, which is generated by residues of Eisenstein series. Each irreducible constituent of $L^2(G,\omega)$ is called an *automorphic* representation, and we have *discrete-spectrum* representations, *cuspidal*, *residual* and *continuous-spectrum* representations. Each such has central character ω. The discrete-spectrum representations occur in L^2_d with finite multiplicities. Similar definitions apply to the groups $\mathbf{H}$, $\mathbf{G}'$ and $\mathbf{H}'$.

By a $\mathbf{G}(\mathbb{A})$-*module* we mean an admissible representation of $\mathbf{G}(\mathbb{A})$. Any irreducible $\mathbf{G}(\mathbb{A})$-module π is a restricted tensor product $\otimes_v\pi_v$ of admissible irreducible representations π_v of $G_v=\mathbf{G}(F_v)$, which are almost all (at most finitely many exceptions) unramified. A G_v-module π_v is called *unramified* if it has a nonzero K_v-fixed vector. It is a rare property for a $\mathbf{G}(\mathbb{A})$-module to be automorphic.

An L-groups homomorphism ${}^LH\to{}^LG$ defines via the Satake transform a lifting $\rho_v\mapsto\pi_v$ of unramified representations. Given an automorphic representation ρ of $\mathbf{H}(\mathbb{A})$, the L-groups homomorphism ${}^LH\to{}^LG$ defines then unramified π_v at almost all places. We say that ρ *quasi-e-lifts* to π if ρ_v e-lifts to π_v for almost all places v. Our preliminary result is an existence result, of π in the following statement.

QUASI-LIFTING. *Every automorphic ρ quasi-e-lifts to an automorphic π.*

Every automorphic π quasi-b-lifts to an automorphic σ-invariant Π on $\mathrm{GL}(3, \mathbb{A}_E)$.

The same result holds for each of the homomorphisms in our diagram.

To be pedantic, under the identification $\mathrm{GL}(3, E) = G'$, $g \mapsto (g, \sigma g)$, we can introduce $\Pi'(g, \sigma g) = \Pi(g)$. Then ${}^\sigma\Pi = {}^\iota\Pi'$, where $\iota(x, y) = (y, x)$. Thus Π is σ-invariant as a $\mathrm{GL}(3, E)$-module iff Π' is ι-invariant as a G'-module (and similarly globally).

Our main global results consist of a refinement of the quasi-lifting to lifting in terms of all places. To state the result we need to define and describe packets of discrete-spectrum $\mathbf{G}(\mathbb{A})$-modules. To introduce the definition, recall that we defined above packets of tempered G_v-modules at each v, as well as quasi-packets, which contain a nontempered representation. If v splits then $G_v = \mathrm{GL}(3, F_v)$ and a (quasi-) packet consists of a single irreducible.

Definition. (1) Given a local packet P_v for all v such that P_v contains an unramified member π_v^0 for almost all v, we define the *global packet P* to be the set of products $\otimes\pi_v$ over all v, where π_v lies in P_v for all v, and $\pi_v = \pi_v^0$ for almost all v.
(2) Given a character μ of $C_E^1 = \mathbb{A}_E^1/E^1$, the quasi-packet $\{\pi(\mu)\}$ is defined as in the case of packets, where P_v is replaced by the quasi-packet $\{\pi(\mu_v)\}$ for all v, and π_v^0 is the unramified $\pi_v^\times$ at the v where E/F and μ are unramified.
(3) The $\mathbf{H}(\mathbb{A})$-module $\rho = \otimes\rho_v$ *endo-lifts* to the $\mathbf{G}(\mathbb{A})$-module $\pi = \otimes\pi_v$ if ρ_v endo-lifts to π_v (i.e. $\{\rho_v\}$ endo-lifts to $\{\pi_v\}$) for all v. Similarly, $\pi = \otimes\pi_v$ *basechange* lifts to the $\mathrm{GL}(3, \mathbb{A}_E)$-module $\Pi = \otimes\Pi_v$ if π_v basechange lifts to Π_v for all v.

A complete description of the packets is as follows.

Global Lifting. *The basechange lifting is a one-to-one correspondence from the set of packets and quasi-packets which contain an automorphic $\mathbf{G}(\mathbb{A})$-module, to the set of σ-invariant automorphic* $\mathrm{GL}(3, \mathbb{A}_E)$-*modules Π which are not of the form $I(\rho')$. Here ρ' is the* $\mathrm{GL}(2, \mathbb{A}_E)$-*module obtained by stable basechange from a discrete-spectrum $\mathbf{H}(\mathbb{A})$-packet $\{\rho\}$.*

As usual, we write $\{\pi(\rho)\}$ for a packet which basechanges to $\Pi = I(\rho' \otimes \kappa)$, where the $\mathbf{H}'(\mathbb{A})$-module ρ' is the stable basechange lift of the $\mathrm{GL}(2, \mathbb{A}_E)$-packet $\{\rho\}$. We conclude:

Description of packets. *Each discrete-spectrum* $\mathbf{G}(\mathbb{A})$*-module* π *lies in one of the following.*
(1) *A packet* $\{\pi(\Pi)\}$ *associated with a discrete-spectrum* σ*-invariant representation* Π *of* $\mathrm{GL}(3, \mathbb{A}_E)$.
(2) *A packet* $\{\pi(\rho)\}$ *associated with a cuspidal* $\mathbf{H}(\mathbb{A})$*-module* ρ.
(3) *A quasi-packet* $\{\pi(\mu)\}$ *associated with an automorphic one-dimensional* $\mathbf{H}(\mathbb{A})$*-module* $\rho = \mu \circ \det$.

Packets of type (1) will be called *stable*, those of type (2) *unstable*, and quasi-packets are unstable too. The terminology is justified by the following result.

Multiplicities. (1) *The multiplicity of a* $\mathbf{G}(\mathbb{A})$*-module* $\pi = \otimes \pi_v$ *from a packet* $\{\pi(\Pi)\}$ *of type* (1) *in the discrete spectrum of* $\mathbf{G}(\mathbb{A})$ *is one. Namely each element* π *of* $\{\pi(\Pi)\}$ *is automorphic, in the discrete spectrum.*
(2) *The multiplicity of* π *from a packet* $\{\pi(\rho)\}$ *or a quasi-packet* $\{\pi(\mu)\}$ *in the discrete spectrum of* $\mathbf{G}(\mathbb{A})$ *is equal to one or zero. This multiplicity is not constant over* $\{\pi(\rho)\}$ *and* $\{\pi(\mu)\}$*. If* π *lies in* $\{\pi(\mu)\}$ *it is given by*

$$m(\mu, \pi) = \frac{1}{2}\left[1 + \varepsilon(\mu', \kappa) \prod_v \varepsilon_v(\mu_v, \pi_v)\right]$$

where $\varepsilon(\mu', \kappa)$ *is a sign* (1 *or* -1) *depending on* μ (*or* $\mu'(x) = \mu(x/\overline{x})$) *and* κ*, and where* $\varepsilon_v(\mu_v, \pi_v) = 1$ *if* $\pi_v = \pi^\times_{\mu_v}$ *and* $\varepsilon_v(\mu_v, \pi_v) = -1$ *if* $\pi_v = \pi^-_{\mu_v}$.

If π *lies in* $\{\pi(\rho)\}$*, and there is a single* ρ *which endo-lifts to* π*, then the multiplicity is*

$$m(\rho, \pi) = \frac{1}{2}\left(1 + \prod_v \varepsilon(\rho_v, \pi_v)\right),$$

where $\varepsilon_v(\rho_v, \pi_v) = 1$ *if* π_v *lies in* $\pi(\rho_v)^+$*, and* $\varepsilon_v(\rho_v, \pi_v) = -1$ *if* π_v *lies in* $\pi(\rho_v)^-$.

Let π *lie in* $\{\pi(\rho_1)\} = \{\pi(\rho_2)\} = \{\pi(\rho_3)\}$ *where* $\{\rho_1\}$, $\{\rho_2\}$, $\{\rho_3\}$ *are distinct* $\mathbf{H}(\mathbb{A})$*-packets. Then the multiplicity of* π *is* $\frac{1}{4}(1 + \sum_{i=1}^3 \langle \varepsilon_i, \pi \rangle)$. *The signs* $\langle \varepsilon_i, \pi \rangle = \prod_v \langle \varepsilon_i, \pi_v \rangle$ *are defined by* $(**)$.

The sign $\varepsilon(\mu', \kappa)$ is likely to be the value at $1/2$ of the ε-factor $\varepsilon(s, \mu'\kappa)$ of the functional equation of the L-function $L(s, \mu'\kappa)$ of $\mu'\kappa$. This is the case when $L(\frac{1}{2}, \mu'\kappa) \neq 0$, in which case $\pi^\times_\mu = \prod_v \pi^\times_{\mu_v}$ is residual and $\varepsilon(\frac{1}{2}, \mu'\kappa)$ is 1. When $L(\frac{1}{2}, \mu'\kappa) = 0$ the automorphic representation $\pi^\times_\mu$ is discrete spectrum (necessarily cuspidal) iff $\varepsilon(\mu',\kappa)=1$. An irreducible π in the quasi-packet of $\pi_\mu^\times$ which is discrete spectrum (thus $m(\mu,\pi)=1$) with at least one component π_v^- is cuspidal since π_v^- is cuspidal.

In particular we have the following

Multiplicity One Theorem. *Each discrete-spectrum automorphic representation of* $\mathbf{G}(\mathbb{A})$ *occurs in the discrete spectrum of* $L^2(\mathbf{G}(\mathbb{A}),\omega)$ *with multiplicity one.*

Rigidity Theorem. *If* π *and* π' *are discrete-spectrum representations of* $\mathbf{G}(\mathbb{A})$ *whose components* π_v *and* π'_v *are equivalent for almost all* v*, then they lie in the same packet, or quasi-packet.*

Genericity. *Each* G_v*- and* $\mathbf{G}(\mathbb{A})$*-packet contains precisely one generic representation. Quasi-packets do not contain generic representations.*

Corollary. (1) *Suppose that* π *is a discrete-spectrum* $\mathbf{G}(\mathbb{A})$*-module which has a component of the form* $\pi_w^\times$*. Then* π *lies in a quasi-packet* $\{\pi(\mu)\}$*, of type* (3)*. In particular its components are of the form* $\pi_v^\times$ *for almost all* v*, and of the form* π_v^- *for the remaining finite set* (*of even cardinality iff* $\varepsilon(\mu',\kappa)$ *is* 1) *of places of* F *which stay prime in* E*.*
(2) *If* π *is a discrete-spectrum* $\mathbf{G}(\mathbb{A})$*-module with an elliptic component at a place of* F *which splits in* E*, or a one-dimensional or Steinberg component at a place of* F *which stay prime in* E*, then* π *lies in a packet* $\{\pi(\Pi)\}$*, where* Π *is a discrete-spectrum* $\mathrm{GL}(3,\mathbb{A}_E)$*-module.*

A cuspidal representation in a quasi-packet $\{\pi(\mu)\}$ of type (3) (for example, one which has a component π_v^-) makes a *counter example to the naive Ramanujan conjecture*: almost all of its components are nontempered, namely $\pi_v^\times$. The Ramanujan conjecture for $\mathrm{GL}(n)$ asserts that all local components of a cuspidal representation of $\mathrm{GL}(n,\mathbb{A})$ are tempered. The Ramanujan conjecture for U(3) should say that all local components of a discrete-spectrum representation π of $\mathbf{U}(3,E/F)(\mathbb{A})$ which basechange lifts to a cuspidal representation of $\mathrm{GL}(3,\mathbb{A})$ are tempered. This can be shown for π with discrete-series components at the archimedean places by using the theory of Shimura varieties associated with U(3).

The discrete-spectrum $\mathbf{G}(\mathbb{A})$-modules π with an elliptic component at a nonarchimedean place v of F which splits in E (such π are stable of type (1)) can easily be transferred to discrete-spectrum $'\mathbf{G}(\mathbb{A})$-modules, where $'\mathbf{G}$ is the inner form of $\mathbf{G}$ which is ramified at v. Thus $'\mathbf{G}$ is the unitary

F-group associated with the central division algebra of rank three over E which is ramified at the places of E over v of F.

Our local results hold for every local nonarchimedean field, of any characteristic, since by the Theorem of [K3] our results can be transfered from the case of characteristic zero to the case of positive characteristic. Consequently (once the σ-twisted trace formula for $\mathrm{GL}(3, \mathbb{A}_E)$ is made available in the function field case) our global results hold for every global field, in particular function fields, not only number fields.

This part is a write-up of our work on the representation theory of the unitary group in three variables, which started with the 1982 Princeton preprint "L-packets and liftings for U(3)", where we introduced the definition of packets and quasi-packets, and explained that contrary to opinions at the time, the lifting from U(2) to U(3) cannot be proven without simultaneously proving the basechange lifting from $\mathrm{U}(3, E/F)$ to $\mathrm{GL}(3, E)$. We were motivated by our then recent work on the symmetric square lifting, SL(2) to PGL(3), where the trace formula twisted by an outer automorphism was stated (a new point was that the twisted trace formula was to be computed by truncation of the kernel at only the parabolic subgroups fixed by the twisting). The twisted trace formula was established in [CLL]. A better exposition of the 1982 preprint was given in [F3;IV], [F3;V], [F3;VI].

The global results were nevertheless restricted to discrete-spectrum representations with two (or one) elliptic component, as we searched for a simple, conceptual proof for the identity of trace formulae for sufficiently general test functions. Such a proof was found in [F3;VII] where we show that using regular spherical functions such a general identity can be established without computing the weighted orbital integrals and the orbital integrals at the singular classes, thus giving a satisfactorily short proof without restricting the generality of the results. This is given in section II.4 here. However our proof works so far only in rank one (and twisted-rank one) cases. It is of great interest to extend this kind of simple proof to the higher case situation.

The fundamental lemma is a prerequisite for deducing any results at all from the trace formulae. This we establish, by means of elementary computations, in [F3;VIII], and in section I.3 here. The proof uses an intermediate double coset decomposition. In addition we record in section I.6 another proof of the fundamental lemma, which J.G.M. Mars wrote to me, confirming my computations. It is pleasing to have different proofs, which agree in the results of rather complicated computations. The fundamental lemma that we prove is for endoscopy, from U(2) to U(3). The fundamental lemma for basechange, from $\mathrm{U}(3, E/F)$ to $\mathrm{GL}(3, E)$, has a satisfactory, general proof (see Kottwitz [Ko4]). These two together imply the lemma for the twisted endoscopic lifting from $\mathrm{U}(2, E/F)$ to $\mathrm{GL}(3, E)$, see section I.2.

The only proof currently known for the multiplicity one theorem is given here in detail in section III.4 (and Proposition III.3.5). It is based on a twisted analogue of Rodier's theorem on the interpretation of the coefficients of regular orbits in the germ expansion of the character near the identity in terms of the number of Whittaker models of the representation in question. This is the local proof sketched in [F3;VI], Proposition 3.5, p. 47. The global proof of [F3;VI], p. 48, is incomplete.

The purpose of this part is then to give a complete and unified exposition to our work. We refer to this part in this book as [F3;I]. We also refer frequently to the papers in [F3] to indicate where notions and techniques were first introduced, although a unified exposition is given in this book. Additional remarks on the development of the area are given in section III.6.

I. LOCAL THEORY

Introduction

The aim of the first section is to classify the conjugacy and stable conjugacy classes in our unitary group G over the field F, as well as the twisted conjugacy classes in $G' = \mathrm{GL}(3, E)$. We give an explicit set of representatives for the classes within a stable class. This is used in section I.3 to compute the orbital integrals and prove the fundamental lemma. Our character relations are stated in terms of these classes, and the trace formula is expressed in terms of integrals over such classes.

In the second section (in this chapter I) we define the orbital integrals, the stable orbital integrals and the unstable ones, as well as the twisted analogues. We state the fundamental lemmas — for the unit elements of the Hecke algebras — for endoscopy, basechange, and twisted endoscopy, as well as the generalized fundamental lemma, for general spherical functions which are corresponding by a map dual to the dual-groups homomorphisms. Further we state that matching test functions exist as a consequence of the fundamental lemmas. We show that the fundamental lemma for twisted endoscopy follows from that for endoscopy, and vice-versa, on using the known fundamental lemma for basechange.

In the third section we prove the fundamental lemma for our (quasi-split) unitary group $\mathrm{U}(3, E/F)$ in three variables associated with a quadratic extension of p-adic fields, and its endoscopic group $\mathrm{U}(2, E/F)$, by means of an elementary technique. This lemma is a prerequisite for an application of the trace formula to classify the automorphic and admissible representations of $\mathrm{U}(3)$ in terms of those of $\mathrm{U}(2)$ and basechange to $\mathrm{GL}(3)$. It compares the (unstable) orbital integral of the characteristic function of the standard maximal compact subgroup K of $\mathrm{U}(3)$ at a regular element (whose centralizer T is a torus), with an analogous (stable) orbital integral on the endoscopic group $\mathrm{U}(2)$. The technique is based on computing the sum over the double coset space $T\backslash G/K$ which describes the integral, by means of an intermediate double coset space $H\backslash G/K$ for a subgroup H of $G = \mathrm{U}(3)$ containing T. The lemma is proven for both ramified and unramified regular elements, for which endoscopy occurs (the stable conjugacy class is not a single orbit). In the sixth section we record an alternative computation of the orbital integrals, due to J.G.M. Mars, based on counting lattices.

In the fourth section we introduce basic results on admissible representations that we need. These concern lifting of induced, one-dimensional and Steinberg representations, characters and twisted characters, Weyl integration formulae, description of reducibility of induced representations of U(3), and properties of modules of coinvariants.

The fifth section describes the representation theory of the real group $U(2,1;\mathbb{C}/R)$.

I.1 Conjugacy classes

1.1 Let $\mathbf{G}$ be a connected reductive group defined over a local or global field F. Fix an algebraic closure $\overline{F}$. Denote by $\overline{G} = \mathbf{G}(\overline{F})$ the group of $\overline{F}$-points on the variety $\mathbf{G}$. Now $\mathrm{Gal}(\overline{F}/F)$ acts on $\overline{G}$. The group $\mathbf{G}(F)$ of fixed points is denoted by G. An F-*torus* $\mathbf{T}$ in $\mathbf{G}$ is a maximal F-subgroup $\overline{F}$-isomorphic to a power of $\mathbb{G}_m$. Its group T of F-points is also called a torus. An element t of G is *regular* if the centralizer $Z_{\mathbf{G}}(t)$ of t in $\mathbf{G}$ is a maximal F-torus $\mathbf{T}$. The elements t, t' of G are *conjugate* if there is g in G with $t' = gtg^{-1}$. They are *stably conjugate* if there is such a g in $\overline{G}$. Tori T and T' are *stably conjugate* if there is g in $\overline{G}$ with $T' = gTg^{-1}$, so that the map $\mathrm{Int}(g) : \mathbf{T} \to \mathbf{T}'$, $\mathrm{Int}(g)(t) = gtg^{-1}$, is defined over F. Then $g_\tau = g^{-1}\tau(g)$ centralizes T for all τ in $\mathrm{Gal}(\overline{F}/F)$, hence lies in $\overline{T}$, since $\mathbf{G}$ is connected and reductive.

Of course the notion of stable conjugacy can be defined by $t' = g^{-1}tg$, which will lead to the definition of the cocycle as $g_\tau = g\tau(g^{-1})$. The change from g to g^{-1} should lead to no confusion, and we use both conventions.

We shall now *list all stable conjugacy classes of tori in* G. Let $\mathbf{T}^*$ be a fixed F-torus, $\mathbf{N}$ its normalizer in $\mathbf{G}$, and $\mathbf{W} = \mathbf{T}^*\backslash\mathbf{N} = \mathbf{N}/\mathbf{T}^*$ the absolute Weyl group. For each $\mathbf{T}$ there is g in $\mathbf{G}(\overline{F})$ with $\mathbf{T} = g\mathbf{T}^*g^{-1}$. Since $\mathbf{T}$ is defined over F, g_τ normalizes $\mathbf{T}^*$, and the cocycle $\tau \mapsto g_\tau$ defines a class in the first cohomology group $H^1(F,\mathbf{N})$ of $\mathrm{Gal}(\overline{F}/F)$ with coefficients in $\mathbf{N}(\overline{F})$. Denote by $\{g'_\tau\}$ the image of $\{g_\tau\}$ under the natural map $H^1(F,\mathbf{N}) \to H^1(F,\mathbf{W})$, obtained from $\mathbf{N} \to \mathbf{W}$.

The stable conjugacy classes are determined by means of the following.

1. Proposition. *The map $T \mapsto \{g'_\tau\}$ injects the set of stable conjugacy classes of tori in G into the image in $H^1(F, \mathbf{W})$ of* $\ker[H^1(F, \mathbf{N}) \to H^1(F, \mathbf{G})]$. *This map is also surjective when $\mathbf{G}$ is quasi-split.*

Proof. If $\mathbf{T} = g\mathbf{T}^* g^{-1}$ and $\mathbf{T}'$ are stably conjugate, then there is x in $\overline{G}$ with $\mathbf{T}' = x\mathbf{T}x^{-1} = xg\mathbf{T}^*(xg)^{-1}$, and $(xg)_\tau = g^{-1}x_\tau g \cdot g_\tau$ has the image g'_τ in $H^1(F, \mathbf{W})$, since $g^{-1}x_\tau g$ lies in $\overline{T}^*$ (x_τ in $\overline{T}$). Hence the map of the proposition is well defined.

Conversely, if $\mathbf{T} = g\mathbf{T}^* g^{-1}$, $\mathbf{T}' = g'\mathbf{T}^* g'^{-1}$, and $g_\tau = a(\tau) g'_\tau$ with $a(\tau)$ in $\overline{T}^*$, then $a(\tau) = g'^{-1}x(\tau)g'$ with $x(\tau)$ in $\overline{T}'$, and the map $t \mapsto gg'^{-1}t(gg'^{-1})^{-1}$ [t in $\overline{T}'$] is defined over F. Hence the map of the proposition is injective.

For the second claim, if $\{g_\tau\}$ lies in $\ker[H^1(F, \mathbf{N}) \to H^1(F, \mathbf{G})]$, then it defines a new $\mathrm{Gal}(\overline{F}/F)$-action by $\widehat{\tau}(h) = g_\tau^{-1}\tau(h)g_\tau$ ($h = t^*$ in $\overline{T}^*$). If h is a fixed $\widehat{\tau}$-invariant regular element, then $\tau(h) = g_\tau h g_\tau^{-1}$, and the conjugacy class of h in $\overline{G}$ is defined over F. When $\mathbf{G}$ is quasi-split, a theorem of Steinberg and Kottwitz [Ko1] implies the existence of h' in G which is conjugate to h in $\overline{G}$, since the field F is perfect. The centralizer of h' in G is a torus whose stable conjugacy class corresponds to $\{g_\tau\}$. Hence the map is surjective. □

Remark. Implicit in the proof is a description — used below — of the action of the Galois group on the torus. Let us make this explicit. All tori are conjugate in $\overline{G}$, thus $\overline{T} = g^{-1}\overline{T}^* g$ for some g in $\overline{G}$. For any t in $\overline{T}$ there is t^* in $\overline{T}^*$ with $t = g^{-1}t^*g$. For t in T, we have

$$\sigma g^{-1}\sigma t^* \sigma g = \sigma t = t = g^{-1}t^*g,$$

hence $\sigma t^* = g_\sigma^{-1}t^*g_\sigma \in \overline{T}^*$. Taking regular t (and t^*), $g_\sigma \in \overline{N}$ is uniquely determined modulo $\overline{T}^*$, namely in $\overline{W}$. For any t^* in $\overline{T}^*$ we then have

$$\sigma(g^{-1}t^*g) = g^{-1}(g\sigma(g^{-1}))\sigma(t^*)(\sigma(g)g^{-1})g,$$

hence the induced action on $\overline{T}^*$ is given by

$$\sigma^*(t^*) = g_\sigma \sigma(t^*) g_\sigma^{-1}.$$

The cocycle $\rho = \rho(T)\colon \Gamma \to \overline{W}$, given by $\rho(\sigma) = g_\sigma \bmod \overline{T}^*$, determines $\mathbf{T}$ up to stable conjugacy.

1.2 Here $A(\mathbf{T}/F)$ is the pointed set of g in $\mathbf{G}(\overline{F})$ so that $\mathbf{T}' = {}^g\mathbf{T} = g\mathbf{T}g^{-1}$ is defined over F. Then the set

$$B(\mathbf{T}/F) = G\backslash A(\mathbf{T}/F)/\mathbf{T}(\overline{F})$$

parametrizes the morphisms of T into G over F, up to inner automorphisms by elements of G. If T is the centralizer of x in G then $B(\mathbf{T}/F)$ *parametrizes the set of conjugacy classes within the stable conjugacy class of* x in G. The map

$$g \mapsto \{\tau \mapsto g_\tau = g^{-1}\tau(g); \tau \in \mathrm{Gal}(\overline{F}/F)\}$$

defines a bijection

$$B(\mathbf{T}/F) \simeq \ker[H^1(F,\mathbf{T}) \to H^1(F,\mathbf{G})].$$

Let $p : \mathbf{G}^{\mathrm{sc}} \twoheadrightarrow \mathbf{G}^{\mathrm{der}}$ denote the simply connected covering group of the derived group $\mathbf{G}^{\mathrm{der}}$ of $\mathbf{G}$. If $\mathbf{T}$ is an F-torus in $\mathbf{G}$, let $\mathbf{T}^{\mathrm{sc}} = p^{-1}(\mathbf{T}^{\mathrm{der}})$ of $\mathbf{T}^{\mathrm{der}} = \mathbf{T} \cap \mathbf{G}^{\mathrm{der}}$. Then $\mathbf{G} = \mathbf{T}\mathbf{G}^{\mathrm{der}}$ and $\mathbf{G}/p(\mathbf{G}^{\mathrm{sc}}) = \mathbf{T}/p(\mathbf{T}^{\mathrm{sc}})$. Then the pointed set $B(\mathbf{T}/F)$ is a subset of the group $C(\mathbf{T}/F)$, defined to be the image of $H^1(F,\mathbf{T}^{\mathrm{sc}})$ in $H^1(F,\mathbf{T})$. If $H^1(F,\mathbf{G}^{\mathrm{sc}}) = \{0\}$, for example when F is a nonarchimedean local field, then $B(\mathbf{T}/F) = C(\mathbf{T}/F)$. If F is a global field with a ring $\mathbb{A}$ of adèles, then we put $C(\mathbf{T}/\mathbb{A}) = \oplus_v C(\mathbf{T}/F_v)$, $B(\mathbf{T}/\mathbb{A}) = \oplus_v B(\mathbf{T}/F_v)$. The sums are pointed. They range over all places v of F.

Let K be a finite Galois extension of F over which $\mathbf{T}$ splits. Denote $H^{-1}(\mathrm{Gal}(K/F), X)$ by $H^{-1}(X)$ and $\mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ by $X_*(\mathbf{T})$. In the local case the Tate-Nakayama duality (see [KS]) identifies $C(\mathbf{T}/F)$ with the image of $H^{-1}(X_*(\mathbf{T}^{\mathrm{sc}}))$ in $H^{-1}(X_*(\mathbf{T}))$. In the global case it yields an exact sequence

$$C(\mathbf{T}/F) \to C(\mathbf{T}/\mathbb{A}) \to \mathrm{Im}[H^{-1}(X_*(\mathbf{T}^{\mathrm{sc}})) \to H^{-1}(X_*(\mathbf{T}))].$$

The last term here is the quotient of the $\mathbb{Z}$-module of μ in $X_*(\mathbf{T}^{\mathrm{sc}})$ with $\sum_\tau \tau\mu = 0$ (sum over τ in $\mathrm{Gal}(K/F)$), by the submodule spanned by $\mu - \tau\mu$, where μ ranges over $X_*(\mathbf{T})$ and τ over $\mathrm{Gal}(K/F)$.

We denote by $W(T)$ the Weyl group of T in G, by $\mathbf{W} = S_3$ the Weyl group of $\mathbf{T}^*$ in $\mathbf{G}$, and by $W'(T)$ the Weyl group of T in $A(\mathbf{T}/F)$. We write σ for the nontrivial element in $\mathrm{Gal}(E/F)$.

1.3 We shall now discuss *the above definitions in our case* where $\mathbf{G} = \mathbf{U}(3, E/F)$. The centralizer E' of T in the algebra $M(3, E)$ of 3×3 matrices over E, is a maximal commutative semisimple subalgebra. Hence it is isomorphic to a direct sum of field extensions of E.

There are three possibilities.

(1) $E' = E \oplus E \oplus E$.

(2) $E' = E'' \oplus E$, $[E'' : E] = 2$.

(3) E' is a cubic extension of E.

The absolute Weyl group $\mathbf{W}$ is the symmetric group on three letters, generated by the reflections (12), (23), (13). Note that $\sigma(12) = (23)$, $\sigma(13) = (13)$. In view of Proposition 1, the stable conjugacy classes are determined by $H^1(F, \mathbf{W})$. We also note that if the eigenvalues of g in G are α, β, γ in K, then τ in $\mathrm{Gal}(K/F)$ whose restriction to E is nontrivial, maps α, β, γ to $\tau\alpha^{-1}$, $\tau\beta^{-1}$, $\tau\gamma^{-1}$. The lattice $X_*(\mathbf{T})$ is the group of $\mu = (x, y, z)$ in $\mathbb{Z}^3$, and $X_*(\mathbf{T}^{\mathrm{sc}})$ is the subgroup of μ with $x + y + z = 0$. Indeed, $\mathbf{G}^{\mathrm{sc}} = \mathbf{SU}(3)$. If $\tau|E \neq 1$ it maps the set $\{x, y, z\}$ to the set $\{-x, -y, -z\}$.

2. Proposition. (1) *There are two stable conjugacy classes of F-tori in $\mathbf{G}$ which split over E. One, named of type* (0), *consists of a single conjugacy class, represented by the torus $\mathbf{T}^*$ with*

$$T^* = \{\mathrm{diag}(a, b, \sigma a^{-1});\ a \in E^\times,\ b \in E^1 = \{x \in E^\times; x\sigma x = 1\}\}.$$

We have $W'(T^*) = W(T^*) = \mathbb{Z}/2$. The other stable conjugacy class, named of type (1), *consists of tori $\mathbf{T}$ with $T = (E^1)^3$, and $C(\mathbf{T}/F) = \{(a, b, c) \in F^\times/NE^\times; abc = 1\}$. We have $W'(T) = S_3$, and this group acts transitively on the nontrivial elements in* (*and characters of*) $C(\mathbf{T}/F)$.

(2) *The stable conjugacy classes of F-tori in $\mathbf{G}$ whose splitting fields are quadratic extensions of E, named of type* (2), *split over biquadratic extensions EL of F. Then $\mathrm{Gal}(EL/F) = \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by σ which fixes L and τ which fixes E; put $K = (EL)^{\sigma\tau}$. Each such torus is $T \simeq \{(a, b, \sigma a^{-1}); a \in (EL/K)^1, b \in E^1\}$. Here $(EL/K)^1 = \{a \in EL; a\sigma\tau a = 1\}$. Further $C(\mathbf{T}/F) = K^\times/N_{EL/K}(EL)^\times = \mathbb{Z}/2$ and $W'(T) = \mathbb{Z}/2$.*

(3) *The stable conjugacy classes of F-tori in $\mathbf{G}$ whose splitting fields are cubic extensions of E, named of type* (3), *are split over cubic extensions ME of E, where M is a cubic extension of F. Each stable class consists of a single conjugacy class. If EM/F is not Galois then $W'(T)$ is trivial. If $\mathrm{Gal}(EM/F) = S_3$ or $\mathbb{Z}/3$ then $W'(T)$ is $\mathbb{Z}/3$.*

Proof. A cocycle in $H^1(\mathrm{Gal}(E/F), \mathbf{W})$ is determined by w_σ in $\mathbf{W} = S_3$ with $1 = w_{\sigma^2} = w_\sigma \sigma(w_\sigma)$. Thus w_σ is 1 or (13), or (12)(23) or (23)(12). As

$$\sigma((23))[(12)(23)](23) = 1 = \sigma((12))[(23)(12)](12),$$

the last two are cohomologous to 1. The cocycle $w_\sigma = 1$ defines the action $\sigma^*(t^*) = \sigma(t^*)$ on $\overline{T}^*$. To determine $C(\mathbf{T}^*/F)$, note that $H^1(F, \mathbf{T}^*) = H^1(\mathrm{Gal}(E/F), \mathbf{T}^*(E))$ is the quotient of the cocycles $t_\sigma = \mathrm{diag}(a, b, c) \in \mathbf{T}^*(E) = E^{\times 3}$, $t_\sigma \sigma(t_\sigma) = t_{\sigma^2} = 1$, thus $t_\sigma = \mathrm{diag}(a, b, \sigma a)$, $a \in E^\times$, $b \in F^\times$, by the coboundaries $t_\sigma \sigma(t_\sigma^{-1}) = \mathrm{diag}(a\sigma c, b\sigma b, c\sigma a)$. Since $\mathbf{G}^{\mathrm{sc}}$ is the subgroup of $\mathbf{G}$ of elements of determinant 1, the cocycles which come from $H^1(F, \mathbf{T}^{*\mathrm{sc}})$ have the form $t_\sigma = \mathrm{diag}(a, 1/a\sigma a, \sigma a)$. These are coboundaries: $u_\sigma \sigma(u_\sigma^{-1})$, with $u_\sigma = (a, 1/a, 1)$, hence $C(\mathbf{T}^*/F)$ is trivial.

The cocycle $w_\sigma = (13)$ defines the action

$$\sigma^*(\mathrm{diag}(a, b, c)) = \mathrm{diag}(\sigma a^{-1}, \sigma b^{-1}, \sigma c^{-1})$$

on $\overline{T}^*$. Then $\mathbf{T} = g^{-1}\mathbf{T}^* g$ for some g in $\overline{G}$ with $g\sigma(g^{-1}) = J \pmod{\overline{T}^*}$, and $T = \mathbf{T}(F) = g^{-1}(E^1)^3 g$. A cocycle $t_\sigma = \mathrm{diag}(a, b, c) \in (E^\times)^3$ of $\mathrm{Gal}(E/F)$ in $\mathbf{T}^*(E)$ satisfies $1 = t_{\sigma^2} = t_\sigma \sigma^*(t_\sigma) = \mathrm{diag}(a/\sigma a, b/\sigma b, c/\sigma c)$, thus $a, b, c \in F^\times$ and it comes from $\mathbf{T}^{*\mathrm{sc}}(E)$ if $abc = 1$. The coboundaries take the form $t_\sigma \sigma^*(t_\sigma)^{-1} = \mathrm{diag}(a\sigma a, b\sigma b, c\sigma c)$, hence $C(\mathbf{T}/F) = \{(a, b, c) \in (F^\times/NE^\times)^3; abc = 1\}$.

Consider next an F-torus $\mathbf{T}$ in $\mathbf{G}$ which splits over a quadratic extension L_1 of E, but not over E. We claim that L_1/F is Galois. Indeed, the involution $\iota(x) = J{}^t\overline{x}J$ stabilizes $T = \mathbf{T}(F)$, and its centralizer $L_1^\times \times E^\times$ in $\mathrm{GL}(3, E)$. It induces on L_1 an automorphism whose restriction to E generates $\mathrm{Gal}(E/F)$. Hence L_1/F is Galois.

We claim that the Galois group of L_1/F is not $\mathbb{Z}/4$. Indeed, had $\mathrm{Gal}(L_1/F) = \mathbb{Z}/4$ been generated by τ, then τ^2 be trivial on E, $(w_{\tau^2})^2 = w_{\tau^4} = 1$ implies $w_{\tau^2} = 1$ or (13) up to coboundaries. But $(13) = w_{\tau^2} = w_\tau \tau(w_\tau) = w_\tau (13) w_\tau (13)$ implies $w_\tau^2 = (13)$, which has no solutions, and $w_{\tau^2} = 1$ implies that T splits over E. Then $\mathrm{Gal}(L_1/F) = \mathbb{Z}/2 \times \mathbb{Z}/2$, and L_1 is the compositum of E and a quadratic extension L of F, not isomorphic to E. There are two such L (up to isomorphism), both ramified if E/F is unramified.

The Galois group $\mathrm{Gal}(LE/F)$ is generated by σ whose restriction to L is trivial, and τ whose restriction to E is trivial. Up to coboundaries, w_τ

is 1 or (13). If $w_\sigma = (13)$, then $w_\tau \neq 1$ is of order 2. Up to coboundary which does not change w_σ, we have $w_\tau = (13)$, and replacing σ by $\sigma\tau$ (thus changing L) we may assume $w_\sigma = 1$. If $w_\sigma = 1$, $w_\tau w_\sigma = w_{\tau\sigma} = w_{\sigma\tau} = w_\sigma \sigma(w_\tau) = w_\sigma(13)w_\tau(13)$ implies that w_τ ($\neq 1$) commutes with (13), hence $w_\tau = (13)$. Up to isomorphism, T consists of $(a,b,c) \in (LE)^{\times 3}$ which are fixed by $\sigma^*(a,b,c) = (\sigma c^{-1}, \sigma b^{-1}, \sigma a^{-1})$ and $\tau^*(a,b,c) = (\tau c, \tau b, \tau a)$. Thus $b = \tau b = \sigma b^{-1}$ lies in E^1, and $c = \sigma a^{-1} = \tau a$, namely $T \simeq \{(a, b, \sigma a^{-1}); a \in (EL/K)^1, b \in E^1\}$, where $(EL/K)^1 = \{a \in EL; a\sigma\tau a = 1\}$.

It is simplest to compute $C(\mathbf{T}/F)$ using Tate-Nakayama duality. Locally, the image of

$$\hat{H}^{-1}(F, X_*(\mathbf{T}^{\mathrm{sc}})) = \{X = (x,y,z) \in \mathbb{Z}^3; x+y+z = 0\}/\langle X - \sigma X, X - \tau X\rangle$$

in

$$\hat{H}^{-1}(F, X_*(\mathbf{T})) = \mathbb{Z}^3/\langle X - \tau\sigma X = (2x, 2y, 2z), \quad X - \tau X = (x-z, 0, z-x)\rangle$$

is $\mathbb{Z}/2$.

Here is an explicit computation of $H^1(\mathrm{Gal}(LE/F), \mathbf{T}(LE))$. We replace $\mathbf{T}$ by $\mathbf{T}^*$ if $\rho \in \mathrm{Gal}(LE/F)$ acts by ρ^*. To compute note that a cocycle in $H^1(\mathrm{Gal}(LE/F), \mathbf{T}^*(LE))$ is defined by $\{t_\sigma, t_\tau, t_{\sigma\tau}\}$ satisfying the cocycle relations. Thus $t_\tau = (a,b,c) \in (EL)^{\times 3}$ satisfies $1 = t_{\tau^2} = t_\tau \tau^*(t_\tau) = (a,b,c)(\tau c, \tau b, \tau a)$. So $b = b'/\tau b'$ and if $g = (a, b', 1)$, replacing our cocycle $\{t_\rho\}$ by its product $\{t_\rho g^{-1}\rho^*(g)\}$ with a coboundary, we may assume that $t_\tau = 1$. If $t_{\tau\sigma} = (u,v,w)$ then

$$1 = t_{(\tau\sigma)^2} = t_{\tau\sigma}(\sigma\tau)^*(t_{\tau\sigma}) = (u,v,w)(\tau\sigma u^{-1}, \tau\sigma v^{-1}, \tau\sigma w^{-1}).$$

Hence $(u,v,w) \in K^{\times 3}$. Here K is the fixed field of $\tau\sigma$ in LE. Further, $t_{\tau\sigma}(\tau\sigma)^*(t_\tau) = t_\sigma = t_\tau \tau^*(t_{\tau\sigma})$. Hence $t_{\tau\sigma} = (u,v,w) = (\tau w, \tau v, \tau u) = (u, v, \tau u)$, $u \in K^\times$, $v \in F^\times$. We can still multiply our cocycle t_ρ by a coboundary $g^{-1}\rho^*(g)$ with $g = \tau(g)$ (to preserve $t_\tau = 1$). Thus $g = (x, y, \tau x)$, $y = \tau y \in E^\times$. Then $g^{-1}(\tau\sigma)^*(g) = (1/u, 1/y\sigma(y), 1/\tau(u))$, $u = x\tau\sigma(x)$. Now $H^1(\mathrm{Gal}(LE/F), \mathbf{T}^{\mathrm{sc}}(LE))$ is spanned by the $t_{\tau\sigma} = (u, v, \tau u)$, $u \in K^\times/N_{EL/K}(EL)^\times$, $vu\tau u = 1$. Then $\mathrm{Im}[H^1(F, \mathbf{T}^{\mathrm{sc}}) \to H^1(F, \mathbf{T})]$ is represented by

$$(u, 1/u\tau u, \tau u), \qquad u \in K^\times/N_{EL/K}(EL)^\times \simeq \mathbb{Z}/2.$$

Consider next an F-torus $\mathbf{T}$ in $\mathbf{G}$ which splits over a cubic extension M_1 of E, but not over E. The involution $\iota(x) = J{}^t\overline{x}J$ stabilizes $T = \mathbf{T}(F)$, and its centralizer $M_1^\times$ in $\mathrm{GL}(3,E)$. It induces on the field M_1 an automorphism, denoted σ, whose restriction to E generates $\mathrm{Gal}(E/F)$. Define M to be the subfield of M_1 whose elements are fixed by σ. It is a cubic extension of F, $M_1 = ME$, and M_1/F is Galois precisely when M/F is. If M' is a Galois closure of M_1/F, then there is τ in $\mathrm{Gal}(M'/F)$ with $\tau(x,y,z) = (z,x,y)$ (up to order). But $\mu - \tau\mu = (x, y, -x-y)$ if $\mu = (x, x+y, 0)$. Hence $C(\mathbf{T}/F)$ is $\{0\}$.

There are two possible actions of the Galois group of the Galois closure of M_1 over F. In both cases we may assume that $\tau^*(x,y,z) = (\tau z, \tau x, \tau y)$. If $\sigma^*(x,y,z) = (\sigma z^{-1}, \sigma y^{-1}, \sigma x^{-1})$ then $\tau\sigma = \sigma\tau^2$, the Galois group is S_3, and T^* consists of $(x, \tau x, \tau^2 x)$, $x \in M_1$ with $x\tau\sigma x = 1$.

If $\sigma^*(x,y,z) = (\sigma x^{-1}, \sigma y^{-1}, \sigma z^{-1})$ then $\tau\sigma = \sigma\tau$, the Galois group is $\mathbb{Z}/2$, and T^* consists of $(x, \tau x, \tau^2 x)$, $x \in M_1$ with $x\sigma x = 1$. □

Here is an explicit realization of the stable conjugacy classes which consist of several conjugacy classes. They are parametrized by the tori $T = (E^1)^3$ and $T = (EL/K)^1 \times E^1$. This is useful for example in computations of orbital integrals.

3. Proposition. *Let $\mathbf{T}^*$ be the diagonal torus. Put $r = \mathrm{diag}(\rho^{-1}, \rho, 1)$ with $\rho \in F - NE$, $T_0 = \mathbf{T}^*(E^1)$, thus*

$$T_0 = \{t_0 = \mathrm{diag}(a,b,c); a,b,c \in E^1\}, \quad h = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ -1 & & 1 \end{pmatrix},$$

$T_1 = h^{-1}T_0 h$ and $T_2 = (hr)^{-1}T_0 hr$. Then T_1 and T_2 are tori in $H \subset G$, $H = Z_G(\mathrm{diag}(1,-1,1))$. A complete set of representatives for the conjugacy classes within the stable conjugacy class of a regular $t_1 = h^{-1}\,\mathrm{diag}(a,b,c)h$ in T_1 (thus $a \neq b \neq c \neq a$), is given by t_i, $1 \leq i \leq 4$, where

$$t_1 = \begin{pmatrix} \frac{1}{2}(a+c) & & \frac{1}{2}(a-c) \\ & b & \\ \frac{1}{2}(a-c) & & \frac{1}{2}(a+c) \end{pmatrix}$$

and $t_2 = r^{-1}h^{-1}\,\mathrm{diag}(a,b,c)hr$. When there is $x \in E$ with $x\overline{x} = 2$, for example when E/F is unramified and $p \neq 2$, we can take

$$t_3 = r^{-1}h^{-1}\,\mathrm{diag}(a,c,b)hr$$

in H, and when there is $x \in E$ with $x\overline{x} = -2$, for example when E/F is unramified and $p \neq 2$, we can take $t_4 = r^{-1}h^{-1}\operatorname{diag}(b,a,c)hr$ in H.

Suppose that $E = F(\sqrt{D}) = (EL)^{\tau}$, $L = F(\sqrt{A}) = (EL)^{\sigma}$, $K = F(\sqrt{AD}) = (EL)^{\sigma\tau}$, are distinct quadratic extensions of F. We write $\mathrm{Gal}(EL/K) = \langle \tau, \sigma \rangle$. We may assume D, A lie in the set $\{u, \boldsymbol{\pi}, u\boldsymbol{\pi}\}$, where u is a nonsquare unit in F. A set of representatives for the conjugacy classes of tori $\simeq (LE/K)^1 \times E^1$ is given by

$$T_H = \left\{ \begin{pmatrix} \alpha & & A\beta/\sqrt{D} \\ & b & \\ \beta\sqrt{D} & & \alpha \end{pmatrix}; b \in E^1;\ \alpha, \beta \in E;\ (\alpha + \beta\sqrt{A})(\overline{\alpha} - \overline{\beta}\sqrt{A}) = 1 \right\}$$

$$= \left\{ h^{-1} \begin{pmatrix} a & & 0 \\ & b & \\ 0 & & \tau a \end{pmatrix} h;\ b \in E^1,\ a = \alpha + \beta\sqrt{A} \in (EL/K)^1 \right\},$$

where

$$h = \begin{pmatrix} 1 & & \sqrt{A/D} \\ & 1 & \\ \frac{-\sqrt{D/A}}{2} & & \frac{1}{2} \end{pmatrix} = \sigma(h).$$

$$= \left\{ d \begin{pmatrix} \alpha & & A\beta/\sqrt{D} \\ & b & \\ \beta\sqrt{D} & & \alpha \end{pmatrix}; b, d \in E^1;\ \alpha, \beta \in F;\ \alpha^2 - \beta^2 A = 1 \right\}$$

$$\subset H = Z_G(\operatorname{diag}(1,-1,1)) = \mathrm{U}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times E^1 \subset G = \mathrm{U}(J),$$

and

$$T_{H'} = \left\{ \begin{pmatrix} \alpha & A\beta & \\ \beta & \alpha & \\ & & b \end{pmatrix}; b \in E^1;\ \alpha, \beta \in E;\ (\alpha + \beta\sqrt{A})(\overline{\alpha} - \overline{\beta}\sqrt{A}) = 1 \right\}$$

$$= \left\{ d \begin{pmatrix} \alpha & A\beta & \\ \beta & \alpha & \\ & & b \end{pmatrix}; b, d \in E^1;\ \alpha, \beta \in F;\ \alpha^2 - \beta^2 A = 1 \right\}$$

$$\subset H' = Z_{G'}(\operatorname{diag}(1,1,-1)) = \mathrm{U}\begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \times E^1 \subset G' = \mathrm{U}(J'),$$

Here $J' = \begin{pmatrix} A & & 0 \\ & -1 & \\ 0 & & -A^{-1} \end{pmatrix}$. *Then* $J = gJ'{}^t\overline{g}$ *with* $g = \begin{pmatrix} \frac{1}{2A} & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & A \end{pmatrix}$, *so that* $G' = g^{-1}Gg$.

PROOF. An F-torus $\mathbf{T}$ within the stable conjugacy class defined by the cocycle $\{\sigma \mapsto (13)\}$ in $H^1(\mathrm{Gal}(E/F), W)$ takes the form $h^{-1}\mathbf{T}^*h$, with h in $\mathbf{G}(E) = \mathrm{GL}(3,E)$ such that $h_\sigma = h\sigma(h^{-1})$ is (13) in W. The h of the proposition satisfies $\sigma(h^{-1}) = h$, and $h^2 = \operatorname{diag}(2,-1,-2)J$.

A stably conjugate $t_2 = g_2^{-1} t_1 g_2 = (hg_2)^{-1} t_0 h g_2$ is defined by $g_2 \in \mathbf{G}(E)$ such that $g_{2\sigma} = g_2 \sigma(g_2)^{-1} = h^{-1} a_{2\sigma} h$. We take the elements of $C(\mathbf{T}_1/F)$ to be represented by $a_{1\sigma} = 1$, $a_{2\sigma} = \mathrm{diag}(\rho, \rho^{-2}, \rho)$, $a_{3\sigma} = \mathrm{diag}(\rho, \rho, \rho^{-2})$, $a_{4\sigma} = \mathrm{diag}(\rho^{-2}, \rho, \rho)$, $\rho \in F - NE$. In this case $h^{-1} a_{2\sigma} h = a_{2\sigma}$. Thus we need to solve $g_2 J {}^t\overline{g}_2 = a_{2\sigma} J$. Bar indicates componentwise action of σ. Clearly $g_2 = r$ is a solution.

The next stably conjugate element is $t_3 = g_3^{-1} t_1 g_3 = (hg_3)^{-1} t_0 h g_3$, where g_3 satisfies $g_{3\sigma} = g_3 \sigma(g_3^{-1}) = h^{-1} a_{3\sigma} h \in T_1$. Thus we need to solve

$$hg_3 J {}^t(h\overline{g}_3) = hg_3 \sigma(hg_3)^{-1} J = a_{3\sigma} h \sigma(h)^{-1} J = \mathrm{diag}(2\rho, -\rho, -2\rho^{-2}).$$

Define g_3 by $hg_3 = u\iota h g_2$, $\iota = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & \end{pmatrix}$, for which

$$hg_3 J {}^t(h\overline{g}_3) = u\iota\, \mathrm{diag}(2\rho, -\rho^{-2}, -2\rho) \iota {}^t\overline{u} = u\, \mathrm{diag}(2\rho, -2\rho, -\rho^{-2}) {}^t\overline{u}.$$

There is u for which this is $\mathrm{diag}(2\rho, -\rho, -2\rho^{-2})$. When E/F is unramified and $p \neq 2$, there is $x \in E$ with $x\overline{x} = 2$. We take $u = \mathrm{diag}(1, x^{-1}, x)$.

For the last case, replace the index 3 by 4, and note that a solution to $hg_4 J {}^t(h\overline{g}_4) = \mathrm{diag}(2\rho^{-2}, -\rho, -2\rho)$ is given by g_4 defined by

$$hg_4 = u \begin{pmatrix} & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} hg_2 \quad \text{with} \quad u \begin{pmatrix} -\rho^{-2} & & \\ & 2\rho & \\ & & -2\rho \end{pmatrix} {}^t\overline{u} = \begin{pmatrix} 2\rho^{-2} & & \\ & -\rho & \\ & & -2\rho \end{pmatrix}.$$

When E/F is unramified and $p \neq 2$, there is $y \in E$ with $y\overline{y} = -2$. We take $u = \mathrm{diag}(y, y^{-1}, 1)$.

To exhibit nonconjugate (in G) tori $\simeq (LE/K)^1 \times E^1$ in G, we construct one (T_H) in the quasi-split subgroup $H = \mathrm{U}(1,1) \times \mathrm{U}(1)$ of G, and another $(T_{H'})$ in the anisotropic subgroup $H' = \mathrm{U}(2) \times \mathrm{U}(1)$ of G. To simplify the notations, we omit the factor E^1 from the notations. To describe T_H, consider the torus

$$\tilde{T}_1 = \left\{ \begin{pmatrix} \alpha & \beta A \\ \beta & \alpha \end{pmatrix} = h_0^{-1} \begin{pmatrix} \alpha+\beta\sqrt{A} & 0 \\ 0 & \alpha-\beta\sqrt{A} \end{pmatrix} h_0 \right\}, \quad h_0 = \begin{pmatrix} 1 & \sqrt{A} \\ \frac{-1}{2\sqrt{A}} & \frac{1}{2} \end{pmatrix}$$

in $\mathrm{GL}(2,F)$. Here α, $\beta \in F$. Note that

$$E^\times \mathrm{GL}(2, E/F) = E^\times \mathrm{U}_2, \quad \mathrm{GL}(2, E/F) = \{x \in \mathrm{GL}(2,F); \det x \in NE^\times\}.$$

Here $\mathrm{U}_2 = \mathrm{U}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The centralizer of $\tilde{T}_1$ in $\mathrm{GL}(2, E)$ is

$$T_1 = \left\{ h_0^{-1} \begin{pmatrix} \alpha+\beta\sqrt{A} & 0 \\ 0 & \alpha-\beta\sqrt{A} \end{pmatrix} h_0 \in \mathrm{GL}(2, E) \right\},$$

thus α, $\beta \in E$. The corresponding torus in U_2 is $\mathrm{U}_2 \cap T_1$. But $H = \mathrm{U}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D_1^{-1}\,\mathrm{U}_2\,D_1$, where $D_1 = \mathrm{diag}(\sqrt{D}, 1)$. Put $h = D_1^{-1} h_0 D_1$. The corresponding torus in H is then

$$T_H = \{h^{-1}\,\mathrm{diag}(a, b, \tau a)h;\ b \in E^1,\ a = \alpha + \beta\sqrt{A} \in (EL/K)^1\}.$$

To describe $T_{H'}$ and H', note that up to F-isomorphism there is only one form of the unitary group in 3 variables associated with a quadratic extension E/F of p-adic fields. We then work with $G' = \mathrm{U}(J')$, which is $g^{-1}Gg$ as stated in the proposition. In this case the anisotropic H' is easily specified as the centralizer $Z_{G'}(\mathrm{diag}(1, 1, -1))$. Note that we could alternatively work with

$$H'' = gH'g^{-1} = Z_G \begin{pmatrix} 0 & & \frac{1}{2A} \\ & 1 & \\ 2A & & 0 \end{pmatrix}.$$

Now H' consists of $\mathrm{diag}(h, b)$, $b \in E^1$, and $h \in \mathrm{GL}(2, E)$ with $h \begin{pmatrix} A & \\ & -1 \end{pmatrix} {}^t\overline{h} = \begin{pmatrix} A & \\ & -1 \end{pmatrix}$. Clearly $\det h = u \in E^1$ ($= \overline{v}/v$ for some $v \in E^\times$). Solving the equation we see that $h = \begin{pmatrix} \alpha & u\overline{\beta}A \\ \beta & u\overline{\alpha} \end{pmatrix}$ with $\alpha\overline{\alpha} - A\beta\overline{\beta} = 1$, $u \in E^1$, or alternatively $h = v^{-1}\begin{pmatrix} a & \overline{c}A \\ c & \overline{a} \end{pmatrix}$ with $a\overline{a} - Ac\overline{c} = v\overline{v}$. Here given a, c, v, put $\alpha = a/v$, $\beta = c/v$, $u = \overline{v}/v$. Given α, β, u, for any v with $u = \overline{v}/v$ put $a = \alpha v$, $c = \beta v$.

A maximal torus splitting over EL, in H', is given by the centralizer in H' of $\mathrm{diag}(h, b)$, $h = \begin{pmatrix} x & yA \\ y & x \end{pmatrix}$, x, $y \in F$. The centralizer in $\mathrm{GL}(3, E)$ consists of $\mathrm{diag}(h, b)$, $h = \begin{pmatrix} x & yA \\ y & x \end{pmatrix}$, x, $y \in E$. Such h has the form $\begin{pmatrix} \alpha & u\overline{\beta}A \\ \beta & u\overline{\alpha} \end{pmatrix}$ with $\alpha\overline{\alpha} - A\beta\overline{\beta} = 1$, $u \in E^1$, precisely when $\alpha = u\overline{\alpha}$, $u\overline{\beta} = \beta$, thus $\alpha\overline{\beta} = \overline{\alpha}\beta$ and so $T_{H'}$ is as asserted.

Note that $\alpha + \beta\sqrt{A}$ lies in $(EL/K)^1$ iff $\alpha\overline{\alpha} - \beta\overline{\beta}A = 1$ and $\alpha\overline{\beta} = \overline{\alpha}\beta$. Any $v \in E^\times$ with $\alpha/\overline{\alpha} = \beta/\overline{\beta} = \overline{v}/v$ has $\alpha + \beta\sqrt{A} = \frac{1}{v}(a + c\sqrt{A})$ with $a = v\alpha$, $c = v\beta$ in F. Here $v \in E^\times$, $a + c\sqrt{A} \in L^\times$. As $N_{E/F}E^\times \cap N_{L/F}L^\times = F^{\times 2}$, there is $r \in F^\times$ with $v\overline{v} = r^2$. Replacing a, c, v by their quotients by r we may assume $v \in E^1$ and $a + b\sqrt{A} \in L^1$, as stated in the proposition. $\square$

REMARK. The Weyl group $W(T)$ of $T = T_1$ in G is S_3 when $p \neq 2$ and E/F is unramified. Indeed, $h^{-1}\begin{pmatrix} y & & 0 \\ & y^{-1} & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} h$ lies in G if $y\overline{y} = -2$. It represents the reflection (12).

All unitary groups $G(\mathbf{J}) = \{g \text{ in } \mathrm{GL}(3,E); g\mathbf{J}{}^t\overline{g} = \mathbf{J}\}$, where $\mathbf{J}$ is any form (symmetric matrix in $\mathrm{GL}(3,F)$), are isomorphic over F. We normally work with $\mathbf{J} = J$ since then the proper parabolic subgroup of $G = G(J)$ is the upper triangular subgroup. Suppose now that $\mathbf{J} = \mathrm{diag}(1,1,j)$, where j lies in $F^\times$, and put $G(j)$ for $G(\mathbf{J})$. Denote the diagonal subgroup of $G(j)$ by $T(j) \simeq (E^1)^3$. It is clear that: (a) If j lies in $NE^\times$ then $W(T(j)) = S_3$. (b) If j lies in $F - NE$ then $W(T(j))$ contains the transposition (12) and $W(T(j)) = \mathbb{Z}/2$.

The Weyl group $W(T^*)$ of T^* in G consists of 1 and (13) only.

1.4 *In the case* of $\mathbf{H} = \mathrm{U}(2)$, each torus $\mathbf{T}$ splits over a biquadratic extension of F, and $C(\mathbf{T}/F)$ is trivial, unless $\mathbf{T}$ splits over E and σ acts by $\sigma(x,y) = (-x,-y)$, where $C(\mathbf{T}/F)$ is $\mathbb{Z}/2$ in the local case.

1.5 We also need a *a twisted analogue of the above discussion.* Let $\mathbf{G}' = \mathrm{R}_{E/F}\,\mathbf{G}$ be the group obtained from $\mathbf{G} = \mathbf{U}(3,E/F)$ upon restricting scalars from E to F. It is defined over F. In fact, $\mathbf{G}'(\overline{F}) = \mathbf{G}(\overline{F}) \times \mathbf{G}(\overline{F})$, and $\mathrm{Gal}(\overline{F}/F)$ acts on $\mathbf{G}'(\overline{F})$ by $\tau(x,y) = (\tau x, \tau y)$ if $\tau|E = 1$, or by $\tau(x,y) = \iota(\tau x, \tau y)$ if $\tau|E \neq 1$. Here $\iota(x,y) = (y,x)$. Further we have $\mathbf{G}'(E) = \mathbf{G}(E) \times \mathbf{G}(E)$, and $G' = \mathbf{G}'(F)$ consists of all $(x, \sigma x)$, x in $\mathbf{G}(E) = \mathrm{GL}(3,E)$. The group G embeds in G' as the diagonal.

Denote by $Z_{\mathbf{G}'}(x\iota)$ the ι-centralizer of $x = (x', x'')$ in $\mathbf{G}'$. It consists of the $y = (y', y'')$ in $\mathbf{G}'$ with $(y',y'')(x',x'') = (x',x'')\iota(y',y'')$. These y satisfy $y'x'x'' = x'x''y'$, $y'' = x'^{-1}y'x'$. If $x = (x', \sigma(x'))$ lies in G', $\mathbf{T} = Z_{\mathbf{G}'}(x\iota)$ is defined over F, since ι is. The group T of F-rational points consists of such y with $y'' = \sigma y'$. The ι-centralizer $\mathbf{T}$ is isomorphic to the σ-centralizer of x' in $\mathbf{G}$.

The elements x and x^1 in G' are called (*stably*) *σ-conjugate* if there is y in G' (resp. $\mathbf{G}'(\overline{F})$) so that $yx = x^1\iota(y)$. In this case $\tau x = x$ for all τ in $\mathrm{Gal}(\overline{F}/F)$, and $\tau(y)x = x^1\iota(\tau y)$. Hence the σ-conjugacy classes within the stable σ-conjugacy class of x are parametrized by the elements $\{\tau \mapsto y_\tau = y^{-1}\tau(y)\}$ of the kernel $B''(\mathbf{T}/F)$ of the natural map from

$H^1(F, \mathbf{T})$ to $H^1(F, \mathbf{G}')$. Here $\mathbf{T}$ denotes the ι-centralizer of $x = (x', x'')$ in $\mathbf{G}'$.

The conjugacy class in $\mathbf{G}(\overline{F})$ of $x'x'' = x'\sigma(x')$ is defined over F. Hence it contains a member Nx of G by [Ko1]. The element Nx is determined only up to stable conjugacy. The group T is isomorphic to the centralizer of Nx in G, over F, by the map $(y', y'') \mapsto y'$. The pointed set $H^1(F, \mathbf{G}')$ is trivial. Hence $B''(\mathbf{T}/F) = H^1(F, \mathbf{T})$.

We introduce the notion of (stable) σ-conjugacy since we shall use below orbital integrals $\int \phi(gx\sigma(g)^{-1})dg/dt$ over $G'/Z_{G'}(x)$ of functions ϕ which transform under the center $Z' = E^\times$ of $G' = \mathrm{GL}(3, E)$ via a character $\omega'(z) = \omega(z/\overline{z})$ of $z \in E^\times$. In particular ϕ transforms trivially on $F^\times$. Hence the actual notion of stable σ-conjugacy that we need is $yx\iota(y)^{-1} = zx$, for z in $F^\times$, viewed as $(z, \sigma(z) = z^{-1})$ in G'.

The map $z \mapsto \{z_\tau = (z, 1)\tau(z, 1)^{-1}\}$ embeds $F^\times$ in $B''(\mathbf{T}/F)$. Here z_τ acts on x in $\mathbf{G}'$ by

$$(z, 1)x\iota(z, 1)^{-1} = zx\,(= (zx', \sigma(zx'))\,\mathrm{if}\, x = (x', \sigma x')).$$

Thus z maps the member $\{y_\tau = y^{-1}\tau(y)\}$ of $B''(\mathbf{T}/F)$ to $\{(zy)_\tau\}$, which sends x to

$$[(z, 1)y]x\iota[(z, 1)y]^{-1} = (z, z^{-1})yx\iota(y^{-1}).$$

The quotient of $B''(\mathbf{T}/F)$ under this action of $F^\times$ is denoted by $B'(\mathbf{T}/F)$. Put

$$B'(\mathbf{T}/\mathbb{A}) = \oplus_v B'(\mathbf{T}/F_v)$$

(pointed sum) if F is global.

The Tate-Nakayama theory implies that $B'(\mathbf{T}/F)$ (in the local case) or $B'(\mathbf{T}/\mathbb{A})/\mathrm{Image}\ B'(\mathbf{T}/F)$ (in the global case), is the quotient of the $\mathbb{Z}$-module of the μ in $X_*(\mathbf{T})$ modulo $\mathbf{Z}$ with $\sum_\tau \tau\mu = 0$ (τ in $\mathrm{Gal}(K/F)$), by the span of $\mu - \tau\mu$ for all μ in $X_*(\mathbf{T})$ and τ in $\mathrm{Gal}(K/F)$, where K is a Galois extension of F over which $\mathbf{T}$ splits.

The map $x \mapsto Nx$ gives a bijection from the set of stable σ-conjugacy classes in G' (parametrized by $B'(\mathbf{T}/F)$), to the set of stable conjugacy classes in G. In fact, for our present work it suffices to consider regular x in G (x with distinct eigenvalues), and σ-regular x in G' (Nx is regular). Hence there are four types of stable σ-conjugacy classes of σ-regular elements in G', denoted by (0), (1), (2), (3) as in the nontwisted case. Using the Tate-Nakayama theory we see (in the local case) that $B'(\mathbf{T}/F)$ is trivial if $\mathbf{T}$ is $\mathbf{T}^*$, and in case (3); it is $\mathbb{Z}/2$ in case (2); it is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ if $\mathbf{T}$ splits over E but $\mathbf{T}$ is not (stably) conjugate to $\mathbf{T}^*$.

To compute orbital integrals, we need explicit representatives.

4. LEMMA. *If $\mathbf{T}$ splits over E but is not $\mathbf{T}^*$,*

$$H^1(F, \mathbf{T})/F^\times = F^{\times 3}/F^\times NE^{\times 3}.$$

If $\mathbf{T}$ splits over a biquadratic extension LE of F, $\mathrm{Gal}(LE/F) = \langle \tau, \sigma \rangle$, $L = (LE)^\sigma$, $E = (LE)^\tau$, $K = (LE)^{\sigma\tau}$ are the quadratic extensions of F in EL, then $H^1(F, \mathbf{T})/F^\times$ is $K^\times/N_{LE/K}(LE)^\times$.

PROOF. If $\mathbf{T}$ splits over E but is not $\mathbf{T}^*$, a cocycle $t_\sigma = (a, b, c)$ in $H^1(E, \mathbf{T}(E))$ satisfies

$$1 = t_{\sigma^2} = t_\sigma \sigma^*(t_\sigma) = (a, b, c)(\sigma a^{-1}, \sigma b^{-1}, \sigma c^{-1}).$$

Thus (a, b, c) lies in $F^{\times 3}$. A coboundary has the form

$$t_\sigma \sigma^*(t_\sigma)^{-1} = (a, b, c)(\sigma a, \sigma b, \sigma c).$$

Hence we get $NE^{\times 3}$, and $H^1(F, \mathbf{T})/F^\times$ is $F^{\times 3}/F^\times NE^{\times 3}$, where $F^\times$ embeds diagonally.

If $\mathbf{T}$ splits over a biquadratic extension LE of F, the group

$$H^1(\mathrm{Gal}(LE/F), \mathbf{T}(LE))$$

is computed in the proof of Proposition 2. Then

$$H^1(\mathrm{Gal}(LE/F), \mathbf{T}(LE))/F^\times$$

is represented by

$$t_{\tau\sigma} = (u, 1, \tau u), \quad u \in K^\times/N_{LE/K}(LE)^\times,$$

which is $\mathbb{Z}/2\mathbb{Z}$. □

We also need an explicit realizations of the twisted stable conjugacy classes in the cases that they contain several twisted conjugacy classes, namely the cases corresponding to the tori $T = (E^1)^3$ and $T = (EL/K)^1 \times E^1$. This is useful in computations of twisted orbital integrals and twisted characters.

5. PROPOSITION. *A set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of x in* $\mathrm{GL}(3,E)$ *with norm in an anisotropic torus which splits over E, thus $Nx = h^{-1}\operatorname{diag}(a/\overline{a}, b/\overline{b}, c/\overline{c})h$ in a torus $T_1 = h^{-1}\mathbf{T}^*(E^1)h$, $h = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ -1 & & 1 \end{pmatrix}$, is given by*

$$x_1 = h^{-1}\operatorname{diag}(a,b,c)h, \quad x_2 = h^{-1}\operatorname{diag}(a,b\rho,c)h,$$

$$x_3 = h^{-1}\operatorname{diag}(a\rho,b,c)h, \quad x_4 = h^{-1}\operatorname{diag}(a,b,c\rho)h,$$

where a, b, c lie in $E^\times$, $\rho \in F - NE$.

A set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of x in $\mathrm{GL}(3,E)$ *with norm in a torus which splits over a biquadratic extension EL of F, where $L = F(\sqrt{A}) = (EL)^\sigma$, $E = (EL)^\tau = F(\sqrt{D})$, and $K = (EL)^{\sigma\tau} = F(\sqrt{DA})$ are the distinct quadratic extensions of F, with $\{A, D, AD\} = \{\boldsymbol{\pi}, u, u\boldsymbol{\pi}\}$ and a unit u in $R_E - R_E^2$, can be realized by*

$$t = h^{-1}\begin{pmatrix} (a+b\sqrt{A})\alpha & & \\ & c & \\ & & (a-b\sqrt{A})\tau(\alpha) \end{pmatrix} h, \qquad h = \begin{pmatrix} 1 & & \sqrt{A/D} \\ & 1 & \\ -\frac{\sqrt{D/A}}{2} & & \frac{1}{2} \end{pmatrix},$$

where a, b, $c \in E^\times$ and $\alpha \in K^\times/N_{EL/K}(EL)^\times$. Then

$$Nt = t\sigma(t) = h^{-1}\operatorname{diag}((a+b\sqrt{A})/(\overline{a}-\overline{b}\sqrt{A}), c/\overline{c}, (a-b\sqrt{A})/(\overline{a}+\overline{b}\sqrt{A}))h.$$

The norm map is surjective.

PROOF. First note that $x_1 = h^{-1}\operatorname{diag}(a,b,c)h$ satisfies

$$Nx_1 = x_1\sigma(x_1) = h^{-1}\operatorname{diag}(a,b,c)h \cdot \sigma(h^{-1})\operatorname{diag}(1/\overline{c}, 1/\overline{b}, 1/\overline{a})\sigma(h).$$

Since $\sigma(h^{-1}) = h$ and $h^2 = \operatorname{diag}(2,-1,-2)J$, this is

$$= h^{-1}\operatorname{diag}(a/\overline{a}, b/\overline{b}, c/\overline{c})\operatorname{diag}(2,-1,-2)^t h^{-1}J.$$

But $\operatorname{diag}(2,-1,-2)^t h^{-1}J = h$. In particular the norm N is onto the torus $T \simeq (E^1)^3$, which we realize as $T_1 = h^{-1}T_0 h$.

The stable ι-conjugates of x_1 are given by $y'x_1y''^{-1}$ where

$$y_\sigma = y'^{-1}\sigma(y'') \in H^1(F, \mathbf{T}_1)/F^\times, \quad \mathbf{T}_1 = h^{-1}\mathbf{T}^*h,$$

where $\mathbf{T}^*$ denotes the diagonal torus. A set of representatives for the stable ι-conjugates of x_1 up to ι-conjugacy is given as y_σ ranges over $h^{-1}th$, where t ranges over $\mathbf{T}^*(F)/\mathbf{Z}(F)N\mathbf{T}^*(E)$; $\mathbf{Z}$ is the diagonal. Choose $\rho \in F - NE$. Thus we may take t to be 1, $\operatorname{diag}(1,\rho,1)$, $\operatorname{diag}(\rho,1,1)$, $\operatorname{diag}(1,1,\rho)$. Taking y'' to be 1, we choose $y' = h^{-1}th$, to get x_i $(1 \le i \le 4)$ of the proposition.

In the case of the torus splitting over EL and isomorphic to $\ker N_{EL/K} \times E^1$, note that $\sigma(h) = h$, and that $\sigma^*(a,b,c) = (\sigma c^{-1}, \sigma b^{-1}, \sigma a^{-1})$. We write $\sigma a = \overline{a}$, and σ fixes $\sqrt{A}$. The σ-conjugacy classes within the stable σ-conjugacy class are parametrized in Lemma 4. □

I.2 Orbital integrals

To write the stable trace formula of $\mathbf{H}(\mathbb{A}) = \mathbf{U}(2,E/F)(\mathbf{A})$ as the unstable part of the stabilized trace formula for $\mathbf{G}(\mathbb{A}) = \mathbf{U}(3,E/F)(\mathbf{A})$, and the stable trace formula of $\mathbf{G}(\mathbb{A})$ as the stable part of the stabilized twisted trace formula for $\mathbf{G}'(\mathbb{A}) = \mathrm{GL}(3,\mathbb{A})$, we shall need to introduce a suitable combination $\Phi^\kappa(x, f dg)$ of orbital integrals of the test measure $f dg = \otimes f_v dg_v$ on $\mathbf{G}(\mathbb{A})$ and express it as the stable orbital integral $\Phi^{\mathrm{st}}(x, {}'f dh)$ of a test measure ${}'f dh = \otimes {}'f_v dh_v$ on $\mathbf{H}(\mathbb{A})$. Similar such definitions are to be made for our other groups.

To formulate the desired local relation, suppose that E/F is a quadratic extension of nonarchimedean local fields. Put $G = \mathbf{G}(F)$, $H = \mathbf{H}(F)$. Let ω be a character of $E^1 = \ker N$, where $N = N_{E/F}$ is the norm from E to F, and $\omega'(z) = \omega(z/\overline{z})$ a character of $E^\times$. Note that up to isomorphism the quasi-split unitary group $\mathbf{U}(2,E/F)$ is unique, so we take here its form $\mathbf{H}$ which is contained in $\mathbf{G}$ as $Z_{\mathbf{G}}(\operatorname{diag}(1,-1,1))$.

Let $C_c^\infty(G,\omega^{-1})$ denote the space of (complex valued) smooth (locally constant in the nonarchimedean case) functions f on G with $f(zg) = \omega(z)^{-1} f(g)$ $(z \in Z,\ g \in G)$ which are compactly supported modulo the center Z of G. Let dg be a Haar measure. Note that $C_c^\infty(G,\omega^{-1})$ is a convolution algebra. Similarly we have $C_c^\infty(H)$ and $C_c^\infty(G',\omega'^{-1})$. These are convolution algebras of functions ${}'f$ on H (compactly supported), and ϕ on G', once Haar measures dh and dg' are chosen.

For almost all places the component f (resp. ${}'f$, ϕ) of the global test function is the unit element f^0 (resp. ${}'f^0$, ϕ_0) in the convolution Hecke algebra $C_c(K\backslash G/K,\omega^{-1})$ (resp. $C_c(K_H\backslash H/K_H)$, $C_c(K'\backslash G'/K',\omega'^{-1})$) of spherical functions of G (resp. H, G'). Thus f^0 is supported on ZK, where $K = \mathbf{G}(R)$ is the maximal compact subgroup of G and Z is the center of G, and $f^0(zk) = \omega(z)^{-1}/|K|$ there, ϕ^0 is supported on $Z'K'$, where $K' = \mathbf{G}'(R)$ is the maximal compact subgroup of G' and Z' is the center of G', and $\phi^0(z'k') = \omega'(z')^{-1}/|K'|$ there, while ${}'f^0$ is the characteristic function of ${}'K = \mathbf{H}(R)$ divided by the volume $|{}'K|$. The volumes are measured using the Haar measures dg (and dh, dg').

For f in $C_c^\infty(G, \omega^{-1})$ and x in G define the orbital integral $\Phi(x, fdg)$ to be $\int f(gxg^{-1})\frac{dg}{dt}$ $(g \in G/T)$, where $T = Z_G(x)$ is the centralizer of x in G. It depends on a choice of Haar measures dg and dt on G and T. We shall be concerned only with regular elements x, those whose centralizer is a torus.

In comparing orbital integrals of measures (such as fdg) on different groups, the measures dt are taken to be compatible using the fact that the centralizers T on both sides are isomorphic. Note that we compare orbital integrals of measures, e.g., fdg and ${}'fdh$ and $\phi dg'$. It is not useful to note separate dependence on the function and on the Haar measure. However, a misleading standard convention, that we shall often follow too, is to omit the Haar measure dg etc. from the notations. In calculations it is sometimes convenient to choose dg which assigns the volume 1 to the maximal compact subgroup of G.

Let x be an element of the subgroup $\tilde{H} = \{(a_{ij});\ a_{ij} = 0 \text{ if } i+j \text{ is odd}\} \simeq H \times E^1$ of G. Its eigenvalues are a, $b = a_{22}$, c. We view H as the subgroup $H \times 1$ $(a_{22} = 1)$ of G. Then $\tilde{H} = HZ$. An element, conjugacy class, or stable conjugacy class in H defines one in G. But note that the 3 distinct stable conjugacy classes in $\tilde{H}$ with eigenvalues a, b, c in E^1 define the same stable conjugacy class in G.

As explained in Proposition I.1.3, there are two types of elliptic regular stable conjugacy classes in G with a representative in H. The type which splits over E has 4 conjugacy classes within the stable conjugacy class x, denoted in Proposition I.1.3 by $t_i \in T_i = Z_G(t_i)$, $1 \le i \le 4$. Write $\Phi(x, f, \kappa)$ or $\Phi_f^\kappa(x)$ for

$$\Phi^\kappa(x, fdg) = \Phi(t_1, fdg) + \Phi(t_2, fdg) - \Phi(t_3, fdg) - \Phi(t_4, fdg),$$

and

$$\Phi^{\mathrm{st}}(x, {}'fdh) = \Phi(t_1, {}'fdh) + \Phi(t_2, {}'fdh),$$

where x indicates the stable conjugacy class and t_i its representative in T_i. The other type of stable conjugacy class splits over a biquadratic extension EL of F. For such a class x, represented by $t \in H$, we put

$$\Phi^\kappa(x, fdg) = \Phi(t, fdg) - \Phi(t', fdg), \qquad \Phi^{\mathrm{st}}(x, {}'fdh) = \Phi(t, {}'fdh),$$

where t' denotes the conjugacy class in the stable conjugacy class of x, which is not in the conjugacy class of t.

Thus κ is the nontrivial character of the quotient $C(\mathbf{T}/F)/\operatorname{Im} C(\mathbf{T_H}/F)$, of the conjugacy classes within a stable conjugacy class in G, by the set of conjugacy classes within the corresponding stable conjugacy class in $\tilde{H} = Z_G(\mathrm{diag}(1, -1, 1))$. The combination $\Phi^\kappa(x, fdg)$ can then be described as the sum over the conjugacy classes t^δ, $\delta \in C(\mathbf{T}/F)$, in the stable conjugacy of t in G, of $\kappa(\delta)\Phi(t^\delta, fdg)$.

Fix a character κ of $E^\times$ which is trivial on $NE^\times$, but nontrivial on $F^\times$. Put $\boldsymbol{\kappa}(x) = \kappa(-(1 - a/b)(1 - c/b))$. If $x = \mathrm{diag}(a, b, c)$ then $c = \overline{a}^{-1}$, and $\boldsymbol{\kappa}(x) = \kappa(a/b)$. Put $\Delta(x) = |1 - \det(\mathrm{Ad}(x))|\mathrm{Lie}(G/Z_G(x))|^{1/2}$ and $\Delta'(x) = |1 - \det(\mathrm{Ad}(x))|\mathrm{Lie}(H/Z_H(x))|^{1/2}$, where $|\varepsilon|^2$ is $|N\varepsilon|$. Then $\Delta(x) = |(\varepsilon - 1)(\varepsilon' - 1)(\varepsilon - \varepsilon')|$ and $\Delta'(x) = |\varepsilon' - \varepsilon|$ if $\varepsilon = a/b$ and $\varepsilon' = c/b$.

In section I.3 we prove the key Fundamental Lemma for the endoscopic lifting e:

1. Proposition. *Suppose that E/F and κ are unramified. Then*

$$\boldsymbol{\kappa}(x)\Delta(x)\Phi(x, f^0dg, \kappa) = \Delta'(x)\Phi^{\mathrm{st}}(x, {}'f^0dh).$$

For the study of the local lifting we will need an approximation argument based on a generalization of the Fundamental Lemma to the context of an arbitrary spherical function. We give this generalization here as it explains the appearance of the lifting. So we fix an unramified quadratic extension E/F, and an unramified character κ of $E^\times/NE^\times$ which is nontrivial on $F^\times$. The Hecke convolution algebra $\mathbb{H}$ consists of K-biinvariant compactly supported functions, named spherical. The Satake isomorphism identifies $\mathbb{H}$ with the algebra $\mathbb{C}[\widehat{G}^0 \times \sigma]^W$ of W-invariant finite Laurent series on the conjugacy classes in the dual group ${}^LG$ (see [Bo2]) of G of the form $t' \times \sigma$, where t' lies in the connected component $\widehat{G} = \mathrm{GL}(3, \mathbb{C})$. The Satake transform $f \mapsto f^\vee$ is given by $f^\vee(t' \times \sigma) = \sum_{n \in \mathbb{Z}} F(x_n, fdg)t^n$, where $t' = \mathrm{diag}(t, 1, 1)$, $t \in \mathbb{C}^\times$ (see, e.g., [F3;II], p. 714).

The spherical function f is completely determined by the coefficients of $f^\vee$. These are the normalized orbital integrals

$$F(x_n, fdg) = \Delta(x_n)\Phi(x_n, fdg)$$

at the diagonal regular elements $x_n = (u\boldsymbol{\pi}^n, 1, \overline{u}^{-1}\boldsymbol{\pi}^{-n})$, where u is a unit, and $\boldsymbol{\pi}$ a uniformizer. This $F(x_n, fdg)$ is independent of u, and we denote it by $F(n, fdg)$.

Note that the dual group ${}^LG$ used here is the semidirect product $\widehat{G} \rtimes W_{E/F}$. The connected component of the identity is denoted by $\widehat{G}$, and $W_{E/F}$ is the Weil group of E/F, namely an extension of $\mathrm{Gal}(E/F)$ by $E^\times$. The nontrivial element σ of $\mathrm{Gal}(E/F)$ has σ^2 in $F - NE$; it acts on $\widehat{G}$ by $\sigma x = J^t x^{-1} J$.

Similarly, we have the Hecke algebra $'\mathbb{H}$ on H and dual group ${}^LH = \widehat{H} \rtimes W_{E/F}$, where σ acts on $\widehat{H} = \mathrm{GL}(2, \mathbb{C})$ by $\sigma x = w^t x^{-1} w^{-1}$. Here $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We write $F(n, 'fdh)$ for the value of $F(x, 'fdh) = \Delta'(x)\Phi(x, 'fdh)$ at $x = (u\boldsymbol{\pi}^n, \overline{u}^{-1}\boldsymbol{\pi}^{-n})$.

To relate f and $'f$ it suffices to relate $F(n, fdg)$ and $F(n, 'fdh)$. We need to observe that when $x = (\varepsilon, 1, \overline{\varepsilon}^{-1})$, we have $\kappa(x) = \kappa(\varepsilon)$. So we want $(-1)^n F(n, fdg) = F(n, 'fdh)$, and in fact use this as a definition of a map $\mathbb{H} \to\ '\mathbb{H}$, $f \mapsto\ 'f$. This map is dual to the *endo-lift* homomorphism $e^* : {}^LH \to {}^LG$, defined by

$$h = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e \right) \mapsto \begin{pmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{pmatrix} = h_1;$$

$$\sigma \mapsto (1, 1, -1) \times \sigma; \quad E^\times \ni z \mapsto (\kappa(z), 1, \kappa(z)) \times z.$$

A standard global argument, applied e.g. in [F2;I], shows that the Fundamental Lemma implies the Generalized Fundamental Lemma

2. Proposition. *For spherical functions f, f' related by the map e^* : $\mathbb{H} \to \mathbb{H}'$ we have*

$$F^\kappa(x, fdg) = F^{\mathrm{st}}(x, 'fdh).$$

Here $F^\kappa(x, fdg)$ is $\boldsymbol{\kappa}(x)\Delta(x)\Phi^\kappa(x, fdg)$, and

$$F^{\mathrm{st}}(x, 'fdh) = \Delta'(x)\Phi^{\mathrm{st}}(x, 'fdh).$$

A theorem of Waldspurger [W3] permits to deduce from the Fundamental Lemma the Matching Orbital Integrals Lemma:

3. Proposition. *For each smooth compactly supported measure fdg on G with f in $C_c^\infty(G,\omega^{-1})$ there exists a smooth compactly supported measure $'fdh$ on H with $'f$ in $C_c^\infty(H)$, and for each $'fdh$ there exists an fdg, so that $F^\kappa(x,fdg) = F^{\mathrm{st}}(x,{}'fdh)$.*

This statement is easy if $\Phi(x,fdg)$ is supported on the regular set. A direct proof can also be given, along the lines of the proof given in [F2;I]. We say that f, $'f$ are *matching* if $F^\kappa(x,fdg) = F^{\mathrm{st}}(x,{}'fdh)$ for all regular x.

The dual group ${}^LG'$ of G' is the semidirect product of the connected component $\widehat{G} = \mathrm{GL}(3,\mathbb{C}) \times \mathrm{GL}(3,\mathbb{C})$ with $W_{E/F}$. The group $W_{E/F}$ acts through its quotient $\mathrm{Gal}(E/F)$, by $\sigma(x,y) = (\theta y, \theta x)$. The diagonal map $b : {}^LG \to {}^LG'$, $x \mapsto (x,x)$, $w \mapsto (1,1)\times w$, indicates a dual map $b^* : \mathbb{H}' \to \mathbb{H}$ of Hecke algebras, called *basechange*.

For a smooth compactly supported modulo center function ϕ on G', in $C_c^\infty(G',\omega'^{-1})$, put

$$\Phi'(x\sigma,\phi dg') = \int_{G'/Z'_{G'}(x\sigma)} \phi(gx\sigma(g)^{-1})\frac{dg}{dt},$$

where

$$Z'_{G'}(x\sigma) = \{y \in G';\, yx\sigma(y)^{-1} = zx,\, z \in F^\times\}.$$

Since $\omega'(z) = \omega(z/\overline{z})$ is trivial on $(z \in)\ F^\times$, $\phi(zg) = \omega'(z)^{-1}\phi(g)$ $(z \in E^\times)$ implies $\phi(zg) = \phi(g)$ for all $z \in F^\times$.

By $\Phi'^{\mathrm{st}}(x\sigma,\phi dg')$ we mean the sum of $\Phi'(x'\sigma,\phi dg')$ over a set of representatives x' for the σ-conjugacy classes within the stable σ-conjugacy class of x. Then we have the (Generalized) Fundamental Lemma as well as the Matching Orbital Integrals Lemma for basechange:

4. Proposition. (1) *Suppose E/F is unramified, and ϕ maps to f under the map $b^* : \mathbb{H}' \to \mathbb{H}$. Then $\Phi'^{\mathrm{st}}(x\sigma,\phi dg') = \Phi^{\mathrm{st}}(Nx,fdg)$ for all σ-regular x in G'. In particular $\Phi'^{\mathrm{st}}(x\sigma,\phi^0 dg') = \Phi^{\mathrm{st}}(Nx,f^0 dg)$.*
(2) *For any quadratic extension E/F, for every ϕ there exists a matching f, and for every f there exists a matching ϕ.*

We say that ϕ, f are *matching* if $\Phi'^{\mathrm{st}}(x\sigma,\phi dg') = \Phi^{\mathrm{st}}(Nx,fdg)$ for all σ-regular x in G'. The general case of (1) again follows as in [F2;I] from the case of the unit elements in the Hecke algebras.

The matching statement (2) follows from (1) by [W3]. A direct proof can perhaps be given too, as in [F2;I]. At a split place v, if $\phi_v = (f'_v, f''_v)$ then $f_v = f'_v * f''_v$.

The case of $(\phi, f) = (\phi^0, f^0)$ is due to Kottwitz (see [Ko4]), except that [Ko4] considers the characteristic function ϕ' of K' in G' instead of our ϕ^0 which is the characteristic functions of $K'Z'$. Note that the center Z of G is contained in K. For $\phi' \in C_c(G')$ the orbital integral is defined by

$$\Phi(x\sigma, \phi' dg) = \int_{G'/Z_{G'}(x\sigma)} \phi'(gx\sigma(g)^{-1}) dg$$

where $Z_{G'}(x\sigma) = \{g \in G'; gx\sigma(g)^{-1} = x\}$ and we write dg for dg' here. In the integral write $g = zg_1$ with $z \in E^\times/E^1$ and $g_1 \in G'/Z'Z_{G'}(x\sigma)$ to get

$$\int_{G'/Z'Z_{G'}(x\sigma)} \int_{NE^\times} \phi'(zgx\sigma(g)^{-1}) dz dg.$$

Now $gx\sigma(g)^{-1} = u\boldsymbol{\pi}^{\mathrm{odd}}x$ $(u \in R_E^\times)$ implies $\boldsymbol{\pi}^{3\,\mathrm{odd}} = N(\det g) = \boldsymbol{\pi}^{\mathrm{even}}$ up to units, a contradiction. Hence in the last integral we may replace $G'/Z'Z_{G'}(x\sigma)$ by $G'/Z'_{G'}(x\sigma)$. In fact the integral over $NE^\times$ can be replaced by an integral over $F^\times$, and even $E^\times = \boldsymbol{\pi}^{\mathbb{Z}} R_E$, since $\phi'(g) \neq 0$ if and only if $\phi'(\boldsymbol{\pi} g) = 0$. Since $\phi^0(g) = \int_{E^\times} \phi'(zg) dz$, we conclude that $\Phi'(x\sigma, \phi^0) = \Phi(x\sigma, \phi' dg)$.

For the local study of unstable σ-invariant local G'-modules, and for complete study of the automorphic $\mathbf{G}(\mathbf{A})$-modules, we need the Hecke algebras σ-endo-lift map $e'^* : \mathbb{H}' \to \mathbb{H}$, $\phi \mapsto {}'\phi$, dual to the dual groups σ-endo-lift homomorphism $e' : {}^L H \to {}^L G'$ by $h \mapsto (h_1, h_1)$ and $\sigma \mapsto [(1,1,-1),(-1,1,1)] \times \sigma$. We denote smooth compactly supported functions on H by ${}'\phi$. Recall that a character κ on $E^\times/NE^\times$ was fixed, as well as factors $\kappa(x)$ on the regular elliptic elements of G of types (1) and (2), and $\Delta(x)$ on G and $\Delta'(x)$ on H.

To state the Fundamental Lemma put

$$F'^{\kappa}(x\sigma, \phi dg') = \boldsymbol{\kappa}(Nx)\Delta(Nx)\Phi'^{\kappa}(x\sigma, \phi dg'),$$

where $\Phi'^{\kappa}(x\sigma, \phi dg')$ is $\sum_x \kappa(x)\Phi'(x\sigma, \phi dg')$, a sum over a set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of

x. Here κ indicates a character on the group $H^1(F,\mathbf{T})/F^\times$ parametrizing the σ-conjugacy classes within the stable σ-conjugacy class of x. Thus $\Phi'^\kappa(x\sigma,\phi dg')$ is $\sum_i \iota(i)\Phi'(x_i\sigma,\phi dg')$ if x is of type (1) and $\iota(i)$ is 1 if $i=1,2$ and -1 if $i=3,4$, and it is $\sum_\alpha \kappa(\alpha)\Phi'(t_\alpha\sigma,\phi dg')$ if x is of type (2) and κ denotes the nontrivial character of $\alpha\in K^\times/N_{LE/K}(LE)^\times$, see Proposition I.1.5.

We say that ϕ and $'\phi$ are *matching* if $F'^\kappa(x\sigma,\phi dg') = F^{\mathrm{st}}(Nx,{}'\phi dh)$ for all σ-regular x in G'. Then we have the (Generalized) Fundamental Lemma as well as the Matching Orbital Integrals Lemma for the σ-endo-lift e':

5. Proposition. (1) *If E/F and κ are unramified then*

$$F'^\kappa(x\sigma,\phi^0 dg') = F^{\mathrm{st}}(Nx,{}'\phi^0 dh)$$

for all σ-regular x in G'. If ϕ maps to $'\phi$ under $\mathbb{H}'\to{}'\mathbb{H}$, then ϕ and $'\phi$ are matching.
(2) *For any quadratic extension E/F and every ϕ there is a matching ϕ', and for every ϕ' there is a matching ϕ.*

As usual, (2) follows here from (1), and the Generalized Fundamental Lemma follows from the Fundamental Lemma. As for the Fundamental Lemma itself we have:

6. Proposition. *The Fundamental Lemma for the endo-lift e* (*of Proposition* 1) *is equivalent to the Fundamental Lemma for the σ-endo-lift e'* (*of Proposition* 5)*.*

Proof. The result of [Ko4] applies with any character κ of the group $\mathrm{Im}[H^1(F,\mathbf{T}^{\mathrm{sc}})\to H^1(F,\mathbf{T})]\simeq H^1(F,\mathbf{T})/F^\times$ (thus

$$\{(a,b,c)\in(F^\times/NE^\times)^3; abc=1\}\simeq (F^\times/NE^\times)^3/F^\times \text{ or } (\mathbb{Z}/2)^2,$$

if x is elliptic of type (1), or $\mathbb{Z}/2\mathbb{Z}$ if x is elliptic of type (2), trivial otherwise) which parametrizes the σ-conjugacy classes within the stable σ-conjugacy class of a σ-regular element x and the conjugacy classes within the stable conjugacy class of Nx. Thus [Ko4] implies that

$$\Phi'^\kappa(x\sigma,\phi^0 dg') = \Phi^\kappa(Nx,f^0 dg)$$

for all σ-regular x. Note that $'f^0$ is $'\phi^0$.

By Proposition 1, the right side multiplied by $\boldsymbol{\kappa}(Nx)\Delta(Nx)$ is $\Delta'(Nx)\Phi^{\mathrm{st}}(Nx,{}'f^0dh)$. This implies Proposition 5 (1).

On the other hand, Proposition 5 (1) asserts that the left side multiplied by the same factor is $\Delta'(Nx)\Phi^{\mathrm{st}}(Nx,{}'\phi^0 dh)$. Proposition 1 follows. □

I.3 Fundamental lemma

A. Introduction

Let E/F be an unramified quadratic extension of p-adic fields, $p > 2$, $\mathbf{G} = \mathbf{U}(2,1; E/F)$ a quasi-split unitary group in 3 variables associated with E/F, and $\mathbf{H} = \mathbf{U}(1,1) \times \mathbf{U}(1)$ a subgroup of $\mathbf{G}$, where $\mathbf{U}(1,1) = \mathbf{U}(1,1; E/F)$ is a quasi-split unitary group in 2 variables and $\mathbf{U}(1) = \mathbf{U}(1; E/F)$ is an anisotropic torus. Let $\mathbf{T}$ be an anisotropic F-torus in $\mathbf{H}$ (and $\mathbf{G}$) which splits over E. Then $\mathbf{T} = \mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1)$. Put $T = \mathbf{T}(F)$, $H = \mathbf{H}(F)$, $G = \mathbf{G}(F)$ for the group of F-points of the F-groups $\mathbf{T}$, $\mathbf{H}$, $\mathbf{G}$. Denote the group of F-points of $\mathbf{U}(1)$ by $E^1 = \{x \in E^\times; Nx = 1\}$, $N = N_{E/F}$ signifies the norm map from E to F. Let K be the hyperspecial maximal compact subgroup $\mathbf{G}(R)$ of G, where R is the ring of integers in F, and 1_K the unit element in the Hecke convolution algebra of K-biinvariant compactly supported functions on G, divided by the volume of K. A choice of a Haar measure on G is implicit.

Let $\kappa \neq 1$ be a suitable character on the group $\mathbb{Z}/2 \times \mathbb{Z}/2$ of conjugacy classes within the stable conjugacy class of a regular ($a \neq b \neq c \neq a$) element $t = (a, b, c)$ in $T = (E^1)^3$. Then the κ-orbital integral $\Phi^\kappa_{1_K}(t)$ is defined to be the sum — weighted by the values of κ — of the orbital integrals of 1_K over the conjugacy classes within the stable conjugacy class of t.

Analogously one has the standard maximal compact subgroup K_H in H, the measure 1_{K_H}, and the stable orbital integral $\Phi^{\mathrm{st}}_{1_{K_H}}(t)$ on H, where "st" (for "stable") indicates $\kappa = 1$.

The "endoscopic fundamental lemma" asserts that $\Delta_{G/H}(t)\Phi^\kappa_{1_K}(t) = \Phi^{\mathrm{st}}_{1_{K_H}}(t)$. In our case the transfer factor $\Delta_{G/H}(t)$ (defined by Langlands [L6], p. 51, and in general by Langlands and Shelstad [LS]) is $(-q)^{-N_1-N_2}$. Here $q = \#(R/\boldsymbol{\pi} R)$ is the residual cardinality of F (R : ring of integers in F, $\boldsymbol{\pi}$: generator of the maximal ideal in R), and $a-b \in \boldsymbol{\pi}^{N_1} R_E^\times$, $c-b \in \boldsymbol{\pi}^{N_2} R_E^\times$, define the nonnegative integers N_1, N_2 (R_E: ring of integers in E).

The other "endoscopic fundamental lemma" concerns the anisotropic F-torus $\mathbf{T}_L$ in $\mathbf{H}$ and $\mathbf{G}$ whose splitting field is a biquadratic extension EL of F. Thus L is a ramified quadratic extension of F. Then $T_L \simeq (EL/K)^1 \times E^1$ consists of scalar multiples (in E^1) of $t = (t_1, 1)$, and t is regular if t_1 ($\in (EL/K)^1 = \{x \in (EL)^\times; N_{EL/K} x = 1\}$, where $N_{EL/K}$ signifies the norm from EL to the quadratic extension K other than E and L of F) does not lie in E^1. Define n by $t_1 - 1 \in \boldsymbol{\pi}_{EL}^n R_{EL}^\times$. The transfer factor $\Delta_{G/H}(t)$ is $(-q)^{-n}$. Once again the "lemma" asserts $\Delta_{G/H}(t)\Phi_{1_K}^\kappa(t) = \Phi_{1_{K_H}}^{\mathrm{st}}(t)$ for a regular t. In this section H' and G' do not indicate $\mathrm{R}_{E/F}\, H$ and $\mathrm{R}_{E/F}\, G$.

B. Explicit realization

To compute the integrals which occur in the fundamental lemma, we need explicit realizations of the tori $T = (E^1)^3$ and $T = (EL/K)^1 \times E^1$. We repeat Proposition I.1.3 here, when E/F is unramified. Then $E = F(\sqrt{D})$, $D \in R - R^2$, A of I.1.3 is $\boldsymbol{\pi}$, $L = F(\sqrt{\boldsymbol{\pi}})$, $K = F(\sqrt{D\boldsymbol{\pi}})$.

1. Proposition. *Put* $r = \mathrm{diag}(\rho^{-1}, \rho, 1)$ *with* $\rho \in F - NE$,

$$T_0 = \{t_0 = \mathrm{diag}(a,b,c); a,b,c \in E^1\}, \quad h = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ -1 & & 1 \end{pmatrix},$$

$T_1 = h^{-1}T_0h$ *and* $T_2 = (hr)^{-1}T_0hr$. *Then* T_1 *and* T_2 *are tori in* G. *A complete set of representatives for the conjugacy classes within the stable conjugacy class of a regular* $t_1 = h^{-1}\,\mathrm{diag}(a,b,c)h$ *in* T_1 *(thus* $a \neq b \neq c \neq a$*), is given by* t_1, $t_2 = r^{-1}h^{-1}\,\mathrm{diag}(a,b,c)hr$,

$$t_3 = r^{-1}h^{-1}\,\mathrm{diag}(a,c,b)hr, \quad t_4 = r^{-1}h^{-1}\,\mathrm{diag}(b,a,c)hr.$$

A set of representatives for the conjugacy classes of tori $\simeq (LE/K)^1 \times E^1$ *is given by*

$$T_H = \left\{ d\begin{pmatrix} \alpha & & \boldsymbol{\pi}\beta/\sqrt{D} \\ & b & \\ \beta\sqrt{D} & & \alpha \end{pmatrix}; b,d \in E^1,\ \alpha,\beta \in F;\ \alpha^2 - \beta^2\boldsymbol{\pi} = 1 \right\}$$

$$\subset H = Z_G(\mathrm{diag}(1,-1,1)) = \mathrm{U}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times E^1 \subset G = \mathrm{U}(J),$$

and

$$T_{H'} = \left\{ d\begin{pmatrix} \alpha & \boldsymbol{\pi}\beta & \\ \beta & \alpha & \\ & & b \end{pmatrix}; b,d \in E^1,\ \alpha,\beta \in F;\ \alpha^2 - \beta^2\boldsymbol{\pi} = 1 \right\}$$

$$\subset H' = Z_{G'}(\mathrm{diag}(1,1,-1)) = \mathrm{U}\begin{pmatrix} \boldsymbol{\pi} & 0 \\ 0 & -1 \end{pmatrix} \times E^1 \subset G' = \mathrm{U}(J').$$

Here $J' = \begin{pmatrix} \boldsymbol{\pi} & & \\ & -1 & \\ & & -\boldsymbol{\pi}^{-1} \end{pmatrix}$ *has* $J = gJ'{}^t\overline{g}$ *with* $g = \begin{pmatrix} 1/2\boldsymbol{\pi} & 0 & -1/2 \\ 0 & 1 & 0 \\ 1 & 0 & \boldsymbol{\pi} \end{pmatrix}$, *so that* $G' = g^{-1}Gg$.

C. Decompositions

Let K be the maximal compact subgroup $\mathbf{G}(R)$ of $\mathbf{G}$ (its entries are in the ring R_E of integers of E). Denote by 1_K the characteristic function of K in G. Fix the Haar measure on G which assigns K the volume 1. Our aim is to compute the orbital integrals

$$\int_{T_\rho\backslash G} 1_K(x^{-1}t_\rho x)dx\,, \qquad t_\rho = \begin{pmatrix} \frac{a+c}{2} & & \frac{a-c}{2}\rho \\ & b & \\ \frac{a-c}{2\rho} & & \frac{a+c}{2} \end{pmatrix},$$

where ρ is 1 or $\boldsymbol{\pi}$. Here $T_\rho = T_1$ if $\rho = 1$ and $T_\rho = T_2$ if $\rho = \boldsymbol{\pi}$. We shall also compute the integrals $\int_{T_H\backslash G} 1_K(x^{-1}t_1x)dx$ and $\int_{T_{H'}\backslash G} 1_K(x^{-1}t_2x)dx$. The measure on each compact torus is chosen to assign it the volume 1. We define $\overline{\rho}$ by $\rho = \boldsymbol{\pi}^{\overline{\rho}}$ ($\overline{\rho} = 0$ or 1). Put H for the centralizer of $\operatorname{diag}(1,-1,1)$ in G. It contains T_ρ and T_H. Let N denote the unipotent upper triangular subgroup of G. It contains

$$u_0' = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad u_0 = \begin{pmatrix} 1 & x & 1 \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x & & 0 \\ & 1 & \\ 0 & & \overline{x}^{-1} \end{pmatrix} u_0' \begin{pmatrix} x & & 0 \\ & 1 & \\ 0 & & \overline{x}^{-1} \end{pmatrix}^{-1}$$

($x\overline{x} = 2$). As in I.1.3, put $H'' = gH'g^{-1} = Z_G\begin{pmatrix} 0 & & \frac{1}{2\boldsymbol{\pi}} \\ 0 & 1 & 0 \\ 2\boldsymbol{\pi} & & 0 \end{pmatrix}$. Our computation of the orbital integral is based on the following decomposition.

2. Proposition. *We have* $G = \bigcup_{m\geq 0} Hu_mK$, *where* $u_m = u_0d_m$, $d_m = \operatorname{diag}(t,1,t^{-1})$, $t = \boldsymbol{\pi}^m$. *Further,* $H_m^K = H \cap u_mKu_m^{-1}$ *consists of*

$$\begin{pmatrix} a_1-b+ta_2 & 0 & b-ta_2+tb_3+2a_3t^2 \\ 0 & a_1 & 0 \\ b & 0 & a_1-b-tb_3 \end{pmatrix} \in H$$

with a_1, a_2, a_3, b, b_3 *in* R_E.

Also $G = \cup_{m\geq 0}H''d_mK$, *and* $H_m' = H' \cap g^{-1}d_mKd_m^{-1}g$ *consists of* $\operatorname{diag}(u^{-1}\left(\begin{smallmatrix} a & c\boldsymbol{\pi} \\ \overline{c} & \overline{a} \end{smallmatrix}\right), e)$, $e \in E^1$, $u \in E^\times$, $a, c \in E$ *with* $a\overline{a} - \boldsymbol{\pi}c\overline{c} = u\overline{u}$ *and* $|a/u - e| \leq |\boldsymbol{\pi}|^{1+2m}$, $|c/u| \leq |\boldsymbol{\pi}|^m$, *or equivalently of scalar multiples by* E^1 *of* $\operatorname{diag}(e\left(\begin{smallmatrix} a & uc\boldsymbol{\pi} \\ \overline{c} & u\overline{a} \end{smallmatrix}\right), 1)$, $e, u \in E^1$, $a, c \in R_E$ *with* $1 = a\overline{a} - \boldsymbol{\pi}c\overline{c}$, $|a-1| \leq |\boldsymbol{\pi}|^{1+2m}$, $|c| \leq |\boldsymbol{\pi}|^m$. *Both decompositions are disjoint.*

Proof. For the decomposition:

$$G = T^*NK = HNK = \bigcup_{m\geq 0}\bigcup_{\varepsilon\in R_E^\times} H\begin{pmatrix} 1 & \varepsilon t^{-1} & \frac{1}{2}\varepsilon\overline{\varepsilon}t^{-2} \\ 0 & 1 & \overline{\varepsilon}t^{-1} \\ 0 & 0 & 1 \end{pmatrix}K$$

$$= \bigcup_{m,\varepsilon} H\begin{pmatrix} \varepsilon t^{-1} & & 0 \\ & 1 & \\ 0 & & \overline{\varepsilon}^{-1}t \end{pmatrix} u_0' \begin{pmatrix} \varepsilon^{-1} t & & 0 \\ & 1 & \\ 0 & & \overline{\varepsilon}t^{-1} \end{pmatrix} K = \bigcup_{m\geq 0} Hu_m'K,$$

$u_m' = u_0' d_m$. It is disjoint since (by matrix multiplication) ${u_m'}^{-1} h u_m'$ lies in K for some h in H only if $n = m$.

The intersection ${H'}_m^K = H \cap u_m' K {u_m'}^{-1}$ consists of (a_i, b_i, c_i in R_E):

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t & & 0 \\ & 1 & \\ 0 & & t^{-1} \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} t^{-1} & & 0 \\ & 1 & \\ 0 & & t \end{pmatrix}\begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & ta_2 & t^2 a_3 \\ t^{-1}b_1 & b_2 & tb_3 \\ t^{-2}c_1 & t^{-1}c_2 & c_3 \end{pmatrix}\begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

in H, thus $c_1 = -tb_1$ and $c_1 = tc_2$, and we define $b \in E$ by $b_1 = -2bt$. Thus $c_1 = 2bt^2$, $c_2 = 2bt$, and we continue with

$$= \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & ta_2 & t^2 a_3 \\ -2b & b_2 & tb_3 \\ 2b & 2b & c_3 \end{pmatrix}\begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & ta_2 - a_1 & \frac{1}{2}a_1 - ta_2 + t^2 a_3 \\ -2b & b_2 + 2b & -b - b_2 + tb_3 \\ 2b & 0 & c_3 - b \end{pmatrix}$$

$$= \begin{pmatrix} a_1 - b & X & \frac{1}{2}b - \frac{1}{2}ta_2 + \frac{1}{2}tb_3 + t^2 a_3 \\ 0 & a_1 - ta_2 & Y \\ 2b & 0 & a_1 - b - ta_2 - tb_3 \end{pmatrix}$$

$$= \begin{pmatrix} x & & 0 \\ & 1 & \\ 0 & & \overline{x}^{-1} \end{pmatrix}^{-1}\begin{pmatrix} a_1 - b & 0 & b - ta_2 + tb_3 + 2a_3 t^2 \\ 0 & a_1 - ta_2 & 0 \\ b & 0 & a_1 - b - ta_2 - tb_3 \end{pmatrix}\begin{pmatrix} x & & 0 \\ & 1 & \\ 0 & & \overline{x}^{-1} \end{pmatrix}.$$

Since this has to be in H, we obtained the relation $X = 0$, thus $a_1 - ta_2 = b_2 + 2b$, which implies that $b \in R_E$, and $Y = 0$, thus $c_3 - b = b + b_2 - tb_3 = a_1 - b - ta_2 - tb_3$. Replacing a_1 by $a_1 + ta_2$, and noting that $H_m^K = \operatorname{diag}(x, 1, \overline{x}^{-1}){H'}_m^K \operatorname{diag}(x, 1, \overline{x}^{-1})^{-1}$, the first part of the proposition follows.

Recall that $G' = g^{-1}Gg$, and note that if $v_0' = (0, 0, 1)$ then

$$\operatorname{Stab}_{G'}(v_0') = \{x' \in G'; v_0' x' = \lambda v_0',\ \lambda \in E^1\}$$

is $H' = Z_{G'}(\mathrm{diag}(1,1,-1))$. Put $v_0 = v_0' g^{-1} = (-1, 0, 1/2\boldsymbol{\pi})$. Then

$$\mathrm{Stab}_G(v_0) = \{x \in G; v_0 x = \lambda v_0,\ \lambda \in E^1\}$$

is

$$H'' = gH'g^{-1} = Z_G\begin{pmatrix} 0 & & 1/2\boldsymbol{\pi} \\ & 1 & \\ 2\boldsymbol{\pi} & & 0 \end{pmatrix}.$$

Embed

$$H''\backslash G \hookrightarrow S = \{v \in E^3; vJ^t\overline{v} = v_0 J^t \overline{v}_0 = -\boldsymbol{\pi}^{-1}\}$$

by $x \mapsto v = v_0 x$. We have a disjoint decomposition $S = \cup_{m \geq 0} v_0 d_m K$, as $v_0 d_m = (-\boldsymbol{\pi}^m, 0, 1/2\boldsymbol{\pi}^{m+1})$, and $v_0 d_m K = \{v \in S; |v| = |\boldsymbol{\pi}|^{-m-1}\}$. Here

$$|(x,y,z)| = \max\{|x|,|y|,|z|\},$$

and the union ranges only over $m \geq 0$ since $\{m, -m-1\} = \{n, -n-1\}$ if $n + m = -1$. The decomposition $G = \cup_{m \geq 0} H'' d_m K$ follows.

To describe H_m', consider the elements of $d_m^{-1} g H' g^{-1} d_m$ in K. Thus

$$\begin{pmatrix} 1/t & & 0 \\ & 1 & \\ 0 & & t \end{pmatrix}\begin{pmatrix} 1/2\boldsymbol{\pi} & & -1/2 \\ & 1 & \\ 1 & & \boldsymbol{\pi} \end{pmatrix}\begin{pmatrix} a/u & c\boldsymbol{\pi}/u & 0 \\ \overline{c}/u & \overline{a}/u & 0 \\ 0 & 0 & e \end{pmatrix}\begin{pmatrix} \boldsymbol{\pi} & & 1/2 \\ & 1 & \\ -1 & & 1/2\boldsymbol{\pi} \end{pmatrix}\begin{pmatrix} t & & 0 \\ & 1 & \\ 0 & & 1/t \end{pmatrix}$$

$$= \begin{pmatrix} (a/u+e)/2 & c/2ut & (a/u-e)/4\boldsymbol{\pi}t^2 \\ \boldsymbol{\pi}t\overline{c}/u & \overline{a}/u & \overline{c}/2ut \\ (a/u-e)\boldsymbol{\pi}t^2 & \boldsymbol{\pi}tc/u & (a/u+e)/2 \end{pmatrix}$$

lies in K precisely when $|c/u| \leq |\boldsymbol{\pi}|^m$, $|a/u - e| \leq |\boldsymbol{\pi}|^{1+2m}$. □

Note that the integrals $\int_{G/K} dx$ and $\int_{H/K^H} dg$ are independent of the choice of the Haar measures dx on G and dh on H. Also, $\int_{H/K_1^H} dh$ equals $[K^H : K_1^H] \int_{H/K^H} dh$ for a compact open subgroup K_1^H of K^H. It is convenient to normalize the measures dx and dh to assign K and K^H the volume one. Then $[K^H : K_1^H] = |K_1^H|^{-1}$.

3. Proposition. *The orbital integral of* 1_K *at a regular* $t \in T \subset H$ ($T = T_\rho$ *or* T_H) *can be expressed as*

$$\int_{G/K} 1_K(x^{-1}tx)dx = \sum_{m \geq 0} \int_{H/H_m^K} 1_K(u_m^{-1}h^{-1}thu_m)dh$$

$$= \sum_{m \geq 0} \int_{H/H_m^K} 1_{H_m^K}(h^{-1}th)dh\,.$$

At a regular $t = gt'g^{-1} \in G$, where $t' \in T_{H'} \subset H' \subset G' = g^{-1}Gg$, we have

$$\int_{G/K} 1_K(x^{-1}tx)dx = \sum_{m\geq 0}\int_{H'/H'_m} 1_{H'_m}(h^{-1}t'h)dh.$$

PROOF. For the last equality of the first assertion, note that $u_m^{-1}h^{-1}thu_m \in K$ implies that $h^{-1}th \in H \cap u_m K u_m^{-1} = H_m^K$.

For the last claim, the left side equals

$$\sum_{m\geq 0}\int_{H''/H''\cap d_m K d_m^{-1}} 1_K(d_m^{-1}h^{-1}thd_m)dh$$

$$= \sum_{m\geq 0}\int_{H'/H'\cap g^{-1}d_m K d_m^{-1} g} 1_K(d_m^{-1}gh'^{-1}t'h'g^{-1}d_m)dh;$$

the displayed equality follows on writing $h = gh'g^{-1}$ and $t' = g^{-1}tg$. The right side is equal to the right side of the equality of the proposition. $\square$

We then need a decomposition for $T_\rho\backslash H/K \cap H$ and $T_H\backslash H/K \cap H$.

Note that $H = \mathrm{U}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \times E^1$. The first factor is the unitary group in two variables which consists of the g in $\mathrm{GL}(2,E)$ with $g\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}{}^t\overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$. Correspondingly we write $T_\rho = T_{H\rho} \times E^1$ and $K \cap H = K_H \times E^1$. Put $r_j^\rho = \mathrm{diag}(\boldsymbol{\pi}^{-(j-\overline{\rho})/2}, \boldsymbol{\pi}^{(j-\overline{\rho})/2})$ for $j \geq 0$, $j \equiv \overline{\rho} \pmod 2$. In the following statement the factors E^1 and $R^\times$ — whose volume is 1 — can be ignored for our purposes. Write $[x]$ for the largest integer $\leq x$.

4. PROPOSITION. *We have $H = \bigcup_{j\geq 0} T_{H\rho}\cdot r_j^\rho \cdot K_H \times E^1$ $(j \equiv \overline{\rho}(2),\ j \geq 0)$, and*

$$(r_j^\rho)^{-1}T_{H\rho}r_j^\rho \cap K_H = (R + \boldsymbol{\pi}^j R_E)^\times/R^\times \times E^1.$$

Further we have $H = \bigcup_{j\geq 0} T_H \cdot r_j \cdot K_H$, and $r_j^{-1}T_H r_j \cap K_H$ is

$$R_L(j)^1 = E^1 \cap R_L(j), \qquad R_L(j) = R + \sqrt{\boldsymbol{\pi}}\boldsymbol{\pi}^j R,$$

where

$$r_j = \begin{pmatrix} 0 & \boldsymbol{\pi} \\ 1 & 0\end{pmatrix}^{j-2[\frac{j}{2}]}\boldsymbol{\pi}^{-[\frac{j+1}{2}]}\begin{pmatrix} 1 & 0 \\ 0 & -\boldsymbol{\pi}\end{pmatrix}^j.$$

PROOF. Note that $E=F(\sqrt{D})$, $D\in R-R^2$. Put $D_1=\operatorname{diag}(\sqrt{D},1)$. Then $\mathrm{U}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)=D_1^{-1}\,\mathrm{U}_2\,D_1$, where U_2 is the unitary group $\mathrm{U}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Since $\operatorname{diag}(a,\overline{a}^{-1})=a\operatorname{diag}(1,1/a\overline{a})$, we have $E^\times\,\mathrm{U}_2=E^\times\,\mathrm{GL}(2,E/F)$, where

$$\mathrm{GL}(2,E/F)=\{g\in\mathrm{GL}(2,F);\det g\in NE^\times\}.$$

Note that $NE^\times=\boldsymbol{\pi}^{2\mathbb{Z}}R^\times$. Note that $T_{1\rho}=\left\{\left(\begin{smallmatrix}u&vD\rho\\v/\rho&u\end{smallmatrix}\right)\in\mathrm{GL}(2,F)\right\}$ lies in $\mathrm{GL}(2,E/F)$, as $u^2-v^2D=\alpha\overline{\alpha}\in NE^\times$ (for $\alpha=u+v\sqrt{D}$ in $E^\times$). The corresponding torus in U_2 is $T_{2\rho}=\left\{\frac{\beta}{\alpha}\left(\begin{smallmatrix}u&v\rho D\\v/\rho&u\end{smallmatrix}\right);\,\beta\in E^1\right\}$, and $T_{H\rho}=D_1^{-1}T_{2\rho}D_1$ is the torus $\left\{\frac{\beta}{\alpha}\left(\begin{smallmatrix}u&v\rho\sqrt{D}\\v\sqrt{D}/\rho&u\end{smallmatrix}\right)\right\}$ in $D_1^{-1}\,\mathrm{U}_2\,D_1=\mathrm{U}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Thus the map $T_{1\rho}\to T_{H\rho}$ takes an element with eigenvalues $\{\alpha,\overline{\alpha}\}$ to one with eigenvalues $\{\beta,\beta\overline{\alpha}/\alpha\}$. From the well-known (see Remark following the present proof) decomposition $\mathrm{GL}(2,F)=\bigcup_{j\geq0}T_{1\rho}\operatorname{diag}(1,\boldsymbol{\pi}^j)\,\mathrm{GL}(2,R)$ we obtain

$$\mathrm{GL}(2,E/F)=\bigcup_j T_{1\rho}r_j^\rho\,\mathrm{GL}(2,R)\qquad(j\geq0,\quad j\equiv\overline{\rho}(2)).$$

Hence $\mathrm{U}_2=\cup T_{2\rho}r_j^\rho K_2$, where $K_2=\mathrm{U}_2\cap\mathrm{GL}(2,R_E)$. Conjugating by D_1 we get the decomposition of the proposition. Finally,

$$(r_j^\rho)^{-1}\cdot T_{H\rho}\cdot r_j^\rho\cap K_H=\left\{\frac{\beta}{\alpha}\begin{pmatrix}u&v\boldsymbol{\pi}^j\sqrt{D}\\v\boldsymbol{\pi}^{-j}\sqrt{D}&u\end{pmatrix}\in K_H;\,\alpha=u+v\sqrt{D}\right\}.$$

The last matrix has eigenvalues $\beta\in E^1$ and $\beta\overline{\alpha}/\alpha$. Since E/F is unramified, $E^\times/F^\times=R_E^\times/R^\times$, we may assume that $\alpha\in R_E^\times$ and conclude that $u\in R$, $v\in\boldsymbol{\pi}^jR$. Thus our intersection is isomorphic to $(R+\boldsymbol{\pi}^jR_E)^\times/R^\times\times E^1$, as asserted.

For the last claim, in the notations of Proposition 3 in the ramified case ($T=(LE/K)^1\times E^1$), we have that

$$\mathrm{GL}(2,F)=\cup_{j\geq0}T_1\operatorname{diag}(1,(-\boldsymbol{\pi})^j)K=\cup_{j\geq0}T_1r_jK,$$

$r_j=t_j\operatorname{diag}(1,(-\boldsymbol{\pi})^j)$, where t_j is $\boldsymbol{\pi}^{-j/2}$ if j is even, and $\boldsymbol{\pi}^{-(j+1)/2}\left(\begin{smallmatrix}0&\boldsymbol{\pi}\\1&0\end{smallmatrix}\right)$ if j is odd. Then $\mathrm{GL}(2,E/F)=\cup_{j\geq0}ZT_0r_jK$, and

$$\mathrm{U}=\mathrm{U}\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\cup_{j\geq0}E^1T_0r_jK_{\mathrm{U}},$$

and $H = \mathrm{U}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = D_1^{-1}\,\mathrm{U}\,D_1$ with $D_1 = \mathrm{diag}(\sqrt{D}, 1)$ has $H = \cup_{j\ge 0} T_H r_j K_H$, where T_H is as described in Proposition 3.

Now $r_j^{-1} T_H r_j \cap K_H$ consists of

$$\delta^{-1}\begin{pmatrix} \alpha & \beta\boldsymbol{\pi}(-\boldsymbol{\pi})^j/\sqrt{D} \\ \beta\sqrt{D}/(-\boldsymbol{\pi})^j & \alpha \end{pmatrix} \in K_H$$

in the case where j is even (replace D by $1/D$ when j is odd), namely $|\beta| \le |\boldsymbol{\pi}|^j$. Thus $r_j^{-1} T_H r_j \cup K_H$ is

$$R_L(j)^1 = E^1 \cap R_L(j), \qquad R_L(j) = R + \sqrt{\boldsymbol{\pi}}\boldsymbol{\pi}^j R,$$

up to factors of the form E^1, whose volume is 1 and is ignored here. $\square$

REMARK. A proof of the well-known decomposition

$$\mathrm{GL}(2,F) = \bigcup_{j\ge 0} T\,\mathrm{diag}(1, \boldsymbol{\pi}^j)\,\mathrm{GL}(2,R)$$

— extracted from a letter of J.G.M. Mars — is as follows. For another proof see [F4;I], Lemma I.I.1. Let E/F be a separable quadratic extension of nonarchimedean local fields. Let V be E considered as a two-dimensional vector space over F. Multiplication in E gives an embedding $E \subset \mathrm{End}_F(V)$ and $E^\times \subset \mathrm{GL}(V)$. The ring of integers R_E is a lattice in V and $K = \mathrm{Stab}(R_E)$ is a maximal compact subgroup of $\mathrm{GL}(V)$.

Let Λ be a lattice in V. Then $R(\Lambda) = \{x \in E;\, x\Lambda \subset \Lambda\}$ is an order. The orders in E are $R_E(j) = R + \boldsymbol{\pi}^j R_E$, $j \ge 0$ ($\boldsymbol{\pi} = \boldsymbol{\pi}_F$). Note that $R_E(j)/R_E(j+1)$ is a one-dimensional vector space over $R/\boldsymbol{\pi}$. If $R(\Lambda) = R_E(j)$, then $\Lambda = zR_E(j)$ for some $z \in E^\times$. Choose a basis 1, w of E such that $R_E = R + Rw$. Define d_j in $\mathrm{GL}(V)$ by $d_j(1) = 1$, $d_j(w) = \boldsymbol{\pi}^j w$. Then $R_E(j) = d_j R_E$. It follows immediately that $\mathrm{GL}(V) = \cup_{j\ge 0} E^\times d_j K$, or, in coordinates with respect to 1, w:

$$\mathrm{GL}(2,F) = \bigcup_{j\ge 0} T\begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^j \end{pmatrix} GL(2,R),$$

with $T = \left\{\begin{pmatrix} a & \alpha b \\ b & a+\beta b \end{pmatrix};\, a,\, b \in F, \text{ not both } 0\right\}$, where $w^2 = \alpha + \beta w$, α, $\beta \in R$.

5. Proposition. *If* $R_E(j) = R + \boldsymbol{\pi}^j R_E$, $j \geq 0$, *then* $[R_E^\times : R_E(j)^\times]$ *is* 1 *if* $j = 0$, *and* $(1 + q^{-1})q^j$ *if* $j \geq 1$. *Further, we have that* $[(R + \sqrt{\boldsymbol{\pi}}R)^1 : (R + \sqrt{\boldsymbol{\pi}}\boldsymbol{\pi}^j R)^1] = q^j$.

Proof. The first index is the quotient of $[R_E^\times : 1 + \boldsymbol{\pi}^j R_E] = (q^2 - 1)q^{2(j-1)}$ by $[R^\times : 1 + \boldsymbol{\pi}^j R] = (q-1)q^{j-1}$ when $j \geq 1$. When $j = 0$, $R_E(j) = R_E$. The last claim follows from the fact that $u^2 - \boldsymbol{\pi} v^2 = 1$ implies $u = 1 + \boldsymbol{\pi} v^2/2 + \cdots$, up to a sign. □

6. Proposition. *We have* $K_H \times E^1 = P_H H_m^K$, *where*

$$P_H = \left\{ \begin{pmatrix} u & & 0 \\ & w & \\ 0 & & \overline{u}^{-1} \end{pmatrix} \begin{pmatrix} 1 & & v\sqrt{D} \\ & 1 & \\ 0 & & 1 \end{pmatrix}; \, u \in R_E^\times, \, w \in E^1, \, v \in R \right\},$$

and $[P_H : P_H \cap H_m^K]$ *is* 1 *if* $m = 0$ *and* $(1 - q^{-2})q^{4m}$ *if* $m \geq 1$.

Proof. Define $u \in R^\times$, $v \in R$, by the equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & cD \\ c & d \end{pmatrix}$ in $\mathrm{GL}(2, R)$. Hence K_H consists of

$$\begin{pmatrix} u & 0 \\ 0 & \overline{u}^{-1} \end{pmatrix} \begin{pmatrix} 1 & v\sqrt{D} \\ 0 & 1 \end{pmatrix} \frac{1}{\alpha} \begin{pmatrix} d & c\sqrt{D} \\ c\sqrt{D} & d \end{pmatrix} \; (u \in R_E^\times, \, v \in R; \, \alpha = d + c\sqrt{D} \in R_E^\times),$$

and $K_H \times E^1 = P_H H_m^K$. The intersection $P_H \cap H_m^K$ is P_H when $m = 0$, but when $m \geq 1$ and $t = \boldsymbol{\pi}^m$, it consists of

$$\begin{pmatrix} a_1 + ta_2 & & -ta_2 + tb_3 + 2a_3t^2 \\ & a_1 & \\ 0 & & a_1 - tb_3 \end{pmatrix}$$

$$= a_1 \begin{pmatrix} 1 + ta_2' & & -ta_2' + tb_3' + 2a_3't^2 \\ & 1 & \\ 0 & & 1 - tb_3' \end{pmatrix},$$

where $a_2' = a_2/a_1$, $b_3' = b_3/a_1$, $a_3' = a_3/a_1$, $a_1\overline{a}_1 = 1$. These satisfy $1 = (1 + t\overline{a}_2')(1 - tb_3')$, namely $b_3' = \overline{a}_2'/(1 + t\overline{a}_2')$. Thus

$$t(b_3' - a_3') = t(\overline{a}_2'/(1 + t\overline{a}_2') - a_2') = t(\overline{a}_2' - a_2' - ta_2'\overline{a}_2')/(1 + t\overline{a}_2').$$

Erasing the prime from a_2, and the middle entry 1, $P_H \cap H_m^K$ consists of the product of $E^1 = \{a_1\}$ and the matrices

$$\begin{pmatrix} 1+ta_2 & t(\overline{a}_2 - a_2 - ta_2\overline{a}_2)(1+t\overline{a}_2)^{-1} + t^2 2a_3' \\ 0 & 1 - t\overline{a}_2(1+t\overline{a}_2)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1+ta_2 & t(\overline{a}_2-a_2)/(1+t\overline{a}_2) \\ 0 & 1-t\overline{a}_2/(1+t\overline{a}_2) \end{pmatrix} \begin{pmatrix} 1 & t^2 a_3'' \sqrt{D} \\ 0 & 1 \end{pmatrix}.$$

Then $[P_H\colon P_H \cap H_m^K]$ is the product of $[R_E^\times\colon 1+\boldsymbol{\pi}^m R_E] = (q^2-1)q^{2(m-1)}$ (for a_2) and $[R\colon \boldsymbol{\pi}^{2m} R] = q^{2m}$ (for a_3). □

Definition. Put $\delta(X) = 1$ if "X" holds, and $\delta(X) = 0$ if "X" does not hold.

Note that $\int_{P_H/P_H \cap K_m^K} f(p)dp = [P_H\colon P_H \cap H_m^K] \int_{P_H} f(p)dp$, if the measure dp assigns the compact P_H the volume one.

7. Corollary. *The orbital integral* $\int_{T_\rho\backslash G} 1_K(x^{-1}t_\rho x)dx$ *is equal to*

$$\sum_{j\geq 0,\ j\equiv\overline{\rho}(2)} [\delta(j=0) + (1+q^{-1})q^j \delta(j\geq 1)]$$

$$\sum_{m\geq 0} \int_{P_H/P_H\cap H_m^K} 1_{H_m^K}(p^{-1}(r_j^\rho)^{-1} t_\rho r_j^\rho p)dp\,.$$

For a regular $t \in T_H$, *the orbital integral* $\int_{T_H\backslash G} 1_K(x^{-1}tx)dx$ *is equal to*

$$\sum_{m\geq 0} |H_m^K|^{-1} \sum_{j\geq 0} \int_{K_H\cap r_j^{-1}T_H r_j\backslash K_H} 1_{H_m^K}(k^{-1}r_j^{-1}tr_jk)dk$$

$$= \sum_{j\geq 0} q^j \sum_{m\geq 0} \int_{P_H/H_m^K\cap P_H} 1_{H_m^K}(p^{-1}r_j^{-1}tr_jp)dp.$$

D. Computations: $j \geq 1$

In computing the integrals

$$\int_{P_H} 1_{H_m^K}(p^{-1}(r_j^\rho)^{-1} t_\rho r_j^\rho p)dp$$

at $t_\rho = r_\rho^{-1}h^{-1}\operatorname{diag}(a,b,c)hr_\rho$, we put $a' = \frac{a}{b} - 1$, $c' = \frac{c}{b} - 1$, define N_1 by $a' \in \boldsymbol{\pi}^{N_1} R_E^\times$, N_2 by $c' \in \boldsymbol{\pi}^{N_2} R_E^\times$, N by $a' - c' \in \boldsymbol{\pi}^N R_E^\times$ and N^+ by $a' + c' \in \boldsymbol{\pi}^{N^+} R_E^\times$. Since γ_ρ is regular, N, N_1 and N_2 are finite nonnegative integers. Put $M = \max(N_1, N_2)$. We shall distinguish between two cases. If $|a'-c'| < |a'|$, then $|a'| = |c'| = |a'+c'|$, thus $N^+ = N_1 = N_2 < N$. If $|a'| \leq |a'-c'|$, then either $|a'| < |a'-c'|$ $(= |c'| = |a'+c'|$, thus $N^+ = N_2 = N < N_1)$, or $|a'| = |a'-c'|$ $(\geq |a'+c'|, |c'|$, thus N^+, $N_2 \geq N_1 = N)$, namely $N \leq N^+$. Put $\nu = N - j$, and denote — as usual — by $[x]$ the maximal integer $\leq x$.

8. PROPOSITION. *If $j \geq 1$, then*

$$\int_{P_H/P_H \cap H_m^K} 1_{H_m^K}(p^{-1}(r_j^\rho)^{-1} t_\rho r_j^\rho p)dp$$

is 1 *if* $m = 0$, $(1-q^{-2})q^{4m}$ *if* $1 \leq m \leq \min\left(\left[\frac{\nu}{2}\right], \left[\frac{N^+}{2}\right]\right)$, *and*

$$(1-q^{-2})q^{4m} \cdot (q-1)^{-1} q^{\nu+1-2m} = (1+q^{-1})q^{\nu+2m} \quad \textit{if} \quad \nu = N^+ < 2m \leq 2\nu.$$

For all other $m \geq 0$ the integral is zero.

For a regular $t = \operatorname{diag}\left(\delta^{-1}\begin{pmatrix} \alpha & \beta\boldsymbol{\pi}/\sqrt{D} \\ \beta\sqrt{D} & \alpha \end{pmatrix}, v\right)$ *in* $T_H \subset H$, *the integral*

$$\int_{P_H/P_H \cap H_m^K} 1_{H_m^K}(p^{-1} r_j^{-1} t r_j p)dp$$

is 1 *if* $m = 0$, $(1-q^{-2})q^{4m}$ *if* $1 \leq m \leq \min([\nu/2],[(1+N_2)/2])$, *and* $(1+q^{-1})q^{\nu+2m}$ *if* $\nu = 1+N_2 < 2m \leq 2+2N_2$, *and* $N_2 < N$. *For all other* $m \geq 0$ *the integral is zero. Here* $\beta = B\boldsymbol{\pi}^N$ $(B \in R^\times)$, *and* $\delta = \delta_1 + i\delta_2 \in E^1$ *with* $\delta_2 = D_2\boldsymbol{\pi}^{N_2}$, $\delta_1, D_2 \in R^\times$.

PROOF. As $P_H \subset H_m^K$ when $m = 0$, we assume $m \geq 1$. We need to compute the volume of solutions in $u \in R_E^\times/(1+tR_E)$ and $v \in R/t^2R$ $(t = \boldsymbol{\pi}^m)$, of the equation

$$\begin{pmatrix} 1 & & -v\sqrt{D} \\ & 1 & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & & (u-\overline{u})/u \\ & 1 & \\ 0 & & \overline{u} \end{pmatrix} \begin{pmatrix} \frac{1}{2}(a+c) & & \frac{1}{2}(a-c)\boldsymbol{\pi}^j \\ & b & \\ \frac{1}{2}(a-c)\boldsymbol{\pi}^{-j} & & \frac{1}{2}(a+c) \end{pmatrix}$$

$$\cdot \begin{pmatrix} u & & (\overline{u}-u)/\overline{u} \\ & 1 & \\ 0 & & \overline{u}^{-1} \end{pmatrix} \begin{pmatrix} 1 & & v\sqrt{D} \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 - b_1 + ta_2 & & b_1 - ta_2 + tb_3 + 2a_3t^2 \\ & a_1 & \\ b_1 & & a_1 - b_1 - tb_3 \end{pmatrix},$$

for $a_1 \in E^1$; $b_1, a_2, a_3, b_3 \in R_E$. To have a solution, a_1 must be equal to b. We then replace a by a/b, c by c/b on the left, and b_1, a_2, b_3, a_3 by their quotients by a_1 on the right, to assume that $a_1 = b = 1$. Put $w = v\sqrt{D} + (\overline{u} - u)/u\overline{u}$, erase 2nd row and column of our matrices, write b for b_1, define $B \in R_E^\times$ by

$$\frac{1}{2}(a-c)\boldsymbol{\pi}^{-j} = B\boldsymbol{\pi}^\nu \qquad (\nu = N - j \leq N),$$

to express our identity as the equality of

$$\begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(a+c) & \frac{1}{2}(a-c)\boldsymbol{\pi}^j/u\overline{u} \\ \frac{1}{2}(a-c)u\overline{u}\boldsymbol{\pi}^{-j} & \frac{1}{2}(a+c) \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(a+c)-wu\overline{u}B\boldsymbol{\pi}^\nu & B\boldsymbol{\pi}^\nu u\overline{u}(\boldsymbol{\pi}^{2j}/(u\overline{u})^2-w^2) \\ B\boldsymbol{\pi}^\nu u\overline{u} & \frac{1}{2}(a+c)+wB\boldsymbol{\pi}^\nu u\overline{u} \end{pmatrix}$$

and

$$\begin{pmatrix} 1-b+ta_2 & b-ta_2+tb_3+2a_3t^2 \\ b & 1-b-tb_3 \end{pmatrix}.$$

Since $b \in R_E$, to have solutions we must have that $\nu \geq 0$ (consider the entry (row, column) $= (2,1)$ in our identity). This is congruent to $\begin{pmatrix} 1-b & b \\ b & 1-b \end{pmatrix}$ modulo $\boldsymbol{\pi}^m$. Considering the entries $(1,1)$ and $(2,2)$, we deduce that $w\boldsymbol{\pi}^\nu \equiv 0 \ (\mathrm{mod}\ \boldsymbol{\pi}^m)$. If $m > \nu$, considering the entries $(1,2)$ and $(2,1)$ we conclude that $j = 0$.

Since $j \geq 1$, we may now assume that $1 \leq m \leq \nu$. Then $b \equiv \boldsymbol{\pi}^\nu \equiv 0 \ (\mathrm{mod}\, \boldsymbol{\pi}^m)$, and from the equality of the entries $(1,1)$ or $(2,2)$, we obtain $\frac{1}{2}(a+c) \equiv 1 \,(\mathrm{mod}\, \boldsymbol{\pi}^m)$. Put $a' = a-1$, $c' = c-1$. Then $a'+c' \equiv 0 \,(\mathrm{mod}\, \boldsymbol{\pi}^m)$. Since also $a'-c' \equiv 0 \,(\mathrm{mod}\, \boldsymbol{\pi}^m)$, we have $a', c' \equiv 0 \,(\mathrm{mod}\, \boldsymbol{\pi}^m)$, and we have $a'' = a'\boldsymbol{\pi}^{-m}$, $c'' = c'\boldsymbol{\pi}^{-m}$, $b' = b\boldsymbol{\pi}^{-m}$ in R_E. Put $\nu' = \nu - m \geq 0$. The matrix identity translates to 4 equations, the first 3 define b, a_2, b_3 and hence are always solvable:

$$B\boldsymbol{\pi}^{\nu'} u\overline{u} = b', \qquad \frac{1}{2}(a''+c'') + (1-w)u\overline{u}B\boldsymbol{\pi}^{\nu'} = a_2,$$

$$\frac{1}{2}(a''+c'') + (1+w)u\overline{u}B\boldsymbol{\pi}^{\nu'} = -b_3\,,$$

$$B''\boldsymbol{\pi}^{\nu''} + B\boldsymbol{\pi}^{\nu'}u\overline{u}(1-Dv_1^2+\boldsymbol{\pi}^{2j}/(u\overline{u})^2) = 2a_3\boldsymbol{\pi}^m,$$

where

$$B''\boldsymbol{\pi}^{\nu''} = a''+c'', \ v_1 = w/\sqrt{D} \in R.$$

If $m \leq \nu', \nu''$, namely $2m \leq \nu$, N^+, any $u \in R_E^\times$, $v_1 \in R$, make a solution (a_3 is defined by the 4th equation). This proves the proposition for m $\left(1 \leq m \leq \min\left(\left[\frac{\nu}{2}\right], \left[\frac{N^+}{2}\right]\right)\right)$.

If $\nu'' < \nu'$, m, there are no solutions in u, v_1.

If $\nu' < \nu''$, m, since $j \geq 1$ and $1 - Dv_1^2 \in R^\times$, there are no solutions either.

It remains to consider the case where $\nu' = \nu'' < m$ $(\leq \nu)$. Write

$$\varepsilon^{-1} = -u\overline{u}(1 - Dv_1^2)B/B''.$$

Then our equation can be written in the form

$$1 - 2a_3\boldsymbol{\pi}^{m-\nu'}/B'' = -u\overline{u}B/B''(1 - Dv_1^2 + \boldsymbol{\pi}^{2j}(u\overline{u})^{-2}) = \varepsilon^{-1}(1 + \zeta\boldsymbol{\pi}^{2j}\varepsilon^2),$$

where $\zeta = (B/B'')^2(1 - Dv_1^2)$, namely

$$\begin{aligned}\varepsilon &\equiv 1 + \zeta\boldsymbol{\pi}^{2j}\varepsilon^2 \equiv 1 + \zeta\boldsymbol{\pi}^{2j}(1 + 2\zeta\boldsymbol{\pi}^{2j}\varepsilon^2 + \rho^2\boldsymbol{\pi}^{4j}\varepsilon^4) \\ &= 1 + \zeta\boldsymbol{\pi}^{2j} + 2\zeta^2\boldsymbol{\pi}^{4j}\varepsilon^2 + \zeta^3\boldsymbol{\pi}^{6j}\varepsilon^4 \pmod{\boldsymbol{\pi}^{m-\nu'}},\end{aligned}$$

so that ε is uniquely determined modulo $\boldsymbol{\pi}^{m-\nu'}$. Thus a choice of v_1 in R determines ζ, and ε in $R^\times/1 + \boldsymbol{\pi}^{m-\nu'}R$, hence $u\overline{u} \in R^\times/1 + \boldsymbol{\pi}^{m-\nu'}R$. The volume of one coset $\operatorname{mod} \boldsymbol{\pi}^{m-\nu'}$ in $R^\times$ is

$$[R^\times : 1 + \boldsymbol{\pi}^{m-\nu'}R]^{-1} = 1/[(q-1)q^{2m-\nu-1}].$$

Multiplying by $[P_H : P_H \cap H_m^K] = (1 - q^{-2})q^{4m}$ we get $(1 + q^{-1})q^{2m+\nu}$.

In the ramified case, the case $m = 0$ is again trivial, so we assume $m \geq 1$. Putting $B_1 = B\overline{\delta}\sqrt{D}(-1)^j \in R_E^\times$, in analogy with the previous case we are led to solve in u and $v_1 = w/\sqrt{D}$ the equation

$$\begin{pmatrix} \alpha\overline{\delta} - wu\overline{u}B_1\boldsymbol{\pi}^\nu & u\overline{u}B_1\boldsymbol{\pi}^\nu(\boldsymbol{\pi}^{2j+1}/D(u\overline{u})^2 - Dv_1^2) \\ u\overline{u}B_1\boldsymbol{\pi}^\nu & \alpha\overline{\delta} + u\overline{u}B_1\boldsymbol{\pi}^\nu \end{pmatrix}$$

$$= \begin{pmatrix} 1-b+ta_2 & b-ta_2+tb_3+2a_3t^2 \\ b & 1-b-tb_3 \end{pmatrix} \equiv \begin{pmatrix} 1-b & b \\ b & 1-b \end{pmatrix} \pmod{\boldsymbol{\pi}^m}.$$

As $b \in R_E$, using $(2,1)$ we have $0 \leq \nu \leq N$. From $(1,1)$ and $(2,2)$, $w\boldsymbol{\pi}^\nu \equiv 0 \pmod{\boldsymbol{\pi}^m}$. If $\nu < m$ then $|w| < 1$, but this contradicts $(1,2)$ and $(2,1)$. Hence $1 \leq m \leq \nu \leq N$. Put $b' = b\boldsymbol{\pi}^{-m}$, $\nu' = \nu - m$. Then

$$B_1u\overline{u}\boldsymbol{\pi}^{\nu'} = b', \quad \alpha'' + (1-w)u\overline{u}B_1\boldsymbol{\pi}^{\nu'} = a_2, \quad \alpha'' + (1+w)u\overline{u}B_1\boldsymbol{\pi}^{\nu'} = -b_3,$$

define b, a_2, b_3. Here $\alpha' = \alpha\overline{\delta} - 1 \equiv 0 \pmod{\boldsymbol{\pi}^m}$ is used to define $\alpha'' = \alpha'\boldsymbol{\pi}^{-m}$. The remaining equation (add all four entries in the matrix equality) is

$$B''\boldsymbol{\pi}^{\nu''} + u\overline{u}B_1\boldsymbol{\pi}^{\nu'}(1 - Dv_1^2 + \boldsymbol{\pi}^{1+2j}/D(u\overline{u})^2) = 2a_3\boldsymbol{\pi}^m,$$

where $2\alpha'' = B''\boldsymbol{\pi}^{\nu''}$, $B'' \in R_E^\times$. If $2\alpha'' = B''\boldsymbol{\pi}^{N^+}$, $N^+ = \nu'' + m$, then

$$N^+ = \min(1 + N_2, 1 + 2N),$$

since $\alpha' = \alpha\overline{\delta} - 1$ is equal to

$$(1 + B^2\boldsymbol{\pi}^{1+2N}/2 + \cdots)(1 + DD_2^2\boldsymbol{\pi}^{2+2N_2}/2 + \cdots - \sqrt{D}D_2\boldsymbol{\pi}^{1+N_2}) - 1$$

$$= -\sqrt{D}D_2\boldsymbol{\pi}^{1+N_2} + B^2\boldsymbol{\pi}^{1+2N}/2 + \cdots \equiv 0(\operatorname{mod}\boldsymbol{\pi}^m).$$

Of course $\alpha \equiv \delta(\operatorname{mod}\boldsymbol{\pi}^m)$ implies $\delta_2 \equiv 0(\operatorname{mod}\boldsymbol{\pi}^m)$, and $m \leq 1 + N_2$.

Returning to the remaining equation, if $1 \leq m \leq \nu', \nu''$, thus $2m \leq \nu, N^+$, and $\nu \leq N$ implies $1 \leq m \leq \min([\nu/2], [(1 + N_2)/2])$, any $u \in R_E^\times$ and $v_1 \in R$ make a solution, a_3 is defined by the equation, and the number of solutions is as stated in the proposition.

If $\nu'' < \nu', m$, or $\nu' < \nu'', m$, there are no solutions, as $1 - Dv_1^2 \in R^\times$.

If $\nu' = \nu'' < m \leq \nu$, namely $\nu = \min(1 + N_2, 1 + 2N) < 2m \leq 2\nu$, but $\nu \leq N$ implies $\nu = 1 + N_2$, so $N_2 < N$, and the number of solutions is computed as in the unramified case to be as asserted in the proposition. □

9. Proposition. *When $\overline{\rho} = 1$ the orbital integral $\int_{T_\rho\backslash G} 1_K(x^{-1}t_\rho x)dx$ is equal to*

$$\frac{q+1}{q^4-1}\left(q^{4[\frac{N+1}{2}]} - 1\right)$$

if $N \leq N_1$, and to

$$-\frac{q+1}{q^4-1}(1 + q^{2+4[N_1/2]}) + \frac{(-q)^{N+N_1}}{q-1} + \delta \cdot \frac{q+1}{q-1}q^{N+2N_1}$$

if $N > N_1$. Here $\delta = \delta(2 \mid N - 1 - N_1)$ (is 1 if $N - N_1 - 1$ is even, 0 if $N - N_1$ is even).

The orbital integral $\int_{T_H\backslash G} 1_K(x^{-1}tx)dx$ is equal to:

(1) *If $N \leq N_2$, it is*

$$(q^{2N+2} - 1)/((q^2 + 1)(q - 1))$$

if N is odd, and if N is even,

$$(q^{2N+4} - 1)/((q^2 + 1)(q - 1)) - q^{1+2N}.$$

(2) *If $N_2 < N$, it is*

$$q^{N+2N_2+3}/(q-1) - (q^{2N_2+2}+1)/((q^2+1)(q-1))$$

if N_2 is even, and if N_2 is odd,

$$-(q^{2N_2+4}+1)/((q^2+1)(q-1)) + q^{N+2N_2+3}/(q-1).$$

PROOF. It suffices to prove the first statement with N_1 replaced by N^+, since $N > N_1$ if and only if $N > N^+$, in which case $N_1 = N^+$. The contribution from the terms $j \geq 1$ is

$$\sum_{\substack{1\leq j\leq N\\ j\equiv\overline{\rho}(2)}} (1+q^{-1})q^j \cdot \left(1 + \sum_{1\leq m\leq \min\left(\left[\frac{\nu}{2}\right],\left[\frac{N^+}{2}\right]\right)} (1-q^{-2})q^{4m} + \sum_{\frac{\nu}{2}=\frac{N^+}{2}<m\leq\nu} (1+q^{-1})q^{\nu+2m}\right).$$

If $\overline{\rho} = 1$, this is the entire orbital integral. In this case we replace j by $2j+1$, and let j range over $0 \leq j \leq (N-1)/2$. If $N \leq N^+$, $\nu = N-1-2j$ is smaller than N^+, and we get

$$\begin{aligned}
&(q+1)\sum_{0\leq j\leq[(N-1)/2]} q^{2j}\left(1+\sum_{1\leq m\leq[(N-1)/2]-j}(1-q^{-2})q^{4m}\right)\\
=&(q+1)\sum_j q^{2j}(1+(1-q^{-2})q^4(q^{4[(N-1)/2]-4j}-1)/(q^4-1))\\
=&\frac{q+1}{q^2+1}\sum_j q^{2j}(1+q^{2+4[(N-1)/2]-4j})\\
=&\frac{q+1}{q^2+1}\left(\frac{q^{2[(N+1)/2]}-1}{q^2-1}+q^{2+4[(N-1)/2]}\cdot\frac{1-q^{-2[(N+1)/2]}}{1-q^{-2}}\right),
\end{aligned}$$

which is equal to the asserted expression.

If ($\overline{\rho} \equiv 1$ and) $N > N^+$, then $\nu = N-1-2j$, and $\frac{\nu}{2} = \frac{N-1}{2} - j > \frac{N^+}{2}$ precisely when $\frac{1}{2}(N-1-N^+) > j$ (same with $<$ or $=$). Note that

$\delta(N^+=\nu)$ is δ. Put $\min = \min\Big(\left[\frac{\nu}{2}\right], \left[\frac{N^+}{2}\right]\Big)$. Our integral is then

$$(q+1)\sum_{0\le j\le[(N-1)/2]} q^{2j}\left(\frac{1}{q^2+1}+\frac{q^{2+4\min}}{q^2+1}\right)$$
$$+\,\delta\frac{q^{N^++1}}{q-1}(q^{2N^+}-q^{2[N^+/2]})$$

$$=\delta *+\frac{q+1}{q^2+1}\frac{q^{2[(N+1)/2]}-1}{q^2-1}$$
$$+\frac{q^2(q+1)}{q^2+1}\cdot\left(\sum q^{4[N^+/2]}q^{2j}+\sum q^{4[(N-1)/2]}q^{-2j}\right),$$

$0\le j\le[(N-1-N^+)/2]$ in the first sum, $[(N-1-N^+)/2]<j\le[(N-1)/2]$ in the second,

$$=\delta *+\frac{q+1}{q^4-1}(q^{2[(N+1)/2]}-1)$$
$$+\frac{q^2(q+1)}{q^2+1}\cdot\Bigg(q^{4[N^+/2]}\frac{q^{2[(N+1-N^+)/2]}-1}{q^2-1}$$
$$+\,q^{4[(N-1)/2]}\frac{q^{-2([(N-1-N^+)/2]+1)}-q^{-2([(N-1)/2]+1)}}{1-q^{-2}}\Bigg)$$

$$=\frac{q+1}{q^4-1}\Bigg(-1-q^{2+4[N^+/2]}+q^{2+4[N^+/2]+2[(N+1-N^+)/2]}$$
$$+\,q^{4[(N+1)/2]-2[(N+1-N^+)/2]}\Bigg)+\delta\frac{q+1}{q-1}(q^{N+2N^+}-q^{N+2[N^+/2]})\,.$$

If $\delta=0$, then N is even iff N^+ is even, and

$$\left[\frac{1}{2}(N+1-N^+)\right]=\frac{1}{2}(N-N^+)=[N/2]-[N^+/2].$$

Hence we obtain

$$-\frac{q+1}{q^4-1}(1+q^{2+4[N^+/2]})$$
$$+\frac{q+1}{q^4-1}q^{2[N^+/2]+2[N/2]}(q^2+q^{4[(N+1)/2]-4[N/2]})$$
$$=-\frac{q+1}{q^4-1}(1+q^{2+4[N^+/2]})+\frac{q^{N^++N}}{q-1}\,.$$

If $\delta = 1$, then N is even iff N^+ is odd, and

$$\left[\frac{1}{2}(N-1-N^+)\right] = \frac{1}{2}(N-1) - \frac{1}{2}N^+ = \left[\frac{1}{2}(N-1)\right] - \left[\frac{1}{2}N^+\right].$$

We get

$$\begin{aligned}
&-\frac{q+1}{q^4-1}(1+q^{2+4[N^+/2]}) - \frac{q+1}{q-1}q^{N+2[N^+/2]} + \frac{q+1}{q-1}q^{N+2N^+} \\
&+\frac{q+1}{q^4-1}(q^{2+2[N^+/2]+2[(N+1)/2]} + q^{2[(N+1)/2]+2[N^+/2]}) \\
&= -\frac{q+1}{q^4-1}(1+q^{2+4[N^+/2]}) + \frac{q+1}{q-1}q^{N+2N^+} \\
&+\frac{q^{2[N^+/2]}}{q-1}(q^{2[(N+1)/2]} - (q+1)q^N)\,.
\end{aligned}$$

The middle term is $-q^{N+N^+}/(q-1)$ since $N+1$ is even iff N^+ is even.

In the ramified case we compute as follows. Suppose that $N \le N_2$. Then the integral is

$$\begin{aligned}
&\sum_{0\le\nu\le N} q^{N-\nu}\left(1 + \sum_{1\le m\le[\nu/2]}(q^4-q^2)q^{4(m-1)}\right) \\
&= \sum_{0\le\nu\le N} q^{\nu}/(q^2+1) + q^{2+N}\sum_{0\le\nu\le N} q^{4[\nu/2]-\nu}/(q^2+1) \\
&= \frac{q^{N+1}-1}{(q^2+1)(q-1)} + \frac{q^{N+2}}{q^2+1}\cdot \\
&\left(\sum_{0\le\nu_1\le[N/2],\nu=2\nu_1} q^{2\nu_1} + \sum_{0\le\nu_1\le[(N-1)/2],\nu=2\nu_1+1} q^{2\nu_1-1}\right) \\
&= \frac{q^{N+2[N/2]+4} + q^{N+2[(N-1)/2]+3} - q - 1}{q^4-1}\,,
\end{aligned}$$

as asserted.

Suppose that $N_2 < N$. Then the integral is

$$\sum_{0\le\nu\le 1+N_2} q^{N-\nu}\left(1 + \sum_{1\le m\le[\nu/2]}(1-q^{-2})q^{4m}\right)$$

$$+ q^{N-N_2-1} \sum_{[(1+N_2)/2]<m\leq 1+N_2} (1+q^{-1}) q^{2m+1+N_2}$$

$$+ \sum_{1+N_2<\nu\leq N} q^{N-\nu} \left(1 + \sum_{1\leq m\leq [(1+N_2)/2]} (1-q^{-2}) q^{4m}\right) .$$

This is the sum of

$$\frac{q^{N+2}}{q^2+1} \sum_{0\leq \nu_1\leq [(N_2+1)/2], \nu=2\nu_1} q^{2\nu_1} ,$$

$$\frac{q^{N+1}}{q^2+1} \sum_{0\leq \nu_1\leq [N_2/2], \nu=2\nu_1+1} q^{2\nu_1} + \frac{q^N}{q^2+1} \cdot \frac{q^{-N_2-2}-1}{q^{-1}-1}$$

and

$$(1+q^{-1}) q^N \frac{q^{2(N_2+2)} - q^{2[(1+N_2)/2]+2}}{q^2-1} + \frac{q^{4[(1+N_2)/2]+2}+1}{q^2+1} \cdot \frac{q^{N-N_2-1}-1}{q-1} .$$

Adding, we get the expression of the proposition. □

10. Proposition. *When $\overline{\rho} = 0$, the contribution to the orbital integral of 1_K at t_ρ from the terms indexed by $j > 0$ is*

$$\frac{(q+1)q}{q^4-1} (q^{4[N/2]} - 1)$$

if $N \leq N^+$; when $N > N^+$, if $N - N^+$ is odd ($\delta = \delta(n \mid N - N^+ > 0)$ is 0) we obtain

$$-\frac{(q+1)q}{q^4-1} (1 + q^{2+4[N^+/2]}) + \frac{q^{N+N^+}}{q-1} ,$$

while if $\delta = 1$ ($N - N^+ > 0$ is even) we obtain

$$- \frac{(q+1)q}{q^4-1} (1 + q^{2+4[N^+/2]}) + \frac{q^{1+2[N^+/2]+2[N/2]}}{q-1}$$

$$+ \frac{q+1}{q-1} q^{N+2N^+} - \frac{q+1}{q-1} q^{N+2[N^+/2]} .$$

PROOF. Put $\nu = N - 2j$, $1 \le j \le [N/2]$. The sum over j is

$$(1+q^{-1}) \sum_{1\le j\le [N/2]} q^{2j} \cdot$$

$$\left(\frac{1}{q^2+1} + \frac{q^{2+4\min}}{q^2+1} + \delta \sum_{\frac{\nu}{2}=\frac{N^+}{2}<m\le\nu} (1+q^{-1})q^{\nu+2m} \right) .$$

If $N \le N^+$, then $\min = [\nu/2] = [N/2] - j$ and $\delta = 0$, so we get

$$\frac{q+1}{q(q^2+1)} \sum_{1\le j\le[N/2]} (q^{2j} + q^{2+4[N/2]-2j})$$

$$= \frac{(q+1)q}{q^2+1} \left(\frac{q^{2[N/2]}-1}{q^2-1} + q^{4[N/2]} \frac{q^{-2}-q^{-2([N/2]+1)}}{1-q^{-2}} \right) ,$$

which is the asserted expression.

If $N > N^+$, then $\nu/2 = N/2 - j > N^+/2$ iff $\frac{1}{2}(N-N^+) > j$, in which case $\min([\nu/2],[N^+/2])$ is $[N^+/2]$ (it is $[N/2]-j$ when $>$ is replaced by $<$). Thus we obtain the sum of

$$\frac{(q+1)q}{q^2+1} \; \frac{q^{2[N/2]}-1}{q^2-1} + \frac{(q+1)q^2}{q(q^2+1)} \cdot$$

$$\left(q^{4[N^+/2]} \sum_{1\le j\le[(N-N^+)/2]} q^{2j} + q^{4[N/2]} \sum_{(N-N^+)/2<j\le[N/2]} q^{-2j} \right)$$

$$= \frac{(q+1)q}{q^2+1} \; \frac{q^{2[N/2]}-1}{q^2-1} + \frac{(q+1)q^2}{q(q^2+1)} \cdot$$

$$\left(q^{4[N^+/2]} \frac{q^{u+2}-q^2}{q^2-1} + q^{4[N/2]} \frac{q^{-u-2}+q^{-2[N/2]-2}}{1-q^{-2}} \right)$$

$$= \frac{(q+1)q}{q^4-1} (-1 + q^{2+4[N^+/2]+u} - q^{2+4[N^+/2]} + q^{4[N/2]-u})$$

and

$$\delta(q+1)^2 q^{N-2} \sum_{N^+/2<m\le N^+} q^{2m} = \delta \frac{q+1}{q-1} q^N (q^{2N^+} - q^{2[N^+/2]}) ,$$

where $u = 2[(N-N^+)/2]$. When $\delta = 0$, $u = 2[(N-N^+)/2] = N - N^+ - 1$, and noting that N is even iff N^+ is odd, the asserted expression is obtained. When $\delta = 1$, N is even iff so is N^+, hence $u = 2[(N-N^+)/2] = N - N^+ = 2[N/2] - 2[N^+/2]$, and again we obtain the asserted expression. □

E. Computations: $j = 1$

To complete the computation of the orbital integral of 1_K at t_ρ, it remains to compute the contribution from the term indexed by $j = 0$, which exists only when $\overline{\rho} = 0$.

11. PROPOSITION. *When $\overline{\rho} = 0 = j$, the nonzero values of the integral*

$$\int_{P_H/P_H \cap H_m^K} 1_{K_m^K}(p^{-1} t_\rho p) dp$$

are: 1 *if* $m = 0$,

(*a*) $(1 - q^{-2}) q^{4m}$ *if* $1 \leq m \leq \min([N/2], [N^+/2])$,

(*b*) $(1 + q^{-1}) q^{2m+2[N/2]}$ *if* $[N/2] + 1 \leq m \leq \min(N, [M/2])$ (*thus* $N \leq N^+$*; recall:* $M = \max(N_1, N_2)$),

(*c*) $(1 + q^{-1})^2 q^{2m+N}$ *if* $[M/2] + 1 \leq m \leq N$ (*thus* $N \leq N^+$) *and* $M - N$ *is even,*

(*d*) $(1 + q^{-1}) q^{2m+2[N/2]}$ *if* $N + 1 \leq m \leq [M/2]$, *and*

(*e*) $(1 + q^{-1})^2 q^{2m+N}$ *if* $\max(N + 1, [M/2] + 1) \leq m \leq [(M + N)/2]$ *and* $M - N$ *is even.*

PROOF. As in Proposition 10, we may assume that $m \geq 1$, and compute the volume of solutions in $u \in R_E^\times / 1 + \boldsymbol{\pi}^m R_E$ and $v \in R/\boldsymbol{\pi}^{2m} R$, $w = v\sqrt{D}$, of the equation (for some $a_2, a_3, b \in R_E$):

$$\begin{pmatrix} \frac{1}{2}(a + c) - wu\overline{u}B\boldsymbol{\pi}^N & u\overline{u}B\boldsymbol{\pi}^N((u\overline{u})^{-2} - Dv^2) \\ u\overline{u}B\boldsymbol{\pi}^N & \frac{1}{2}(a + c) + wu\overline{u}B\boldsymbol{\pi}^N \end{pmatrix}$$

$$= \begin{pmatrix} 1 - b + ta_2 & b - ta_2 + tb_3 + 2a_3 t^2 \\ b & 1 - b - tb_3 \end{pmatrix}.$$

Consider first the case where $m > N$. Since the matrix on the right is congruent mod $\boldsymbol{\pi}^m$ to $\left(\begin{smallmatrix} 1-b & b \\ b & 1-b \end{smallmatrix}\right)$, considering the entries $(1, 1)$ and $(2, 2)$ of the equality, we get that $w = v\sqrt{D}$, $v = v_1 \boldsymbol{\pi}^{m-N}$, $v_1 \in R$. The identities of the entries $(1, 2)$ and $(2, 1)$ imply that $u\overline{u} \equiv \pm 1(\boldsymbol{\pi}^{m-N})$. If $u\overline{u} \equiv 1(\boldsymbol{\pi}^{m-N})$, put $u\overline{u} = 1 + \varepsilon' \boldsymbol{\pi}^{m-N}$. The matrix identity becomes four equations:

$b = (a' - c')/2 + \varepsilon' B \boldsymbol{\pi}^m$ (always solvable, defines b),

$a_2 = a'' + \varepsilon' B - B\sqrt{D} v_1 u\overline{u}$ (is solvable precisely when $a'' = a' \boldsymbol{\pi}^{-m} \in R_E$, namely $m \leq N_1$),

$-b_3 = a'' + \varepsilon' B + B\sqrt{D} v_1 u\overline{u}$ (solvable when $m \le N_1$), and
$2a' + B\boldsymbol{\pi}^N u\overline{u}(1 + (u\overline{u})^{-2} - 2(u\overline{u})^{-1} - Dv_1^2\boldsymbol{\pi}^{2m-2N}) = 2a_3\boldsymbol{\pi}^{2m}$.

Thus the 2nd and 3rd equations are solvable when $N < m \le N_1$ if $u\overline{u} \equiv 1$, and when $N < m \le N_2$ if $u\overline{u} \equiv -1$. Hence we are led to consider m in the range $N = N^+ = \min(N_1, N_2) < m \le M = \max(N_1, N_2)$. Defining $\varepsilon_1 \in R$ by $(u\overline{u})^{-1} = 1 + \varepsilon_1\boldsymbol{\pi}^{m-N}$, the remaining, 4th equation, takes the form

$$2a''/B + (2a''/B)\varepsilon_1\boldsymbol{\pi}^{m-N} + \boldsymbol{\pi}^{m-N}(\varepsilon_1^2 - Dv_1^2) \in \boldsymbol{\pi}^m R_E,$$

or

$$2a''/B + \boldsymbol{\pi}^{m-N}((\varepsilon_1 + a''/B)^2 - (a''/B)^2 - Dv_1^2) \in \boldsymbol{\pi}^m R_E,$$

and finally

$$(2a''/B)(1 - (a''/2B)\boldsymbol{\pi}^{m-N}) + \boldsymbol{\pi}^{m-N}(\varepsilon^2 - Dv_1^2) \in \boldsymbol{\pi}^m R_E,$$

where $\varepsilon = \varepsilon_1 + a''/B$. Note that when $u\overline{u} \equiv -1$, a has to be replaced by c in these equations.

We claim that to have a solution, we must have $2m \le N + M$. Indeed, $\varepsilon^2 - Dv_1^2 \in R$. Put $\operatorname{Im} x = x - \overline{x}$ for $x \in R_E$. Recall that $a\overline{a} = 1 = c\overline{c}$. Then $\operatorname{Im}(a-1)/(a-c) = -a'c'/(a'-c') \in \boldsymbol{\pi}^M R_E^\times$, hence

$$\operatorname{Im}(a''/B) = \boldsymbol{\pi}^{N-m}\operatorname{Im}(a'/(a'-c')) \in \boldsymbol{\pi}^{M+N-m} R_E^\times,$$

and our equation will have no solution unless $M + N - m \ge m$. For such m we may regard a''/B as lying in R, rather then R_E. There are two subcases.

If $N < m \le M/2$, thus $m \le M - m$, our equation reduces to $\varepsilon^2 - Dv_1^2 \in \boldsymbol{\pi}^N R$. Then ε, $v_1 \in \boldsymbol{\pi}^{[(N+1)/2]}R$, thus

$$(u\overline{u})^{-1} = 1 + (\varepsilon - a''/B)\boldsymbol{\pi}^{m-N} \in 1 + \alpha\boldsymbol{\pi}^{M-N} + \boldsymbol{\pi}^{m-N+[(N+1)/2]}R.$$

Let us compute the number of solutions u, v. First, note that for $0 < k \le m$ we have

$$\#\{u \in R_E^\times/1 + \boldsymbol{\pi}^m R_E\,;\ u\overline{u} \in 1 + \boldsymbol{\pi}^k R\}$$

$$= \frac{[R_E^\times : 1 + \boldsymbol{\pi}^m R_E]}{[R^\times : 1 + \boldsymbol{\pi}^m R]}[\boldsymbol{\pi}^k R : \boldsymbol{\pi}^m R] = (1 + q^{-1})q^m \cdot q^{m-k}\,.$$

Hence

$$\#\{u \in R_E^\times/1+\boldsymbol{\pi}^m R_E\,;\; (u\overline{u})^{-1} \in 1+\alpha\boldsymbol{\pi}^{M-N}+\boldsymbol{\pi}^{m-N+[(N+1)/2]}R\}$$
$$= (1+q^{-1})q^{m+N-[(N+1)/2]}\,.$$

Further, the cardinality of the set of $v \in R/\pi^{2m}R$ such that $v = v_1\boldsymbol{\pi}^{m-N}$, $v_1 \in \pi^{[(N+1)/2]}R$, thus $v \in \boldsymbol{\pi}^{m-N+[(N+1)/2]}R$, is $q^{m+N-[(N+1)/2]}$. Hence the number of solutions is $(1+q^{-1})q^{2m+2N-2[(N+1)/2]}$, as asserted in *case* (d) of the proposition.

If $M-m<m$, thus $2N,\ M<2m\le M+N$, we need to solve the equation

$$\varepsilon^2 - Dv_1^2 \in \alpha\boldsymbol{\pi}^{M+N-2m}+\boldsymbol{\pi}^N R = \alpha\boldsymbol{\pi}^{M+N-2m}(1+\boldsymbol{\pi}^{2m-M}R).$$

Since $F(\sqrt{D})/F$ is unramified, there is a solution precisely when $M+N$ is even. Put

$$\varepsilon = \boldsymbol{\pi}^{\frac12(M+N)-m}\varepsilon_2, \qquad v_1 = \boldsymbol{\pi}^{\frac12(M+N)-m}v_2.$$

So we need to solve $\varepsilon_2^2 - Dv_2^2 \in 1+\boldsymbol{\pi}^{2m-M}R$. Namely we count the pairs

$$\{(u \in R_E^\times/1+\boldsymbol{\pi}^m R_E\,;\; v = v_1\boldsymbol{\pi}^{m-N} = \boldsymbol{\pi}^{(M-N)/2}v_2 \in R/\boldsymbol{\pi}^{2m}R)\}$$

such that

$$(u\overline{u})^{-1} = 1+\varepsilon_1\boldsymbol{\pi}^{m-N} = 1+(\varepsilon - a''/B)\boldsymbol{\pi}^{m-N}+\pi^{(M-N)/2}\varepsilon_2$$

and $\varepsilon_2^2 - Dv_2^2 \in 1+\boldsymbol{\pi}^{2m-M}R$. The relation $\varepsilon_2^2 - Dv_2^2 \in 1+\boldsymbol{\pi}^{2m-M}R$ can be replaced by $\varepsilon_2^2 - Dv_2^2 \in R^\times$ if we multiply the cardinality by the factor $[R^\times\colon 1+\boldsymbol{\pi}^{2m-M}R]^{-1}$, and it can be replaced by $\varepsilon_2 \in R$ and $v_2 \in R$ if we further multiply by the quotient $[R_E\colon R_E^\times]$ of the volume of R_E by that of $R_E^\times$. Then the number of u is

$$([R_E^\times\colon 1+\boldsymbol{\pi}^m R_E]/[R^\times\colon 1+\boldsymbol{\pi}^m R])[\boldsymbol{\pi}^{(M-N)/2}R\colon \boldsymbol{\pi}^m R],$$

and the number of v is $[\boldsymbol{\pi}^{(M-N)/2}R\colon \boldsymbol{\pi}^{2m}R]$. The product is

$$\begin{aligned}
=&([R_E^\times\colon 1+\boldsymbol{\pi}^m R_E]/[R^\times\colon 1+\boldsymbol{\pi}^m R])[\boldsymbol{\pi}^{(M-N)/2}R\colon \boldsymbol{\pi}^m R]\\
&\cdot[\boldsymbol{\pi}^{(M-N)/2}R\colon \boldsymbol{\pi}^{2m}R][R_E\colon R_E^\times][R^\times\colon 1+\boldsymbol{\pi}^{2m-M}R]^{-1}\\
=&(1+q^{-1})q^m\cdot q^{m-(M-N)/2}\cdot q^{2m-(M-N)/2}\\
&\cdot(1-q^{-2})\cdot((1-q^{-1})q^{2m-M})^{-1}\\
=&(1+q^{-1})^2q^{2m+N}\,.
\end{aligned}$$

This completes *case* (e) of the proposition.

It remains to consider $1 \le m \le N$. Then $\boldsymbol{\pi}^N \equiv 0 \pmod{\boldsymbol{\pi}^m}$, thus $a' - c' \equiv 0 \pmod{\boldsymbol{\pi}^m}$. Considering the entries $(1,1)$ and $(2,2)$ of our matrix identity, we get $(a+c)/2 \equiv 1 (\bmod\, \boldsymbol{\pi}^m)$ (since $b \equiv 0 \pmod{\boldsymbol{\pi}^m}$). Then $a' + c' \equiv 0 \pmod{\boldsymbol{\pi}^m}$, and $a'' = a'\boldsymbol{\pi}^{-m}$, $c'' = c'\boldsymbol{\pi}^{-m} \in R_E$. Denoting $b' = b\boldsymbol{\pi}^{-m}$, $N' = N - m$, we see that the first three equations are always solvable:

$b' = u\overline{u}B\boldsymbol{\pi}^{N'}$,

$a_2 = (a'' + c'')/2 + u\overline{u}B\boldsymbol{\pi}^{N'}(1 - w)$,

$-b_3 = (a'' + c'')/2 + u\overline{u}B\boldsymbol{\pi}^{N'}(1 + w)$

(these equations simply define b, a_2, b_3). The remaining equation is

$$a' + c' + \frac{1}{2}(a' - c')u\overline{u}(1 + (u\overline{u})^{-2} - Dv^2) = 2a_3\boldsymbol{\pi}^{2m}.$$

When $2m \le N, N^+$ every u, v makes a solution. This completes *case* (a) of the proposition. If $N^+ < N$, $2m$, then there are no solutions.

It remains to deal with the case where $N \le N^+$ and $N < 2m$. Put $\varepsilon = (u\overline{u})^{-1} \in R^\times$, $x = (a' + c')/(a' - c')$. We have to solve the equation $\varepsilon^2 + 1 - Dv^2 + 2\varepsilon x \in \boldsymbol{\pi}^{2m-N}R_E$. Note that $\operatorname{Im}(x) \in \boldsymbol{\pi}^{N_1+N_2-N}R_E^\times$. Since $N \le N^+$, we have $N = \min(N_1, N_2)$, and $2m \le 2N \le N_1 + N_2 = N + M$. Hence $\operatorname{Im}(x) \in \boldsymbol{\pi}^{2m-N}R_E$, and we may assume that $x \in R$. Thus we need to solve

$$(\varepsilon + x)^2 - Dv^2 \in x^2 - 1 + \boldsymbol{\pi}^{2m-N}R,$$

for a fixed $x \in \boldsymbol{\pi}^{N^+-N}R^\times \subset R$. Once we find a solution, in $\varepsilon \in R$, then $\varepsilon \in R^\times$; otherwise $\varepsilon \in \boldsymbol{\pi}R$, hence $Dv^2 \in 1 + \boldsymbol{\pi}R$, but $D \notin R^{\times 2}$. Note that $x \pm 1$ is $2a'/(a' - c')$ or $2c'/(a' - c')$, so

$$x^2 - 1 = 4a'c'/(a' - c')^2 \in \boldsymbol{\pi}^{N_1+N_2-2N}R_E^\times = \boldsymbol{\pi}^{M-N}R_E^\times.$$

We distinguish between two cases.

If $N/2 < m \le \min(N, [M/2])$ and $N \le N^+$, then $M - N \ge 2m - N > 0$, and we must have $N = N^+$ (thus $|x| = 1$). Thus we need to count the $\varepsilon = (u\overline{u})^{-1} \in -x + \boldsymbol{\pi}^{m-[N/2]}R$ and $v \in \boldsymbol{\pi}^{m-[N/2]}R/\boldsymbol{\pi}^{2m}R$. Then

$$\#\{u \in R_E^\times/1 + \boldsymbol{\pi}^m R_E\,;\ u\overline{u} \in 1 + \boldsymbol{\pi}^{m-[N/2]}R\}$$

is $(1+q^{-1})q^{m+[N/2]}$, while the number of the v is $q^{m+[N/2]}$. This completes *case* (b) of the proposition.

If $M/2 < m \le N(\le N^+)$, thus $M - N < 2m - N$, we need to solve $(\varepsilon + x)^2 - Dv^2 \in \alpha\boldsymbol{\pi}^{M-N} + \boldsymbol{\pi}^{2m-N}R = \alpha\boldsymbol{\pi}^{M-N}(1 + \boldsymbol{\pi}^{2m-M}R)$ (for some $\alpha \in R^\times$). There is a solution precisely when $M - N$ is even (as $NR_E^\times = R^\times$). As noted above, given a solution, ε must be in $R^\times$. To compute the volume of solutions, fix measures with

$$\int_{R_E^\times} d^\times u = \int_{R^\times} d^\times \varepsilon$$

and $d^\times\varepsilon = (1 - q^{-1})^{-1} d\varepsilon$ (thus $\int_{R^\times} d^\times\varepsilon = \int_R d\varepsilon$). Put

$$A = \delta(\{(u\overline{u} + x)^2 - Dv^2 \in \alpha\boldsymbol{\pi}^{M-N}(1 + \boldsymbol{\pi}^{2m-M}R)\}),$$

$$B = \delta(\{\varepsilon^2 - Dv^2 \in \boldsymbol{\pi}^{M-N}\alpha(1 + \boldsymbol{\pi}^{2m-M}R)\}).$$

Then the volume is

$$\begin{aligned}
&(1 - q^{-2})q^{4m} \int_{u \in R_E^\times} \int_{v \in R} A d^\times u dv \\
&= (1 - q^{-2})q^{4m}(1 - q^{-1})^{-1} \int_{\varepsilon \in R} \int_{v \in R} B d\varepsilon dv \\
&= (1 - q^{-2})(1 - q^{-1})^{-1} q^{4m} q^{-(M-N)} \int_{z \in R_E} \delta(\{Nz \in 1 + \boldsymbol{\pi}^{2m-M}R\}) dz\,.
\end{aligned}$$

The last integral ranges only over $R_E^\times$, and there $dz/|z| = (1 - q^{-2})d^\times z$. Now

$$\int_{R^\times} \delta(\{z \in 1 + \boldsymbol{\pi}^{2m-M}R\}) d^\times z$$

is the inverse of

$$[R^\times : 1 + \boldsymbol{\pi}^{2m-M}R] = (1 - q^{-1})q^{2m-M}\,.$$

Altogether we get

$$(1 - q^{-2})^2(1 - q^{-1})^{-2} q^{4m+N-M-2m+M} = (1 + q^{-1})^2 q^{2m+N},$$

completing *case* (c), and the proposition.

An alternative volume computation is as follows. The cardinality of

$$\{(u \in R_E^\times / 1 + \boldsymbol{\pi}^m R_E\,,\ v \in R/\boldsymbol{\pi}^{2m}R)\,;$$

$$(u\overline{u}+x)^2 - Dv^2 \in \alpha\boldsymbol{\pi}^{M-N}(1+\boldsymbol{\pi}^{2m-M}R)\}$$

is $(1+q^{-1})q^m$ times

$$\#\{(\varepsilon \in R^\times/1+\boldsymbol{\pi}^m R,\ v \in \dots);\ (\varepsilon + x)^2 - Dv^2 \in \dots\},$$

and since ε must be in $R^\times$ to have a solution, this # is equal to

$$\#\{(\varepsilon \in R/\boldsymbol{\pi}^m R,\ v \in R/\boldsymbol{\pi}^{2m} R);\ \varepsilon^2 - Dv^2 \in \alpha\boldsymbol{\pi}^{M-N}(1+\cdots)\}.$$

As $\varepsilon = \varepsilon_1 \boldsymbol{\pi}^{(M-N)/2}$, $v = v_1\boldsymbol{\pi}^{(M-N)/2}$, this product is

$$(1+q^{-1})q^m \cdot q^{m-(M-N)/2} \cdot q^{2m-(M-N)/2}$$

$$\cdot \operatorname{vol}\{z \in R_E;\ Nz \in 1 + \boldsymbol{\pi}^{2m-M}R\},$$

which equals $(1+q^{-1})^2 q^{2m+N}$, as required. □

12. Proposition. *When $\overline{\rho} = 0$ the orbital integral $\int_{T_\rho\backslash G} 1_K(g^{-1}t_\rho g)dg$ is equal, if $N_1 < N$, in which case $N^+ = N_1 = N_2$, to*

$$-\frac{q+1}{q^4-1}(1+q^{2+4[N_1/2]}) - \frac{(-q)^{N+N_1}}{q-1} + \delta(2 \mid N+N^+)\frac{q+1}{q-1}q^{2N_1+N},$$

and if $N \leq N_1$ to

$$-\frac{q+1}{q^4-1}(1+q^{2+4[N/2]}) - \frac{(-q)^{M+N}}{q-1} + \delta(2 \mid M-N)\frac{q+1}{q-1}q^{2N+M}.$$

Proof. It suffices to prove this with N_1 replaced by N^+, as $N_1 < N$ precisely when $N^+ < N$, in which case $N^+ = N_1$. If $N^+ < N$, $j = 0$ contributes

$$1 + \sum_{1 \leq m \leq \min([N/2],[N^+/2])} (1-q^{-2})q^{4m} = \frac{q^2-1}{q^4-1}(1+q^{2+4[N^+/2]}).$$

The $j > 0$ contributes, when $\delta = 0$, thus $N + N^+$ is odd, the expression:

$$-\frac{q^2+q}{q^4-1}(1+q^{2+4[N^+/2]}) + \frac{q^{N+N^+}}{q-1},$$

while when $\delta = 1$, thus $N + N^+$ is even, the $j > 0$ contribute to the orbital integral:

$$-\frac{q^2+q}{q^4-1}(1+q^{2+4[N^+/2]}) + \frac{1}{q-1}\Big(q^{1+2[N^+/2]+2[N/2]} + (q+1)q^{N+2N^+} - (q+1)q^{N+2[N^+/2]}\Big).$$

The sum is as stated in the proposition.

If $N \leq N^+$, the sum is (when $M/2 < N$ and also when $M/2 \geq N$)

$$\begin{aligned}
&\frac{q^2+q}{q^4-1}(q^{4[N/2]}-1) + 1 + q^2(q^2-1) \sum_{0\leq m<[N/2]} q^{4m} \\
&+ (1+q^{-1})q^{2[N/2]} \sum_{[N/2]+1\leq m\leq[M/2]} q^{2m} \\
&+ \delta(2 \mid M-N)(1+q^{-1})^2 q^N \sum_{[M/2]+1\leq m\leq[(M+N)/2]} q^{2m} \\
&= -\frac{q+1}{q^4-1} + \frac{q^4+q}{q^4-1}q^{4[N/2]} + q^{2[N/2]+1}\cdot\frac{q^{2[M/2]}-q^{2[N/2]}}{q-1} \\
&+ \delta\frac{q+1}{q-1}q^N(q^{M+N}-q^{2[M/2]}),
\end{aligned}$$

which is easily seen to be the expression of the proposition (consider separately the cases of even ($\delta = 1$) and odd ($\delta = 0$) values of $M - N$). □

F. Conclusion

Put $\Phi(t) = \int_{Z(t)\backslash G} 1_K(g^{-1}tg)dg$. In the notations of Proposition 1 for anisotropic tori which split over E, the κ-orbital integral is

$$\Phi^\kappa_{1_K}(t_0) = \Phi(t_1) + \Phi(t_2) - \Phi(t_3) - \Phi(t_4).$$

The tori $T_1 = Z(t_1)$ and $T_2 = Z(t_2)$ ($Z(t)$ is the centralizer of t in G) embed as tori in H. Denote by K_H the maximal compact subgroup $H \cap K$ of H, by 1_{K_H} its characteristic function in H, choose on H the Haar measure which assigns K_H the volume 1, introduce the stable orbital integral $\Phi^{\mathrm{st}}_{1_{K_H}}(t_0) =$

$\Phi^H(t_1)+\Phi^H(t_2)$, where $\Phi^H(t)=\int_{Z_H(t)\backslash H} 1_{K_H}(h^{-1}th)dh$ and $Z_H(t)$ is the centralizer in H of a regular t in H. It is well known (see, e.g., [F2;I], Proposition II.5) that $\Phi^{\mathrm{st}}_{1_{K_H}}(t_0)=(q^N(q+1)-2)/(q-1)$ (where E/F is unramified).

REMARK. A proof of the last equality — extracted from Mars' letter mentioned in the Remark following the proof of Proposition 4 — is as follows. Thus $G=\mathrm{GL}(V)$ and $K=\mathrm{Stab}(R_E)$, dg on G assigns K the volume 1, dt on $E^\times$ assigns $R_E^\times$ the volumes 1, and $\gamma\in E^\times-F^\times$. Then

$$\int_{E^\times\backslash G} 1_K(g^{-1}\gamma g)dg/dt \quad \text{is} \quad \sum_{E^\times\backslash G/K}|K|/|E^\times\cap gKg^{-1}|1_K(g^{-1}\gamma g).$$

But $E^\times\backslash G/K$ is the set of $E^\times$-orbits on the set of all lattices in E. Representatives are the lattices $R_E(j)$, $j\geq 0$. So our sum is the sum of $|R_E^\times|/|R_E(j)^\times|=[R_E^\times : R_E(j)^\times]$ over the $j\geq 0$ such that $\gamma\in R_E(j)^\times$. As $[R_E^\times : R_E(j)^\times]$ is 1 if $j=0$ and $q^{j+1-f}(q^f-1)/(q-1)$ if $j>0$, putting N for the maximum of the j with $\gamma\in R_E(j)^\times$, the integral equals $(q^N(q+1)-2)/(q-1)$ if $e=1$, and $(q^{N+1}-1)/(q-1)$ if $e=2$ ($ef=2$). Of course, the integral vanishes for γ not in $R_E^\times$. If $\gamma=a+bw\in R_E^\times$, then N is the order of b. Note that the stable orbital integral on the unitary group H in two variables is just the orbital integral on GL(2).

Put $\Delta_{G/H}(t_0)=(-q)^{-N_1-N_2}$. The fundamental lemma is the following.

13. THEOREM. *For a regular t_0 we have $\Delta_{G/H}(t_0)\Phi^\kappa_{1_K}(t_0)=\Phi^{\mathrm{st}}_{1_{K_H}}(t_0)$.*

PROOF. Note that $\Phi(t_2)$ depends only on N_1, N_2, N, so we write $\Phi(t_2)=\varphi(N_1,N_2,N)$, and so $\Phi(t_3)=\varphi(N,N_2,N_1)$ and $\Phi(t_4)=\varphi(N_1,N,N_2)$. If $N=N_2<N_1$, $\Phi(t_2)=\Phi(t_4)$, hence $\Phi^K(t_0)=\Phi(t_1)-\Phi(t_3)$, and this difference is

$$-\frac{2}{q-1}(-q)^{N_2+N_1}+(\delta(2\mid N_1-N_2)-\delta(2\mid N_1-1-N_2))\frac{q+1}{q-1}q^{N_1+2N_2},$$

as required.

If $N=N_1\leq N_2$, $\Phi(t_2)=\Phi(t_3)$, hence $\Phi^\kappa(t_0)=\Phi(t_1)-\Phi(t_4)$, and this difference is

$$-\frac{2}{q-1}(-q)^{N_1+N_2}+(\delta(2\mid N_2-N_1)-\delta(2\mid N_2-1-N_1))\frac{q+1}{q-1}q^{N_2+2N_1},$$

as required.

If $N_1 = N_2 < N$, $\Phi^\kappa(t_0)$ is the sum of

$$\Phi(t_1) = -\frac{q+1}{q^4-1}(1+q^{2+4[N_1/2]}) - \frac{(-q)^{N+N_1}}{q-1} + \delta(2 \mid N+N_1)\frac{q+1}{q-1}q^{N+2N_1},$$

$$\Phi(t_2) = -\frac{q+1}{q^4-1}(1+q^{2+4[N_1/2]}) + \frac{(-q)^{N+N_1}}{q-1} + \delta(2 \mid N-1-N_1)\frac{q+1}{q-1}q^{N+2N_1},$$

and

$$-\Phi(t_3) - \Phi(t_4) = -2\frac{q+1}{q^4-1}(q^{4[N_1+2)/2]} - 1).$$

This sum is $-\frac{2q^{2N_1}}{q-1} + \frac{q+1}{q-1}q^{N+2N_1}$, as required.

Since the two minimal numbers among N_1, N_2, N are equal, we are done. □

We now turn to the ramified case. It remains to deal with regular t' in the torus $T_{H'} \subset H' \subset G'$ of Proposition 1.

14. Proposition. *The integral* $\int_{H'/H'_m} 1_{H'_m}(h^{-1}t'h)dh$ *of Proposition* 3 *is equal to*

$$(q+1)q^{4m} \qquad \text{if} \qquad 0 \le m \le \min([N/2],[N_2/2]),$$

and to

$$(q+1)q^{N+2m} \qquad \text{if} \qquad N \le N_2 \qquad \text{and} \qquad [N/2] < m \le N.$$

Here

$$t' = \operatorname{diag}(\delta^{-1}\begin{pmatrix}\alpha & \beta\boldsymbol{\pi}\\ \beta & \alpha\end{pmatrix}, 1), \qquad \delta\overline{\delta} = \alpha^2 - \boldsymbol{\pi}\beta^2 = 1, \qquad \beta = B\boldsymbol{\pi}^N$$

and $\delta = \delta_1 + \delta_2\sqrt{D}$, $\delta_2 = D_2\boldsymbol{\pi}^{1+N_2}$, *and* B, D_2, δ_1, $\alpha \in R^\times$.

Proof. We need to compute the number of $c \in R_E/\boldsymbol{\pi}^m R_E$, and $a \in R_E^\times/1+\boldsymbol{\pi}^{1+2m}R_E$, for which

$$\begin{pmatrix}\overline{a} & -c\boldsymbol{\pi}\\ -\overline{cu} & a\overline{u}\end{pmatrix}\overline{\delta}\begin{pmatrix}\alpha & \beta\boldsymbol{\pi}\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}a & uc\boldsymbol{\pi}\\ \overline{c} & u\overline{a}\end{pmatrix} = \overline{\delta}\begin{pmatrix}\alpha+\boldsymbol{\pi}\beta(\overline{ac}-ac) & \beta\boldsymbol{\pi}u(\overline{a}^2-\boldsymbol{\pi}c^2)\\ a^2\beta\overline{u}-\boldsymbol{\pi}\beta\overline{c}^2\overline{u} & \alpha+\boldsymbol{\pi}\beta(ac-\overline{ac})\end{pmatrix}$$

lies in H'_m. Using the description of H'_m in Proposition 4, this is equivalent to solving two equations: $|\beta(a^2-\boldsymbol{\pi}\overline{c}^2)| \le |\boldsymbol{\pi}|^m$, which means $0 \le m \le N$ since $a \in R_E^\times$, $c \in R_E$, $\beta \in \boldsymbol{\pi}^N R^\times$ (note that there is no constraint on $u \in E^1$, and the volume of E^1 is 1), and $|\alpha + \boldsymbol{\pi}\beta(\overline{ac}-ac)-\delta| \le |\boldsymbol{\pi}|^{1+2m}$. Replacing c by c/a, the equations simplify to $a\overline{a}-\boldsymbol{\pi}c\overline{c}/a\overline{a}=1$, and $|\alpha + \boldsymbol{\pi}\beta(\overline{c}-c)-\delta| \le |\boldsymbol{\pi}|^{1+2m}$. The last equation implies $\alpha-\delta_1 \in \boldsymbol{\pi}^{1+2m}R$. Since $\alpha^2 = 1+B^2\boldsymbol{\pi}^{1+2N}$, and $1=\delta\overline{\delta}=\delta_1^2-D\delta_2^2$, we conclude that $\delta_2^2 \in \boldsymbol{\pi}^{1+2m}R$, hence $\delta_2 = D_2\boldsymbol{\pi}^{1+N_2} \in \boldsymbol{\pi}^{1+m}R$, and $m \le N_2$. Put

$$c=c_1+c_2 i, \quad i=\sqrt{D}, \quad \overline{c}-c=-2ic_2, \quad c_2=C_2\boldsymbol{\pi}^{n_2} \quad (C_2 \in R^\times).$$

Then our equation becomes $-2BC_2\boldsymbol{\pi}^{N+n_2}-D_2\boldsymbol{\pi}^{N_2} \in \boldsymbol{\pi}^{2m}R$.

We shall now determine the number of c. If $0 \le m \le [N/2]$, then $2m \le N$, hence $2m \le N_2$ (if there are solutions to our equation), namely $m \le [N_2/2]$, and any (C_2 and) c is a solution. The number of c is $\#R_E/\boldsymbol{\pi}^m R_E = q^{2m}$. If $[N/2] < m \le N$, thus $m \le N < 2m$, we consider two subcases. If $m \le [N_2/2]$, or $2m \le N_2$, then $N < N_2$, and there are solutions C_2 precisely when $n_2 \ge 2m-N$, and any C_2 is a solution. Then

$$c_2 = C_2\boldsymbol{\pi}^{n_2} \in \boldsymbol{\pi}^{2m-N}R/\boldsymbol{\pi}^m R \simeq R/\boldsymbol{\pi}^{N-m}R$$

has q^{N-m} possibilities, $c_1 \in R/\boldsymbol{\pi}^m R$ has q^m, and $\#c = q^N$. If $m > [N_2/2]$, or $N_2 < 2m$, there are solutions only when $n_2 = N_2 - N$ ($n_2 \ge 0$ implies $N \le N_2$), and the solutions are given by $C_2 \in -D_2/2B + \boldsymbol{\pi}^{2m-N_2}R$, and again c_2 is determined modulo

$$\boldsymbol{\pi}^{n_2}\boldsymbol{\pi}^{2m-N_2}R/\boldsymbol{\pi}^m R = R/\boldsymbol{\pi}^{N-m}R.$$

Given $c \in R_E/\boldsymbol{\pi}^m R_E$, we need to solve in $a \in R_E^\times/1+\boldsymbol{\pi}^{1+2m}R_E$ the equation

$$(a\overline{a})^2 - a\overline{a} + 1/4 = 1/4 - \boldsymbol{\pi}c\overline{c}, \text{ namely } (a\overline{a}-1/2)^2 = (1-2\boldsymbol{\pi}c\overline{c}+\cdots)^2/4,$$

or $a\overline{a} = 1/2 \pm (1-2\boldsymbol{\pi}c\overline{c}+\cdots)/2$. There are no solutions for the negative sign, and there exists a solution for the positive sign. The number of

$$a \in R_E^\times/1+\boldsymbol{\pi}^{1+2m}R_E \qquad \text{with} \qquad a\overline{a} \in v+\boldsymbol{\pi}^{1+2m}R \qquad (v \in R^\times)$$

is $\#(R_E^\times/1+\boldsymbol{\pi}^{1+2m}R_E)/\#(R^\times/1+\boldsymbol{\pi}^{1+2m}R)$

$$= ((q^2-1)q^{2\cdot 2m}/(q-1)q^{2m}) = (q+1)q^{2m},$$

as asserted. □

15. PROPOSITION. *The last orbital integral of Proposition* 3, *of* 1_K *at a regular* $t = gt'g^{-1} \in G$, *where* $t' \in T_{H'} \subset H' \subset G'$, *is*

$$(q^{4+4\min} - 1)/((q^2+1)(q-1)) + \delta(N \le N_2)q^N(q^{2N+2} - q^{2[N/2]+2})/(q-1).$$

Here $\min = \min([N/2],[N_2/2])$, *and* N, N_2 *are defined in Proposition* 14.

PROOF. The integral is equal to

$$\sum_{0\le m\le\min}(q+1)q^{4m} + \delta(N \le N_2)\sum_{[N/2]<m\le N}(q+1)q^{N+2m},$$

which is equal to the asserted expressions. □

The κ-orbital integral $\Phi^\kappa_{1_K}(t)$ of 1_K on the stable conjugacy class of a regular $t \in T_H \subset H \subset G$ is the difference of

$$\Phi(t) = \int_{T_H\backslash G} 1_K(x^{-1}tx)dx \qquad \text{and} \qquad \Phi'(t) = \int_{Z_G(t'')\backslash G} 1_K(x^{-1}t''x)dx,$$

where $t'' = gt'g^{-1} \in G$ is stably conjugate to t (and $t' \in T_{H'} \subset H' \subset G' = g^{-1}Gg$). The stable conjugacy class of t in H consists of a single conjugacy class, and it is well known (see Remark before Theorem 13) that $\Phi^{\rm st}_{1_{K_H}}(t) = \Phi^H(t) = (q^N-1)/(q-1)$, where N is defined in Proposition 14. The transfer factor $\Delta_{G/H}(t)$ is $(-q)^{-n}$, where if $t = (t_1, 1) \in (EL/K)^1 \times E^1$, the n is defined by $t_1 - 1 \in \boldsymbol{\pi}^n_{EL}R^\times_{EL}$.

16. THEOREM. *For a regular* t *we have* $\Delta_{G/H}(t)\Phi^\kappa_{1_K}(t) = \Phi^{\rm st}_{1_{K_H}}(t)$.

PROOF. Since $t = (\alpha+\beta\sqrt{\boldsymbol{\pi}})(\delta_1 - i\delta_2)$ is $(1+B^2\boldsymbol{\pi}^{1+2N}/2+\cdots+B\sqrt{\boldsymbol{\pi}}\boldsymbol{\pi}^N)$ times

$$(1 + DD_2^2\boldsymbol{\pi}^{2+2N_2} + \cdots - \sqrt{D}D_2\boldsymbol{\pi}^{1+N_2}),$$

namely $1 + B\boldsymbol{\pi}^{N+1/2} - \sqrt{D}D_2\boldsymbol{\pi}^{1+N_2} + \cdots$, we have that n is equal to $\min(1+2N, 2+2N_2)$. If $N \le N_2$, we then need to show that

$$\Phi^\kappa_{1_K}(t) = -q^{1+2N}(q^{N+1}-1)/(q-1).$$

When $N_2 < N$, we have to show that

$$\Phi^\kappa_{1_K}(t) = q^{2+2N_2}(q^{N+1}-1)/(q-1).$$

Proposition 9 gives an explicit expression for $\Phi(t)$. Proposition 15 gives an explicit expression for $\Phi'(t)$. The difference, $\Phi^\kappa_{1_K}(t)$, is easily seen to be equal to $\Phi^H(t)$. □

G. Concluding remarks

Langlands — who stated the fundamental lemma and explained its importance to the study of automorphic forms by means of the trace formula — suggested a proof based on counting vertices of the Bruhat-Tits building of G. Such a proof ([LR], p. 360 [by Kottwitz, in the EL — or ramified — case], and p. 388 [by Blasius-Rogawski, in the E — or unramified — case]; both cases are attributed by [L6], p. 52 to the last author [who claimed them in the last page of his thesis]) presumes building expertise, which I do not have. This technique has not yet been applied in rank > 1 unstable cases.

Since the orbital integrals are just integrals, our idea is simply to perform the integration in a naive fashion, using the fact that $T \subset H$, and using a double coset decomposition $H \backslash G/K$, which we easily establish here. We then obtain a direct and elementary proof, using no extraneous notions. The integrals which we compute are nevertheless nontrivial, and this is reflected in our computations. We have used this direct approach to give a simple proof of the fundamental lemma for the symmetric square lifting [F2;VI] from SL(2) to PGL(3) (in the stable and unstable cases), and a proof [F4;I] of this lemma for the lifting from GSp(2) to GL(4), a rank-two case, by developing and combining twisted analogues of ideas of Kazhdan [K1] and Weissauer [We], who had dealt with endoscopy for GSp(2) (an alternative approach — using lattices — was later found by J. G. M. Mars; see section I.6 below). The importance of the fundamental lemma led us to test this technique in our case. Thus here we apply our direct approach to give an elementary and self contained proof in the unitary case.

I.4 Admissible representations

4.1 Induced representations

The diagram of dual groups homomorphisms implies a diagram of liftings of unramified representations, and of representations induced from characters of the diagonal (minimal Levi) subgroup. When E/F, κ and ω are unramified, this is done via the Satake transform. Let us review these basic facts.

If P is a parabolic subgroup of a connected reductive group G, and (η, V_1) is a representation of a Levi subgroup M of P, *the representation* $\pi = I(\eta)$ *of* G *normalizedly induced from* η is the G-module whose space consists of all functions $\varphi : G \to V_1$ with $\varphi(mngu) = \delta_P^{1/2}(m)\eta(m)\varphi(g)$ for all $m \in M$, $n \in N$ (the unipotent radical of P), $g \in G$ and $u \in U_\varphi$, an open compact subgroup of G. Here $\delta_P(m) = |\det(\mathrm{Ad}(m))|\mathfrak{n}|$, $\mathfrak{n}$ is the Lie algebra of N. Normalized induction means the presence of $\delta_P^{1/2}$ in the transformation formula satisfied by φ. It secures the unitarizability of $I(\eta)$ when η is unitarizable. The action of G is by right shifts: $(\pi(g)\varphi)(h) = \varphi(hg)$. When η is admissible, which means that that each vector in its space is fixed by some open subgroup, and that for each open subgroup the dimension of the space of vectors fixed by it is finite, then $I(\eta)$ is admissible too ([BZ1]). If Ind indicates unnormalized induction then $I(\eta) = \mathrm{Ind}(\delta_P^{1/2}\eta)$.

Here are the cases of concern in this part. In the case of $H = \mathrm{U}(2, E/F)$, a character of the diagonal has the form $\mathrm{diag}(a, \overline{a}^{-1}) \mapsto \mu(a)$ (a in $E^\times$), the corresponding (normalizedly) induced module is denoted by $\rho = {}'I(\mu)$, and $\delta(\mathrm{diag}(a, \overline{a}^{-1})) = |a\overline{a}|_F = |a|_E$, as $N_H = \left\{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right); x \in F \right\}$.

On $G = \mathrm{U}(3, E/F)$, a character of the diagonal whose restriction to the center is ω is given by $\mathrm{diag}(a, b, \overline{a}^{-1}) \mapsto \mu(a)(\omega/\mu)(b)$. The associated normalizedly induced G-module is denoted by $I(\mu)$. Here $\delta(\mathrm{diag}(a, b, \overline{a}^{-1})) = |a|_E^2$. If $i \in E - F$ has $i + \overline{i} = 0$, then

$$N = \left\{ \left(\begin{smallmatrix} 1 & x & iy+\frac{1}{2}x\overline{x} \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1 \end{smallmatrix}\right); x \in E, \, y \in F \right\}.$$

Further, $I(\eta)$ denotes the $G' = \mathrm{GL}(3, E)$-module normalizedly induced from the character η of the diagonal subgroup $E^\times \times E^\times \times E^\times$ of G'. The restriction of η to the center Z' is taken to be ω'. Here $\delta(\mathrm{diag}(a, b, c)) = |a/c|^2$.

Let us recall what we need from the Satake transform. Fix a Haar measure dg on G. Let π be an admissible representation of G with central character ω. If f is a function in $C_c^\infty(G, \omega^{-1})$, the convolution operator $\pi(f dg) = \int_{G/Z} f(g)\pi(g) dg$ has finite rank, hence has a finite trace. Such f is called *spherical* if it is biinvariant under the maximal compact subgroup K of G. Here E/F is assumed to be unramified, R denotes the ring of integers of F and R_E that of E, and $K = \mathrm{U}(3, E/F)(R)$ is the group of $g \in \mathrm{GL}(3, R_E)$ in G. An admissible representation π is called *unramified* if its space contains a nonzero K-fixed vector. If π is irreducible and unramified, such a vector is unique up to a scalar multiple. Thus if f is spherical, $\operatorname{tr}\pi(fdg)$ is zero unless π is unramified.

Denote by $f^\vee$ the function $f^\vee(t) = \sum_\chi F_f(\chi)\chi(t)$ on $t \in \widehat{T}^W$, where W is the Weyl group of the torus $\widehat{T}$ in $\widehat{G}$ (fixed in the definition of the dual group), as well as of the maximally split torus $\mathbf{T}$ in $\mathbf{G}$. The sum ranges over $\chi \in X^*(\widehat{T})^W \simeq X_*(\mathbf{T})^W$. For a regular $u \in T = \mathbf{T}(F)$, put $F(u, fdg) = \Delta(u)\Phi(u, fdg)$. Here the Jacobian $\Delta(u)$ is given by $|\det((1-\operatorname{Ad}(u))|\mathfrak{n})|^{1/2}$. Further, $\Phi(u, fdg)$ denotes the orbital integral of fdg at u. A simple change of variables formula shows that $F(u, fdg)$ is $\delta_B(u)^{1/2}\int_N\int_K f(k^{-1}unk)dkdn$, where B is a Borel (minimal parabolic) subgroup of G (and N is its unipotent radical), hence it depends only on the image χ of u in $\mathbf{T}(F)/\mathbf{T}(R) \to X_*(\mathbf{T})$. Hence we denote it by $F(\chi, fdg)$. The $F(\chi, fdg)$ determine the spherical f completely, and the Satake transform is an isomorphism $f \mapsto f^\vee$ from the Hecke convolution algebra $\mathbb{H}$ of spherical functions to the algebra $\mathbb{C}[X_*(\mathbf{T})]^W$ of W-invariant polynomials on $X_*(\mathbf{T})$.

If π is unramified there is a unique conjugacy class in $\widehat{G}$, represented by $t = t(\pi)$ in $\widehat{T}/W$, such that $\operatorname{tr}\pi(fdg) = f^\vee(t)$ (note that $F(fdg)$ depends too on the choice of measure dg). Note that each irreducible unramified representation is the unique unramified irreducible constituent in the unramified representation normalizedly induced from the unramified character $u \mapsto \chi_u(t(\pi))$ of B/N.

Now our diagram and the Satake transform formally imply a lifting of unramified representations. For example, $e : {}^LH \to {}^LG$ implies $t \mapsto e(t)$, that is $\pi_H(t) \mapsto \pi(e(t))$. Moreover, a dual group map gives rise to a dual map, e.g. $e^* : \mathbb{H} \to \mathbb{H}_H$, of Hecke convolution algebras of spherical functions: $e^*(f) = {}'f$ is defined by $\operatorname{tr}\pi_H(t)({}'fdh) = {}'f^\vee(t) = f^\vee(e(t)) = \operatorname{tr}\pi(e(t))(fdg)$.

Let us make explicit the liftings of unramified representations, or rather the unramified induced representations, implied by our diagram, and the Satake transform. Put $\overline{\mu}$ for $\overline{\mu}(x) = \mu(\overline{x})$.

1. Proposition. (1) *Basechange, b, maps $I(\mu)$ to $I(\mu, \omega'\overline{\mu}/\mu, \overline{\mu}^{-1})$.*
(2) *The endo lifting map e maps ${}'I(\mu)$ to $I(\kappa\mu)$.*
(3) *The endo basechange map e' maps ${}'I(\mu)$ to $I(\mu, \omega'\overline{\mu}/\mu, \overline{\mu}^{-1})$.*
(4) *The functor i indicates induction: the H'-module τ maps to the*

G'-module $I(\tau)$.
(5) *The unstable basechange map b' maps $'I(\mu)$ to the H'-module $'I(\mu,\overline{\mu}^{-1})\otimes\kappa$.*
(6) *The stable basechange map b'' maps $'I(\mu)$ to $'I(\mu,\overline{\mu}^{-1})$.*

This Proposition deals with the case where E/F, κ and ω are unramified. But the result is valid under no restriction. To explain this, let E/F be a quadratic extension of local fields, and denote by π, Π and ρ representations of $\mathrm{U}(3,E/F)$, $\mathrm{PGL}(3,E)$, $\mathrm{U}(2,E/F)$, or of $\mathrm{GL}(3,F)$, $\mathrm{GL}(3,F)\times\mathrm{GL}(3,F)$, $\mathrm{GL}(2,F)$ if $E=F\oplus F$.

Definition. Let (π,Π), (ρ,π) or (ρ,Π) be a pair of induced representations. We say that π *basechange lifts* to Π, ρ *endo-lifts* to π, or ρ *e'-lifts* to Π, if for all matching pairs $(fdg,\phi dg')$, $('fdh,fdg)$ and $('\phi dh,\phi dg')$ of measures (see I.2), we have $\mathrm{tr}\,\pi(fdg)=\mathrm{tr}\,\Pi(\phi dg'\times\sigma)$, $\mathrm{tr}\,\pi(fdg)=\mathrm{tr}\,\rho('fdh)$, $\mathrm{tr}\,\Pi(\phi dg'\times\sigma)=\mathrm{tr}\,\rho('\phi dh)$.

Similar statements hold with respect to the maps b', b'', as discussed in [F3;II]. These relations in the induced case give a hint to be pursued in the general case.

Using the definition of matching of functions in I.2, and the standard computation [F2;I] or [F4], of characters of induced modules (and the twisted character of $I(\eta)$ when η is a σ-invariant character), it is easy to check that:

2. Proposition. *We have*: (1) *$\pi=I(\mu)$ basechange lifts to $\Pi=I(\mu,\omega'\overline{\mu}/\mu,\overline{\mu}^{-1})$;*
(2) *$\rho={}'I(\mu)$ endo-lifts to $\pi=I(\kappa\mu)$;*
(3) *$\rho={}'I(\mu)$ e'-lifts to $\Pi=I(\mu,\omega'\overline{\mu}/\mu,\overline{\mu}^{-1})$;*
(4) *$'I(\mu)$ b'-lifts to $'I(\mu,\overline{\mu}^{-1})\otimes\kappa$ and b''-lifts to $'I(\mu,\overline{\mu}^{-1})$.*

The definition of lifting given above extends to the case of basechange of one-dimensional and Steinberg representations, as follows. A representation of $\mathrm{U}(3,E/F)$ of dimension one has the form $\mu_G: g\mapsto\mu(\det g)$, where μ is a character of E^1. A one-dimensional representation of $\mathrm{GL}(3,E)$ has the form $\mu'_{G'}: g\mapsto\mu'(\det g)$, where μ' is a character of $E^\times$.

Now μ_G is the unique nontempered irreducible constituent (in fact a quotient) in the composition series of the induced representation $I(\mu\nu)$ of $\mathrm{U}(3,E/F)$. The only other constituent, in fact a subrepresentation, denoted

$\mathrm{St}_G(\mu)$, is square integrable, named the *Steinberg* representation (see 4.3 below).

Similarly, $\mu'_{G'}$ is the unique irreducible quotient in the composition series of the induced representation $\Pi = I(\mu'\nu, \mu', \mu'\nu^{-1})$ of $\mathrm{GL}(3, E)$. This Π has a unique irreducible subrepresentation, which is square integrable, denoted $\mathrm{St}_{G'}(\mu')$ and named the *Steinberg* representation. There are two other irreducible constituents in the composition series of Π, nontempered and non-σ-invariant, which are mapped to each other by σ. Both $\mu'_{G'}$ and $\mathrm{St}_{G'}(\mu')$ are σ-invariant.

PROPOSITION 3. *For each character μ of E^1, the representation μ_G of H basechange lifts to $\mu'_{G'}$, where $\mu'(x) = \mu(x/\overline{x})$, and $\mathrm{St}_G(\mu)$ lifts to $\mathrm{St}_{G'}(\mu')$.*

PROOF. It follows from the Weyl integration formula of 4.2 below that $\mathrm{tr}\, \mu'_{G'}(\phi dg' \times \sigma) = \mathrm{tr}\, \mu_G(f dg)$ for all matching measures $f dg$ and $\phi dg'$. Proposition 2 implies the statement for the Steinberg representations since $\mathrm{tr}\, I(\mu'\nu, \mu', \mu'\nu^{-1}; \phi dg' \times \sigma)$ equals the sum of $\mathrm{tr}\, \mu'_{G'}(\phi dg' \times \sigma)$ and

$$\mathrm{tr}\, \mathrm{St}_{G'}(\mu')(\phi dg' \times \sigma).$$

Indeed, the other two constituents in the composition series of Π are not σ-invariant, hence have twisted-trace zero. □

Analogous definitions and results apply in the case where $E = F \oplus F$. Let us briefly recall the lifting in the case where *the place v splits in E* (see [F1;III], section 1.5, for a fuller discussion in the case of basechange). In this case $E_v = F_v \oplus F_v$ and H, G, G' are $\mathrm{GL}(2, F_v)$, $\mathrm{GL}(3, F_v)$ and $\mathrm{GL}(3, F_v) \times \mathrm{GL}(3, F_v)$. We now omit v for brevity. The generator σ of $\mathrm{Gal}(E/F)$ acts on $\mathbf{G}'(F) = \mathbf{G}(E)$ by mapping (x, x') to $(\theta x', \theta x)$ where $\theta x = J\,{}^t x^{-1} J$ for x in G. The component at v of the global character κ is a character of $E^\times = F^\times \times F^\times$ invariant under σ. It is a pair (κ, κ^{-1}) of characters of $F^\times$.

The notion of local lifting which we use when $E = F \oplus F$ is again defined via character relations, thus e.g. π basechange lifts to Π if $\mathrm{tr}\, \pi(f dg) = \mathrm{tr}\, \Pi(\phi dg' \times \sigma)$ for all matching $f dg$, $\phi dg'$. Recall that matching functions is a relation defined in this case in [F1;III], section 1.5. It is then easy to check ([F1;III], section 1.5, in the case of basechange; computation of the character of an induced G-module in the endo-cases), that

PROPOSITION 4. (1) *π lifts to $\Pi = \pi \oplus \sigma\pi$ by basechange;*
(2) *τ lifts to $I(\tau \otimes \kappa)$ in the case of endo-lifting, where κ is the character of $F^\times$ fixed in the definition of the endo-lifting;*
(3) *τ lifts to $I(\tau) \oplus I(\sigma\tau) = I(\tau \oplus \sigma\tau)$ in the case of σ-endo-lifting.*

Here $(\sigma\pi)(x) = \pi(\sigma x)$, and $(\sigma\tau)(x) = \tau(\sigma x)$, as usual.

4.2 Characters

Our study of the lifting is based on the Harish-Chandra theory [HC2] of characters, which we briefly now record. Let π be a representation of a connected reductive p-adic group G. Suppose it is irreducible. By Schur's lemma it has a central character, say ω. Suppose it is also admissible. Then for each test function f in $C_c^\infty(G, \omega^{-1})$ and Haar measure dg, the convolution operator $\pi(f dg) = \int_{G/Z} f(g)\pi(g)dg$ has finite rank. Hence the trace $\operatorname{tr}\pi(f dg)$ is defined. Then [HC2] asserts the following.

PROPOSITION 1. *There exists a complex valued function χ_π on G which is locally constant on the regular set of G, conjugacy invariant, transforms by $\chi_\pi(zg) = \omega(z)\chi_\pi(g)$ under Z, and is locally integrable, such that for all f in $C_c^\infty(G, \omega^{-1})$ we have*

$$\operatorname{tr}\pi(f dg) = \int_{G/Z} \chi_\pi(g) f(g) dg.$$

The method of [HC2] applies in the twisted case too. Let Π be an irreducible σ-invariant G'-module. Thus ${}^\sigma\Pi \simeq \Pi$, where ${}^\sigma\Pi(g) = \Pi(\sigma(g))$. Then there is a nonzero intertwining operator $A : \Pi \to {}^\sigma\Pi$ with $A\Pi(g) = \Pi(\sigma(g))A$. In particular A^2 is a scalar by Schur's lemma, since Π is irreducible. Replacing A by its product with the complex number $\sqrt{A^2}^{-1}$ we see that $A^2 = 1$ and A is unique up to sign. This sign can be fixed by requiring, when Π is generic, that A acts on the Whittaker model by $AW(g) = W(\sigma(g))$, and when Π is unramified, that A fixes the unique (up to scalar) K-fixed vector. These normalizations are clearly compatible. Put $\Pi(\sigma) = A$. Denote by $\Pi(\phi dg' \times \sigma)$ the convolution operator $\int_{G'/Z'} \phi(g')\Pi(g' \times \sigma)dg'$, dg' is a Haar measure. Harish-Chandra's theory [HC2] extends to the twisted case (see [Cl2]) to assert:

Proposition 2. *Given an admissible irreducible σ-invariant G'-module* Π *with central character ω', there exists a locally-integrable function χ_Π^σ on G', which transforms by ω' on Z', satisfies $\chi_\Pi^\sigma(g) = \chi_\Pi^\sigma(xg\sigma(x)^{-1})$ for all x and g in G', and is smooth on the σ-regular set, such that* $\operatorname{tr}\Pi(\phi dg' \times \sigma)$ *is equal to $\int_{G'/Z'} \chi_\Pi^\sigma(g')\phi(g')dg'$ for all ϕ.*

Also we use the *Weyl integration formula*. Let $\{T\}$ be a set of representatives for the conjugacy classes of tori in G. An element of G is called regular if its centralizer in G is a torus. Write $\operatorname{Int}(g)t = gtg^{-1}$. Then the regular set G^{reg} of G is the disjoint union $\cup_{\{T\}} \operatorname{Int}(G/T)T^{\text{reg}}$. The Weyl integration formula asserts:

Proposition 3. *For all $f \in C_c^\infty(G/Z)$ we have*

$$\int_{G/Z} f(g)dg = \sum_T [W(T)]^{-1} \int_{T/Z} \Delta(t)^2 \int_{G/T} f(\operatorname{Int}(g)t)\frac{dg}{dt}dt.$$

Here $\Delta(t)^2$ is the Jacobian $|\det(I - \operatorname{Ad}(t)|\mathfrak{g}/\mathfrak{t}))|$, $\mathfrak{t}$ is the Lie algebra of T, $\mathfrak{g}$ of G, and $[W(T)]$ indicates the cardinality of the Weyl group $W(T)$ (normalizer of T in G, quotient by the centralizer).

In the twisted case we say that $g \in G'$ is *σ-regular* if $g\sigma(g)$ is regular. Write $G'^{\sigma-\text{reg}}$ for the set of such elements. Let $\{T\}_s$ indicate the set of representatives of stable conjugacy classes of tori T of G. For each T in this set, write T' for its centralizer $Z_{G'}(T)$ in G'. Then T' is a σ-invariant torus in G' and $T = T'^\sigma = \{t \in T; \sigma(t) = t\} = T' \cap G'$. Write $\operatorname{Int}^\sigma(g)t = gt\sigma(g)^{-1}$. Proposition I.1.5 shows that

$$\begin{aligned} G'^{\sigma-\text{reg}}/Z' &= \cup_{\{T\}_s} \operatorname{Int}^\sigma(G'/T')(T'^{\sigma-\text{reg}}/Z') \\ &= \cup_{\{T\}_s} \operatorname{Int}^\sigma(G'/T)(T'^{\sigma-\text{reg}}/Z'T'^{1-\sigma}). \end{aligned}$$

Here $T'^{1-\sigma} = \{t\sigma(t)^{-1};\, t \in T'\}$. Note that $T'^{\sigma-\text{reg}}/Z'T'^{1-\sigma}$ contains a set of representatives for the σ-conjugacy classes within each stable σ-conjugacy class represented in T'. Put $W^\sigma(T')$ for the quotient of the σ-normalizer $\{n \in G'; nT'\sigma(n)^{-1} \subset T'\}$ of T' in G', by the σ-centralizer of T' in G'. Write $\Delta(t\times\sigma)^2$ for the Jacobian $|\det(I - \operatorname{Ad}(t\times\sigma)|\mathfrak{g}'/\mathfrak{t}'))|$.

The twisted Weyl integration formula asserts:

Proposition 4. *For any $\phi \in C_c^\infty(G'/Z')$ we have $\int_{G'/Z'} \phi(g\times\sigma)dg'$*

$$= \sum_{\{T\}_s} [W^\sigma(T')]^{-1} \int_{T'/T'^{1-\sigma}} \Delta(t\times\sigma)^2 \int_{G'/TZ'} \phi(\operatorname{Int}(g)(t\times\sigma))\frac{dg'}{dt}dt.$$

4.3 Reducibility

Suppose that E/F is a quadratic extension of local fields, and ν is the valuation character $\nu(x) = |x|$ on $E^\times$. Suppose μ' is a unitary character of $E^\times$, and s a real number. The induced representations $I(\mu'\nu^s)$ and $I(\overline{\mu'}^{-1}\nu^{-s})$ have equal traces, hence equivalent composition series. In particular they are equivalent if they are irreducible. Hence we assume $s \geq 0$.

There are three cases in which an induced G-module is reducible [Ke]. The composition series in these cases has length two (since $[W(A)]$=2, where A denotes the diagonal torus), and μ' is then a character of $E^\times$ which is trivial on $F^\times$ (thus $\mu'(x) = \mu(x/\overline{x})$ for some μ on E^1). The cases are listed in the

Proposition. (1) *If $\mu'^3 \neq \omega'$, then $I(\mu')$ is the direct sum of tempered non-discrete-series G-modules denoted by π^+ and π^-. Namely the condition for reducibility is that the restriction to $A \cap \mathrm{SL}(3, E)$, of the character* $\mathrm{diag}(a, b, \overline{a}^{-1}) \mapsto \mu'(a)(\omega/\mu')(b)$ *which defines $I(\mu')$ (thus $b = \overline{a}/a$), is nontrivial.*
(2) *$I(\mu'\kappa\nu^{1/2})$ has a nontempered component $\pi^\times_{\mu'}$ and a discrete-series component $\pi^+_{\mu'}$.*
(3) *If $\omega = \theta^3$, and $\mu' = \theta/\overline{\theta}$ for a character θ of E^1, then $I(\mu'\nu)$ has the nontempered one-dimensional component $\pi(\mu'\nu)$, and the Steinberg square-integrable component $St(\mu'\nu)$.*

Otherwise the induced $I(\mu'\nu^s)$ is irreducible.

4.4 Coinvariants

Some of our proofs below are inductive on the rank, and depend on reduction to the elliptic set of smaller Levi subgroup.

In our rank-one case there is only one induction step, and here we set up the required notations. Let E/F be a quadratic extension of local fields.

Denote by A the diagonal subgroup, by N the unipotent upper triangular subgroup of G, and by K the maximal compact subgroup $\mathbf{G}(R)$ of G, so that $G = ANK$; R is the ring of integers in F. We use the analogous notations $'A$, $'N$, $'K$ in the case of H, and A', N', K' in the case of G', the even drop the primes if no confusion is likely to occur.

Definition. (1) If $g = ank$, $a = \operatorname{diag}(\alpha, \beta, \overline{\alpha}^{-1}) \in A$, $n \in N$, $k \in K$, put $\delta(g) = |\alpha|^2$. This is the modulus function on G.
(2) For a function f on G, and $a = \operatorname{diag}(\alpha, \beta, \overline{\alpha}^{-1}) \in A$, put

$$f_N(a) = \delta(a)^{1/2} \int_K \int_N f(k^{-1}ank) dn dk.$$

(3) Let (π, V) be a G-module. The quotient V_N of V by the span of the vectors $\pi(n)v - v$ (n in N, v in V) is an A-module $\tilde{\pi}_N$. The *normalized A-module* (π_N, V_N) *of N-coinvariants* of π is the tensor product of $(\tilde{\pi}_N, V_N)$ with $\delta^{1/2}$.
(4) The central characters of the irreducible constituents in π_N, $N \neq \{1\}$, are called *central exponents* of π.

In our case π_N consists of up to two characters of A, thus π has at most two central exponents. In general, if π is admissible, then so is π_N (see [BZ1]).

A theorem of Deligne [D6] and Casselman [C1] asserts

Lemma. *At $a = \operatorname{diag}(\alpha, \beta, \overline{\alpha}^{-1})$ with $|\alpha| < 1$ we have $\chi_\pi(a) = \chi_{\tilde{\pi}_N}(a)$. Hence $\Delta\chi_\pi(a) = \chi_{\pi_N}(a)$, where $\Delta(a) = |(\alpha - \beta)(\beta - \overline{\alpha}^{-1})|$ ($= |\alpha|^{-1}$ if $|\alpha| < 1$).*

Consequently, if f is supported on the conjugacy classes of the a with $|\alpha| < 1$, the Weyl integration formula implies that

$$\operatorname{tr} \pi(f dg) = \operatorname{tr} \pi_N(f_N da).$$

Similar definitions apply in the cases of H and G'-modules.

Definition. A G-module π is called *cuspidal* if π_N is $\{0\}$. A G-module π is called *tempered* if its central exponents are bounded, and *square integrable* if its central exponents are strictly less than 1 on the a with $|\alpha| < 1$. In particular, a square-integrable π has at most one central exponent in π_N.

An alternative definition is as follows. An admissible irreducible G-module π is called *square integrable*, or *discrete series*, if it has a coefficient $f(g) = \langle \pi(g)v, v' \rangle$ which is absolutely square integrable on G/Z, where Z is the center of G. Such a π is called *cuspidal* if there is a compactly supported (modulo center) such a coefficient, in which case the property holds for every coefficient.

REMARK. Harish-Chandra used the terminology "cuspidal" for what is currently called square integrable (or discrete series), and he used the terminology "supercuspidal" for what we (and [BZ1]) call cuspidal. It is unnecessary to use the term "supercuspidal" when there is no term "cuspidal".

I.5 Representations of U(2,1; $\mathbb{C}/\mathbb{R}$)

Here we record well-known results concerning the representation theories of the groups of this part in the case of the archimedean quadratic extension $\mathbb{C}/\mathbb{R}$. For proofs we refer to [Wh], §7, to [BW], Ch. VI for cohomology, and to [Cl1], [Sd] for character relations. This is then used in conjunction with Theorem III.5.2.1 and its corollaries to determine all automorphic $\mathbf{G}(\mathbb{A})$-modules with nontrivial cohomology outside of the middle dimension.

We first recall some notations. Denote by σ the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Put $\overline{z} = \sigma(z)$ for z in $\mathbb{C}$, and $\mathbb{C}^1 = \{z/|z|;\ z \text{ in } \mathbb{C}^\times\}$. Put $H' = \mathrm{GL}(2,\mathbb{C}), G' = \mathrm{GL}(3,\mathbb{C})$,

$$H = \mathrm{U}(1,1) = \left\{h \text{ in } H'; hw^t\overline{h} = w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

and

$$G = \mathrm{U}(2,1) = \left\{g \text{ in } G'; gJ^t\overline{g} = J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}\right\}.$$

The center Z of G is isomorphic to $\mathbb{C}^1$; so is that of H. Fix an integer $\mathbf{w}$ and a character $\omega(z/|z|) = (z/|z|)^{\mathbf{w}}$ of $\mathbb{C}^1$. Put $\omega'(z) = \omega(z/\overline{z})$. Any representation of any subgroup of G which contains Z will be assumed below to transform under Z by ω.

The diagonal subgroup A_H of H will be identified with the subgroup of the diagonal subgroup A of G consisting of $\mathrm{diag}(z, z', \overline{z}^{-1})$ with $z' = 1$. For any character χ_H of A_H there are complex a, c with $a + c$ in $\mathbb{Z}$ such that

$$\chi_H(\mathrm{diag}(z, \overline{z}^{-1})) = (\text{“}z^a(\overline{z}^{-1})^c = \text{”})|z|^{a-c}(z/|z|)^{a+c}.$$

The character χ_H extends uniquely to a character χ of A whose restriction to Z is ω. In fact $b = \mathbf{w} - a - c$ is integral, and $\chi = \chi(a,b,c)$ is defined by

$$\chi(\mathrm{diag}(z, z', \overline{z}^{-1})) = z'^b|z|^{a-c}(z/|z|)^{a+c}.$$

A character κ of $\mathbb{C}^\times$ which is trivial on the multiplicative group $\mathbb{R}_+^\times$ of positive real numbers but is nontrivial on $\mathbb{R}^\times$ is of the form $\kappa(z) = (z/|z|)^{2k+1}$, where k is integral.

The H-module $I(\chi_H) = I(\chi_H; B_H, H) = \mathrm{Ind}(\delta_H^{1/2}\chi_H; B_H, H)$ normalizedly induced from the character χ_H of A_H extended trivially to the upper triangular subgroup B_H of H, is irreducible unless a, c are equal with $a+c$ an odd integer, or are distinct integers. If $a = c$ and $a+c \in 1+2\mathbb{Z}$ then χ_H is unitary and $I(\chi_H)$ is the direct sum of two tempered representations. If a, c are distinct integers the sequence $JH(I(\chi_H))$ of constituents, repeated with their multiplicities, in the composition series of $I(\chi_H)$, consists of (1) an irreducible finite-dimensional H-module $F_H = F_H(\chi_H) = F_H(a,c)$ of dimension $|a-c|$ (and central character $z \mapsto z^{a+c}$), and (2) the two irreducible square-integrable constituents of the packet $\rho = \rho(a,c)$ (of highest weight $|a-c|+1$) on which the center of the universal enveloping algebra of H acts by the same character as on F_H.

The Langlands classification [L7] (see also [BW], Ch. IV) defines a bijection between the set of packets and the set of $\widehat{H}$-conjugacy classes of homomorphisms from the Weil group

$$W_{\mathbb{C}/\mathbb{R}} = \langle z, \sigma; z \text{ in } \mathbb{C}^\times, \sigma z = \overline{z}\sigma, \sigma^2 = -1\rangle$$

to the dual group ${}^L H = \widehat{H} \rtimes W_{\mathbb{C}/\mathbb{R}}$ ($W_{\mathbb{C}/\mathbb{R}}$ acts on the connected component $\widehat{H} = \mathrm{GL}(2,\mathbb{C})$ by $\sigma(h) = w^t h^{-1} w^{-1}$ $(= \frac{1}{\det h} h)$), whose composition with the second projection is the identity. Note that $W_{\mathbb{C}/\mathbb{R}}$ is the subgroup $\mathbb{C}^\times \cup \mathbb{C}^\times j$ of $\mathbb{H}^\times$, where $\mathbb{H}$ is the Hamiltonian quaternions, and σ is j. The norm $\mathbb{H} \to \mathbb{R}_{\geq 0}$ defines a norm $W_{\mathbb{C}/\mathbb{R}} \to \mathbb{R}_{>0}^\times$. Such homomorphism is called *discrete* if its image is not conjugate by $\widehat{H}$ to a subgroup of $\widehat{B}_H = B_H \rtimes W_{\mathbb{C}/\mathbb{R}}$. The packet $\rho(a,c) = \rho(c,a)$ corresponds to the homomorphism $y(\chi_H) = y(a,c)$ defined by

$$z \mapsto \begin{pmatrix} (z/|z|)^a & 0 \\ 0 & (z/|z|)^c \end{pmatrix} \times z, \quad \sigma \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \sigma.$$

It is discrete if and only if $a \neq c$.

The composition $y(a,b,c)$ of $y(\chi_H \otimes \kappa^{-1}) = y(a-2k-1, c-2k-1)$ with the endo-lift $e : {}^L H \to {}^L G$ is the homomorphism $W_{\mathbb{C}/\mathbb{R}} \to {}^L G$ defined by

$$z \mapsto \begin{pmatrix} (z/|z|)^a & & 0 \\ & (z/|z|)^b & \\ 0 & & (z/|z|)^c \end{pmatrix} \times z, \quad \sigma \mapsto J \times \sigma.$$

Here $b = \mathbf{w} - a - c$ is determined by a, c, and the central character, thus $\mathbf{w}$. The corresponding G-packet $\pi = \pi(a,b,c)$ depends only on the set $\{a,b,c\}$. It consists of square integrables if and only if a, b, c are distinct.

The irreducible representations of $\mathrm{SU}(2,1)$ (up to equivalence) are described in [Wh], §7. We proceed to summarize these results, but in the standard notations of normalized induction, which are used for example in [Kn], and in our p-adic theory. Thus [Wh], (1) on p. 181, defines the induced representation

π_Λ on space of functions transforming by $f(gma) = e^{\Lambda(a)} f(g)$,

while [Kn] defines the induced representation

I_Λ on space of functions transforming by $f(gma) = e^{(-\Lambda-\rho)(a)} f(g)$. Thus

$$\pi_\Lambda = I_{-\Lambda-\rho}, \qquad \pi_{-\Lambda-\rho} = I_\Lambda,$$

and ρ is half the sum of the positive roots. Note that the convention in representation theory of real groups is that G acts on the left: $(I_\Lambda(h)f)(g) = f(h^{-1}g)$, while in representation theory of p-adic groups the action is by right shifts: $(I(\Lambda)(h)f)(g) = f(gh)$, and f transforms on the left: "$f(mag) = e^{(\Lambda+\rho)(ma)} f(g)$". We write $I(\Lambda)$ for right shift action, which is equivalent to the left shift action I_Λ of, e.g., [Kn].

To translate the results of [Wh], §7, to the notations of [Kn], and ours, we simply need to replace Λ of [Wh] by $-\Lambda-\rho$. Explicitly, we choose the basis $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,1,-1)$ of simple roots in the root system Δ of $\mathfrak{g}_\mathbb{C} = \mathrm{sl}(3,\mathbb{C})$ relative to the diagonal $\mathfrak{h}$ (note that in the definition of Δ^+ in [Wh], p. 181, h should be H). The basic weights for this order are $\Lambda_1 = \left(\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\right)$, $\Lambda_2 = \left(\frac{1}{3},\frac{1}{3},-\frac{2}{3}\right)$, [Wh] considers π_Λ only for "*G-integral*" $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$ (thus $k_i \in \mathbb{C}$, $k_1 - k_2 \in \mathbb{Z}$), and $\rho = (1,0,-1) = \alpha_1 + \alpha_2 = \Lambda_1 + \Lambda_2$. Then [Wh], 7.1, asserts that I_Λ is reducible iff $\Lambda \neq 0$ and Λ is *integral* ($k_i \in \mathbb{Z}$), and [Wh], 7.2, asserts that I_Λ is unitarizable iff $\langle \Lambda, \rho\rangle \in i\mathbb{R}$. The normalized notations I_Λ are convenient as the infinitesimal character of $I_{s\Lambda}$ for any element s in the Weyl group $W_\mathbb{C} = S_3$ is the $W_\mathbb{C}$-orbit of Λ. In the unnormalized notations of [Wh], p. 183, l. 13, one has $\chi_\Lambda = \chi_{s(\Lambda+\rho)-\rho}$ instead. The Weyl group $W_\mathbb{C}$ is generated by the reflections $s_i\Lambda = \Lambda - \langle \Lambda, \alpha_i^\vee\rangle\alpha_i$, where $\alpha_i^\vee = 2\alpha_i/\langle\alpha_i,\alpha_i\rangle$ is α_i. Put $w_0 = s_1s_2s_1 = s_2s_1s_2$ for the longest element.

For integral $k_i = \langle \Lambda, \alpha_i\rangle < 0$ ($i = 1,2$), [Wh], p. 183, l. -3, shows that I_Λ contains a finite-dimensional representation F_Λ. Thus F_Λ is a quotient of $I_{w_0\Lambda}$, and has infinitesimal character $w_0\Lambda$ and highest weight $w_0\Lambda-\rho$. Note that $\mathcal{F}$ in midpage 183 and $\mathcal{F}^+$ in 7.6 of [Wh] refer to integral and not G-integral elements. For such Λ the set of discrete-series representations sharing infinitesimal character $(W_{\mathbb{C}}\cdot\Lambda)$ with F_Λ consists of $D^+_{s_1s_2\Lambda}$, $D^-_{s_2s_1\Lambda}$, $D_{w_0\Lambda}$ ([Wh], 7.6, where "G" should be "$\hat{G}$"). The holomorphic discrete series $D^+_{s_2w_0\Lambda}$ is defined in [Wh], p. 183, as a subrepresentation of $I_{s_2w_0\Lambda}$, and it is a constituent also of $I_{w_0s_2w_0\Lambda}=I_{s_1\Lambda}$ ([Wh], 7.10) but of no other $I_{\Lambda'}$. The antiholomorphic discrete series $D^-_{s_1w_0\Lambda}$ is defined as a sub of $I_{s_1w_0\Lambda}$ and it is a constituent of $I_{s_2\Lambda}=I_{w_0s_1w_0\Lambda}$, but of no other $I_{\Lambda'}$. The nonholomorphic discrete series $D_{w_0\Lambda}$ is defined as a sub of $I_{w_0\Lambda}$ and it is a constituent of $I_{s\Lambda}$ for all $s\in W_{\mathbb{C}}$, but of no other $I_{\Lambda'}$.

Let us repeat this with Λ positive: $k_i=\langle\Lambda,\alpha_i\rangle>0$ $(i=1,2)$ (we replace Λ by $w_0\Lambda$).
F_Λ is a quotient of I_Λ;
$D^+_{s_2\Lambda}$ lies (only) in $I_{s_2\Lambda}$, $I_{w_0s_2\Lambda}$;
$D^-_{s_1\Lambda}$ lies (only) in $I_{s_1\Lambda}$, $I_{w_0s_1\Lambda}$;
D_Λ lies in $I_{s\Lambda}$ for all $s\in W_{\mathbb{C}}$.

The induced I_Λ is reducible and unitarizable iff $\Lambda\neq 0$ and $\langle\Lambda,\rho\rangle=0$, thus $k_1+k_2=0$, $k_i\neq 0$ in $\mathbb{Z}$, and $\Lambda=k_1(\Lambda_1-\Lambda_2)=k_1s_2\Lambda_2=-k_1s_1\Lambda_1$. The composition series has length two ([Wh], (i) and (ii) on p. 184, and 7.11). We denote them by $\pi^\pm_\Lambda$ (corresponding to $\pi^\pm_{-\Lambda-\rho}$ in [Wh]). These $\pi^\pm_\Lambda$ do not lie in any other $I_{\Lambda'}$ than indicated next.

If $k_1<0$ then $\Lambda=-k_1s_1\Lambda_1$, π^-_Λ lies in I_Λ and π^+_Λ in $I_{s\Lambda}$ for all $s\in W_{\mathbb{C}}$.

Thus $\pi^-_{s_1\Lambda}$ lies in $I_{s_1\Lambda}$ and $\pi^+_{s_1\Lambda}$ in $I_{s\Lambda}$ for all $s\in W_{\mathbb{C}}$, where $\Lambda\geq 0$ has $k_2=0$, $k_1>0$.

If $k_1>0$ then $\Lambda=k_1s_2\Lambda_2$, π^+_Λ lies in I_Λ and π^-_Λ in $I_{s\Lambda}$ for all $s\in W_{\mathbb{C}}$.

Thus $\pi^+_{s_2\Lambda}$ lies in $I_{s_2\Lambda}$ and $\pi^-_{s_2\Lambda}$ in $I_{s\Lambda}$ for all $s\in W_{\mathbb{C}}$, where $\Lambda\geq 0$ has $k_1=0$, $k_2>0$.

There are also nontempered unitarizable non one-dimensional representations $J^\pm_k$ $(k\geq -1)$. J^+_k is defined in [Wh], p. 184, as a sub of $I_{-k\Lambda_1-\rho}$, thus a constituent of $I_{-w_0(k\Lambda_1+\rho)}=I_{\Lambda_1+(k+1)\Lambda_2}$, and it is a constituent also of $I_{-s_1(k\Lambda_1+\rho)}$ and $I_{-s_1s_2(k\Lambda_1+\rho)}$ but of no other $I_{\Lambda'}$, unless $k=-1$ where J^+_{-1} is a constituent of $I_{s\Lambda_1}$ for all $s\in W_{\mathbb{C}}$.

Similarly J^-_k is a sub of $I_{-k\Lambda_2-\rho}$ and a constituent of $I_{-w_0(k\Lambda_2+\rho)}=I_{(k+1)\Lambda_1+\Lambda_2}$, and a constituent of $I_{-s_2(k\Lambda_2+\rho)}$, $I_{-s_2s_1(k\Lambda_2+\rho)}$ but of no other $I_{\Lambda'}$, unless $k=-1$ where J^-_{-1} is a constituent of $I_{s\Lambda_2}$ for all $s\in W_{\mathbb{C}}$ (see [Wh], 7.12, where in (1) Λ_2 should be Λ_1).

Let us express this with $\Lambda > 0$.

If $k_1 = 1$, $k_2 = k+1 \geq 0$, $J_k^+ = J_{s_2\Lambda}^+$ is a constituent of I_Λ, $I_{w_0\Lambda}$, $I_{s_2\Lambda}$, $I_{s_2 s_1\Lambda}$.

If $k_2 = 1$, $k_1 = k+1 \geq 0$, $J_k^- = J_{s_1\Lambda}^-$ is a constituent of I_Λ, $I_{w_0\Lambda}$, $I_{s_1\Lambda}$, $I_{s_1 s_2\Lambda}$.

To compare the parameters k_1, k_2 of I_Λ with the (a,b,c) of our induced $I(\chi)$, which is $\mathrm{Ind}(\delta_G^{1/2}\chi; B, G)$, note that $\Lambda(\mathrm{diag}(x, y/x, 1/y)) = x^{k_1} y^{k_2}$ and $\chi(\mathrm{diag}(x, y/x, 1/y)) = x^{a-b} y^{b-c}$. Thus $k_1 = a-b$, $k_2 = b-c$. We then write $I(a,b,c)$ for I_Λ with $k_1 = a-b$, $k_2 = b-c$, extended to U(2,1) with central character $\mathbf{w} = a+b+c$. If $gJ^t\overline{g} = J$ and $z = \det g$, then $z\overline{z} = 1$, thus $z = e^{i\theta}$, $-\pi < \theta \leq \pi$, then $x = e^{i\theta/3}$ has that $h = x^{-1}g$ satisfies $hJ^t\overline{h} = J$ and $x\overline{x} = 1$, and $\det h = 1$. Note that $I_{s_1\Lambda}$ gives $I(b,a,c)$ and $I_{s_2\Lambda}$ gives $I(a,c,b)$.

Here is a list of all irreducible unitarizable representations with infinitesimal character $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$, integral $k_i \geq 0$, $\Lambda \neq 0$.

$k_1 = k_2 = 1$: F_Λ, J_0^+, J_0^-, $D_{s_2\Lambda}^+$, $D_{s_1\Lambda}^-$, D_Λ.

$k_1 > 1$, $k_2 > 1$: F_Λ, $D_{s_2\Lambda}^+$, $D_{s_1\Lambda}^-$, D_Λ.

$k_1 > 1$, $k_2 = 1$: F_Λ, $J_{k_1-1}^-$, $D_{s_2\Lambda}^+$, $D_{s_1\Lambda}^-$, D_Λ.

$k_1 = 1$, $k_2 > 1$: F_Λ, $J_{k_2-1}^+$, $D_{s_2\Lambda}^+$, $D_{s_1\Lambda}^-$, D_Λ.

$k_1 = 0$, $k_2 > 1$: $\pi_{k_2 s_2\Lambda_2}^+$, $\pi_{k_2 s_2\Lambda_2}^-$.

$k_1 > 1$, $k_2 = 0$: $\pi_{k_1 s_1\Lambda_1}^+$, $\pi_{k_1 s_1\Lambda_1}^-$.

$k_1 = 0$, $k_2 = 1$: J_{-1}^-, $\pi_{s_2\Lambda_2}^+$, $\pi_{s_2\Lambda_2}^-$.

$k_1 = 1$, $k_2 = 0$: J_{-1}^+, $\pi_{s_1\Lambda_1}^+$, $\pi_{s_1\Lambda_1}^-$.

Here is a list of composition series. $\Lambda \geq 0 \neq \Lambda$.

I_Λ has F_Λ, $J_{s_2\Lambda}^+$ (unitarizable iff $k_1 = 1$, $k_2 \geq 0$), $J_{s_1\Lambda}^-$ (unitarizable iff $k_2 = 1$, $k_1 \geq 0$), D_Λ.

$I_{s_1\Lambda}$ has $J_{s_1\Lambda}^-$ (unitarizable iff $k_2 = 1$, $k_1 \geq 0$), $D_{s_1\Lambda}^-$, D_Λ.

$I_{s_2\Lambda}$ has $J_{s_2\Lambda}^+$ (unitarizable iff $k_1 = 1$, $k_2 \geq 0$), $D_{s_2\Lambda}^+$, D_Λ.

$k_1 = 0$, $k_2 = 1$: $I_{s_1\Lambda_2}$ has $J_{s_1\Lambda_2}^-$, $\pi_{s_2\Lambda_2}^-$.

$k_1 = 1$, $k_2 = 0$: $I_{s_2\Lambda_1}$ has $J_{s_2\Lambda_1}^+$, $\pi_{s_1\Lambda_1}^+$.

To fix notations in a manner consistent with the nonarchimedean case, note that if μ is a one-dimensional H-module then there are unique integers $a \geq b \geq c$ with $a+b+c = \mathbf{w}$ and either (i) $a = b+1, \mu = F_H(a,b)$, or (ii) $b = c+1, \mu = F_H(b,c)$. If the central character on the U(1,1)-part is $z \mapsto z^{2k+1}$, case (i) occurs when $\mathbf{w} - 3k \leq 1$, while case (ii) occurs if $\mathbf{w} - 3k \geq 2$.

If, in addition, $a > b > c$, put $\pi_\mu^\times = J^+_{s_2\Lambda}$, $\pi_\mu^- = D^-_{s_1\Lambda}$, and $\pi_\mu^+ = D_\Lambda \oplus D^+_{s_2\Lambda}$ in case (i), $\pi_\mu^\times = J^-_{s_1\Lambda}$, $\pi_\mu^- = D^+_{s_2\Lambda}$ and $\pi_\mu^+ = D_\Lambda \oplus D^-_{s_1\Lambda}$ in case (ii).

The motivation for this choice of notations is the following character identities. Put

$$\rho = \rho(a,c) \otimes \kappa^{-1}, \qquad \rho^- = \rho(b,c) \otimes \kappa^{-1}, \qquad \rho^+ = \rho(a,b) \otimes \kappa^{-1}.$$

Then $\{\rho, \rho^+, \rho^-\}$ is the set of H-packets which lift to the G-packet $\pi = \pi(a,b,c)$ via the endo-lifting e. As noted above, ρ, ρ^+ and ρ^- are distinct if and only if $a > b > c$, equivalently π consists of three square-integrable G-modules. Moreover, every square-integrable H-packet is of the form ρ, ρ^+ or ρ^- for unique $a \geq b \geq c$, $a > c$.

If $a = b = c$ then $\rho = \rho^+ = \rho^-$ is the H-packet which consists of the constituents of $I(\chi_H(a,\, c) \otimes \kappa^{-1})$, and $\pi = I(\chi(a,\, b,\, c))$ is irreducible.

If $a > b = c$ put $\langle \rho, \pi^+ \rangle = 1$, $\langle \rho, \pi^- \rangle = -1$.

If $a = b > c$ put $\langle \rho, \pi^+ \rangle = -1$, $\langle \rho, \pi^- \rangle = 1$.

If $a > b > c$ put $\langle \tilde\rho, D_\Lambda \rangle = 1$ for $\tilde\rho = \rho$, ρ^+, ρ^-, and:
$\langle \rho, D^+_{s_2\Lambda} \rangle = -1$, $\langle \rho, D^-_{s_1\Lambda} \rangle = -1$;
$\langle \rho^+, D^+_{s_2\Lambda} \rangle = 1$, $\langle \rho^+, D^-_{s_1\Lambda} \rangle = -1$;
$\langle \rho^-, D^+_{s_2\Lambda} \rangle = -1$, $\langle \rho^-, D^-_{s_1\Lambda} \rangle = 1$.

5.1 PROPOSITION ([Sd]). *For all matching measures fdg on G and $'fdh$ on H, we have*

$$\operatorname{tr} \tilde\rho('fdh) = \sum_{\pi' \in \pi} \langle \tilde\rho, \pi' \rangle \operatorname{tr} \pi'(fdg) \quad (\tilde\rho = \rho,\ \rho^+ \text{ or } \rho^-).$$

From this and the character relation for induced representations we conclude the following

5.2 COROLLARY. *For every one-dimensional H-module μ and for all matching measures fdg on G and $'fdh$ on H we have*

$$\operatorname{tr} \mu('fdh) = \operatorname{tr} \pi_\mu^\times(fdg) + \operatorname{tr} \pi_\mu^-(fdg).$$

Further, if ρ is a tempered H-module, π the endo-lift of ρ (then π is a G-packet), ρ' is the basechange lift of ρ (thus ρ' is a σ-invariant H'-module), and $\pi' = I(\rho')$ is the G'-module normalizedly induced from ρ' (we regard H' as a Levi subgroup of a maximal parabolic subgroup of G'), then we have

5.3 PROPOSITION ([Cl1]). *We have* $\operatorname{tr}\pi(fdg) = \operatorname{tr}\pi'(\phi dg' \times \sigma)$ *for all matching* fdg *on* G *and* $\phi dg'$ *on* G'.

From this and the character relation for induced representations we conclude the following

5.4 COROLLARY. *For all matching measures* fdg *on* G *and* $\phi dg'$ *on* G' *and every one-dimensional* H*-module* μ *we have*

$$\operatorname{tr} I(\mu'; \phi dg' \times \sigma) = \operatorname{tr}\pi_\mu^\times(fdg) - \operatorname{tr}\pi_\mu^-(fdg).$$

Our next aim is to determine the $(\mathfrak{g}, K)$-cohomology of the G-modules described above, where $\mathfrak{g}$ denotes the complexified Lie algebra of G. For that we describe the K-types of these G-modules, following [Wh], §7, and [BW], Ch. VI. Note that $G = \mathrm{U}(2,1)$ can be defined by means of the form

$$J' = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix}$$

whose signature is also (2,1) and it is conjugate to

$$J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix} \qquad \text{by} \qquad \mathbf{B} = \begin{pmatrix} 2^{-1/2} & 0 & 2^{-1/2} \\ 0 & 1 & 0 \\ 2^{-1/2} & 0 & -2^{-1/2} \end{pmatrix}$$

of [Wh], p. 181. To ease the comparison with [Wh] we now take G to be defined using J'. In particular we now take A to be the maximal torus of G whose conjugate by $\mathbf{B}$ is the diagonal subgroup of $G(J)$. A character χ of A is again associated with (a, b, c) in $\mathbb{C}^3$ such that $a + c$ and b are integral, and $I(\chi)$ denotes the G-module normalizedly induced from χ extended to the standard Borel subgroup B. The maximal compact subgroup K of G is isomorphic to $\mathrm{U}(2) \times \mathrm{U}(1)$; it consists of the matrices $\begin{pmatrix} \alpha u & 0 \\ 0 & \mu \end{pmatrix}$; u in $\mathrm{SU}(2)$; α, μ in $\mathrm{U}(1) = \mathbb{C}^1$. Note that $A \cap K$ consists of $\gamma \operatorname{diag}(\alpha, \alpha^{-2}, \alpha)$, and the center of K consists of $\gamma \operatorname{diag}(\alpha, \alpha, \alpha^{-2})$.

Let π_K denote the space of K-finite vectors of the admissible G-module π. By Frobenius reciprocity, as a K-module $I(\chi)_K$ is the direct sum of the irreducible K-modules $\mathfrak{h}$, each occurring with multiplicity

$$\dim[\operatorname{Hom}_{A \cap K}(\chi, \mathfrak{h})].$$

The $\mathfrak{h}$ are parametrized by (a', b', c') in $\mathbb{Z}^3$, such that $\dim \mathfrak{h} = a' + 1$, and the central character of $\mathfrak{h}$ is $\gamma \operatorname{diag}(\mu, \mu, \mu^{-2}) \mapsto \mu^{b'} \gamma^{c'}$; hence $b' \equiv c' (\operatorname{mod} 3)$ and $a' \equiv b' (\operatorname{mod} 2)$. In this case we write $\mathfrak{h} = \mathfrak{h}(a', b', c')$. For any integers a, b, c, p, q with p, $q \geq 0$ we also write

$$\mathfrak{h}_{p,q} = \mathfrak{h}(p + q, 3(p - q) - 2(a + c - 2b), a + b + c).$$

5.5 Lemma. *The K-module $I(\chi)_K$, $\chi = \chi(a, b, c)$, is isomorphic to $\oplus_{p,q \geq 0} \mathfrak{h}_{p,q}$.*

Proof. The restriction of $\mathfrak{h} = \mathfrak{h}(a', b', c')$ to the diagonal subgroup

$$D = \{\gamma \operatorname{diag}(\beta\alpha, \beta/\alpha, \beta^{-2})\}$$

of K is the direct sum of the characters $\alpha^n \beta^{b'} \gamma^{c'}$ over the integral n with $-a' \leq n \leq a'$ and $n \equiv a' (\operatorname{mod} 2)$. Hence the restriction of $\mathfrak{h}$ to $A \cap K$ is the direct sum of the characters $\gamma \operatorname{diag}(\alpha, \alpha^{-2}, \alpha) \mapsto \alpha^{(3n - b')/2} \gamma^{c'}$. On the other hand, the restriction of $\chi = \chi(a, b, c)$ to $A \cap K$ is the character $\lambda \operatorname{diag}(\alpha, \alpha^{-2}, \alpha) \mapsto \alpha^{a+c-2b} \lambda^{a+b+c}$. If $-a \leq n \leq a'$ and $n \equiv a'$ (mod 2), there are unique p, $q \geq 0$ with $a' = p + q$, and $n = p - q$. Then

$$\mathfrak{h}(a', b', c') | (A \cap K)$$

contains $\chi(a, b, c) | (A \cap K)$ if and only if there are p, $q \geq 0$ with

$$a' = p + q, \qquad b' = 3(p - q) - 2(a + c - 2b) \qquad c' = a + b + c.$$

□

Definition. For integral a, b, c put $\chi = \chi(a, b, c)$, $\chi^- = \chi(b, a, c)$, $\chi^+ = \chi(a, c, b)$. Also write

$$\mathfrak{h}^-_{p,q} = \mathfrak{h}(p + q, 3(p - q) - 2(b + c - 2a), a + b + c),$$

and

$$\mathfrak{h}^+_{h,q} = \mathfrak{h}(p + q, 3(p - q) - 2(a + b - 2c), a + b + c).$$

Lemma 5.5 implies that (the sum are over p, $q \geq 0$)

$$I(\chi)_K = \oplus \mathfrak{h}_{p,q}, \quad I(\chi^+)_K = \oplus \mathfrak{h}^+_{p,q}, \quad I(\chi^-)_K = \oplus \mathfrak{h}^-_{p,q}.$$

Definition. Write $JH(\pi)$ for the unordered sequence of constituents of the G-module π, repeated with their multiplicities.

If $a>b>c$ then $JH(I(\chi))=\{F,J^+,J^-,D\}$. By [Wh], 7.9, the K-type decomposition of the constituents is of the form $\oplus\mathfrak{h}_{p,q}$. The sums range over: (1) $p<a-b$, $q<b-c$ for F; (2) $p\geq a-b$, $q<b-c$ for J^-; (3) $p<a-b$, $q\geq b-c$ for J^+; (4) $p\geq a-b$, $q\geq b-c$ for D.

Next, $JH(I(\chi^-))=\{J^-,D^-,D\}$. The K-types are of the form $\oplus\mathfrak{h}^-_{p,q}$, with sums over: (1) $p\geq 0$, $a-b\leq q<a-c$ for J^-; (2) $p\geq 0$, $q<a-b$ for D^-; (3) $p\geq 0$, $q\geq a-c$ for D.

Finally, $JH(I(\chi^+))=\{J^+,D^+,D\}$. The K-types are of the form $\oplus\mathfrak{h}^+_{p,q}$, with sums over: (1) $b-c\leq p<a-c$, $q\geq 0$ for J^+; (2) $p<b-c$, $q\geq 0$ for D^+; (3) $p\geq a-c$, $q\geq 0$ for D.

Recall that J^- is unitary if and only if $b-c=1$, and J^+ is unitary if and only if $a-b=1$.

If $a>b=c$ (resp. $a=b>c$) then χ^- (resp. χ^+) is unitary, and $I(\chi^-)$ (resp. $I(\chi^+)$) is the direct sum of the unitary G-modules π^+ and π^-. The K-type decomposition is $\pi^+_K=\oplus\mathfrak{h}^+_{p,q}$ ($p\geq 0$, $q\geq a-b$), $\pi^-_K=\oplus\mathfrak{h}^+_{p,q}$ ($p\geq 0$, $q<a-b$) if $a>b=c$, and $\pi^+_K=\oplus\mathfrak{h}^-_{p,q}$ ($p\geq b-c$, $q\geq 0$), $\pi^-_K=\oplus\mathfrak{h}^-_{p,q}$ ($p<b-c$, $q\geq 0$) if $a=b>c$. Moreover, $JH(I(\chi))$ is $\{\pi^\times=J^+,\ \pi^+\}$ if $a>b=c$, and $\{\pi^\times=J^-,\ \pi^-\}$ if $a=b>c$. The corresponding K-type decompositions are $J^-=\oplus\mathfrak{h}_{p,q}$ ($p<a-b$, $q\geq 0$), $J^+=\oplus\mathfrak{h}_{p,q}$ ($p\geq 0$, $q<b-c$).

As noted above, J^+ is unitary if and only if $a-1=b\geq c$; J^- is unitary if and only if $a\geq b=c+1$.

Next we define holomorphic and anti-holomorphic vectors, and describe those G-modules which contain such vectors. We have the vector spaces of matrices

$$P^+=\left\{\begin{pmatrix}0&0&0\\0&0&0\\x&y&0\end{pmatrix}\right\},\qquad P^-=\left\{\begin{pmatrix}0&0&x\\0&0&y\\0&0&0\end{pmatrix}\right\},$$

in the complexified Lie algebra $\mathfrak{g}=M(3,\mathbb{C})$. These P^+, P^- are K-modules under the adjoint action of K, clearly isomorphic to $\mathfrak{h}(1,3,0)$ and $\mathfrak{h}(1,-3,0)$.

Definition. A vector in the space π_K of K-finite vectors in a G-module π is called *holomorphic* if it is annihilated by P^-, and *anti-holomorphic* if it is annihilated by P^+.

5.6 Lemma. *If $I(\chi)$ is irreducible then $I(\chi)_K$ contains neither holomorphic nor anti-holomorphic vectors.*

Proof. The K-modules $P^+ = \mathfrak{h}(1,3,0)$ and $P^- = \mathfrak{h}(1,-3,0)$ act by

$$\mathfrak{h}(1,3,0) \otimes \mathfrak{h}(a,b,c) = \mathfrak{h}(a+1,b+3,c) \oplus \mathfrak{h}(a-1,b+3,c)$$

and

$$\mathfrak{h}(1,-3,0) \otimes \mathfrak{h}(a,b,c) = \mathfrak{h}(a+1,b-3,c) \oplus \mathfrak{h}(a-1,b-3,c).$$

Hence the action of P^+ on $I(\chi)_K$ maps $\mathfrak{h}_{p,q}$ to $\mathfrak{h}_{p+1,q} \oplus \mathfrak{h}_{p,q-1}$, and that of P^- maps $\mathfrak{h}_{p,q}$ to $\mathfrak{h}_{p,q+1} \oplus \mathfrak{h}_{p-1,q}$. Consequently if $\mathfrak{h}_{p',q'}$ is annihilated by P^+, then $\oplus\mathfrak{h}_{p,q}$ $(p \geq p', q \leq q')$ is a $(\mathfrak{g}, K)$-submodule of $I(\chi)$, and if P^- annihilates $\mathfrak{h}_{p',q'}$ then $\oplus\mathfrak{h}_{p,q}$ $(p \leq p',\ q \geq q')$ is a $(\mathfrak{g}, K)$-submodule of $I(\chi)$. The lemma follows. □

Definition. Denote by π_K^{hol} the space of holomorphic vectors in π_K, and by π_K^{ah} the space of anti-holomorphic vectors.

The above proof implies also the following

5.7 Lemma. (i) *The irreducible unitary G-modules with holomorphic vectors are*

(1) $\pi = D^+(a,b,c)$, *where* $a > b > c$; *then* $\pi_K^{\mathrm{hol}} = \mathfrak{h}(a-b-1, a+b-2c+3, a+b+c)$;

(2) $\pi = J^-(a,b,b-1)$, *with* $a \geq b$; *then* $\pi_K^{\mathrm{hol}} = \mathfrak{h}(a-b, a-b+2, a+2b-1)$;

(3) $\pi = \pi^+(a,b,b)$, $a > b$; *then* $\pi_K^{\mathrm{hol}} = \mathfrak{h}(a-b-1, a-b+3, a+2b)$.

(ii) *The irreducible unitary G-modules with antiholomorphic vectors are*

(1) $\pi = D^-(a,b,c)$, $a > b > c$; *then* $\pi_K^{\mathrm{ah}} = \mathfrak{h}(b-c-1, b+c-2a-3, a+b+c)$;

(2) $\pi = J^+(b+1,b,c)$, $b \geq c$; *then* $\pi_K^{\mathrm{ah}} = \mathfrak{h}(b-c, c-b-2, 2b+c+1)$;

(3) $\pi = \pi^-(a,a,c)$, $a > c$; *then* $\pi_K^{\mathrm{ah}} = \mathfrak{h}(a-c-1, c-a-3, 2a+c)$.

We could rename the $J^{\pm}$, but decided to preserve the notations induced from [Wh].

Let $F = F(a,b,c)$ be the irreducible finite-dimensional G-module with highest weight $\mathrm{diag}(x,y,z) \mapsto x^{a-1}y^b z^{c+1}$. It is the unique finite dimensional quotient of $I(\chi)$, $\chi = \chi(a,b,c)$, $a > b > c$. Let $\widetilde{F}$ denote the contragredient of F. Let π be an irreducible unitary G-module. Denote by $H^j(\mathfrak{g}, K; \pi\otimes\widetilde{F})$ the $(\mathfrak{g}, K)$-cohomology of $\pi\otimes\widetilde{F}$. This cohomology vanishes, by [BW], Theorem 5.3, p. 29, unless π and F have equal infinitesimal characters, namely π is associated with the triple (a, b, c) of F. It follows from the K-type computations above that one has (cf. [BW], Theorem VI.4.11, p. 201) the following

5.8 PROPOSITION. *If $H^j(\pi \otimes \widetilde{F}) \neq 0$ for some j then π is one of the following.*

(1) *If π is $D(a, b, c)$, $D^+(a, b, c)$ or $D^-(a, b, c)$ then $H^j(\pi \otimes \widetilde{F})$ is $\mathbb{C}$ if $j = 2$ and 0 if $j \neq 2$. Such π have Hodge types $(1, 1)$, $(2, 0)$, $(0, 2)$, respectively.*

(2) *If π is $J^+(a, b, c)$ with $a - b = 1$ or $J^-(a, b, c)$ with $b - c = 1$ then $H^j(\pi \otimes \widetilde{F})$ is $\mathbb{C}$ if $j = 1$, 3 and 0 if $j \neq 1$, 3. Such π have Hodge types $(0, 1)$, $(0, 3)$ and $(1, 0)$, $(3, 0)$, respectively.*

(3) *$H^j(F \otimes \widetilde{F})$ is 0 unless $j = 0$, 2, 4 when it is $\mathbb{C}$. The Hodge types of F are $(0, 0)$, $(1, 1)$, $(2, 2)$.*

I.6 Fundamental lemma again

The following is a computation of the orbital integrals for GL(2), SL(2), and our U(3), for the characteristic function 1_K of K in G, leading to a proof of the fundamental lemma for (U(3),U(2)), due to J.G.M. Mars (letter to me, dated June 30, 1997).

Case of SL(2)

1. Let E/F be a (separable) quadratic extension of nonarchimedean local fields. Denote by $\mathcal{O}_E$ and $\mathcal{O}$ their rings of integers. Let $\boldsymbol{\pi} = \boldsymbol{\pi}_F$ be a generator of the maximal ideal in $\mathcal{O}$. Then $ef = 2$ where e is the degree of ramification of E over F. Let $V = E$, considered as a two-dimensional vector space over F. Multiplication in E gives an embedding $E \subset \mathrm{End}_R(V)$ and $E^\times \subset \mathrm{GL}(V)$. The ring of integers $\mathcal{O}_E$ is a lattice (free $\mathcal{O}$-module of maximal rank, namely which spans V over F) in V and $K = \mathrm{Stab}(\mathcal{O}_E)$ is a maximal compact subgroup of $\mathrm{GL}(V)$.

Let Λ be a lattice in V. Then $R = R(\Lambda) = \{x \in E | x\Lambda \subset \Lambda\}$ is an order. The orders in E are $R(m) = \mathcal{O} + \boldsymbol{\pi}^m\mathcal{O}_E$, $m \geq 0$ of F. This is well known and easy to check. The quotient $R(m)/R(m+1)$ is a one-dimensional vector space over $\mathcal{O}/\boldsymbol{\pi}$. If $R(\Lambda) = R(m)$, then $\Lambda = zR(m)$ for some $z \in E^\times$.

Choose a basis 1, w of E such that $\mathcal{O}_E = \mathcal{O} + \mathcal{O}w$. Define $d_m \in \mathrm{GL}(V)$ by $d_m(1) = 1$, $d_m(w) = \boldsymbol{\pi}^m w$. Then $R(m) = d_m\mathcal{O}_E$. It follows immediately that $\mathrm{GL}(V) = \bigcup_{m\geq 0} E^\times d_m K$, or, in coordinates with respect to 1, w:

$$\mathrm{GL}(2,F) = \bigcup_{m\geq 0} T \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{pmatrix} \mathrm{GL}(2,\mathcal{O}),$$

with $T = \left\{ \begin{pmatrix} a & \alpha b \\ b & a+\beta b \end{pmatrix};\ a,\ b \in F, \text{ not both } = 0 \right\}$, where $w^2 = \alpha + \beta w$, α, $\beta \in \mathcal{O}$.

2. Put $G = \mathrm{GL}(V)$, $K = \mathrm{Stab}(\mathcal{O}_E)$. Choose the Haar measure dg on G such that $\int_K dg = 1$, and dt on $E^\times$ such that $\int_{\mathcal{O}_E} dt = 1$. Choose $\gamma \in E^\times$, $\gamma \notin F^\times$. Let 1_K be the characteristic function of K in G. Then

$$\int_{E^\times\backslash G} 1_K(g^{-1}\gamma g)\frac{dg}{dt} = \sum_{E^\times\backslash G/K} \frac{\mathrm{vol}(K)}{\mathrm{vol}(E^\times \cap gKg^{-1})} 1_K(g^{-1}\gamma g).$$

Now $E^\times\backslash G/K$ is the set of $E^\times$-orbits on the set of all lattices in E. Representatives are the lattices $R(m)$, $m \geq 0$. So our sum is

$$\sum_{m\geq 0, \gamma\in R(m)^\times} \frac{\mathrm{vol}(\mathcal{O}_E^\times)}{\mathrm{vol}(R(m)^\times)} = \sum_{m\geq 0, \gamma\in R(m)^\times} (\mathcal{O}_E^\times : R(m)^\times).$$

Note that $(\mathcal{O}_E^\times : R(m)^\times) = 1$ if $m = 0$, $\quad = q^{m+1-f}\frac{q^f-1}{q-1}$ if $m > 0$.

Put $M = \max\{m | \gamma \in R(m)^\times\}$. Then the integral equals

$$q^M\frac{q+1}{q-1} - \frac{2}{q-1} \quad \text{if} \quad e = 1, \qquad\qquad \frac{q^{M+1}-1}{q-1} \quad \text{if} \quad e = 2.$$

(If $\gamma \notin \mathcal{O}_E^\times$, then $\int = 0$). If $\gamma = a + bw \in \mathcal{O}_E^\times$, then $M = \mathfrak{v}_F(b)$, the order-valuation at b.

3. Let $G = \mathrm{SL}(V)$, $K = \mathrm{Stab}(\mathcal{O}_E) \cap G$, $E^1 = E^\times \cap G$. Choose the Haar measure dg on G such that $\int_K dg = 1$, and dt on E^1 such that $\int_{E^1} dt = 1$. Let $\gamma \in E^1$, $\gamma \neq \pm 1$. Then

$$\int_{E^1\backslash G} 1_K(g^{-1}\gamma g)\frac{dg}{dt} = \int_G 1_K(g^{-1}\gamma g)dg = \sum_{G/K} 1_K(g^{-1}\gamma g)$$

is the number of lattices in the G-orbit of $\mathcal{O}_E$ fixed by γ.

Let Λ be a lattice in E. If $R(\Lambda) = \mathcal{O}_E$, then $\Lambda \in G \cdot \mathcal{O}_E \Leftrightarrow \Lambda = \mathcal{O}_E$. And $\gamma\mathcal{O}_E = \mathcal{O}_E$ if γ fixes Λ. If $R(\Lambda) = R(m)$ with $m > 0$, then $\Lambda = zR(m) \in G \cdot \mathcal{O}_E \Leftrightarrow N_{E/F}(z)\boldsymbol{\pi}^m \in \mathcal{O}^\times \Leftrightarrow f\mathfrak{v}_E(z) = -m$ and $\gamma\Lambda = \Lambda \Leftrightarrow \gamma \in R(m)^\times$.

Suppose $e = 1$. Then m must be even and $\Lambda = \boldsymbol{\pi}^{-\frac{m}{2}} uR(m)$, $u \in \mathcal{O}_E^\times \bmod R(m)^\times$. This gives $(\mathcal{O}_E^\times : R(m)^\times) = q^{m-1}(q+1)$ lattices, if $\gamma \in R(m)^\times$.

Suppose $e = 2$. Then $\Lambda = \boldsymbol{\pi}_E^{-m} uR(m)$, $u \in \mathcal{O}_E^\times \bmod R(m)^\times$. This gives $(\mathcal{O}_E^\times : R(m)^\times) = q^m$ lattices, if $\gamma \in R(m)^\times$.

Put $N = \max\{m | \gamma \in R(m)^\times,\ m \equiv 0(f)\}$. Then the integral equals

$$\frac{q^{N+1} - 1}{q - 1}.$$

For $K = \mathrm{Stab}(R(1)) \cap G$ one find $\frac{q^{N'+1}-1}{q-1}$ with N' defined as N, but with $m \equiv 1(f)$.

4. Notations as in 3. Choose $\boldsymbol{\pi} = N_{E/F}(\boldsymbol{\pi}_e)$ if $e = 2$. The description of the lattices in $G \cdot \mathcal{O}_E$ above gives the following decomposition for $\mathrm{SL}(2, F)$.

Choose a set A_m of representations for $N_{E/F}\mathcal{O}_E^\times / N_{E/F}R(m)^\times$ and for each $\varepsilon \in A_m$ choose b_ε such that $N_{E/F}(b_\varepsilon) = \varepsilon$. For $m = 0$ we may take $A_0 = \{1\}$, $b_1 = 1$.

$$\mathrm{SL}(2, F) = \bigcup_{m \geq 0, \text{even}} \bigcup_{\varepsilon \in A_m} E^1 b_\varepsilon^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}^{-\frac{m}{2}} & 0 \\ 0 & \boldsymbol{\pi}^{\frac{m}{2}} \end{pmatrix} K \qquad \text{if } e = 1,$$

$$\mathrm{SL}(2, F) = \bigcup_{m \geq 0} \bigcup_{\varepsilon \in A_m} E^1 b_\varepsilon^{-1} \boldsymbol{\pi}_E^{-m} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{pmatrix} K \qquad \text{if } e = 2.$$

Remark. If $e = 1$, $m > 0$, then

$$N_{E/F}\mathcal{O}_E^\times / N_{E/F}R(m)^\times = \mathcal{O}^\times / \mathcal{O}^{\times 2}(1 + \boldsymbol{\pi}^m \mathcal{O})$$

(two elements, when $|2| = 1$). If $|2| = 1$ and $e = 2$, then $N_{E/F}R(m)^\times = N_{E/F}\mathcal{O}_E^\times$ for all m.

Case of U(3)

1.1 Let E/F be a separable quadratic extension and V a three-dimensional vector space over E. Let (x, y) be an Hermitian form on $V \times V$ with discriminant one. Let G be its unitary group. Then G is the set of points over F of the algebraic group $\mathbf{G}$.

The relation $(ux, y) = (x, {}^{\iota}uy)$ defines an involution ι of the second kind of $A = \mathrm{End}_E(V)$ and $G = \{u \in A | {}^{\iota}uu = 1\}$.

Let γ be a regular semisimple element of G. Let Y denote the centralizer of γ in A and $T = G \cap Y$. Then $T = \mathbf{T}(F)$ where $\mathbf{T}$ is an algebraic torus over F. Now Y is a three-dimensional E-algebra. This Y is semisimple and is the direct product $\prod Y_i$ of separable extensions of E. The space V is isomorphic to Y as a Y-module. It decomposes as $V = \oplus V_i$, where V_i is a one-dimensional vector space over Y_i. The algebra Y is stable under the involution ι. If $\mathbf{T}$ is F-anisotropic then each Y_i is stable under ι. If $\iota Y_i = Y_i$, then $V_i \perp V_j$ for $i \neq j$.

Let $\mathbf{C}$ denote the conjugacy class of γ in $\mathbf{G}$ and $C = \mathbf{C}(F)$.

We have bijections

$$G\backslash C \leftrightarrow G\backslash\{h \in A^\times | h\gamma h^{-1} \in C\}/Y^\times \quad \underset{h \mapsto {}^{\iota}hh}{\overset{\text{bij}}{\rightarrow}}$$

$$\{u \in Y^\times | {}^{\iota}u = u, \det(u) \in N_{E/F}E^\times\}/\{{}^{\iota}uu | u \in Y^\times\}\,.$$

1.2 Assume F is a nonarchimedean local field. If Λ is a lattice in V, the dual lattice is $\Lambda^* = \{x \in V | (x, y) \in \mathcal{O}_E \text{ for all } y \in \Lambda\}$. There is a bijective semilinear map $\Lambda^* \to \mathrm{Hom}_{\mathcal{O}_E}(\Lambda, \mathcal{O}_E)$ (a "lattice" will always be an $\mathcal{O}_E$-module).

If $g \in \mathrm{GL}_E(V)$, then $(g\Lambda)^* = {}^{\iota}g^{-1}\Lambda^*$, in particular $(g\Lambda)^* = g\Lambda^*$ if $g \in G$ and $(c\Lambda)^* = \overline{c}^{-1}\Lambda^*$ if $c \in E^\times$.

The lattices which coincide with their dual form one orbit of G. We have to compute $\mathrm{Card}\{\Lambda | \Lambda^* = \nu\Lambda,\ \gamma\Lambda = \Lambda\}$ for $\nu \in Y^\times$, ${}^{\iota}\nu = \nu$, $\det(\nu) \in N_{E/F}E^\times$ (ν modulo $\{{}^{\iota}uu | u \in Y^\times\}$).

2.1 Notations of 1.1 and 1.2 with $Y = E \times Y_1$, $[Y_1 : E] = 2$. Let σ denote the restriction of ι to Y_1. Let L be the field of fixed points of σ. Then $L \neq E$ and $Y_1 \simeq E \underset{F}{\otimes} L$ is EL.

We assume E/F to be unramified. Then L/F is ramified. The quotient $G\backslash C$ consists of two elements: Let $(\mu, \nu) \in F^\times \times E_1^\times$ such that $\mu N_{L/F}(\nu) \in N_{E/F}E^\times$. The latter condition means that $\mathfrak{v}_F(\mu) \equiv \mathfrak{v}_L(\nu) \bmod 2$. The pair (μ, ν) has to be taken modulo $N_{E/F}E^\times \times N_{EL/L}Y_1^\times$. There are two classes, determined by $\mathfrak{v}_F(\mu) + 2\mathbb{Z} = \mathfrak{v}_L(\nu) + 2\mathbb{Z}$. Here $\mathfrak{v}$ denotes the order valuation.

From now on we assume that $|2| = 1$, $E = F(\sqrt{D})$, $L = F(\sqrt{\boldsymbol{\pi}})$, $D \in \mathcal{O}_F^\times - \mathcal{O}_F^{\times 2}$, $\boldsymbol{\pi} = \boldsymbol{\pi}_F$ a generator of the maximal ideal $\mathfrak{p}_F$ in the ring $\mathcal{O}_F$ of integers of F.

We have $(x, y) = ax\overline{y}$ if $x, y \in E$, with $a \in F^\times$, and

$$(x, y) = \mathrm{tr}_{EL/E}(bx\sigma(y))$$

if $x, y \in EL$, with $b \in L^\times$. The discriminant is

$$-4\boldsymbol{\pi} a N_{L/F}(b) \, (\mathrm{mod}\, N_{E/F}E^\times).$$

This discriminant is one if $\mathfrak{v}_F(a) + \mathfrak{v}_L(b)$ is odd. We may choose arbitrary a and b satisfying that condition. We take $a = 1$, $b = \frac{1}{\sqrt{\boldsymbol{\pi}}}$.

We have $EL = E(w)$, $\mathcal{O}_{EL} = \mathcal{O}_E + \mathcal{O}_E w$, where $w = \sqrt{\boldsymbol{\pi}}$. Now $(1, 1) = (w, w) = 0$ and $(1, w) = (w, 1) = 2$.

The orders in EL are $\mathcal{O}_{EL}(n) = \mathcal{O}_E + \mathcal{O}_E \boldsymbol{\pi}^n w$ $(n \geq 0)$. The lattices in EL are of the form $z\mathcal{O}_{EL}(n)$, $z \in Y_1^\times$, $n \geq 0$. The dual to $z\mathcal{O}_{EL}(n)$ is $\sigma(z)^{-1}\boldsymbol{\pi}^{-n}\mathcal{O}_{EL}(n)$.

Let Λ be a lattice in $V = Y = E \oplus EL$. Then Λ *is determined by lattices* $M_1 \subset N_1 \subset E$, $M_2 \subset N_2 \subset EL$ *and an isomorphism of* $\mathcal{O}_E$*-modules* $\varphi : N_1/M_1 \simeq N_2/M_2$. The dual lattice Λ^* corresponds to $N_1^* \subset M_1^*$, $N_2^* \subset M_2^*$ and $-(\varphi^*)^{-1} : M_1^*/N_1^* \to M_2^*/N_2^*$.

Fix (μ, ν) as above. We have

$$\Lambda^* = (\mu, \nu)\Lambda \Leftrightarrow N_1 = \mu^{-1}M_1^*, \quad N_2 = \nu^{-1}M_2^*, \quad \nu \circ \varphi \circ \mu^{-1} = -(\varphi^*)^{-1}.$$

If $\gamma = (s, t)$, $\quad s \in E^\times$, $\quad N_{E/F}(s) = 1$, $\quad t \in Y_1^\times$, $\quad N_{EL/L}(t) = 1$, then

$$\gamma\Lambda = \Lambda \Leftrightarrow sM_1 = M_1,\, sN_1 = N_1,\, tM_2 = M_2,\, tN_2 = N_2,\, t \circ \varphi \circ s^{-1} = \varphi$$

$$\Leftrightarrow tM_2 = M_2, \quad tN_2 = N_2 \text{ and } t \text{ is multiplication by } s \text{ on } N_2/M_2.$$

We may assume $s = 1$.

The *number of lattices with the same* M_1, N_1, M_2, N_2 is equal to the number of isomorphisms $\varphi : N_1/M_1 \to N_2/M_2$ satisfying $\nu \circ \varphi \circ \mu^{-1} = -(\varphi^*)^{-1}$. If $N_1/M_1 \simeq N_2/M_2 = 0$, there is only one φ. If $N_1/M_1 \simeq N_2/M_2 \simeq \mathcal{O}_E/\boldsymbol{\pi}^{n_1}\mathcal{O}_E$ $(n_1 > 0)$, then φ is given by an element u of $\mathcal{O}_E^\times (\mathrm{mod}\, \boldsymbol{\pi}^{n_1})$. The condition $\nu \circ \varphi \circ \mu^{-1} = -(\varphi^*)^{-1}$ amounts to a congruence $N_{E/F}(u) \equiv$ some element of $\mathcal{O}_F^\times \,\mathrm{mod}\, \boldsymbol{\pi}^{n_1}$. So the number of φ is $q^{n_1 - 1}(q + 1)$.

Let $M_1 = \mathfrak{p}_E^m$. Then $N_1 = \mu^{-1} M_1^* = \mu^{-1}\mathfrak{p}_E^{-m}$ and $2m + \mathfrak{v}_F(\mu) \geq 0$.

Let $M_2 = z\mathcal{O}_{EL}(n)$ with $z \in Y_1^\times$, $n \geq 0$. Then $N_2 = \nu^{-1} M_2^* = \nu^{-1}\sigma(z)^{-1}\boldsymbol{\pi}^{-n}\mathcal{O}_{EL}(n)$.

Since $N_2 \supset M_2$, we must have $\boldsymbol{\pi}^n \nu N_{EL/L}(z) \in \mathcal{O}_{EL}(n) \cap L = \mathcal{O}_L(n)$.

Now $N_1/M_1 \simeq \mathcal{O}_E/\mu\boldsymbol{\pi}^{2m}\mathcal{O}_E$ and $N_2/M_2 \simeq \mathcal{O}_{EL}(n)/c\mathcal{O}_{EL}(n)$, where $c = \boldsymbol{\pi}^n \nu N_{EL/L}(z)$.

These two $\mathcal{O}_E$-modules are isomorphic if and only if $c \notin \boldsymbol{\pi}\mathcal{O}_L(n)$ and $\mathfrak{v}_L(c) = 2m + \mathfrak{v}_F(\mu)$ (this follows easily from a computation of the elementary divisors of the $\mathcal{O}_E$-module $c\mathcal{O}_{EL}(n)$ with respect to $\mathcal{O}_{EL}(n)$).

So m, n and z must satisfy

(1) $2n + \mathfrak{v}_L(\nu) + 2\mathfrak{v}_{EL}(z) = 2m + \mathfrak{v}_F(\mu) \geq 0$ and $c \in \mathcal{O}_L(n), c \notin \boldsymbol{\pi}\mathcal{O}_L(n)$,

where $c = \boldsymbol{\pi}^n \nu N_{EL/L}(z)$.

Moreover, $\gamma\Lambda = \Lambda$ gives the conditions

$$t \in \mathcal{O}_{EL}(n)^\times \text{ and } t - 1 \in c\mathcal{O}_{EL}(n). \tag{2}$$

2.2 We take $\mu = \nu = 1$ when $\mathfrak{v}_F(\mu)$ and $\mathfrak{v}_L(\nu)$ are even,
$\mu = \boldsymbol{\pi}$, $\nu = w$ when $\mathfrak{v}_F(\mu)$ and $\mathfrak{v}_L(\nu)$ are odd.

We compute $\sum_{m,n,z} \mathrm{Card}\{\varphi\}$, *where* m, n, z *satisfy* (1) *and* (2) *above.* In the summation z is taken modulo $\mathcal{O}_{EL}(n)^\times$. We know from 2.1 that $\mathrm{Card}\{\varphi\} = 1$ if $2m + \mathfrak{v}_F(\mu) = 0$ and $\mathrm{Card}\{\varphi\} = q^{2m+\mathfrak{v}_F(\mu)-1}(q+1)$ if $2m + \mathfrak{v}_F(\mu) > 0$.

If $2m + \mathfrak{v}_F(\mu) = 0$, we have by assumption $\mu = 1$ and $m = 0$. Conditions (1) and (2) are now: $\mathfrak{v}_{EL}(z) = -n$, $\boldsymbol{\pi}^n N_{EL/L}(z) \in \mathcal{O}_L(n)$, $t \in \mathcal{O}_{EL}(n)$. Put $z = w^{-n} z_1$, $z_1 \in \mathcal{O}_{EL}^\times$. Then $N_{EL/L}(z_1) \in \mathcal{O}_L(n) = \mathcal{O}_F + \mathcal{O}_F\boldsymbol{\pi}^n w$ has q^n solutions $\bmod\, \mathcal{O}_{EL}(n)^\times$ [write $z_1 = y(1 + xw)$ with $y \in \mathcal{O}_E^\times$, $x \in \mathcal{O}_E$; $N_{EL/L}(z_1) = y\overline{y}(1 + (x + \overline{x})w + x\overline{x}\boldsymbol{\pi})$. The condition is that $\mathrm{tr}_{E/F}(x) \in \boldsymbol{\pi}^n\mathcal{O}_F$, i.e. $x \in \mathcal{O}_F\sqrt{D} + \mathcal{O}_E\boldsymbol{\pi}^n$]. This gives:

In the case that $\mu = \nu = 1$, *the number of lattices with* $m = 0$ *is*

$$\sum_{n \geq 0, t \in \mathcal{O}_{EL}(n)} q^n = \frac{q^{B+1} - 1}{q - 1} \text{ with } B = \max\{n | t \in \mathcal{O}_{EL}(n)\}.$$

Now consider the lattices with $2m + \mathfrak{v}_F(\mu) > 0$. There are two cases: $\mu = \nu = 1$ and $m > 0$ (case 1), and: $\mu = \boldsymbol{\pi}$, $\nu = w$ and $m \geq 0$ (case 2).

In case 1 we have the conditions

(1) $\mathfrak{v}_{EL}(z) = m - n$, put $z = w^{m-n} z_1$, $z_1 \in \mathcal{O}_{EL}^\times / \mathcal{O}_{EL}(n)^\times$;
$N_{EL/L}(z_1) \in \mathcal{O}_F^\times + \mathcal{O}_F^\times \boldsymbol{\pi}^{n-m} w.$

(2) $$t \in \mathcal{O}_{EL}(n), \qquad t - 1 \in \boldsymbol{\pi}^m N_{EL/L}(z_1) \mathcal{O}_{EL}(n).$$

Condition (1) implies that $m \leq n$.

In case 2 we have

(1) $$m = n, \qquad z \in \mathcal{O}_{EL}^\times / \mathcal{O}_{EL}(n)^\times.$$

(2) $$t \in \mathcal{O}_{EL}(n), \qquad t - 1 \in \boldsymbol{\pi}^n w N_{EL/L}(z) \mathcal{O}_{EL}(n).$$

[Condition (1) gives $\mathfrak{v}_{EL}(z) = m - n$ and $\boldsymbol{\pi}^n w N_{EL/L}(z) \in \mathcal{O}_F \boldsymbol{\pi} + \mathcal{O}_F^\times \boldsymbol{\pi}^n w$. Now $\mathfrak{v}_L(\boldsymbol{\pi}^n w N_{EL/L}(z)) = 2m+1$ and any element of F has even valuation in L, hence $2m + 1 = \mathfrak{v}_L(\boldsymbol{\pi}^n w) = 2n + 1$. There is no other condition on z left than $z \in \mathcal{O}_{EL}^\times$].

Let $t = t_1 + t_2 w$ with $t_1, t_2 \in \mathcal{O}_E$, $t_1 \bar{t}_1 + \boldsymbol{\pi} t_2 \bar{t}_2 = 1$, $t_1, \bar{t}_2 + t_2 \bar{t}_1 = 0$. Since t is regular, $t_2 \neq 0$.

Assuming that condition (1) is satisfied we write
in case 1: $N_{EL/L}(z_1) = \xi + \eta \boldsymbol{\pi}^{n-m} w$ with ξ, $\eta \in \mathcal{O}_F^\times$ (here $0 < m \leq n$),
in case 2: $w N_{EL/L}(z) = \xi + \eta w$ with $\xi \in \mathfrak{p}_F$, $\eta \in \mathcal{O}_F^\times$ (here $m = n \geq 0$).

In both cases condition (2) becomes: $n \leq \mathfrak{v}_E(t_2)$ and

$$t - 1 \in (\xi \boldsymbol{\pi}^m + \eta \boldsymbol{\pi}^n w) \mathcal{O}_{EL}(n).$$

The latter is equivalent to

(*) $$\xi \eta^{-1} \boldsymbol{\pi}^{m-n} t_2 \equiv t_1 - 1 \bmod \boldsymbol{\pi}^{2m} \mathcal{O}_E \quad \text{in case 1,}$$
$$\bmod \boldsymbol{\pi}^{2n+1} \mathcal{O}_E \quad \text{in case 2.}$$

Case 1. If $m + n \leq \mathfrak{v}_E(t_2)$, (*) reduces to $2m \leq \mathfrak{v}_E(t_1 - 1)$. (Notice that $t_1 \neq 1$). The number of z_1 $(\bmod\, \mathcal{O}_{EL}(n)^\times)$ is then $q^{m+n-1}(q-1)$ [$z_1 = y(1 + xw)$ must satisfy $\mathfrak{v}_F(\mathrm{tr}_{E/F}(x)) = n - m$, i.e. $x \in \mathcal{O}_F^\times \boldsymbol{\pi}^{n-m} + \mathcal{O}_F \sqrt{D}$].

The contribution to our sum is

$$\sum_{\substack{0<m\le n\\ m+n\le \mathfrak{v}_E(t_2)\\ 2m\le \mathfrak{v}_E(t_1-1)}} q^{3m+n-2}(q^2-1)=(q+1)\left\{q^{B+1}\frac{q^{2C}-1}{q^2-1}-q^2\frac{q^{4C}-1}{q^4-1}\right\}$$

where $A=\mathfrak{v}_E(t_1-1)$, $B=\mathfrak{v}_E(t_2)$, $C=\min\left(\left[\frac{A}{2}\right],\left[\frac{B}{2}\right]\right)$.

If $m+n>\mathfrak{v}_E(t_2)$, $(*)$ implies that $n-m=\mathfrak{v}_E(t_2)-\mathfrak{v}_E(t_1-1)$ and necessarily $\mathfrak{v}_E(t_1-1)\le\mathfrak{v}_E(t_2)$. Moreover $n\le\mathfrak{v}_E(t_2)$ and $m\le\mathfrak{v}_E(t_1-1)$.

From now on we write $\mathfrak{v}$ for $\mathfrak{v}_E$.

Lemma. a) *Let $m\in\mathbb{Z}$. Then*

$$\frac{t_1-1}{t_2}\in F+\mathcal{O}_E\boldsymbol{\pi}^m \Leftrightarrow \mathfrak{v}\left(\frac{t_1-1}{t_2}-\frac{\bar t_1-1}{\bar t_2}\right)\ge m$$

$$\Leftrightarrow \mathfrak{v}((t_1-1)\bar t_2 t_2^{-1}-(\bar t_1-1))\ge m+\mathfrak{v}(t_2).$$

b) $\mathfrak{v}((t_1-1)\bar t_2 t_2^{-1}-(\bar t_1-1))=\min(2\mathfrak{v}(t_1-1),2\mathfrak{v}(t_2)+1)$.

Proof. a) is trivial and b) follows from $(t_1-1)\bar t_2 t_2^{-1}-(\bar t_1-1)=(t_1-1)^2\bar t_1 t_1^{-1}+\boldsymbol{\pi} t_2\bar t_2$. □

We continue case 1 with the extra assumption $m+n>\mathfrak{v}_E(t_2)$. We have $\mathfrak{v}(t_1-1)\le\mathfrak{v}(t_2)$, hence $\mathfrak{v}((t_1-1)\bar t_2 t_2^{-1}-(\bar t_1-1))=2\mathfrak{v}(t_1-1)\ge 2m$ by *b)* of the lemma, and by *a)* there is $\delta\in F$ such that $t_1-1\in\delta t_2+\mathcal{O}_E\boldsymbol{\pi}^{2m}$. Since $\mathfrak{v}(t_1-1)=\mathfrak{v}(t_2)+m-n<2m$, we have $\mathfrak{v}(\delta t_2)=\mathfrak{v}(t_1-1)$ and $\mathfrak{v}(\delta)=m-n$. Put $\delta=\varepsilon\boldsymbol{\pi}^{m-n}$, $\varepsilon\in\mathcal{O}_F^\times$. Now z_1 must satisfy $\xi\eta^{-1}\equiv\varepsilon \bmod \boldsymbol{\pi}^{m+n-\mathfrak{v}(t_2)}$. The number of z_1 $(\bmod\, \mathcal{O}_{EL}(n)^\times)$ is $q^{\mathfrak{v}(t_2)}$ [$z_1=y(1+xw)$ must satisfy $1+x\bar x\boldsymbol{\pi}\equiv\varepsilon\boldsymbol{\pi}^{m-n}(x+\bar x)\bmod \boldsymbol{\pi}^{m+n-\mathfrak{v}(t_2)}$. This congruence has $q^{m+n-\mathfrak{v}(t_2)}$ solutions for $x \bmod \boldsymbol{\pi}^{m+n-\mathfrak{v}(t_2)}\mathcal{O}_E$, as one sees writing $x=x_1\,\boldsymbol{\pi}^{n-m}+x_2\sqrt{D}$ with x_1, $x_2\in\mathcal{O}_F$, hence $q^{\mathfrak{v}(t_2)}$ solutions $\bmod\, \boldsymbol{\pi}^n\mathcal{O}_E$]. This gives the contribution

$$\sum_{\frac12\mathfrak{v}(t_1-1)<m\le\mathfrak{v}(t_1-1)} q^{2m-1+\mathfrak{v}(t_2)}(q+1)=q^{B+2C+1}\ \frac{q^{2A-2C}-1}{q-1},$$

if $\mathfrak{v}(t_1-1)\le\mathfrak{v}(t_2)$ (so $C=\left[\frac{A}{2}\right]$).

Case 2. If $2n\le\mathfrak{v}_E(t_2)$, $(*)$ reduces to $2n+1\le\mathfrak{v}_E(t_1-1)$. The number of z is $(\mathcal{O}_{EL}^\times:\mathcal{O}_{EL}(n)^\times)=q^{2n}$.

If $2n > \mathfrak{v}_E(t_2)$, it follows from (∗) that we must have $\mathfrak{v}_E(t_1-1) > \mathfrak{v}_E(t_2)$. Then $\mathfrak{v}((t_1-1)\bar{t}_2 t_2^{-1} - (\bar{t}_1 - 1)) = 2\mathfrak{v}(t_2)+1 \geq 2n+1$ by b) of the lemma, and by a) there is $\delta \in F$ such that $t_1 - 1 \in \delta t_2 + \mathcal{O}_E \boldsymbol{\pi}^{2n+1}$. Obviously $\delta \in \mathfrak{p}_F$. The condition on z is: $\xi\eta^{-1} \equiv \delta \bmod \boldsymbol{\pi}^{2n+1-\mathfrak{v}(t_2)}$. The number of z is $q^{\mathfrak{v}(t_2)}$ [$z = y(1+xw)$ must satisfy $x + \overline{x} \equiv \boldsymbol{\pi}^{-1}\delta(1+x\overline{x}\boldsymbol{\pi}) \bmod \boldsymbol{\pi}^{2n-\mathfrak{v}(t_2)}$]. Thus we have the contributions

$$\sum_{\substack{0 \leq 2n \leq \mathfrak{v}(t_2) \\ 2n+1 \leq \mathfrak{v}(t_1-1)}} q^{4n}(q+1) = (q+1)\frac{q^4 C' - 1}{q^4 - 1}$$

with

$$C' = \min\left(\left[\frac{A+1}{2}\right], \left[\frac{B}{2}\right] + 1\right)$$

and, if $\mathfrak{v}(t_1 - 1) > \mathfrak{v}(t_2)$,

$$\sum_{\frac{1}{2}\mathfrak{v}(t_2) < n \leq \mathfrak{v}(t_2)} q^{2n+\mathfrak{v}(t_2)}(q+1) = q^{B+2C+2}\,\frac{q^{2B-2C}-1}{q-1} \qquad \left(\text{here } C = \left[\frac{B}{2}\right]\right).$$

3.1 Notations of 1.1 and 1.2 with $Y = E \times E \times E$.

We assume E/F unramified and $|2| = 1$.

It suffices to consider the Hermitian form $(x,y) = x_1\overline{y}_1 + x_2\overline{y}_2 + x_3\overline{y}_3$. Let $\nu = (\nu_1, \nu_2, \nu_3)$, $\nu_i \in F^\times$, $\nu_1\nu_2\nu_3 \in N_{E/F}E^\times$. There are four classes modulo $(N_{E/F}E^\times)^3$, determined by $(\mathfrak{v}(\nu_i)+2\mathbb{Z})$ with $\mathfrak{v}(\nu_1)+\mathfrak{v}(\nu_2)+\mathfrak{v}(\nu_3) \in 2\mathbb{Z}$.

Let Λ be a lattice in $V = V_1 \oplus V_2 \oplus V_3$. The lattice Λ is determined by lattices $M_1 \subset N_1 \subset V_1$, $M_{23} \subset N_{23} \subset V_2 \oplus V_3$ and an isomorphism of $\mathcal{O}_E$-modules $\varphi : N_1/M_1 \simeq N_{23}/M_{23}$.

We have $\Lambda^* = \nu\Lambda \Leftrightarrow N_1 = \nu_1^{-1}M_1^*, \quad N_{23} = \nu_{23}^{-1}M_{23}^*, \quad \nu_{23} \circ \varphi \circ \nu_1^{-1} = -(\varphi^*)^{-1}$.

We have $\gamma\Lambda = \Lambda \Leftrightarrow t_{23}M_{23} = M_{23}, \quad t_{23}N_{23} = N_{23}, \quad t_{23} =$ multiplication by t_1 on N_{23}/M_{23}. Here ν_{23}, t_{23} denote the linear maps multiplication by (ν_2, ν_3), (t_2, t_3).

We put $\gamma = (t_1, t_2, t_3)$ with $t_i \in E^\times$, $t_i\bar{t}_i = 1$. We may assume $t_1 = 1$.

If $N_1/M_1 \simeq N_{23}/M_{23} = 0$, there is only one φ. If $N_1/M_1 \simeq N_{23}/M_{23} \simeq \mathcal{O}_E/\boldsymbol{\pi}^{n_1}\mathcal{O}_E$ $(n_1 > 0)$, the number of φ satisfying $\nu_{23} \circ \varphi \circ \nu_1^{-1} = -(\varphi^*)^{-1}$ is $q^{n_1-1}(q+1)$, as in 2.1.

Let $M_1 = \mathfrak{p}_E^{m_1}$, $N_1 = \nu_1^{-1}\mathfrak{p}_E^{-m_1}$ with $n_1 = 2m_1 + \mathfrak{v}(\nu_1) \geq 0$. Then $N_1/M_1 \simeq \mathcal{O}_E/\boldsymbol{\pi}^{n_1}\mathcal{O}_E$. Now we have to look for lattices $M_{23} \subset V_2 \oplus V_3$ with the properties:

a) $N_{23} = \nu_{23}^{-1}M_{23}^* \supset M_{23}$ and $N_{23}/M_{23} \simeq \mathcal{O}_E/\boldsymbol{\pi}^{n_1}\mathcal{O}_E$;

b) $t_{23}M_{23} = M_{23}$ and $t_{23}=$ id on N_{23}/M_{23}. Note: $t_{23}M_{23} = M_{23} \Rightarrow t_{23}N_{23} = N_{23}$.

The lattice M_{23} is given by lattices $\mathfrak{p}_E^{m_2} \subset \mathfrak{p}_E^{m_2'} \subset V_2$, $\quad \mathfrak{p}_E^{m_3} \subset \mathfrak{p}_E^{m_3'} \subset V_3$ and an isomorphism $\mathfrak{p}_E^{m_2'}/\mathfrak{p}_E^{m_2} \simeq \mathfrak{p}_E^{m_3'}/\mathfrak{p}_E^{m_3}$. We must have $m_2 - m_2' = m_3 - m_3' \geq 0$. The isomorphism in question corresponds to elements of $(\mathcal{O}_E/\boldsymbol{\pi}^{m_2-m_2'}\mathcal{O}_E)^\times$, $\quad \boldsymbol{\pi}^{m_2'} + \mathfrak{p}_E^{m_2} \mapsto u\boldsymbol{\pi}^{m_3'} + \mathfrak{p}_E^{m_3}$.

The lattice $N_{23} = \nu_{23}^{-1}M_{23}$ is given by $\nu_2^{-1}\mathfrak{p}_E^{-m_2'} \subset \nu_2^{-1}\mathfrak{p}_E^{-m_2}$, $\nu_3^{-1}\mathfrak{p}_E^{-m_3'} \subset \nu_3^{-1}\mathfrak{p}_E^{-m_3}$ and the isomorphism $\nu_2^{-1}\boldsymbol{\pi}^{-m_2} + \nu_2^{-1}\mathfrak{p}_E^{-m_2'} \mapsto -\nu_3^{-1}\overline{u}^{-1}\boldsymbol{\pi}^{-m_3} + \nu_3^{-1}\mathfrak{p}_E^{-m_3'}$ from $\nu_2^{-1}\mathfrak{p}_E^{-m_2}/\nu_2^{-1}\mathfrak{p}_E^{-m_2'}$ onto $\nu_3^{-1}\mathfrak{p}_E^{-m_3}/\nu_3^{-1}\mathfrak{p}_E^{-m_3'}$.

Property a) means that M_{23} should have the elementary divisors $\boldsymbol{\pi}^{n_1}$ and 1 with respect to N_{23}. The exponents of the elementary divisors are $m_2 + m_2' + m_3 + m_3' + \mathfrak{v}(\nu_2) + \mathfrak{v}(\nu_3)$ and

$$\min[m_2 + m_2' + \mathfrak{v}(\nu_2),\, m_3 + m_3' + \mathfrak{v}(\nu_3),$$
$$\mathfrak{v}(\nu_3\boldsymbol{\pi}^{2m_3'}N_{E/F}(u) + \nu_2\boldsymbol{\pi}^{m_2+m_2'+m_3'-m_3})]$$

[use, e.g., the basis $(\boldsymbol{\pi}^{m_2'}, \boldsymbol{\pi}^{m_3'}u)$, $(0, \overline{u}^{m_3})$ of M_{23} and the basis

$$(\nu_2^{-1}\boldsymbol{\pi}^{-m},\, -\nu_3^{-1}\boldsymbol{\pi}^{-m_3}\overline{u}^{-1}),\, (0, \nu_3^{-1}\boldsymbol{\pi}^{-m_3'})$$

of N_{23}]. Thus *a*) means

$$\begin{aligned} &m_2 + m_2' + m_3 + m_3' = 2m_1 + \mathfrak{v}(\nu_1) - \mathfrak{v}(\nu_2) - \mathfrak{v}(\nu_3), \\ &\min[m_2 + m_2' + \mathfrak{v}(\nu_2),\, m_3 + m_3' + \mathfrak{v}(\nu_3), \\ &\quad \mathfrak{v}(\nu_3\boldsymbol{\pi}^{2m_3'}N_{E/F}(u) + \nu_2\boldsymbol{\pi}^{2m_2'})] = 0. \end{aligned}$$

Consider property b). We have $t_{23}M_{23} = M_{23} \Leftrightarrow t_{23}M_{23} \subset M_{23}$

$$\Leftrightarrow (t_2\boldsymbol{\pi}^{m_2'}, t_3\boldsymbol{\pi}^{m_3'}u) \in M_{23} \Leftrightarrow \mathfrak{v}(t_2 - t_3) \geq m_3 - m_3'.$$

Moreover $(t_{23} - 1)N_{23} \subset M_{23} \Leftrightarrow \mathfrak{v}(t_2 - 1) \geq m_2 + m_2' + \mathfrak{v}(\nu_2)$,

$$\mathfrak{v}(t_3 - 1) \geq m_3 + m_3' + \mathfrak{v}(\nu_3),$$

$$(t_2-1)\nu_2^{-1}\boldsymbol{\pi}^{-2m_2}N_{E/F}(n)+(t_3-1)\nu_3^{-1}\boldsymbol{\pi}^{-2m_3}\in\mathcal{O}_E.$$

Put $n_i = 2m_i + \mathfrak{v}(\nu_i)$. It follows from $m_2 - m_2' = m_3 - m_3'$ and a) that

$$m_2' = \frac{1}{2}(n_1 - n_3 - \mathfrak{v}(\nu_2)), \qquad m_3' = \frac{1}{2}(n_1 - n_2 - \mathfrak{v}(\nu_3))$$

and properties a) and b), together with $m_2 - m_2' \geq 0$, are:

$$\begin{cases} n_2+n_3\geq n_1, \qquad n_1+n_3\geq n_2, \qquad n_1+n_2\geq n_3, \\ \frac{1}{2}(n_2+n_3-n_1)\leq \mathfrak{v}(t_2-t_3), \qquad \frac{1}{2}(n_1+n_3-n_2)\leq \mathfrak{v}(t_3-1), \\ \frac{1}{2}(n_1+n_2-n_3)\leq \mathfrak{v}(t_2-1), \\ N_{E/F}(u')\in -\nu_2\nu_3^{-1}\boldsymbol{\pi}^{n_2-n_3-\mathfrak{v}(\nu_2)+\mathfrak{v}(\nu_3)}+\boldsymbol{\pi}^{n_2-n_1}\mathcal{O}_E, \\ \quad (+\boldsymbol{\pi}^{n_2-n_1}\mathcal{O}_E^\times \text{ if } n_1+n_2>n_3 \text{ and } n_1+n_3>n_2), \\ (t_2-1)N_{E/F}(u)+(t_3-1)\nu_2\nu_3^{-1}\boldsymbol{\pi}^{n_2-n_3-\mathfrak{v}(\nu_2)+\mathfrak{v}(\nu_3)}\in\boldsymbol{\pi}^{n_2}\mathcal{O}_E. \end{cases}$$

We have $n_i \equiv \mathfrak{v}(\nu_i) \bmod 2$. The ν_i satisfy $\mathfrak{v}(\nu_1)+\mathfrak{v}(\nu_2)+\mathfrak{v}(\nu_3)\in 2\mathbb{Z}$. Here u is to be considered as an element of $(\mathcal{O}_E/\boldsymbol{\pi}^{\frac{1}{2}(n_2+n_3-n_1)}\mathcal{O}_E)^\times$.

We compute $\sum_{n_1,n_2,n_3} \mathrm{Card}\{\varphi\}\cdot\mathrm{Card}\{u\}$. (For $\mathrm{Card}\{\varphi\}$: see 3.1 above).

3.2 (Computation of $\mathrm{Card}\{u\}$).

We may take $\nu_i = 1$ or $\boldsymbol{\pi}$, so that $\nu_2\nu_3^{-1}\boldsymbol{\pi}^{-\mathfrak{v}(\nu_2)+\mathfrak{v}(\nu_3)} = 1$. If $n_2 + n_3 = n_1$, the conditions are: $0 \leq n_2 \leq \mathfrak{v}(t_2-1)$, $0 \leq n_3 \leq \mathfrak{v}(t_3-1)$ and $n_2 = 0$ or $n_3 = 0$. There is one u.

Assume $n_2 + n_3 > n_1$.

The congruence $N_{E/F}(u) \in -\boldsymbol{\pi}^{n_2-n_3} + \boldsymbol{\pi}^{n_2-n_1}\mathcal{O}_E^\times$ (resp. $\boldsymbol{\pi}^{n_2-n_1}\mathcal{O}_E^\times$).

If $n_1 + n_2 = n_3$ or $n_1 + n_3 = n_2$, then $n_1 = 0$, $n_2 = n_3 > 0$. The congruence $N_{E/F}(u) \equiv -1 \bmod \boldsymbol{\pi}^{n_2}$ has $q^{n_2-1}(q+1)$ solutions modulo $\boldsymbol{\pi}^{n_2}$.

If $n_1 + n_2 > n_3$ and $n_1 + n_3 > n_2$, we get $N_{E/F}(u) \in -\boldsymbol{\pi}^{n_2-n_3} + \boldsymbol{\pi}^{n_2-n_1}\mathcal{O}_F^\times$. We have the following cases.

$n_1 > n_3$. Then $n_2 = n_3$. This gives $0 < n_1 < n_2 = n_3$, $N_{E/F}(u) \in -1 + \boldsymbol{\pi}^{n_2-n_1}\mathcal{O}_F^\times$.

$n_1 > n_3$. Then $n_1 = n_2$. This gives $0 < n_3 < n_1 = n_2$, u arbitrary.

$n_1 = n_3$. Then $n_1 \geq n_2$. This gives $0 < n_2 < n_1 = n_3$, u arbitrary, and

$n_1 = n_2 = n_3 > 0$, $\quad N_{E/F}(u) \not\equiv -1 \bmod \mathfrak{p}_F$.

The congruence $(t_2-1)N_{E/F}(u) + (t_3-1)\boldsymbol{\pi}^{n_2-n_3} \in \boldsymbol{\pi}^{n_2}\mathcal{O}_E$.

If $\mathfrak{v}(t_2-1) \geq n_2$ and $\mathfrak{v}(t_3-1) \geq n_3$, u is arbitrary.

If $\mathfrak{v}(t_2-1) \geq n_2$ and $\mathfrak{v}(t_3-1) < n_3$, or $\mathfrak{v}(t_2-1) < n_2$ and $\mathfrak{v}(t_3-1) \geq n_3$, there is no solution.

If $\mathfrak{v}(t_2-1)<n_2$ and $\mathfrak{v}(t_3-1)<n_3$, we must have $\mathfrak{v}(t_2-1)-\mathfrak{v}(t_3-1)=n_2-n_3$. Then

$$N_{E/F}(u)\equiv -\frac{t_3-1}{t_2-1}\boldsymbol{\pi}^{n_2-n_3} \bmod \frac{\boldsymbol{\pi}^{n_2}}{t_2-1}\mathcal{O}_E$$

is equivalent to

$$\begin{cases} \mathfrak{v}(t_3-1)+\mathfrak{v}(t_2-t_3)\geq n_3, \\ N_{E/F}(u)\equiv -\boldsymbol{\pi}^{n_2-n_3}\frac{t_3-1}{t_2-1}\ \frac{t_2+t_3}{2t_3} \bmod \boldsymbol{\pi}^{n_2-\mathfrak{v}(t_2-1)}. \end{cases}$$

$\Big[\frac{t_3-1}{t_2-1}\ \frac{t_2+t_3}{t_3}=\frac{t_3-1}{t_2-1}+\frac{\bar{t}_3-1}{\bar{t}_2-1}$. We have $\mathfrak{v}(t_2-t_3)\geq\frac{1}{2}(n_2+n_3-n_1)>0$, so $\mathfrak{v}(t_2+t_3)=0$. The right hand side is the congruence for $N_{E/F}(u)$ is an element of $\mathcal{O}_F^\times\Big]$.

The inequality $\mathfrak{v}(t_3-1)+\mathfrak{v}(t_2-t_3)\geq n_3$ is a consequence of the inequalities for $\mathfrak{v}(t_2-t_3)$ and $\mathfrak{v}(t_3-1)$ (see 3.1).

If $\mathfrak{b}(t_2-1)<n_2$ and $\mathfrak{v}(t_3-1)<n_3$, the two congruences together give the following.

I) $n_1=0$, $n_2=n_3>0$. Then $\mathfrak{v}(t_2-1)=\mathfrak{v}(t_3-1)<n_2\leq\mathfrak{v}(t_2-t_3)$. Further,

$$N(u)\equiv -1 \bmod \boldsymbol{\pi}^{n_2}, \qquad \text{and} \quad N(u)\equiv -\frac{t_3-1}{t_2-1}\frac{t_2+t_3}{2t_3} \bmod \boldsymbol{\pi}^{n_2-\mathfrak{v}(t_2-1)}.$$

The element u is to be taken $\bmod\, \boldsymbol{\pi}^{n_2}$.

From $\frac{t_3-1}{t_2-1}\ \frac{t_2+t_3}{2t_3}-1=\frac{(t_3+1)(t_3-t_2)}{2t_3(t_2-1)}$ and $\mathfrak{v}(t_2-t_3)\geq n_2$ we see that the second congruence for $N(u)$ is a consequence of the first one.

So there are $q^{n_2-1}(q+1)$ solutions for u.

II) $0<n_1<n_2=n_3$. Then

$$\frac{1}{2}n_1\leq\mathfrak{v}(t_2-1)=\mathfrak{v}(t_3-1)<n_2, \qquad \mathfrak{v}(t_2-t_3)\geq n_2-\frac{1}{2}n_1.$$

Further $N(u)\equiv -1 \bmod \boldsymbol{\pi}^{n_2-n_1}$, $\quad\not\equiv -1 \bmod \boldsymbol{\pi}^{n_2-n_1+1}$,

$$N(u)\equiv -\frac{t_3-1}{t_2-1}\ \frac{t_2+t_3}{2t_3} \bmod \boldsymbol{\pi}^{n_2-\mathfrak{v}(t_2-1)}.$$

The element u is to be taken modulo $\boldsymbol{\pi}^{n_2-\frac{1}{2}n_1}$.

a) If $\mathfrak{v}(t_2-1) \geq n_1$, there is no solution unless $\mathfrak{v}(t_2-t_3) \geq n_2$ and in that case the conditions for u are $N(u) \equiv -1 \bmod \boldsymbol{\pi}^{n_2-n_1}$, $\not\equiv -1 \bmod \boldsymbol{\pi}^{n_2-n_1+1}$. There are $q^{n_2-2}(q^2-1)$ solutions.

b) If $\mathfrak{v}(t_2-1) < n_1$, necessary for solvability is that $\mathfrak{v}(t_2-t_3) - \mathfrak{v}(t_2-1) = n_2 - n_1$ and then only the last congruence for $N(u)$ is left. There are $q^{\mathfrak{v}(t_2-t_3)-1}(q+1)$ solutions.

III) $0 < n_3 < n_2 = n_1$. Then $\frac{1}{2}n_3 \leq \mathfrak{v}(t_3-1) < n_3$, $n_2 - \frac{1}{2}n_3 \leq \mathfrak{v}(t_2-1) < n_2$, $\mathfrak{v}(t_2-1) - \mathfrak{v}(t_3-1) = n_2 - n_3$. We have only the second congruence.

There are $q^{\mathfrak{v}(t_3-1)-1}(q+1)$ solutions for $u \bmod \boldsymbol{\pi}^{\frac{1}{2}n_3}$.

IV) $0 < n_2 < n_3 = n_1$. Then $\frac{1}{2}n_2 \leq \mathfrak{v}(t_2-1) < n_2$, $n_3 - \frac{1}{2}n_2 \leq \mathfrak{v}(t_3-1) < n_3$, $\mathfrak{v}(t_2-1) - \mathfrak{v}(t_3-1) = n_2 - n_3$. Again only the second congruence counts.

There are $q^{\mathfrak{v}(t_2-1)-1}(q+1)$ solutions for $u \bmod \boldsymbol{\pi}^{\frac{1}{2}n_2}$.

V) $n_1 = n_2 = n_3 > 0$. Then $\frac{1}{2}n_1 \leq \mathfrak{v}(t_2-1) = \mathfrak{v}(t_3-1) < n_1$. Further,

$$N(u) \equiv -\frac{t_3-1}{t_2-1}\,\frac{t_2+t_3}{2t_3} \bmod \boldsymbol{\pi}^{n_1-\mathfrak{v}(t_2-1)},$$

and $N(u) \not\equiv -1 \bmod \boldsymbol{\pi}$. The element u is to be taken modulo $\boldsymbol{\pi}^{\frac{1}{2}n_1}$.

Necessary for solvability is that $\frac{t_3-1}{t_2-1}\,\frac{t_2+t_3}{2t_3} \not\equiv 1 \bmod \boldsymbol{\pi}$, i.e. $\mathfrak{v}(t_2-t_3) = \mathfrak{v}(t_2-1)$. Then $q^{\mathfrak{v}(t_2-1)-1}(q+1)$ solutions.

If $\mathfrak{v}(t_2-1) \geq n_2$, $\mathfrak{v}(t_3-1) \geq n_3$, the number of u is in the different cases:

$$\begin{aligned}
n_1 = 0, n_2 = n_3 > 0: &\qquad q^{n_2-1}(q+1)\\
0 < n_1 < n_2 = n_3: &\qquad q^{n_2-2}(q^2-1)\\
0 < n_3 < n_1 = n_2: &\qquad q^{n_3-2}(q^2-1)\\
0 < n_2 < n_1 = n_3: &\qquad q^{n_2-2}(q^2-1)\\
n_1 = n_2 = n_3 > 0: &\qquad q^{n_1-2}(q+1)(q-2)
\end{aligned}$$

3.3 Notations: $A = \mathfrak{v}(t_2-t_3)$, $B = \mathfrak{v}(t_1-t_3)$, $C = \mathfrak{v}(t_1-t_2)$, $M = \min(A,B,C)$, $N = \max(A,B,C)$. If $A > B$, then $B = C$, etc. $F(\nu,t) = \sum_{n_1,n_2,n_3} \operatorname{Card}\{\varphi\} \cdot \operatorname{Card}\{\nu\}$ is the sum of the following sums (where always $n_i \equiv \mathfrak{v}(\nu_i) \bmod 2$).

1) $$\sum_{n_1=n_2=n_3=0} 1 = 1 \qquad \text{if all } \mathfrak{v}(\nu_i) \equiv 0, \text{ otherwise } 0.$$

2) $$\sum_{n_2=0, 0<n_1=n_3\le B} q^{n_1-1}(q+1) = \frac{q(q^{2\left[\frac{B}{2}\right]}-1)}{q-1} \qquad \text{if all } \mathfrak{v}(\nu_i) \equiv 0,$$
$$= \frac{q^{2\left[\frac{B+1}{2}\right]}-1}{q-1} \qquad \text{if } \mathfrak{v}(\nu_2) \equiv 0,\ \mathfrak{v}(\nu_1) \equiv \mathfrak{v}(\nu_3) \equiv 1.$$

3) $$\sum_{n_3=0, 0<n_1=n_2\le C} q^{n_1-1}(q+1) = \frac{q(q^{2\left[\frac{C}{2}\right]}-1)}{q-1} \qquad \text{if all } \mathfrak{v}(\nu_i) \equiv 0,$$
$$= \frac{q^{2\left[\frac{C+1}{2}\right]}-1}{q-1} \qquad \text{if } \mathfrak{v}(\nu_3) \equiv 0,\ \mathfrak{v}(\nu_1) \equiv \mathfrak{v}(\nu_2) \equiv 1.$$

4) $$\sum_{0=n_1<n_2=n_3\le M} q^{n_2-1}(q+1) = \frac{q(q^{2\left[\frac{M}{2}\right]}-1)}{q-1} \qquad \text{if all } \mathfrak{v}(\nu_i) \equiv 0,$$
$$= \frac{q^{2\left[\frac{M+1}{2}\right]}-1}{q-1} \qquad \text{if } \mathfrak{v}(\nu_1) \equiv 0,\ \mathfrak{v}(\nu_2) \equiv \mathfrak{v}(\nu_3) \equiv 1.$$

5) $$\sum_{0<n_1<n_2=n_3\le M} q^{n_1+n_2-3}(q+1)(q^2-1)$$
$$= \frac{q(q+1)(q^{4\left[\frac{M}{2}\right]}-1)}{q^4-1} - \frac{q(q^{2\left[\frac{M}{2}\right]}-1)}{q-1}$$
$$\text{if all } \mathfrak{v}(\nu_i) \equiv 0,$$
$$= \frac{q^4(q+1)(q^{4\left[\frac{M-1}{2}\right]}-1)}{q^4-1} - \frac{q^2(q^{2\left[\frac{M-1}{2}\right]}-1)}{q-1}$$
$$\text{if } \mathfrak{v}(\nu_1) \equiv 0, \quad \mathfrak{v}(\nu_2) \equiv \mathfrak{v}(\nu_3) \equiv 1.$$

6) $$\sum_{\substack{0<n_3<n_2=n_1\le C\\ n_3\le B}} q^{n_1+n_3-3}(q+1)(q^2-1)$$

$$= \frac{q^{2\left[\frac{C}{2}\right]+1}(q^{2\left[\frac{M}{2}\right]}-1)}{q-1} - \frac{q^3(q+1)(q^{4\left[\frac{M}{2}\right]}-1)}{q^4-1} \quad \text{if all } \mathfrak{v}(\nu_i)\equiv 0,$$

$$= \frac{q^{2\left[\frac{C+1}{2}\right]}(q^{2\left[\frac{M}{2}\right]}-1)}{q-1} - \frac{q^2(q+1)(q^{4\left[\frac{M}{2}\right]}-1)}{q^4-1} \quad \text{if } \mathfrak{v}(\nu_3)\equiv 0, \quad \mathfrak{v}(\nu_1)\equiv\mathfrak{v}(\nu_2)\equiv 1.$$

7) $$\sum_{\substack{0<n_2<n_3=n_1\le B\\ n_2\le C}} q^{n_1+n_2-3}(q+1)(q^2-1)$$

$$= \frac{q^{2\left[\frac{B}{2}\right]+1}(q^{2\left[\frac{M}{2}\right]}-1)}{q-1} - \frac{q^3(q+1)(q^{4\left[\frac{M}{2}\right]}-1)}{q^4-1} \quad \text{if all } \mathfrak{v}(\nu_i)\equiv 0,$$

$$= \frac{q^{2\left[\frac{B+1}{2}\right]}(q^{2\left[\frac{M}{2}\right]}-1)}{q-1} - \frac{q^2(q+1)(q^{4\left[\frac{M}{2}\right]}-1)}{q^4-1} \quad \text{if } \mathfrak{v}(\nu_2)\equiv 0, \quad \mathfrak{v}(\nu_1)\equiv\mathfrak{v}(\nu_3)\equiv 1.$$

8) $$\sum_{0<n_1=n_2=n_3\le M} q^{2n_1-3}(q-2)(q+1)^2$$

$$= \frac{q(q-2)(q+1)^2(q^{4\left[\frac{M}{2}\right]}-1)}{q^4-1} \qquad \text{if all } \mathfrak{v}(\nu_i)\equiv 0.$$

9) $$\sum_{n_1=0,B<n_2=n_3\le A} q^{n_2-1}(q+1) = \frac{q^{2\left[\frac{M}{2}\right]+1}(q^{2\left[\frac{N}{2}\right]-2\left[\frac{M}{2}\right]}-1)}{q-1}$$

$$\text{if all } \mathfrak{v}(\nu_i)\equiv 0,\ A>B,$$

$$= \frac{q^{2\left[\frac{M+1}{2}\right]}(q^{2\left[\frac{N+1}{2}\right]-2\left[\frac{M+1}{2}\right]}-1)}{q-1}$$

$$\text{if } \mathfrak{v}(\nu_1)\equiv 0, \quad \mathfrak{v}(\nu_2)\equiv\mathfrak{v}(\nu_3)\equiv 1,\ A>B.$$

10) $$\sum_{0<n_1\le B<n_2=n_3\le A} q^{n_1+n_2-3}(q^2-1)(q+1)$$

$$= \frac{q^{2\left[\frac{M}{2}\right]+1}(q^{2\left[\frac{M}{2}\right]}-1)(q^{2\left[\frac{N}{2}\right]-2\left[\frac{M}{2}\right]}-1)}{q-1}$$

if all $\mathfrak{v}(\nu_i)\equiv 0,\ \ A>B,$

$$= \frac{q^{2\left[\frac{M+1}{2}\right]}(q^{2\left[\frac{M}{2}\right]}-1)(q^{2\left[\frac{N+1}{2}\right]-2\left[\frac{M+1}{2}\right]}-1)}{q-1}$$

if $\mathfrak{v}(\nu_1)\equiv 0,\ \ \mathfrak{v}(\nu_2)\equiv\mathfrak{v}(\nu_3)\equiv 1,\quad A>B.$

11) $$\sum_{\substack{B<n_1\le 2B\\ n_3=n_2=n_1+A-B>n_1}} q^{n_1+A-2}(q+1)^2 = \frac{q^{N+2\left[\frac{M}{2}\right]}(q+1)(q^{2\left[\frac{M+1}{2}\right]}-1)}{q-1}$$

if $\mathfrak{v}(\nu_1)\equiv 0,\quad \mathfrak{v}(\nu_2)\equiv\mathfrak{v}(\nu_3)\equiv A-B,\quad A>B.$

12) $$\sum_{\substack{B<n_3\le 2B\\ n_1=n_2=n_3+C-B>n_3}} q^{n_1+B-2}(q+1)^2 = \frac{q^{N+2\left[\frac{M}{2}\right]}(q+1)(q^{2\left[\frac{M+1}{2}\right]}-1)}{q-1}$$

if $\mathfrak{v}(\nu_3)\equiv 0,\quad \mathfrak{v}(\nu_1)\equiv\mathfrak{v}(\nu_2)\equiv C-B,\quad B<C.$

13) $$\sum_{\substack{C<n_2\le 2C\\ n_1=n_3=n_2+B-C>n_2}} q^{n_1+C-2}(q+1)^2$$

$$= \frac{q^{N+2\left[\frac{M}{2}\right]}(q+1)(q^{2\left[\frac{M+1}{2}\right]}-1)}{q-1}$$

if $\mathfrak{v}(\nu_2)\equiv 0,\quad \mathfrak{v}(\nu_1)\equiv\mathfrak{v}(\nu_3)\equiv B-C,\quad B>C.$

14) $$\sum_{\substack{A<n_1=n_2=n_3\le 2A\\ A=B=C}} q^{n_1+A-2}(q+1)^2 = \frac{q^{M+2\left[\frac{M}{2}\right]}(q+1)(q^{2\left[\frac{M+1}{2}\right]}-1)}{q-1}$$

if all $\mathfrak{v}(\nu_i)\equiv 0,\quad A=B=C.$

If $\mathfrak{v}(\nu_1) \equiv 0$, $\mathfrak{v}(\nu_2) \equiv \mathfrak{v}(\nu_3) \equiv 1$, $F(\nu, t)$ is the sum of (4) + (5), (9) + (10) (if $A > B$) and (11) (if $A \not\equiv B$ and $A > B$).

If $\mathfrak{v}(\nu_2) \equiv 0$, $\mathfrak{v}(\nu_1) \equiv \mathfrak{v}(\nu_3) \equiv 1$, $F(\nu, t)$ is the sum of (2) + (7) and (13) (if $B \not\equiv C$ and $B > C$).

If $\mathfrak{v}(\nu_3) \equiv 0$, $\mathfrak{v}(\nu_1) \equiv \mathfrak{v}(\nu_2) \equiv 1$, $F(\nu, t)$ is the sum of (3) and (6) and (12) (if $B \not\equiv C$ and $C > B$).

We can make the symmetry in the answer explicit by some computations.

$$(4) + (5) = \frac{q(q+1)(q^{4\left[\frac{M}{2}\right]} - 1)}{q^4 - 1} \qquad \text{if all } \mathfrak{v}(\nu_i) \equiv 0,$$

$$= \frac{(q+1)(q^{4\left[\frac{M+1}{2}\right]} - 1)}{q^4 - 1} \qquad \text{if } \mathfrak{v}(\nu_1) \equiv 0, \quad \mathfrak{v}(\nu_2) \equiv \mathfrak{v}(\nu_3) \equiv 1.$$

$$(9) + (10) = \frac{q^{4\left[\frac{M}{2}\right]+1}(q^{2\left[\frac{N}{2}\right]-2\left[\frac{M}{2}\right]} - 1}{q-1} \qquad \text{if all } \mathfrak{v}(\nu_i) \equiv 0 \text{ and } M \neq A,$$

$$= \frac{q^{2m}(q^{2\left[\frac{N+1}{2}\right]-2\left[\frac{M+1}{2}\right]} - 1)}{q-1} \qquad \text{if } \mathfrak{v}(\nu_1) \equiv 0,\ \mathfrak{v}(\nu_2) \equiv \mathfrak{v}(\nu_3) \equiv 1 \text{ and } M \neq A.$$

$$(2) + (7) = \frac{q(q+1)(q^{4\left[\frac{M}{2}\right]} - 1)}{q^4 - 1} \qquad \text{if } M = B,$$

$$= \text{idem } + \frac{q^{4\left[\frac{M}{2}\right]+1}(q^{2\left[\frac{N}{2}\right]-2\left[\frac{M}{2}\right]} - 1)}{q-1} \qquad \text{if } M \neq B,$$

if all $\mathfrak{v}(\nu_i) \equiv 0$;

$$(2) + (7) = \frac{(q+1)(q^{4\left[\frac{M+1}{2}\right]} - 1)}{q^4 - 1} \qquad \text{if } M = B,$$

$$= \text{idem } + \frac{q^{2M}(q^{2\left[\frac{N+1}{2}\right]-2\left[\frac{M+1}{2}\right]} - 1)}{q-1} \qquad \text{if } M \neq B,$$

if $\mathfrak{v}(\nu_2) \equiv 0$, $\mathfrak{v}(\nu_1) \equiv 1$, $\mathfrak{v}(\nu_3) \equiv 1$.

(3) + (6) = same formulas, but the different cases are $M = C$ (resp. $M \neq C$) and all $\mathfrak{v}(\nu_i) \equiv 0$ (resp. $\mathfrak{v}(\nu_3) \equiv 0$, $\mathfrak{v}(\nu_1) \equiv \mathfrak{v}(\nu_2) \equiv 1$).

The final result is:
If $\mathfrak{v}(\nu_1) \equiv 0$, $\mathfrak{v}(\nu_2) \equiv \mathfrak{v}(\nu_3) \equiv 1$, then $F(\nu, t)$ is equal to

$$(q+1)\frac{q^{4\left[\frac{M+1}{2}\right]}-1}{q^4-1} \qquad \text{if } M = A,$$

$$\text{idem} + q^{2M}\frac{q^{2\left[\frac{N+1}{2}\right]-2\left[\frac{M+1}{2}\right]}-1}{q-1} \qquad \text{if } M \neq A \text{ and } M \equiv N \bmod 2,$$

$$\text{idem } + \text{idem} + q^{N+2\left[\frac{M}{2}\right]}(q+1)\frac{q^{2\left[\frac{M+1}{2}\right]}-1}{q-1} \quad \text{if } M \neq A,\ M \not\equiv N(2).$$

If $\mathfrak{v}(\nu_2) \equiv 0$, $\mathfrak{v}(\nu_1) \equiv \mathfrak{v}(\nu_3) \equiv 1$: the same formulas, read B instead of A.
If $\mathfrak{v}(\nu_3) \equiv 0$, $\mathfrak{v}(\nu_1) \equiv \mathfrak{v}(\nu_2) \equiv 1$: the same formulas, read C instead of A.
If all $\mathfrak{v}(\nu_i) \equiv 0$, then $F(\nu, t) =$

$$1 + q(q^3+1)\frac{q^{4\left[\frac{M}{2}\right]}-1}{q^4-1} + q^{4\left[\frac{M}{2}\right]+1}\frac{q^{2\left[\frac{N}{2}\right]-2\left[\frac{M}{2}\right]}-1}{q-1}$$
$$+ q^{N+2\left[\frac{M}{2}\right]}(q+1)\frac{q^{2\left[\frac{M+1}{2}\right]}-1}{q-1} \qquad (M \equiv N \bmod 2).$$

The last term occurs when $M \equiv N \bmod 2$ only.

II. TRACE FORMULA

II.1 Stable trace formula

1.1 Let F be a global field with a ring $\mathbb{A} = \mathbb{A}_F$ of adèles. Denote by E a quadratic field extension, and by $\mathbb{A}^1$ the group of idèles of E whose norm from E to F is 1. The center $\mathbf{Z}(\mathbb{A})$ of $\mathbf{G}(\mathbb{A}) = \mathbf{U}(3, E/F)(\mathbb{A})$ is isomorphic to $\mathbb{A}^1$. Fix a character ω of $\mathbf{Z}(\mathbb{A})/Z$ (Z is $\mathbf{Z}(F)$). Denote the action of ($\sigma \neq 1 \in$) $\mathrm{Gal}(E/F)$ on the idèle x in $\mathbb{A}_E^\times$ by $\overline{x}$. Then $\omega'(x) = \omega(\overline{x}/x)$ defines a character of the center $\mathbf{Z}'(\mathbb{A}) = \mathbb{A}_E^\times$ of $\mathbf{G}'(\mathbb{A}) = \mathbf{G}(\mathbb{A}_E)$, which is trivial on $E^\times\mathbb{A}^\times$.

For each place v of F, let f_v be a smooth (this means locally constant in the nonarchimedean case) complex-valued function on $G_v = \mathbf{G}(F_v)$, which satisfies $f_v(zx) = \omega_v(z)^{-1}f_v(x)$ for all z in Z_v, x in G_v, where ω_v is the component of ω at v. Further, the support of f_v is compact modulo Z_v. At v which splits in E we have $G_v = \mathrm{GL}(3, F_v)$. If v is nonarchimedean let R_v be the ring of integers in F_v and R_{Ev} that of $E_v = F_v \otimes_F E$. Let K_v be the hyperspecial maximal compact subgroup $\mathbf{G}(R_v)$ of G_v. That is, it is the group of $\mathrm{Gal}(E/F)$-fixed points on $\mathbf{G}(R_{Ev})$. At almost all v the character ω_v is unramified, and we take f_v to be the function f_v^0, which attains the value $\omega_v(z)^{-1}/|K_v/K_v \cap Z_v|$ at zk in Z_vK_v and 0 elsewhere. Here $|K_v|$ denotes the volume of K_v with respect to a Haar measure fixed below. Put $f = \otimes f_v$.

Let $L = L^2$ be the space of complex valued functions ψ on $G\backslash\mathbf{G}(\mathbb{A})$ with $\psi(zg) = \omega(z)\psi(g)$ ($z \in Z\backslash\mathbf{Z}(\mathbb{A})$) which are square integrable on $G\mathbf{Z}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$. The group $\mathbf{G}(\mathbb{A})$ acts on L by right translation, thus $(r(g)\psi)(h) = \psi(hg)$. Each irreducible constituent of the $\mathbf{G}(\mathbb{A})$-module L is called an *automorphic* $\mathbf{G}(\mathbb{A})$-*module* (or *representation*). Fix a Haar measure $dg = \otimes dg_v$ on $\mathbf{G}(\mathbb{A})/\mathbf{Z}(\mathbb{A})$ such that $\prod_v |K_v/K_v \cap Z_v|$ converges. Let f be any smooth complex valued function on $\mathbf{G}(\mathbb{A})$ which transforms by ω^{-1} under $\mathbf{Z}(\mathbb{A})$ and is compactly supported on $\mathbf{G}(\mathbb{A})/\mathbf{Z}(\mathbb{A})$. Let $r(fdg)$ be the (convolution) operator on L which maps ψ to

$$(r(fdg)\psi)(h) = \int f(g)\psi(hg)dg \qquad (g \in \mathbf{G}(\mathbb{A})/\mathbf{Z}(\mathbb{A})).$$

This is

$$\int_{\mathbf{G}(\mathbb{A})/\mathbf{Z}(\mathbb{A})} f(h^{-1}g)\psi(g)dg = \int_{G\mathbf{Z}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})} K(h,g)\psi(g)dg.$$

Hence $r(fdg)$ is a convolution operator with kernel

$$K(h,g) = K_f(h,g) = \sum_{\gamma\in G/Z} f(h^{-1}\gamma g). \tag{1.1.1}$$

The theory of Eisenstein series provides a direct sum decomposition of the $\mathbf{G}(\mathbb{A})$-module L as $L_d \oplus L_c$. The "continuous spectrum", L_c, is a direct integral of irreducibles. The "discrete spectrum", L_d, is the sum of the irreducible submodules of L. It splits as the direct sum of the cuspidal spectrum L_0 and the residual spectrum L_r. It is a direct sum $\oplus_\pi m(\pi)L_\pi$ of irreducible $\mathbf{G}(\mathbb{A})$-modules (π, L_π) occurring with finite multiplicities $m(\pi)$. If $\{\phi_i^\pi\}$ is an orthonormal basis of L_π then the kernel of $r(fdg)$ on L_d is

$$K_d(k,g) = \sum_\pi m(\pi) \sum_{\phi_i^\pi\in L_\pi} \int_h f(h^{-1}k)\overline{\phi_i^\pi}(h)dh\cdot\phi_i^\pi(g),$$

h in $G\mathbf{Z}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$. Indeed,

$$\begin{aligned}
(r(fdg)\phi)(g) &= \sum_{\pi,\phi_i^\pi} m(\pi)\langle r(fdg)\phi,\phi_i^\pi\rangle\cdot\phi_i^\pi(g)\\
&= \sum m(\pi)\int_h (r(fdg)\phi)(h)\overline{\phi_i^\pi}(h)dh\cdot\phi_i^\pi(g)\\
&= \sum m(\pi)\int_h\int_{k\in\mathbf{G}(\mathbb{A})/\mathbf{Z}(\mathbb{A})} f(k)\phi(hk)dk\cdot\overline{\phi_i^\pi}(h)dh\cdot\phi_i^\pi(g)\\
&= \int_k\left[\sum m(\pi)\int_h f(h^{-1}k)\overline{\phi_i^\pi}(h)dh\cdot\phi_i^\pi(g)\right]\phi(k)dk.
\end{aligned}$$

The trace of $r(fdg)$ over the discrete spectrum is the integral of K_d over the diagonal $k=g$ in $\mathbf{Z}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$:

$$\begin{aligned}
&\sum_\pi\sum_{\phi^\pi{}_i} m(\pi)\int_g\int_h \overline{\phi_i^\pi}(h)f(h^{-1}g)\phi_i^\pi(g)dhdg\\
&= \sum\sum m(\pi)\int_h\int_g \overline{\phi_i^\pi}(h)f(g)\phi_i^\pi(hg)dgdh\\
&= \sum\sum m(\pi)\int_h [r(fdg)\phi_i^\pi](h)\overline{\phi_i^\pi}(h)dh\\
&= \sum_\pi m(\pi)\sum_{\phi_i^\pi}\langle\pi(fdg)\phi_i^\pi,\phi_i^\pi\rangle = \sum_\pi m(\pi)\,\mathrm{tr}\,\pi(fdg),
\end{aligned}$$

where $\pi(fdg)$ denotes the restriction of $r(fdg)$ to π.

The contribution to the trace formula from the complement of L_d in L^2 is described using Eisenstein series; we describe this spectral side below. This side will be used to study the representations π whose traces occur in the sum.

The Selberg trace formula is an identity obtained on (essentially) integrating the spectral and geometric expressions for the kernel over the diagonal $g = h$. To get a useful formula one needs to change the order of summation and integration. This is possible if $\mathbf{G}$ is anisotropic over F or if f has a cuspidal component and a component supported on the regular elliptic set, or is regular in the sense of [FK2]. In general one needs to truncate the two expressions for the kernel in order to be able to change the order of summation and integration.

We now turn to the geometric side of the trace formula.

The geometric side of the trace formula is obtained on integrating over the diagonal $g = h \in \mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})$ the kernel of the convolution operator $r(fdg)$ on L^2:

$$(r(fdg)\phi)(h) = \int_{\mathbf{G}(\mathbb{A})/\mathbf{Z}(\mathbb{A})} f(h^{-1}g)\phi(g)dg$$

$$= \int_{\mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})} \left[\sum_{\gamma\in G/Z} f(h^{-1}\gamma g)\right]\phi(g)dg.$$

We consider only the subsum

$$K_e(h,g) = \sum_{x\in G_e/Z} f(h^{-1}xg)$$

over the set G_e of semisimple, regular and elliptic elements x in G.

A semisimple element x of G is called *regular* if its centralizer $\mathbf{Z_G}(x)$ in $\mathbf{G}$ is a torus, and x is called *elliptic* if it lies in an anisotropic torus. In our global case *anisotropic* means that $\mathbf{T}(\mathbb{A})/T\mathbf{Z}(\mathbb{A})$ is compact, and in the local case it means that T_v/Z_v is compact, where $T = \mathbf{T}(F)$ and $T_v = \mathbf{T}(F_v)$. If x is elliptic regular, T is an elliptic torus.

The integral over $h = g$ in $\mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})$ of $K_e(g,g)dg$ is the sum over a set of representatives x for the conjugacy classes in G_e/Z of orbital integrals:

$$\begin{aligned}&\sum_x \int_{Z_G(x)\mathbf{Z}(\mathbb{A})\backslash \mathbf{G}(\mathbb{A})} f(g^{-1}xg)dg \\ &= \sum_x \mathrm{vol}_{dt}[Z_G(x)\mathbf{Z}(\mathbb{A})\backslash Z_{\mathbf{G}}(x)(\mathbb{A})] \int_{Z_{\mathbf{G}}(x)(\mathbb{A})\backslash \mathbf{G}(\mathbb{A})} f(g^{-1}xg)\frac{dg}{dt}.\end{aligned}$$

1.2 The conjugacy-class analysis of I.1 is motivated by the appearance in the trace formula of the absolutely convergent sum that we just obtained:

$$\sum_x \delta(x)^{-1}|\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|\Phi(x, fdg) \tag{1.2.1}$$

over all conjugacy classes x of regular elliptic elements in G modulo Z. Here $\delta(x)$ is the index $[Z_{G/Z}(x) : T/Z]$ of T/Z in the centralizer $Z_{G/Z}(x)$ of x in G/Z, and $\mathbf{T}$ is the centralizer $Z_{\mathbf{G}}(x)$ of x in $\mathbf{G}$. The volume $|\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|$ of the quotient (with respect to a Tamagawa measure) is finite since x is an elliptic regular element. We fix differential forms of highest degree defined over F on $\mathbf{G}/\mathbf{Z}$ and $\mathbf{T}/\mathbf{Z}$, and define Haar measures dg and dt on G_v/Z_v and T_v/Z_v at all v. The factor $\Phi(x, f_v dg_v)$ is the orbital integral $\int f_v(gxg^{-1})dg/dt$ (over G_v/T_v) if x is regular with centralizer T_v. We put $\Phi(x, fdg) = \prod \Phi(x, f_v dg_v)$ for regular x in G (with centralizer T).

1.3 The sum (1.2.1) can be written as a sum over the conjugacy classes in G of elliptic tori T, and a sum over the regular x in T/Z. But we have to note that $\delta(x)$ equals the number of w in the Weyl group $W(T)$ of T in G with $wxw^{-1} = zx$ for some z in Z, and the conjugacy class of x in G/Z intersects T/Z precisely $[W(T)]/\delta(x)$ times. So we have

$$\sum_T \frac{|\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|}{[W(T)]} {\sum_x}' \quad \Phi(x, fdg) \qquad (x \text{ in } T/Z)$$

where $\sum_x'$ indicates sum over regular elements. This is equal to

$$ {\sum_T}' \frac{|\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|}{[W'(T)]} {\sum_x}' \sum_{b \text{ in } B(\mathbf{T}/F)} \Phi(x^b, fdg). \tag{1.3.1}$$

Here $\sum_T'$ indicates sum over (a set of representatives for the) stable conjugacy classes of elliptic T. The group $W'(T)$ is the Weyl group of T in

$A(\mathbf{T}/F)$. The element x^b is $\mathbf{b}^{-1}x\mathbf{b}$, where $\mathbf{b}$ is a representative of b in $\mathbf{G}(\overline{F})$. Note that $\Phi(x^b, fdg)$, as a function of $\mathbf{b}$, depends only on the projection of $\mathbf{b}$ in $B(\mathbf{T}/F)$.

1.4 For a fixed regular x the sum over b is finite. The pointed set $B(\mathbf{T}/F)$ is a subset of the group $C(\mathbf{T}/F)$. We extend the sum to $C(\mathbf{T}/F)$, setting $\Phi(x^b, fdg) = 0$ if b lies in $C(\mathbf{T}/F) - B(\mathbf{T}/F)$. Note that if the image in $C(\mathbf{T}/\mathbb{A})$, of b in $C(\mathbf{T}/F)$, lies in $B(\mathbf{T}/\mathbb{A})$, then b lies in $B(\mathbf{T}/F)$. Since $\Phi(x^b, fdg) = \prod_v \Phi(x^b, f_v dg_v)$, it depends only on the image of b in $C(\mathbf{T}/\mathbb{A})$. It remains to note that in our case the map $C(\mathbf{T}/F) \to C(\mathbf{T}/\mathbb{A})$ is injective (in general the kernel is finite).

Definition. (1) If κ_v is the restriction of κ to $C(\mathbf{T}/F_v)$ we put

$$\Phi^{\kappa_v}(x, f_v dg_v) = \sum \kappa_v(b)\Phi(x^b, f_v dg_v) \qquad (b \text{ in } C(\mathbf{T}/F_v)),$$

where we set $\Phi(x^b, f_v dg_v) = 0$ if b lies in $C(\mathbf{T}/F_v) - B(\mathbf{T}/F_v)$. Let $\Phi^\kappa(x, fdg)$ be the product over all places v of the local sums (which are almost all trivial).

(2) When κ is trivial, put $\Phi^{\mathrm{st}}(x, fdg)$ for $\Phi^1(x, fdg)$, and $\Phi^{\mathrm{st}}(x, f_v dg_v)$ for $\Phi^{1_v}(x, f_v dg_v)$. These are called *stable* orbital integrals.

(3) Let $k(T)$ be the finite group of characters of $C(\mathbf{T}/\mathbb{A})/C(\mathbf{T}/F)$.

We obtain a sum

$$[k(T)]^{-1} \sum_\kappa \Phi^\kappa(x, fdg).$$

Here κ ranges over the finite group $k(T)$, which is described in I.1.

1.5 The group $k(T)$ is trivial unless T is quadratic, when $[k(T)] = 2$, or T splits over E, when $[k(T)] = 4$. We obtain the sum of

$$\sum_T{}' \quad \frac{|\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|}{[W'(T)][k(T)]} \sum_x{}' \quad \Phi^{\mathrm{st}}(x, fdg) \tag{1^*}$$

and

$$\frac{1}{2} \sum_T{}'' \quad \frac{|\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|}{[W'(T)][k'(T)]} \sum_{\kappa \neq 1} \sum{}_x' \quad \Phi^\kappa(x, fdg). \tag{1^{**}}''$$

$\sum_T''$ ranges over the T with even $[k(T)]$, where we put $[k'(T)] = [k(T)]/2$.

Consider the stable conjugacy class of the elliptic T which splits over E. Fix $\kappa \neq 1$.

LEMMA. *We have* $\sum_x' \Phi^{\kappa'}(x, fdg) = \sum_x' \Phi^{\kappa}(x, fdg)$ *for any* $\kappa' \neq 1$.

PROOF. The group $W'(T)$ acts (transitively) on the group

$$\mathrm{Im}[H^{-1}(F, X_*(\mathbf{T}^{\mathrm{sc}})) \to H^{-1}(F, X_*(\mathbf{T}))],$$

hence on its dual group $k(T)$. For b in $B(\mathbf{T}/\mathbb{A})$ and w in $W'(T)$, we have

$$(bw)_\tau = (bw)^{-1}\tau(bw) = w^{-1}b_\tau w \cdot w_\tau \quad (w_\tau = w^{-1}\tau(w)).$$

If $\kappa^w(\{b_\tau\}) = \kappa(\{w^{-1}b_\tau w\})$, then

$$\Phi(x^w, fdg, \kappa^w) = \sum_b \kappa(\{w^{-1}b_\tau w\})\Phi(x^{bw}, fdg)$$

$$= \kappa(\{w_\tau\})^{-1} \sum_b \kappa(\{b_\tau\})\Phi(x^b, fdg) = \Phi^\kappa(x, fdg).$$

The last equality follows from the triviality of κ on $C(\mathbf{T}/F)$. □

1.6 Note that there is a bijection between the stable conjugacy classes of T in $(1^{**})''$, and the stable conjugacy classes of elliptic tori in $H = \mathrm{U}(2) \simeq \mathrm{U}(2) \times \mathrm{U}(1)/Z$ (where $\mathrm{U}(1) \simeq Z \simeq E^1$). If T is quadratic (its splitting field is a biquadratic extension of F), then $[k'(T_H)] = 1$, and $[W'(T_H)] = 2$ is the cardinality of the Weyl group of T_H in $A(\mathbf{T}_H/F)$ with respect to H. If T splits over E, there are three $\kappa \neq 1$ in $(1^{**})''$, $[k'(T)] = 2$ and $[W'(T)] = 6$. With respect to H, $[k(T_H)] = 2$ and $[W'(T_H)] = 2$. Hence we can write $(1^{**})''$ in the form

$$\frac{1}{2}\sum_{T_H}{}'' \quad \frac{|\mathbf{T}_H(\mathbb{A})/T_H|}{[W'(T_H)][k(T_H)]} \sum_x{}' \quad \Phi^\kappa(x, fdg). \qquad (1^{**})'$$

$\sum_{T_H}''$ now indicates the sum over the stable conjugacy classes of elliptic T_H in H. The groups $W'(T_H)$ and $k(T_H)$ are defined with respect to H, and $\sum_x'$ is the sum over all regular x in T with eigenvalues not equal to 1. The character κ is nontrivial.

The Fundamental Lemma and the Matching Orbital Integrals Lemma of I.2 imply that we can put $(1^{**})'$ in the form

$$\frac{1}{2}\sum_{T_H}{}'' \quad \frac{|\mathbf{T}_H(\mathbb{A})/T_H|}{[W'(T_H)][k(T_H)]} \sum_x{}'' \quad \Phi^{\mathrm{st}}(x, {}'fdh). \qquad (1^{**})$$

Indeed, we choose a global character κ of $\mathbb{A}_E^\times/E^\times N\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}^\times$ is nontrivial. At v which splits in E we take

$$'f(x) = f_M(x)/\kappa(-(\varepsilon-1)(\varepsilon'-1)).$$

As usual, f_M is defined by $f_M(m) = \delta_P(m)^{1/2}\int_N f(mn)dn$ where $P = MN$ is the standard parabolic subgroup with Levi factor M and unipotent radical N, $m \in M$ has eigenvalues 1, ε, ε', and $\delta_P(m) = |\det(\mathrm{Ad}(m)|\mathfrak{n})|$.

The sum of (1^{**}) is the stabilized elliptic regular part of the trace formula of $\mathbf{H}(\mathbb{A})$.

II.2 Twisted trace formula

2.1 Analogous discussion has to be given in the twisted case. Again F is a global field, and E is a quadratic field extension. We fix a character $\omega'(x) = \omega(x/\overline{x})$ on $\mathbf{Z}'(\mathbb{A})/Z'$, namely on $\mathbb{A}_E^\times/E^\times$, which is trivial on $\mathbb{A}^\times$. We use a test function $\phi = \otimes\phi_v$ on $\mathbf{G}'(\mathbb{A}) = \mathbf{G}(\mathbb{A}_E) = \mathrm{GL}(3,\mathbb{A}_E)$, where $\mathbf{G}' = \mathrm{R}_{E/F}\,\mathbf{G}$. The component ϕ_v is smooth, transforms under $Z'_v = \mathbf{Z}'(F_v) = \mathbf{Z}(E_v)$ by $\omega_v'^{-1}$, and is compactly supported modulo the center. For almost all v the component ϕ_v is ϕ_v^0, the function supported on $Z'_vK'_v$, whose value on $K'_v = \mathbf{G}'(R_v)$ is the volume $|K'_v/K'_v \cap Z'_v|^{-1}$. When v splits we take $\phi_v = (f_v, f_v^0)$ if f_v is spherical; otherwise f_v^0 is a measure of volume one with $f_v = f_v * f_v^0$. So for almost all split v, we have $\phi_v^0 = (f_v^0, f_v^0)$.

The trace formula, twisted by σ, is developed in close analogy with the nontwisted case. Let L' be the space of complex valued functions ψ' on $G'\backslash\mathbf{G}'(\mathbb{A})$ which transform under $\mathbf{Z}'(\mathbb{A})$ via ω', and are square integrable on $G'\mathbf{Z}'(\mathbb{A})\backslash\mathbf{G}'(\mathbb{A})$. The group $\mathbf{G}'(\mathbb{A})$ acts on L' by right translation, thus $(r(g)\psi')(h) = \psi'(hg)$. Each irreducible constituent of the $\mathbf{G}'(\mathbb{A})$-module L' is called an *automorphic* $\mathbf{G}'(\mathbb{A})$-module (or representation). Let σ be the involution of $\mathbf{G}(\mathbb{A}_E)$ given by $\sigma(g) = J\,{}^t\overline{g}^{-1}J$. This is the group of points avatar of the algebraic involution $\iota(x,y) = (y,x)$ of the F-group $\mathbf{G}' = \mathrm{R}_{E/F}\,\mathbf{G}$. Put $\mathbf{G}''(\mathbb{A}) = \mathbf{G}'(\mathbb{A}) \rtimes \langle\sigma\rangle$ for the semidirect product of $\mathrm{GL}(3,\mathbb{A}_E)$ and the group $\mathrm{Gal}(E/F) = \langle\sigma\rangle$. Thus $\mathbf{G}'' = \mathbf{G}' \rtimes \langle\iota\rangle$. Extend r to a representation of $\mathbf{G}''(\mathbb{A})$ on L' by putting $(r(\sigma)\psi')(h) = \psi'(\sigma(h))$. Fix a Haar measure $dg' = \otimes dg'_v$ on $\mathbf{G}'(\mathbb{A})$. Let ϕ be any smooth complex valued compactly supported modulo $\mathbf{Z}'(\mathbb{A})$ function on $\mathbf{G}'(\mathbb{A})$ which transforms under the center according to ω'^{-1}. Let $r(\phi dg')$ be the (convolution) operator on L' which maps ψ' to

$$(r(\phi dg')\psi')(h) = \int \phi(g)\psi'(hg)dg' \qquad (g \in \mathbf{G}'(\mathbb{A})).$$

Then $r(\phi dg')r(\sigma)$, which we also denote by $r(\phi dg' \times \sigma)$, is the operator on L' which maps ψ' to $(r(\phi dg')r(\sigma)\psi')(h)$

$$= \int_{\mathbf{G}'(\mathbb{A})/\mathbf{Z}'(\mathbb{A})} \phi(g)(r(\sigma)\psi')(hg)dg' = \int \phi(g)\psi'(\sigma(hg))dg'$$

$$= \int \phi(h^{-1}g)\psi'(\sigma(g))dg' = \int_{\mathbf{G}'(\mathbb{A})/\mathbf{Z}'(\mathbb{A})} \phi(h^{-1}\sigma(g))\psi'(g)dg'$$

$$= \int_{G'\mathbf{Z}'(\mathbb{A})\backslash\mathbf{G}'(\mathbb{A})} K_\phi(h,g)\psi'(g)dg',$$

where

$$K_\phi(h,g) = \sum_{x\in G'/Z'} \phi(h^{-1}x\sigma(g)). \tag{2.1.1}$$

The σ-twisted trace formula is obtained on integrating over the diagonal $g = h$ in $\mathbf{G}'(\mathbb{A})$ the geometric and spectral expressions for the kernel of our convolution operator $r(\phi dg' \times \sigma)$, and changing the order of the summation and integration. For this change we need to truncate both expressions for the kernel. However, the truncation does not affect the σ-regular elliptic part of the geometric side (nor does it affect the discrete part of the spectral spectrum). Thus as in the nontwisted case, we begin by analyzing the σ-elliptic regular part of the geometric expression for the kernel, namely its integral over the diagonal.

Thus we begin with a sum

$$\sum_x \delta'(x)^{-1}|\mathbf{T}(\mathbb{A})/T\mathbf{Z}(\mathbb{A})|\Phi(x\sigma, \phi dg'),$$

over the σ-conjugacy classes x of σ-regular σ-elliptic elements in G'/Z'. The group T is the σ-centralizer of x in G'; $\delta'(x)$ is the index of TZ' in the σ-centralizer of x in G'/Z'. Here $\Phi(x\sigma, \phi dg')$ is the integral $\int \phi(yx\sigma(y^{-1}))dy$ over $\mathbf{G}'(\mathbb{A})/\mathbf{T}(\mathbb{A})\mathbf{Z}'(\mathbb{A})$. As ϕ transforms by ω'^{-1}, we have $\phi(z\bar{z}x) = \phi(x)$ for z in $\mathbf{Z}'(\mathbb{A}) \simeq \mathbb{A}_E^\times$. The orbital integral $\Phi(x\sigma, \phi dg')$ is a product of local orbital integrals $\Phi(x\sigma, \phi_v dg'_v)$.

Lemma. *Let $\sum_T'$ indicate the sum over the stable conjugacy classes of elliptic T in G, and $\sum_x'$ the sum over the regular x' in T/Z. Then our sum is*

$$\sum_T{}' \quad \frac{|\mathbf{T}(\mathbb{A})/T\mathbf{Z}(\mathbb{A})|}{[W'(T)]} \sum_{x'}{}' \sum_{b \text{ in } B'(\mathbf{T}/F)} \Phi((x\sigma)^b, \phi dg').$$

Proof. The sum over b is defined to be 0 unless there is x in G' with $Nx = x'$. If $Nx = x'$, we let $W'(x')$ be the set of g in $\overline{G}/Z_{\overline{G}}(x')$ with $gx'g^{-1} = zx'$ for some z in Z; and $W'(x)$ the set of g in $\overline{G}'/F^\times Z_{\overline{G}'}(x\iota)$ with $gx\iota(g^{-1}) = zx'$ for some z in Z'. Here $\mathbf{Z}_{G'}(x\iota)$ is the ι-centralizer of x in $\mathbf{G}'$, and $F^\times$ is the group of (z, z^{-1}), z in $F^\times$. It is clear that the map $W'(x) \to W'(x')$, by $g = (g', g'') \mapsto g'$, is an isomorphism. Also we put $W(x)$ for the g in $G'/Z'Z_{G'}(x\iota)$. It is clear that $\delta'(x) = [W(x)]$, and that $W(x) \to W'(x)$ is injective. Further we note that the stable conjugacy class of x' intersects T/Z in $[W'(T)]/[W'(x)]$ points. If $\delta''((x\iota)^b)$ is the number of b' in $B'(\mathbf{T}/F)$ with $(x\iota)^{b'}$ conjugate to $z(x\iota)^b$ for some z in Z', it remains to show that $[W'(x)]$ is $\delta'((x\iota)^b)\delta''((x\iota)^b)$, or $\delta''(x\iota) = [W'(x) : W(x)]$, as we can take $b = 1$. But this is clear. Note that it suffices to deal only with x so that $W'(x')$, $W'(x)$ are trivial, by virtue of our assumptions below about the support of ϕ. □

2.2 The sum over b can be replaced by the quotient by $[k''(T)]$ of the sum over κ in $k''(T)$ of $\Phi^\kappa(x, \phi dg')$. The group $k''(T)$ is the dual group of the quotient of $B'(\mathbf{T}/\mathbb{A})$ by (the image of) $B'(\mathbf{T}/F)$, computed above. Note that $[k''(T)] = [k(T)]$. Hence we obtain the twisted analogue of (1^*) and $(1^{**})'$, namely

$$\sum_T{}' \quad \frac{|\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|}{[W'(T)][k(T)]} \sum_x{}' \quad \Phi^{\mathrm{st}}(x\sigma, \phi dg'), \tag{2^*}$$

and

$$\frac{1}{2} \sum_{T_H}{}'' \quad \frac{|\mathbf{T}_H(\mathbb{A})/T_H|}{[W'(T_H)][k(T_H)]} \sum_x{}' \quad \Phi^\kappa(x\sigma, \phi dg'). \tag{$(2^{**})'$}$$

The notations in $(2^{**})'$ are taken with respect to H.

2.3 The twisted and nontwisted stable terms (1^*), (2^*) are related by the basechange map. The twisted unstable sum $(2^{**})'$ can be related to the stable sum of the elliptic terms in the trace formula of H, as in the case of the nontwisted unstable sum $(1^{**})'$. For that we need both the matching and the fundamental Lemmas of I.2.

Assuming that ϕ and $'\phi$ are global matching functions, $(2^{**})'$ can be put in the form

$$\frac{1}{2}\sum_{T_H}{}'' \quad \frac{|\mathbf{T}_H(\mathbb{A})/T|}{[W'(T_H)][k(T_H)]}\sum_{x}{}' \quad \Phi^{\mathrm{st}}(x,'\phi dh). \tag{2^{**}}$$

This is the stabilized elliptic part of the trace formula for $\mathbf{H}(\mathbb{A})$ and $'\phi dh$.

II.3 Restricted comparison

3.1 The theory of Eisenstein series decomposes the module $L' = L(G')$ of automorphic forms into a direct sum of two submodules, L'_d and L'_c. The $\mathbf{G}'(\mathbb{A})$-module L'_d is the submodule of L' consisting of all $\mathbf{G}'(\mathbb{A})$-submodules Π of L'. Each such Π appears with finite multiplicity in $L'_d \subset L'$, and is called *discrete-series* representation. The $\mathbf{G}'(\mathbb{A})$-module L'_c decomposes as a direct integral. The $\mathbf{G}'(\mathbb{A})$-module L'_d further decomposes as the direct sum of the space L'_0 of cusp forms, and the space L'_r of residual forms.

The theory of Eisenstein series provides an alternative, spectral expression for the kernel of the convolution operator $r(\phi dg')r(\sigma)$ of section II.2. The Selberg trace formula is an identity obtained on (essentially) integrating the two expressions for the kernel over the diagonal $g = h$. To get a useful formula one needs to change the order of summation and integration. This is possible if $\mathbf{G}$ is anisotropic over F or if f has some special properties (see, e.g., [FK2]). In general one needs to truncate the two expressions for the kernel in order to be able to change the order of summation and integration.

The discussion above holds for any automorphism σ of finite order of a reductive connected F-group $\mathbf{G}$. When σ is trivial, the truncation introduced by Arthur involves a term for each standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. For $\sigma \neq 1$ it was suggested in our 1981 IHES preprint "The adjoint lifting from SL(2) to PGL(3)" (in the context of the symmetric square lifting) to truncate only with the terms associated with σ-invariant $\mathbf{P}$, and to use a certain normalization of a vector which is used in the definition of truncation. The consequent (nontrivial) computation of the resulting twisted (by σ) trace formula is carried out in [CLL] for general $\mathbf{G}$ and σ. We proceed to record the expression of [CLL] for the analytic side of the trace formula, which involves Eisenstein series.

Let $\mathbf{P}_0$ be a minimal σ-invariant F-parabolic subgroup of $\mathbf{G}$, with Levi subgroup $\mathbf{M}_0$. Let $\mathbf{P}$ be any standard (containing $\mathbf{P}_0$) F-parabolic subgroup of $\mathbf{G}$. Denote by $\mathbf{M}$ the Levi subgroup which contains $\mathbf{M}_0$ and by $\mathbf{A}$ the split component of the center of $\mathbf{M}$. Then $\mathbf{A} \subset \mathbf{A}_0 = \mathbf{A}(\mathbf{M}_0)$. Let $X^*(\mathbf{A})$ be the lattice of rational characters of $\mathbf{A}$, $\mathcal{A}_M = \mathcal{A}_P$ the vector space $X_*(\mathbf{A}) \otimes \mathbb{R} = \operatorname{Hom}(X^*(\mathbf{A}), \mathbb{R})$, and $\mathcal{A}^*$ the space dual to $\mathcal{A}$. Let $W_0 = W(A_0, G)$ be the Weyl group of A_0 in G. Both σ and every s in W_0 act on $\mathcal{A}_0$. The truncation and the general expression to be recorded depend on a vector T in $\mathcal{A}_0 = \mathcal{A}_{M_0}$. In the case considered in this part this T becomes a real number, the expression is linear in T, and we record further below only the value at $T = 0$.

Proposition [CLL]. *The analytic side of the trace formula is equal to a sum over*

(1) *The set of Levi subgroups* $\mathbf{M}$ *which contain* $\mathbf{M}_0$ *of* F*-parabolic subgroups of* $\mathbf{G}'$.

(2) *The set of subspaces* $\mathcal{A}$ *of* $\mathcal{A}_0$ *such that for some* s *in* W_0 *we have* $\mathcal{A} = \mathcal{A}_M^{s\times\sigma}$, *where* $\mathcal{A}_M^{s\times\sigma}$ *is the space of* $s \times \sigma$*-invariant elements in the space* $\mathcal{A}_M$ *associated with a* σ*-invariant* F*-parabolic subgroup* $\mathbf{P}$ *of* $\mathbf{G}'$.

(3) *The set* $W^{\mathcal{A}}(\mathcal{A}_M)$ *of distinct maps on* $\mathcal{A}_M$ *obtained as restrictions of the maps* $s \times \sigma$ *(*s *in* W_0*) on* $\mathcal{A}_0$ *whose space of fixed vectors is precisely* $\mathcal{A}$.

(4) *The set of discrete-spectrum representations* τ *of* $\mathbf{M}(\mathbb{A})$ *with* $(s \times \sigma)\tau \simeq \tau$.

The terms in the sum are equal to the product of
$\frac{[W_0^M]}{[W_0]}(\det(1 - s \times \sigma)|_{\mathcal{A}_M/\mathcal{A}})^{-1}$ *and*

$$\int_{i\mathcal{A}^*} \operatorname{tr}[\mathcal{M}_{\mathcal{A}}^T(P,\lambda) M_{P|\sigma(P)}(s,0) I_{P,\tau}(\lambda;\phi dg' \times \sigma)]|d\lambda|.$$

Here $[W_0^M]$ is the cardinality of the Weyl group $W_0^M = W(A_0, M)$ of A_0 in M; $\mathbf{P}$ is an F-parabolic subgroup of $\mathbf{G}'$ with Levi component $\mathbf{M}$; $M_{P|\sigma(P)}$ is an intertwining operator; $\mathcal{M}_{\mathcal{A}}^T(P,\lambda)$ is a logarithmic derivative of intertwining operators, and $I_{P,\tau}(\lambda)$ is the $\mathbf{G}'(\mathbb{A})$-module normalizedly induced from the $\mathbf{M}(\mathbb{A})$-module $m \mapsto \tau(m)e^{\langle\lambda, H(m)\rangle}$ (in standard notations).

REMARK. The sum of the terms corresponding to $\mathbf{M} = \mathbf{G}'$ in (1) is equal to the sum $I = \sum \operatorname{tr} \Pi(\phi dg' \times \sigma)$ over all discrete-spectrum representations Π of $\mathbf{G}'(\mathbb{A})$ which are σ-invariant.

We write $\operatorname{tr} \Pi(\phi dg')$ for the trace of the trace class convolution operator $\Pi(\phi dg') = \int \phi(g')\Pi(g')dg'$ (g' in $\mathbf{G}'(\mathbb{A})/\mathbf{Z}'(\mathbb{A})$; dg' is a Tamagawa measure, often omitted from the notations), for an admissible Π.

The spectral side of the nontwisted trace formula for $\mathbf{G}(\mathbb{A})$ is described by the Proposition above, where σ is replaced by the identity and $\mathbf{G}'$ by $\mathbf{G}$.

The trace formula for $\mathbf{G}(\mathbb{A})$ and the trace formula for $\mathbf{G}'(\mathbb{A})$ twisted with respect to σ, are compared in II.4 below for measures $f dg$ and $\phi dg'$ sufficiently general to derive our lifting results. Here we consider an easy special case. Fix two places u, u' of F. We shall work here with global functions f, ϕ, $'f$, $'\phi$ whose components at u, u' have (twisted in the case of ϕ) orbital integrals which vanish on the (resp. σ-) regular split set. An element is called *split* if its conjugacy class intersects the diagonal torus nontrivially. Further, we fix a nonarchimedean place u'', and require that the (resp. σ-) orbital integral of the component at u'' be zero on the (resp. σ-) semisimple singular set. These conditions imply that the geometric expression for the kernel contains only terms indexed by (resp. σ-) conjugacy classes of rational (resp. σ-) elliptic elements in G (resp. G').

Under the above restrictions at u, u', u'' on the test function f on $\mathbf{G}(\mathbb{A})$ (and the matching ϕ on $\mathbf{G}'(\mathbb{A})$), the trace formula for f on $\mathbf{G}(\mathbb{A})$ asserts

LEMMA. *The sum* $\sum \operatorname{tr} \pi(f dg)$ *over all discrete-spectrum representations* π *of* $\mathbf{U}(3, E/F)(\mathbb{A}) = \mathbf{G}(\mathbb{A})$ *is equal to the sum of* (1^*), (1^{**}) (*where* $'f dh$ *is a test measure on* $\mathbf{H}(\mathbb{A})$ *matching* $f dg$ *on* $\mathbf{G}(\mathbb{A})$), *and*

$$-\frac{1}{4}\sum_{\mu} \operatorname{tr} M(\mu) I(\mu, f dg).$$

All sums here are absolutely convergent.

The new sum extends over all characters μ of $\mathbb{A}_E^\times / E^\times N \mathbb{A}_E^\times$. The diagonal subgroup $\mathbf{A}(\mathbb{A})$ of $\mathbf{G}(\mathbb{A})$ consists of $\operatorname{diag}(a, b, \overline{a}^{-1})$, a in $\mathbb{A}_E^\times$, b in $\mathbb{A}^1$. Any character of $\mathbf{A}(\mathbb{A})/A$ whose restriction to $\mathbf{Z}(\mathbb{A})$ is ω, is of the form $\operatorname{diag}(a, b, \overline{a}^{-1}) \mapsto \mu(a)(\omega/\mu)(b)$, where μ is a character of $\mathbb{A}_E^\times/E^\times$. We denote the $\mathbf{G}(\mathbb{A})$-module normalizedly induced from the character of $\mathbf{A}(\mathbb{A})$

by $I(\mu)$. We shall also use the analogous notations in the local case. The intertwining operator $M(\mu)$ is defined in the theory of Eisenstein series.

3.2 The twisted trace formula of our group $\mathbf{G}'(\mathbb{A})$ is to be discussed next. The center $\mathbf{Z}'(\mathbb{A})$ of $\mathbf{G}'(\mathbb{A}) = \mathrm{GL}(3, \mathbb{A}_E)$ is isomorphic to $\mathbb{A}_E^\times$. The norm map N takes z in $\mathbf{Z}'(\mathbb{A})$ to $z/\overline{z}$ in $\mathbf{Z}(\mathbb{A})$. The restriction to $\mathbb{A}^\times$ of the character $\omega' = \omega \circ N$ of $\mathbf{Z}'(\mathbb{A})$ is trivial. Let $L(G')$ once again be the space of complex valued functions ψ on $G' \backslash \mathbf{G}'(\mathbb{A})$, which transform under $\mathbf{Z}'(\mathbb{A})$ by ω', and are absolutely square integrable on $G'\mathbf{Z}'(\mathbb{A}) \backslash \mathbf{G}'(\mathbb{A})$. The group $\mathbf{G}'(\mathbb{A})$ acts on $L(G')$ by right translation. The irreducible constituents Π are called automorphic. The discrete and continuous spectra are invariant under the action of σ, which maps ψ to $\sigma\psi$, where $(\sigma\psi)(x) = \psi(\sigma x)$. We say that the $\mathbf{G}'(\mathbb{A})$-module Π is *σ-invariant* if Π is equivalent to ${}^\sigma\Pi$, where $({}^\sigma\Pi)(x) = \Pi(\sigma x)$. In this case there is an intertwining operator $\Pi(\sigma)$ of Π with ${}^\sigma\Pi$, whose square is the identity. We write $\mathrm{tr}\,\Pi(\phi d'x \times \sigma)$ for the trace of the operator $\Pi(\phi\, d'x \times \sigma) = \int \phi(x)\Pi(x)\Pi(\sigma)d'x$ (x in $\mathbf{G}'(\mathbb{A})/\mathbf{Z}'(\mathbb{A})$; $d'x$ is a Tamagawa measure, often omitted from the notations).

As usual normalizedly induced $\mathbf{G}'(\mathbb{A})$-modules are denoted by $I(\eta)$. Here $\eta = (\mu, \mu', \mu'')$ is a character of the diagonal subgroup $\mathbf{A}'(\mathbb{A})$ of $\mathbf{G}'(\mathbb{A})$. The μ, μ', μ'' are characters of $\mathbb{A}_E^\times/E^\times$. For each element w of the Weyl group W of A in G, there is an intertwining operator $M(w, \eta)$ from $I(\eta)$ to $I(w\eta)$, where $(w\eta)(a) = \eta(waw^{-1})$. The $I(\eta)$ which appear in the trace formula are those whose central character $\mu\mu'\mu''$ is equal to ω'.

Suppose τ is an irreducible $\mathbf{H}'(\mathbb{A}) = \mathbf{H}(\mathbb{A}_E)$-module, where $\mathbf{H}'$ is $\mathrm{R}_{E/F}\, \mathbf{U}(2, E/F)$, thus $H' = \mathrm{GL}(2, E)$. Denote by $I(\tau)$ the $\mathbf{G}'(\mathbb{A})$-module normalizedly induced from the $\mathbf{H}'(\mathbb{A}) \times \mathbb{G}_m(\mathbb{A}_E)$ module $\tau \otimes \omega_\tau$, where ω'/ω_τ is the central character of τ. The central character of $I(\tau)$ is then ω'. The representation $I(\tau)$ is σ-invariant if and only if $\tau \simeq w\tau$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(w\tau)(x) = \tau(w {}^t\overline{x}^{-1} w^{-1})$, and $\omega_\tau(a\overline{a}) = 1$ for all a in $\mathbb{A}_E^\times$.

Recall that the twisted orbital integrals of the components of ϕ at u, u' are assumed to be zero on the σ-regular split set. The integral of $\phi_{u''}$ at the σ-semisimple-singular elements, is assumed to be 0. Then the twisted trace formula for $\mathbf{G}'(\mathbb{A})$ and ϕ asserts the following. (For a similar case see [F2;I].)

LEMMA. *The sum* $\sum \mathrm{tr}\,\Pi(\phi dg' \times \sigma)$ *over the σ-invariant representations* Π *of* $\mathbf{G}'(\mathbb{A})$ *in the discrete spectrum is equal to the sum of* (2^*), (2^{**}), *and*

$$-\frac{1}{4}\sum \operatorname{tr} I(\eta, \phi dg' \times \sigma) + \frac{1}{8}\sum \operatorname{tr} I(\eta, \phi dg' \times \sigma)$$
$$+\frac{3}{8}\sum \operatorname{tr} I(\eta, \phi dg' \times \sigma) - \frac{1}{2}\sum \operatorname{tr} I(\tau, \phi dg' \times \sigma).$$

All $\mathbf{G}'(\mathbb{A})$-modules $I(\eta)$, $I(\tau)$ here are σ-invariant. The characters μ, μ', μ'' in η are trivial on $E^\times N\mathbb{A}_E^\times$. The first sum is over all unordered triples of pairwise distinct μ, μ', μ''. The second is over all (μ, μ', μ), $\mu' \neq \mu$. In the third $\mu = \mu' = \mu''$. The $I(\eta)$, $I(\tau)$ here are all irreducible.

In fact the way in which the $I(\eta)$ appear in the trace formula is as

$$\frac{1}{24}\sum \operatorname{tr} M((13), \eta) I(\eta dg', \phi \times \sigma) + \frac{1}{6}\sum_{w=(12),(23)}\sum_{\eta} \operatorname{tr} M(w, \eta) I(\eta, \phi dg' \times \sigma).$$

The nonzero contributions are given by the η for which η, acted upon by σ and then the reflection w is equal to η. Thus the first sum is over the η with μ, μ', μ'' trivial on $N\mathbb{A}_E^\times$; the others are over the η with $\mu = \mu' = \mu''$, μ trivial on $N\mathbb{A}_E^\times$.

The intertwining operators $M(w, \eta)$ can be written as local products $m(w, \eta) \otimes_v R(w, \eta_v)$ (see [Sh]). Here $R(w, \eta_v)$ are the local normalized intertwining operators. They are trivial in our case. The normalizing factors $m(w, \eta)$ are given by $m((12), \eta) = L(1, \mu'/\mu)/L(1, \mu/\mu')$,

$$m((23), \eta) = L(1, \mu''/\mu')/L(1, \mu'/\mu''),$$

and $m((13), \eta)$ is

$$[L(1, \mu''/\mu')/L(1, \mu'/\mu'')][L(1, \mu''/\mu)/L(1, \mu/\mu'')][L(1, \mu'/\mu)/L(1, \mu/\mu')].$$

If at least two of the μ's are equal, $m(w, \eta)$ has to be evaluated as a limit; the value is -1. If the μ are all distinct, then $m((13), \eta)$ is 1. Indeed, $L(1, \mu) = L(1, \overline{\mu})$, and here $\overline{\mu} = \mu^{-1}$. Up to equivalence each $I(\eta)$ appears in the first sum 6 times if the μ are distinct, 3 times if exactly two of the μ are equal, and once if $\mu = \mu' = \mu''$. Whence the expression of the lemma.

3.3 The character μ of $\mathbb{A}_E^\times/E^\times$ defines a character of the diagonal subgroup $'\mathbf{A}(\mathbb{A})$ of $\mathbf{H}(\mathbb{A})$, by $\operatorname{diag}(a, \overline{a}^{-1}) \mapsto \mu(a)$, and an induced representation $I(\mu)$. Under the usual restriction on $'f$ at u, u' and u'', the trace formula for $\mathbf{H}(\mathbb{A})$ and $'f$ asserts the following (see [F3;II]).

LEMMA. *The sum* $\sum n(\rho)\,\mathrm{tr}\{\rho\}('fdh)$ *over all automorphic packets* $\{\rho\}$ *of* $\mathbf{H}(\mathbb{A})$, *is equal to the sum of* (1^{**}) (*times* 2), *and* $\frac{1}{4}\sum_\mu \mathrm{tr}\, I(\mu,'fdh)$. *The sum over* μ *is taken over all characters of* $\mathbb{A}_E^\times/E^\times\mathbb{A}^\times$.

The automorphic, and local, packets of $\mathbf{H}(\mathbb{A})$-modules, and the global multiplicities $n(\rho)$ $(= 1$ or $1/2)$, are defined in [F3;II].

3.4 We now obtain an identity of trace formulae. Let E/F be a global quadratic extension, and $'\phi dh$, $\phi dg'$, fdg, $'fdh$ matching measures on $\mathbf{H}(\mathbb{A})$, $\mathbf{G}'(\mathbb{A})$, $\mathbf{G}(\mathbb{A})$, $\mathbf{H}(\mathbb{A})$. We assume that the (twisted) orbital integrals of the components at u, u' are 0 on the (σ-) regular split set, and that the corresponding integral of the component at the nonarchimedean place u'' vanishes on the (σ-)semisimple singular set. Combining the Lemmas 3.1, 3.2 and 3.3, we deduce

PROPOSITION. *In the above notations, we have*

$$\sum\prod \mathrm{tr}\,\Pi_v(\phi_v dg'_v\times\sigma)+\frac{1}{2}\sum\prod \mathrm{tr}\, I(\tau_v;\phi_v dg'_v\times\sigma)$$
$$+\frac{1}{4}\sum\prod \mathrm{tr}\, I(\eta_v;\phi_v dg'_v\times\sigma)-\frac{1}{8}\sum\prod \mathrm{tr}\, I((\mu_v,\mu'_v,\mu_v);\phi_v dg'_v\times\sigma)$$
$$-\frac{3}{8}\sum\prod \mathrm{tr}\, I(\mu_v,\mu_v,\mu_v);\phi_v dg'_v\times\sigma)$$
$$=\sum\prod \mathrm{tr}\,\pi_v(f_v dg_v)-\frac{1}{2}\sum n(\rho)\prod \mathrm{tr}\{\rho_v\}('f_v dh_v)$$
$$+\frac{1}{2}\sum n(\rho)\prod \mathrm{tr}\{\rho_v\}('\phi_v dh_v)$$
$$\frac{1}{4}\sum m(\eta)\prod \mathrm{tr}\, R(\mu_v)I(\mu_v,f_v dg_v)$$
$$+\frac{1}{8}\sum\prod \mathrm{tr}\, I(\mu_v,'f_v dh_v)-\frac{1}{8}\sum\prod \mathrm{tr}\, I(\mu_v,'\phi_v dh_v).$$

The products $\prod$ are taken over all places v of F. It is useful to fix a finite set V of places, which includes u, u', u'', the archimedean places and those places which ramify in E/F, such that $'\phi_v$, ϕ_v, f_v, $'f_v$ are spherical outside V. Then the components Π_v, π_v and ρ_v are unramified, and correspond to the conjugacy classes $t'_v\times\sigma$, $t_v\times\sigma$, $'t_v\times\sigma$ in the dual groups ${}^LG'$, ${}^LG$, ${}^LH$, by the definition of the Satake transform. For each v outside V we fix $t_v\times\sigma$, and let $t'_v\times\sigma$ be its image under the basechange map ${}^LG\to{}^LG'$, $'t_v\times\sigma$ the pullback via the endo-map ${}^LH\to{}^LG$, and $'t'_v\times\sigma$ the pullback of $t'_v\times\sigma$ via the σ-endo-map ${}^LH \to {}^LG'$. A standard approximation argument, based on (1) the fact that the sums in the Proposition are absolutely convergent, and (2) the matching result of I.2 and I.3, for corresponding spherical functions, implies the following

Corollary. *Fix $\{t_v \times \sigma; v$ outside $V\}$. Then all products in the Proposition extend over V. The sums range over Π, π, ρ whose component at v outside V is parametrized by $t_v \times \sigma$.*

The rigidity theorem for $G' = \mathrm{GL}(3)$ of [JS] implies that at most one of the first five sums involving G'-modules is nonempty, and this sum consists of a single G'-module by multiplicity one theorem.

II.4 Trace identity

Summary. The identity of trace formulae is proven for arbitrary matching functions, under no restriction on any component. The method requires no detailed analysis of weighted orbital integrals, or of orbital integrals of singular classes.

4.1 Introduction

Let E/F be a quadratic extension of global fields. Put G' for $\mathbf{G}(E) = \mathrm{GL}(3,E)$. Denote by $G = \mathbf{G}(F)$ the quasi-split unitary group in three variables. It consists of all g in G' with $\sigma g = g$, where we write $\sigma x = J{}^t\overline{x}^{-1}J$ for x in G' : $\overline{x}$ is $(\overline{x}_{ij})$ if $x = (x_{ij})$, the bar indicating the action of the nontrivial element of the Galois groups $\mathrm{Gal}(E/F)$, and

$$J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarly we put $\sigma x = w{}^t\overline{x}^{-1}w^{-1}$ for x in H', and introduce $H' = \mathbf{H}(E) = \mathrm{GL}(2,E)$ and $H = \mathbf{H}(F) = \{g \in H'; \sigma g = g\}$. Then $\mathbf{G} = \mathbf{U}(3,E/F)$, $\mathbf{H} = \mathbf{U}(2,E/F)$. We use the following smooth complex-valued functions.

(1) $'f = \otimes' f_v$ and $'\phi = \otimes'\phi_v$ are compactly supported on $\mathbf{H}(\mathbb{A})$ ($\mathbb{A} = \mathbb{A}_F$ indicates the ring of adèles of F).

(2) $f = \otimes f_v$ on $\mathbf{G}(\mathbb{A})$ transforms under the center $\mathbf{Z}(\mathbb{A})$ ($\simeq \mathbb{A}_E^1$: E-idèles of norm 1 in $\mathbb{A}_F^\times$) of $\mathbf{G}(\mathbb{A})$ by a fixed character ω^{-1}, where ω is a character

of $\mathbb{A}_E^1/E^1$ ($E^1 = \{x \in E; N_{E/F}x = 1\}$); f is compactly supported modulo $\mathbf{Z}(\mathbb{A})$.

(3) $\phi = \otimes \phi_v$ is a function on $\mathbf{G}'(\mathbb{A}) = \mathbf{G}(\mathbb{A}_E)$ which transforms under the center $\mathbf{Z}'(\mathbb{A}) = \mathbf{Z}(\mathbb{A}_E)$ ($\simeq \mathbb{A}_E^\times$) of $\mathbf{G}'(\mathbb{A})$ by ω'^{-1}, where $\omega'(x) = \omega(x/\overline{x})$, x in the group $A_E^\times$ of idèles.

The local components of $'fdh$, fdg, $\phi dg'$, $'\phi dh$ are taken to be matching, namely their orbital integrals are related in a certain way, specified in II.2.

Our purpose here is to prove the following:

THEOREM. *Let $'fdh$, fdg, $\phi dg'$, $'\phi dh$ be matching measures. Then the identity displayed in Proposition* II.3.3 *holds.*

We abbreviate this identity to:

$$\sum_{\Pi} m(\Pi)\,\mathrm{tr}\,\Pi(\phi dg' \times \sigma) - \frac{1}{2}\sum_{\{\rho\}} n(\rho)\,\mathrm{tr}\,\{\rho\}('\phi dh)$$
$$= \sum_{\pi} m(\pi)\,\mathrm{tr}\,\pi(fdg) - \frac{1}{2}\sum_{\{\rho\}} n(\rho)\,\mathrm{tr}\{\rho\}('fdh).$$

Here the sum over Π ranges over various automorphic σ-invariant $\mathbf{G}'(\mathbb{A})$-modules and $m(\Pi)$ is 1 if Π is discrete spectrum, $\frac{1}{2}$ if $\Pi = I(\tau)$ and $\frac{1}{4}$, $-\frac{1}{8}$ or $-\frac{3}{8}$ if $\Pi = I(\eta)$. The π are automorphic $\mathbf{G}(\mathbb{A})$-modules which may be discrete spectrum or induced and the $m(\pi)$ are integers, $\frac{1}{2}$ or $\frac{1}{4}$. The $\{\rho\}$ are automorphic $\mathbf{H}(\mathbb{A})$-packets, and the $n(\{\rho\}) = n(\rho)$ are again rational numbers. Proposition II.3.3 asserts the theorem under the additional assumption that two local components of (each of) $'f$, f, ϕ, $'\phi$ are elliptic (= discrete). Our purpose here is to prove the theorem unconditionally, and by a simple technique.

To simplify the notations we fix the Haar measures dg', dg, dh, and refer to the functions ϕ, f, etc., rather than the measures $\phi dg'$, fdg, etc.

Trace identity as in the Theorem, for general test functions f, ϕ, ... on two (or more) groups G, G', ..., appears already in (Chapter 16 of) [JL]. But attention to the problem was drawn by Langlands' study [L5] of the first nontrivial case, namely the comparison needed for the completion of the cyclic basechange theory for GL(2), initiated by Saito and Shintani.

Langlands proved the required identity for GL(2) on (1) computing the weighted orbital integrals and orbital integrals of singular classes which appear in the trace formulae, (2) analyzing the asymptotic behavior of the weighted integrals, (3) applying the Poisson summation formula, and so on.

The method presented here is entirely different. The principle is that it suffices to check the identity of the Theorem only for a small class of convenient test functions, and then use the fact that we deal with characters of representations to conclude that the identity holds in general. It is not necessary to deal with arbitrary f, ϕ, $\dots$ at the initial stage. In fact, it is shown below that for a suitable choice of test functions (whose definitions we leave to the text itself), the weighted orbital integrals and the orbital integrals at the singular classes are equal to zero. In particular they need not be further computed and transformed. The proof turns out to be rather simple, once the right track is found.

The present method applies also in the case considered in [L5] to yield a simple and short proof of the trace identity needed for the comparison of basechange for GL(2). It makes a crucial use of the existence of a place u of F which splits in E.

The observation underlying our approach is that the subgroup $F^\times$ of rationals is discrete in the group $\mathbb{A}_F^\times$ of idèles. That this simple fact can actually be used to annihilate the undesirable terms in the trace formula was suggested by Drinfeld's use of spherical functions related to powers of the Frobenius, in the course of the work, [FK2], [FK3] with D. Kazhdan, on the Ramanujan conjecture for automorphic forms with a cuspidal component of GL(n) over a function field.

In the present section admissible spherical functions are used to establish the theorem by our simple approach. This technique is developed in [FK1] to establish the metaplectic and simple algebra correspondences in the context of arbitrary rank and cusp forms with a single cuspidal component. A different variant of the approach, based on the use of regular Iwahori biinvariant functions, is applied in [F1;IV] to give a simple proof of cyclic basechange for GL(2) with no restriction on any component, in [F2;VI] to prove the absolute form of the symmetric square lifting from SL(2) to PGL(3), and in [F1;V] to establish by simple means cyclic base change for cusp forms with at least one cuspidal component on GL(n).

To complete this introduction we now sketch the proof which is given below. We deal with four trace formulae for test functions f, ϕ, $'f$, $'\phi$, on the groups $\mathbf{G}(\mathbb{A}) = \mathbf{U}(3, E/F)(\mathbb{A})$, $\mathbf{G}'(\mathbb{A}) = \mathrm{GL}(3, \mathbb{A}_E)$, and (twice) $\mathbf{H}(\mathbb{A}) = \mathbf{U}(2, E/F)(\mathbb{A})$. Put $\mathbf{q}$ for the quadruple $(f,\ \phi,\ 'f,\ '\phi)$. Each trace formula is an equality of distributions in the test function. These distributions are as follows. OI involves "good" orbital integrals, on the set of rational regular elliptic elements. WI involves "bad" orbital integrals, on the set of rational elements which are not regular elliptic; these "bad" integrals are mostly weighted and noninvariant as distributions in the test function. RD is a (discrete) sum of traces of automorphic representations; these occur with coefficients which may be negative when the representation is not cuspidal. RC is an integral (continuous sum) of traces of induced representations; these traces are often weighted, and the distributions which make up RC are mostly noninvariant. The trace formula takes the form I = R, where R = RD + RC is the representation theoretic side, and I = OI + WI is the geometric side (orbital integrals) of the formula.

We shall be interested in a linear combination of the four formulae. Put

$$\mathrm{RD}(\mathbf{q}) = \left[\mathrm{RD}(\phi) - \frac{1}{2}\,\mathrm{RD}({}'\phi)\right] - \left[\mathrm{RD}(f) - \frac{1}{2}\,\mathrm{RD}({}'f)\right],$$

and introduce $\mathrm{OI}(\mathbf{q})$, $\mathrm{RC}(\mathbf{q})$ analogously. From now on *we always choose* the four components of $\mathbf{q}$ to have matching orbital integrals. This choice implies the vanishing of $\mathrm{OI}(\mathbf{q})$. Hence

$$\mathrm{RD}(\mathbf{q}) = \mathrm{WI}(\mathbf{q}) - \mathrm{RC}(\mathbf{q}).$$

In these notations, the Theorem can be restated as follows.

THEOREM. *For any quadruple* $\mathbf{q}$ *of matching functions we have* $\mathrm{RD}(\mathbf{q}) = 0$.

Fix a nonarchimedean place u of F which splits in E. Then

$$\mathbf{G}(F_u) = \mathrm{GL}(3, F_u), \quad \mathbf{G}'(F_u) = \mathrm{GL}(3, F_u) \times \mathrm{GL}(3, F_u),$$

$\mathbf{H}(F_u) = \mathrm{GL}(2, F_u)$. Fix a quadruple $\mathbf{q}^u = (f^u, \phi^u, {}'f^u, {}'\phi^u)$ of the components outside u of $\mathbf{q}$. Put $\mathrm{RC}(\mathbf{q}_u)$ for $\mathrm{RC}(\mathbf{q}_u \otimes \mathbf{q}^u)$, where

$$\mathbf{q}_u = (f_u, \phi_u, {}'f_u, {}'\phi_u).$$

As the first step in the proof we explicitly construct for any f_u a quadruple $\mathbf{q}_u = \mathbf{q}(f_u)$ which has the property that $\mathrm{RC}(\mathbf{q}(f_u))$ depends only on the orbital integrals of f_u.

For the second step of the proof, we say that a function f'_u on $\mathbf{G}(F_u)$ is n_0-*admissible* (for some $n_0 > 0$) if it is spherical and its orbital integrals on the split regular set vanish at a distance $\leq n_0$ from the walls [namely, on the orbits with eigenvalues of valuations n_1, n_2, n_3 such that $|n_i - n_j|$ is at most n_0 for some $i \neq j$ (i, $j = 1, 2, 3$)] We prove: For any quadruple $\mathbf{q}^u$ of matching f^u, ϕ^u, $'f^u$, $'\phi^u$, which vanish on the adèles-outside-u orbits of the singular-rational elements, there exists an integer $n_0 = n_0(\mathbf{q}^u)$ such that $\mathrm{WI}(\mathbf{q}(f'_u)) = 0$ for every n_0-admissible f'_u. Note that in this case all of four components of $\mathbf{q}(f'_u)$ are spherical.

To prove this we show in the Proposition that, given f^u which vanishes on the $\mathbf{G}(\mathbb{A}^u)$-orbits of the singular set in $\mathbf{G}(F)$, there exists $n_0 = n_0(f^u) > 0$, such that for every n_0-admissible f'_u there exists a function f_u with the same orbital integrals as f'_u with the property that $f^u \otimes f_u$ is zero on the $\mathbf{G}(\mathbb{A})$-orbits of all "bad" rational elements. In particular $\mathrm{WI}(f^u \otimes f_u) = 0$. The function f_u is obtained by replacing f'_u by zero on a small neighborhood of finitely many split orbits where the orbital integral of f'_u is zero. Choosing n_0 sufficiently large, depending on $\mathbf{q}^u$, and noting that the construction of $\mathbf{q}(f_u)$ is such that its components are zero on the image of the split regular orbits where f_u is zero, we conclude that for every n_0-admissible f'_u there is f_u with orbital integrals equal to those of f'_u such that $\mathrm{WI}(\mathbf{q}(f_u)) = 0$. Consequently $\mathrm{WI}(\mathbf{q}(f'_u)) = 0$ for every n_0-admissible f'_u, since

$$\mathrm{WI}(\mathbf{q}(f_u)) = \mathrm{RD}(\mathbf{q}(f_u)) + \mathrm{RC}(\mathbf{q}(f_u))$$

and $\mathrm{RC}(\mathbf{q}(f_u))$ depends (by Step 1) only on the orbital integrals of f_u (which are equal to those of f'_u).

The third step asserts that since $\mathrm{RD}(\mathbf{q}(f'_u)) = -\mathrm{RC}(\mathbf{q}(f'_u))$ for every n_0-admissible f'_u we have $\mathrm{RD}(\mathbf{q}(f'_u)) = \mathrm{RC}(\mathbf{q}(f'_u)) = 0$ for *every* spherical f'_u. This follows from the final Proposition in [FK2], where this claim is stated and proven in the context of an arbitrary reductive p-adic group.

Fix a nonarchimedean place u' of F which splits in E. It follows from Step 3 that for any $\mathbf{q}_{u'}$ whose 4 components vanish on the singular sets, we have $\mathrm{RD}(\mathbf{q}^{u'} \otimes \mathbf{q}_{u'}) = 0$ for all $\mathbf{q}^{u'}$. The fourth step is to show that this holds for any spherical $\mathbf{q}_{u'}$. The proof is the same as in III.1.2. We will recall the argument here.

Write $\mathrm{RD}(\mathbf{q})$ as a sum $\sum_\chi \mathrm{RD}(\mathbf{q}, \chi)$ over all infinitesimal characters χ, of the partial sums $\mathrm{RD}(\mathbf{q}, \chi)$ of $\mathrm{RD}(\mathbf{q})$ taken only over those automorphic representations whose infinitesimal character is χ. Since the archimedean components of $\mathbf{q}$ are arbitrary, a standard argument of "linear independence of characters" implies that since $\mathrm{RD}(\mathbf{q}) = 0$, for every χ we have $\mathrm{RD}(\mathbf{q}, \chi) = 0$ if $\mathbf{q}_{u'} = 0$ on the singular set. Fix $\mathbf{q}^{u'}$, and consider $\mathrm{RD}(\mathbf{q}, \chi)$ as a functional on the space of Iwahori quadruples $\mathbf{q}_{u'}$ (i.e., quadruples whose components are biinvariant under the standard Iwahori subgroups). There are only finitely many automorphic representations with a fixed infinitesimal character, fixed ramification at each finite place $\neq u'$, whose component at u' has a nonzero vector fixed under the action of an Iwahori subgroup. Hence as a functional in the Iwahori quadruple $\mathbf{q}_{u'}$, $\mathrm{RD}(\mathbf{q}, \chi)$ is a finite sum of characters. As it is zero on all $\mathbf{q}_{u'}$ which vanish on the singular set, and our groups are GL(2) and GL(3), it is identically zero. In particular $\mathrm{RD}(\mathbf{q}, \chi)$ vanishes on the spherical quadruples $\mathbf{q}_{u'}$, from which the Theorem easily follows. This completes our outline of the proof of the Theorem.

4.2 Conjugacy classes

Let v be a place of F. Denote by F_v the completion of F at v, and put $E_v = E \otimes_F F_v$. If v stays prime in E, then E_v/F_v is a quadratic field extension. If v splits into v', v'' in E, then $E_v = E_{v'} \times E_{v''}$, where $E_{v'} \simeq E_{v''} \simeq F_v$. In this case

$$G'_v = \mathbf{G}(E_v) = \mathrm{GL}(3, F_v) \times \mathrm{GL}(3, F_v),$$

and

$$G_v = \mathbf{G}(F_v) = \{(g, \sigma g); g \in \mathrm{GL}(3, F_v)\} \simeq \mathrm{GL}(3, F_v).$$

Here $\sigma g = J {}^t g^{-1} J$, as $\mathrm{Gal}(E/F)$ maps $g = (g', g'')$ in G'_v to $\overline{g} = (g'', g')$. Let u be a fixed nonarchimedean place of F which splits in E. Put $f^u = \otimes_{v \neq u} f_v$, where at each place $v \neq u$ of F we take the function f_v to be fixed. The component f_u is a locally constant function on $G_u = \mathbf{G}(F_u) = \mathrm{GL}(3, F_u)$. We choose u such that the central character ω has an unramified component ω_u at u. Replacing ω by its product with an unramified (global) character we may assume that $\omega_u = 1$. Then $f_u(zg) = f_u(g)$ for g in G_u, z in the center Z_u of G_u, and f_u is compactly supported on $Z_u \backslash G_u$. Let $F(g, f_u) = \Delta(g)\Phi(g, f_u)$ be the normalized orbital integral of f_u. Let R_u be the ring of integers in F_u. Put $K_u = \mathbf{G}(R_u)$; it is a maximal compact subgroup of G_u. A spherical function is a K_u-biinvariant function. The theory of the Satake transform implies that a spherical f_u on G_u is determined by its orbital integral on the split set. Let $|\cdot|$ be the (normalized) valuation on F_u, put $q = q_u$ for the cardinality of the residue field of F_u, and val for the additive valuation, defined by $|a| = q^{\mathrm{val}(a)}$ for a in $F_u^\times$. Let $\mathbf{n} = (n_1, n_2, n_3)$ be a triple of integers. Let f'_u be the spherical function on G_u for which $F(g, f'_u)$ is zero at the regular diagonal element $g = (a, b, c)$, unless up to conjugation and modulo the center we have (val a, val b, val c) $= \mathbf{n}$, in which case we require $F(g, f'_u)$ to be equal to one. Embed $\mathbb{Z}$ in $\mathbb{Z}^3$ diagonally. The symmetric group S_3 on three letters acts on $\mathbb{Z}^3$. Denote by $\mathbb{Z}^3/S_3\mathbb{Z}$ the quotient space. Then f'_u depends only on the image of $\mathbf{n}$ in $\mathbb{Z}^3/S_3\mathbb{Z}$. We write $f'_u = f'_u(\mathbf{n})$ to indicate the dependence of f'_u on $\mathbf{n}$.

Definitions. (1) The function f_u on G_u is called *pseudo-spherical* if there exists a spherical function f'_u with $F(g, f_u) = F(g, f'_u)$ for all g in G_u. We write $f_u(\mathbf{n})$ for f_u if $f'_u = f'_u(\mathbf{n})$.

(2) Let n_0 be a nonnegative integer. An element $\mathbf{n} = (n_1, n_2, n_3)$ of $\mathbb{Z}^3/S_3\mathbb{Z}$ is called n_0*-admissible* if $|n_i - n_j| \geq n_0$ for all $i \neq j$; $i, j = 1, 2, 3$.

We also fix a place u' of F which stays prime in E such that $E_{u'}/F_{u'}$ is unramified, and a positive integer n'. Let $S = S(u', n')$ be the set of g in $G_{u'}$ which are conjugate to some diagonal matrix $\mathrm{diag}(a, b, \overline{a}^{-1})$ with $|a|_{u'} = q_{u'}^{n'}$ (and $|b|_{u'} = 1$); $a \in E_{u'}^\times$ and $b \in E_{u'}^1$. We shall assume from now on that the component $f_{u'}$ is a (compactly supported, locally constant) function on $G_{u'}$ such that $F(g, f_{u'})$ is the characteristic function of S. Since S is open and closed we may and do take $f_{u'}$ to be supported on S.

Proposition. *There exists an integer $n_0 \geq 0$ depending on f^u, such that for any n_0-admissible $\mathbf{n}$ there is a pseudo-spherical $f_u = f_u(\mathbf{n})$ with the property that $f = f^u \otimes f_u$ satisfies the following. If γ lies in $\mathbf{G}(F)$, x in $\mathbf{G}(\mathbb{A})$, and $f(x^{-1}\gamma x) \neq 0$, then γ is elliptic regular.*

Proof. If $f_{u'}(x^{-1}\gamma x) \neq 0$ then γ lies in S, hence it is regular in $G_{u'}$ and also in G. If γ is not elliptic, then we may assume that it is the diagonal element $\mathrm{diag}(a, b, \overline{a}^{-1})$ with a in $E^\times$ and b in $E^1 = \{b \text{ in } E^\times;\ b\overline{b} = 1\}$. Modulo the center we may assume that $b = 1$. Also we have $a\overline{a} \neq 1$. At the split place u we have $a = (\alpha, \beta)$, with α, β in $F_u^\times$. Hence γ is $\mathrm{diag}(\alpha, 1, \beta^{-1})$ in G_u. Since f^u is fixed, there are $C_v \geq 1$ for all $v \neq u$, with $C_v = 1$ for almost all v, such that $C_v^{-1} \leq |a|_v \leq C_v$ for all $v \neq u$

if $f^u(x^{-1}\gamma x) \neq 0$ for some x in $\mathbf{G}(\mathbb{A})$. Here $|a|_v = |N_{E/F}a|_v$. Since a lies in $E^\times$, and $N_{E/F}a$ in $F^\times$, the product formula on $F^\times$ implies that $|\alpha\beta|_u = |N_{E/F}a|_u = |a|_u$ lies between $C_u = \prod_{v\neq u} C_v$ and C_u^{-1}. We take n_0 with $q_u^{n_0} > C_u$. Consider an n_0-admissible $\mathbf{n}$ and the spherical $f'_u = f'_u(\mathbf{n})$. If $f'_u(x^{-1}\gamma x) \neq 0$ for some x in G_u, then there is some $C'_u > 1$ such that $|\alpha|_u$ and $|\beta|_u$ are bounded between C'_u and C'^{-1}_u, so that a lies in the discrete set $E^\times$ and in a compact of $\mathbb{A}_E^\times$, hence in a finite set. Hence γ lies in finitely many conjugacy classes modulo the center; let $\gamma_1, \dots, \gamma_t$ be a set of representatives. Put $\gamma_i = \operatorname{diag}(\alpha_i, 1, \beta_i^{-1})$. By definition of f'_u, if $F(\gamma_i, f'_u) \neq 0$ then we have that $|\alpha_i\beta_i|$ or $|\alpha_i\beta_i|^{-1}$ is bigger than $q_u^{n_0}$, hence $f(x^{-1}\gamma_i x) = 0$ for all x and i. We conclude that $F(\gamma_i, f'_u) = 0$ for all i. Let S_i be the characteristic function of the complement of a small open closed neighborhood of the orbit of γ_i in G_u. Then the function $f_u = f'_u \prod_i S_i$ on G_u has the required properties. □

Let $L(G)$ denote the space of automorphic functions on $\mathbf{G}(\mathbb{A})$; these are the square-integrable functions on $\mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})$ which transform on $\mathbf{Z}(\mathbb{A})$ by ω and are right invariant by some compact open subgroups; see [BJ] and [Av]. The group $\mathbf{G}(\mathbb{A})$ acts on $L(G)$ by right translation: $(r(g)\Psi)(h) = \Psi(hg)$. Then r is an integral operator with kernel $K_f(x,y) = \sum_\gamma f(x^{-1}\gamma y)$, where γ ranges over $Z\backslash G$. In view of the Proposition, the integral of $K_f(x,y)$ on the diagonal $x = y$ in $\mathbf{G}(\mathbb{A})/\mathbf{Z}(\mathbb{A})$ is precisely the sum (1.2.1), which is stabilized and analyzed in II.1. The remarkable phenomenon to be noted is that for f with a component f_u as in the Proposition, the only conjugacy classes which contribute to the trace formula are elliptic regular. The weighted orbital integrals and the orbital integrals of the singular classes are zero, for our function f. Moreover, the truncation which is usually used to obtain the trace formula is trivial, for our f.

Each component $\phi_v dg'_v$ of the measure $\phi dg' = \otimes\phi_v dg'_v$ on $\mathbf{G}'(\mathbb{A}) = \mathbf{G}(\mathbb{A}_E)$ is taken to be matching $f_v dg_v$ in the terminology of I.2. In particular we take $\phi_u dg'_u$ to be $(f_u dg_u, f_u^0 dg_u)$, where $f_u^0 dg_u$ is a unit element of the Hecke algebra. Namely the pseudo-spherical $f_u dg_u$ is biinvariant under some σ-invariant compact open subgroup I_u of G_u, where $\sigma(g) = J^t g^{-1} J$, and f_u^0 is taken to be the characteristic function of $Z_u I_u$, divided by the volume of $I_u Z_u/Z_u$. Then $f_u dg_u = f_u^0 dg_u * f_u dg_u = f_u dg_u * f_u^0 dg_u$. An immediate twisted analogue of the proof of the Proposition establishes the following.

Proposition. *If* $\mathbf{n}$ *is* n_0*-admissible,* δ *lies in* G'*,* x *in* $\mathbf{G}(\mathbb{A}_E)$ *and* $\phi(x^{-1}\delta\sigma(x)) \neq 0$*, then* $N\delta$ *is elliptic regular in* G*.*

Here N denotes the norm map from the set of stable σ-conjugacy classes in G' (and $\mathbf{G}(\mathbb{A}_E)$) onto the set of stable conjugacy classes in G (and $\mathbf{G}(\mathbb{A})$) (see I.1.5 and [Ko1]). Again we can introduce the space $L(G')$ of automorphic functions on $G' \backslash \mathbf{G}'(\mathbb{A})$ which transform on $\mathbf{Z}'(\mathbb{A})$ by ω' and the right action r' of $\mathbf{G}(\mathbb{A}_E)$ on $L(G')$. The Galois group $\mathrm{Gal}(E/F)$ acts on $L(G')$ by $(r'(\sigma)\Psi)(g) = \Psi(\sigma g)$. The operator $r'(\phi \times \sigma)$ is an integral operator with kernel $K_\phi(x,y) = \sum_\delta \phi(x^{-1}\delta\sigma(y))$ (δ in $Z' \backslash G'$). The Proposition shows that the integral of K_ϕ along the diagonal $x = y$ in $\mathbf{Z}(\mathbb{A}_E) \backslash \mathbf{G}(\mathbb{A}_E)$ is precisely the sum which is stabilized and discussed in II.2.

The functions $'f$ and $'\phi$ on $\mathbf{H}(\mathbb{A})$ are taken to be matching with f and ϕ, as defined in I.2. Their components at u can be taken to be pseudo-spherical, and the Proposition and its applications hold for $'f$ and $'\phi$ as well. It remains to consider the contribution to the trace formulae from the representation theoretic side.

4.3 Intertwining operators

For brevity we denote by J the difference of the two sides in the equality of our theorem. Then J is the difference of the two sides in the equality of II.3. These are the invariant representation theoretic terms in our trace formulae. The work of II.1 and II.2 concerns the stabilization of the orbital integrals on the elliptic regular conjugacy classes which appear in the trace formulae. It implies that for arbitrary matching functions $'f$, f, ϕ, $'\phi$ the difference J can be expressed as a sum of integrals of logarithmic derivatives of certain intertwining operators, which we momentarily describe, and weighted and singular orbital integrals which vanish for functions as considered in 4.2. In II.3 we concluded from this that $J = 0$ if the functions f, ϕ, ... have two elliptic (= discrete) components. To deal with the case of arbitrary f, ϕ, ... we now record an expression for J, excluding the weighted and singular orbital integrals, as follows. The expression consists of four terms, one for each of ϕ, f, $'f$, $'\phi$. These are the terms involving integrals (over $i\mathbb{R}$) in the trace formulae. They are analogous to the terms (vi), (vii), (viii) of [JL], p. 517. We use the notations of II.3.1, which are standard notations.

The term $J(\phi dg')$, from the twisted formula for $\mathbf{G}'(\mathbb{A})$, is the sum of three expressions, equal to each other. The coefficient $[W_0^M]/[W_0](\det(1-s\times\sigma|_{\mathcal{A}_M|\mathcal{A}})$ of II.3.1 (and [A2], Thm 8.2, p. 1324) is $\frac{1}{12}$ (here $M=M_0$ is the diagonal subgroup A; the Lie algebra $\mathcal{A}$ is one-dimensional). Hence we obtain

$$J(\phi dg') = \frac{1}{4}\sum_{\tau}\int_{i\mathbb{R}} \operatorname{tr}[\mathcal{M}(\lambda,0,-\lambda)I_{P_0\tau}((\lambda,0,-\lambda);\phi dg'\times\sigma)]d\lambda.$$

The sum is over all connected components (with representatives $\tau=(\mu_1,\mu_2,\mu_3)$) of characters of $\mathbf{A}(\mathbb{A}_E)/\mathbf{A}(E)$, with $\sigma\tau=\tau$. More precisely, let ν be the character $\nu(x)=|x|$ of $\mathbb{A}_E^\times$. Note that $\mathbf{A}\simeq\mathbb{G}_m^3$. The connected component of τ consists of $\tau_\lambda=(\mu_1\nu^\lambda,\mu_2,\mu_3\nu^{-\lambda})$, λ in $i\mathbb{R}$. The μ_j are unitary characters of $\mathbb{A}_E^\times/E^\times$, and $\mu_1\mu_2\mu_3=\omega'$. We put $I_{P_0,\tau}((\lambda,0,-\lambda))$ for the $\mathbf{G}(\mathbb{A}_E)$-module normalizedly induced from τ_λ; τ_λ is regarded as a character of the upper triangular subgroup $\mathbf{P}_0(\mathbb{A})$ which is trivial on the unipotent radical of $\mathbf{P}_0(\mathbb{A})$. The action of σ takes τ to $(\overline{\mu}_3^{-1},\overline{\mu}_2^{-1},\overline{\mu}_1^{-1})$, where $\overline{\mu}(x)=\mu(\overline{x})$. Hence $\sigma\tau=\tau$ implies $\tau=(\mu,\omega'\overline{\mu}/\mu,\overline{\mu}^{-1})$, where $\mu=\mu_1$.

The operator $\mathcal{M}$ is a logarithmic derivative of an operator $M=m\otimes_v R_v$, where R_v denotes a local normalized intertwining operator. The normalizing factor $m=m(\lambda)=m(\lambda,\tau)$ is an easily specified (see [F2;I], C2.2) quotient of L-functions, which has neither zeroes nor poles on the domain $i\mathbb{R}$ of integration. Then the logarithmic derivative $\mathcal{M}$ is

$$m'(\lambda)/m(\lambda)+(\otimes R_v^{-1})\frac{d}{d\lambda}(\otimes R_v),$$

and we obtain $J(\phi dg')=J'(\phi dg')+\sum_v J_v(\phi dg')$, where

$$J'(\phi dg') = \frac{1}{4}\sum_{\tau}\int_{i\mathbb{R}}\frac{m'(\lambda)}{m(\lambda)}\Big[\prod_v \operatorname{tr} I_{\tau_v}(\lambda;\phi_v dg'_v\times\sigma)\Big]d\lambda$$

and $J_v(\phi dg')$ is

$$\frac{1}{4}\sum_{\tau}\int_{i\mathbb{R}}[\operatorname{tr} R_{\tau_v}(\lambda)^{-1}R_{\tau_v}(\lambda)'I_{\tau_v}(\lambda;\phi_v dg'_v\times\sigma)]\cdot\prod_{w\neq v}\operatorname{tr} I_{\tau_w}(\lambda;\phi_w dg'_w\times\sigma)d\lambda.$$

The abbreviated notations are standard. The sum over v is finite. It extends over the places v where ϕ_v is not spherical, since when ϕ_v is spherical the operator $I_{\tau_v}(\lambda;\phi_v dg'_v\times\sigma)$ factors through the projection on the one-dimensional subspace (if τ_v is unramified) of $K_v=\mathrm{GL}(3,R_v)$-fixed vectors, on which $R_{\tau_v}(\lambda)$ acts as the scalar one, so that $R_{\tau_v}(\lambda)'=0$.

Next we have to record the analogous term $J(fdg)$ of the trace formula for $\mathbf{G}(\mathbb{A})$. Again we use the notations of 3.1, with $\sigma=1$. This rank-one nontwisted case is well known (see [JL], pp. 516-517). We take $M=M_0$, and $\mathcal{A}=\mathcal{A}_M$ is one dimensional. The element s of the Weyl group is $s=\mathrm{id}$; it lies in $W^{\mathcal{A}}(\mathcal{A}_M)$. The Weyl group W_0 has cardinality two, and $[W_0^M]=1$, and $\mathcal{A}_M/\mathcal{A}=\{0\}$. Hence the coefficient of $J(fdg)$ is $\frac{1}{2}$, and

$$J(fdg)=\frac{1}{2}\sum_\mu\int\limits_{i\mathbb{R}}\operatorname{tr}\mathcal{M}(\lambda)I(\mu\otimes\lambda;fdg)d\lambda.$$

The sum ranges over all connected components with representatives μ, where $\mu(a,b,\overline{a}^{-1})=\mu(a/b)\omega(b)$. Here a lies in $\mathbb{A}_E^\times$, b in $\mathbb{A}_E^1$, μ is a character of $\mathbb{A}_E^\times/E^\times$, and the connected component of μ consists of $\mu\otimes\lambda$, where μ is replaced by $\mu\nu^\lambda$, for λ in $i\mathbb{R}$. The induced $\mathbf{G}(\mathbb{A})$-module $I(\mu\otimes\lambda)$ lifts (see Proposition I.4.1) to the induced $\mathbf{G}(\mathbb{A}_E)$-module $I_\tau(\lambda)$, where $\tau=(\mu,\omega'\overline{\mu}/\mu,\overline{\mu}^{-1})$. This relation defines a bijection $\mu\leftrightarrow\tau$ between the sets over which the sums of $J(\phi dg')$ and $J(fdg)$ are taken. Here $\mathcal{M}(\lambda)$ is again a logarithmic derivative of an operator $M=m\otimes_v R_v$, and $J(fdg)$ is the sum of $J'(fdg)$ and $\sum_v J_v(fdg)$, where

$$J'(fdg)=\frac{1}{2}\sum_\mu\int\limits_{i\mathbb{R}}\frac{m'(\lambda)}{m(\lambda)}\left[\prod_v\operatorname{tr}I(\mu_v\otimes\lambda;f_v dg_v)\right]d\lambda$$

and $J_v(fdg)$ is

$$\frac{1}{2}\sum_\mu\int\limits_{i\mathbb{R}}\operatorname{tr}[R_{\mu_v}(\lambda)^{-1}R_{\mu_v}(\lambda)'I(\mu_v\otimes\lambda;f_v dg_v)]\cdot\prod_{w\neq v}\operatorname{tr}I(\mu_v\otimes\lambda;f_v dg_v)d\lambda.$$

Note that here the normalizing factors $m(\lambda)$ depend on μ, while those of $J'(\phi dg')$ depend on τ. It is clear (see Proposition I.4.1) that for matching measures $f_v dg_v$ and $\phi_v dg'_v$ we have

$$\operatorname{tr}I(\mu_v\otimes\lambda;f_v dg_v)=\operatorname{tr}I_{\tau_v}(\lambda;\phi_v dg'_v\times\sigma),\quad\text{if}\quad\tau_v=(\mu_v,\omega'_\mu\overline{\mu}_v/\mu_v,\overline{\mu}_v^{-1}).$$

It can be shown directly that $2m'(\lambda,\mu)/m(\lambda,\mu)=m'(\lambda,\tau)/m(\lambda,\tau)$, and hence that $J'(fdg)=J'(\phi dg')$, but we do not need this observation. The fundamental observation which we do require is the following.

LEMMA. *For our choice of $f_u dg_u$, $\phi_u dg'_u = (f_u dg_u, f_u^0 dg_u)$ we have $J_u(\phi dg') = J_u(f dg)$.*

PROOF. This is precisely Lemma 16, p. 47, of [F1;III], in the case $l = 2$. Note that the proof of this Lemma 16 is elementary and self-contained. To see that this Lemma 16 applies in our case, recall that we choose f_u^0 to be the characteristic function (up to a scalar multiple) of $Z_u I_u$, where I_u is a σ-invariant open compact subgroup of G_u. Then

$$f_u^0(\sigma g) = f_u^0(g), \quad {}^\sigma\pi_u(f_u^0) = \pi_u(f_u^0) \quad \text{and} \quad f_u = f_u *^\sigma f_u^0 = f_u * f_u^0$$

in the notations of [F1;III], (1.5.2), p. 42, *l.* 7. In fact this Lemma 16 of [F1;III] asserts that

$$\operatorname{tr} R_{\tau_u}(\gamma)^{-1} R_{\tau_u}(\tau)' I_{\tau_u}(\gamma; \phi_u dg'_u \times \sigma) = \ell \operatorname{tr} R_{\mu_u}(\gamma)' I(\mu_u \otimes \gamma; f_u dg_u)$$

in our notations, where $l = 2$. This is precisely the factor needed to match the $\frac{1}{4}$ of $J_u(\phi dg')$ with the $\frac{1}{2}$ of $J_u(f dg)$. Our lemma follows. □

It remains to deal with the terms of $J('f dh)$ and $J('\phi dh)$. Since this case of U(2) is well known (see [F3;II]) we do not write out the expressions here, but simply note the following.

(1) We may assume that the place u is such that the component κ_u of the character κ on $\mathbb{A}_E^\times / E^\times N \mathbb{A}_E^\times$ is unramified.

(2) We may and do multiply κ by an unramified (global) character to assume that $\kappa_u = 1$.

(3) If $'f_v dh_v$ and $'\phi_v dh_v$ are matching measures on H_v in the notations of I.2, and

$$\rho_v = {}'I(\mu_v), \qquad \rho'_v = {}'I(\mu_v \kappa_v)$$

in the same notations, then $\operatorname{tr} \rho_v('f_v dh_v) = \operatorname{tr} \rho'_v('\phi_v dh_v)$ by Proposition I.4.1.

(4) At the split place u we take the components $'f_u$ and $'\phi_u$ to be defined directly by the same formula (of I.4.4) in terms of f_u; they are equal to each other. We conclude:

LEMMA. *In the above notations, we have $J_u('f dh) = J_u('\phi dh)$.*

PROOF. This follows from (3) and (4). Indeed, the sets of μ parametrizing the sums which appear in $J('f dh)$ and $J('\phi dh)$ are isomorphic. The isomorphism $('I(\mu) \to' I(\mu\kappa))$ is defined by the dual group diagram and by Proposition I.4.1. □

REMARK. $J'('fdh)$ and $J'('\phi dh)$ are given by precisely the same formulae, hence they are equal to each other by (3). We do not use this remark below.

4.4 Approximation

We conclude that for $f = f^u \otimes f_u$ with fixed f^u and $f_u = f_u(\mathbf{n})$ where $\mathbf{n}$ is n_0-admissible for some $n_0 = n_0(f^u)$, we have the identity

$$J = J'(\phi dg') - J'(fdg) + J'('fdh) - J'('\phi dh)$$

$$+ \sum_v [J_v(\phi dg') - J_v(fdg) + J_v('fdh) - J_v('\phi dh)]. \qquad (1)$$

The sum over v is finite and ranges over $v \neq u$. On the left J represents a sum with complex coefficients (depending on f^u but not on f_u) of traces of the form $\operatorname{tr}\pi_u(f_u dg_u)$, $\operatorname{tr}\Pi_u(\phi_u dg_u' \times \sigma)$, $\operatorname{tr}\{\rho_u\}('f_u dh_u)$ or $\operatorname{tr}\{\rho_u\}('\phi_u dh_u)$. This is an invariant distribution in $f_u dg_u$; it depends only on the orbital integrals of $f_u dg_u$. On the right we have a sum over the connected components (represented by μ_u) of the manifold of characters mentioned in§2, of integrals over $i\mathbb{R}$. The integrands are of the form $c(\lambda)\operatorname{tr} I(\mu_u \otimes \lambda; f_u dg_u)$. The right side of (1) is therefore also an invariant distribution in $f_u dg_u$, depending only on the orbital integrals of $f_u dg_u$. We conclude

LEMMA. *The identity* (1) *holds with the pseudo-spherical function* $f_u = f_u(\mathbf{n})$ *replaced by the spherical function* $f_u' = f_u'(\mathbf{n})$.

PROOF. By definition $f_u(\mathbf{n})$ and $f_u'(\mathbf{n})$ have equal orbital integrals. □

From now on we denote by f_u a spherical function of the form $f_u'(\mathbf{n})$ with n_0-admissible $\mathbf{n}$. The identity (1) holds for our $f = f^u \otimes f_u$. Since f_u is spherical, $\operatorname{tr}\pi_u(f_u dg_u) \neq 0$ only when π_u is unramified. The theory of the Satake transform establishes an isomorphism from the set of unramified irreducible G_u-modules π_u, to the variety $\mathbb{C}^{\times 3}/S_3$: the unordered triple $\mathbf{z} = (z_1, z_2, z_3)$ of nonzero complex numbers corresponds to the unramified subquotient $\pi_u(\mathbf{z})$ of the G_u-module $I_u(\mathbf{z})$ normalizedly induced from the unramified character $(a_{ij}) \mapsto \prod_i z_i^{\operatorname{val}(a_{ii})}$ of the upper triangular subgroup.

The central character of $\pi_u(\mathbf{z})$ is trivial if and only if $z_1z_2z_3 = 1$. For z in $\mathbb{C}^\times$ and $\mathbf{z}$ in $\mathbb{C}^{\times 3}$ we write $z\mathbf{z}$ for $(z_1z,\ z_2,\ z_3z^{-1})$. We conclude that there are (a) $\mathbf{t}_i$ in $\mathbb{C}^{\times 3}/S_3$ $(i \geq 0)$ and $\mathbf{z}_i$ in $\mathbb{C}^{\times 3}$ $(i \geq 0)$ with $t_{i1}t_{i2}t_{i3} = 1$, $z_{i1}z_{i2}z_{i3} = 1$ and $|z_{ij}| = 1$, and (b) complex numbers c_i, and integrable functions $c_i(z)$ on $|z| = 1$, such that (1) takes the form

$$\sum_i c_i \operatorname{tr}(\pi_u(t_i))(f_u dg_u) = \sum_j \int_{|z|=1} c_j(z) \operatorname{tr}(\pi_u(\mathbf{z}_j z))(f_u dg_u) d^\times z. \tag{2}$$

The Satake transform $f_u \mapsto f_u^\vee$, defined by $f_u^\vee(\mathbf{z}) = \operatorname{tr}(\pi_u(\mathbf{z}))(f_u dg_u)$, in an isomorphism from the convolution algebra of spherical functions f_u on G_u to the algebra of Laurent series $f_u^\vee$ of $\mathbf{z}$ in $\mathbb{C}^{\times 3}/S_3$ with $z_1z_2z_3 = 1$. Then (2) can be put in the form

$$\sum_i c_i f_u^\vee(t_i) = \sum_j \int_{|z|=1} c_j(z) f_u^\vee(\mathbf{z}_j z) d^\times z. \tag{3}$$

Our aim is to show that $c_i = 0$ for all $i \geq 0$. For that we note that all sums and products in the trace formula are absolutely convergent for any f_u, in particular for the function with $f_u^\vee = 1$. Hence $\sum_i |c_i|$ is finite, and $\sum_i \int |c_i(z)||dz|$ is finite. Moreover, let X be the set of $\mathbf{z}$ in $\mathbb{C}^{\times 3}/S_3$ with $z_1z_2z_3 = 1$, $\overline{\mathbf{z}}^{-1} = \mathbf{z}$, and $q^{-1} \leq |z_i| \leq q$ for each entry z_i of $\mathbf{z}$. Since all representations which contribute to the trace formula are unitary, the $\mathbf{t}_i$ and $\mathbf{z}_i z$ lie in X. But then the case where $n = 3$ of the final Proposition in [FK2], where the analogous problem is rephrased and solved for an arbitrary reductive group, implies that all c_i in (3) are zero. The theorem follows. □

II.5 The σ-endo-lifting e'

Summary. Let $\mathbf{G} = \mathbf{U}(3, E/F)$ be the quasi-split unitary group in three variables defined using a quadratic extension E/F of number fields. Complete local and global results are obtained for the σ-endo-(unstable) lifting from the group of F-rational and $\mathbb{A}$-points of $\mathbf{U}(2, E/F)$ to the corresponding group $\mathrm{GL}(3, E)$ or $\mathrm{GL}(3, \mathbb{A}_E)$. This is used to establish quasi-(endo-) lifting for automorphic forms from U(2) to U(3) by means of basechange

from U(3) to GL$(3, E)$. Basechange quasi-lifting is also proven. Our diagram is:

$$\begin{array}{ccccc} {}^LG & \xrightarrow{b} & {}^LG' & & \\ e\uparrow & & i\uparrow & \nwarrow e' & \\ {}^LH & \xrightarrow[b']{} & {}^LH' & \xleftarrow[b'']{} & {}^LH \end{array}$$

5.1 Quasi-lifting

The notion of local lifting in the unramified case is defined in I.4. A preliminary, weak, definition of global lifting, is given next in terms of almost all places.

Definition. Let J, J' be a pair of groups as above (H, G, etc.) for which the local notion of lifting is defined in the unramified case. If $\pi = \otimes\pi_v$ and $\pi' = \otimes\pi'_v$ are automorphic $\mathbf{J}(\mathbb{A})$- and $\mathbf{J}'(\mathbb{A})$-modules, and π_v lifts to π'_v for almost all v, then we say that π *quasi-lifts* to π'.

We shall later define the strong notion of global lifting, in terms of all places. This has been done in [F3;II] in the case of the basechange liftings b' and b''. The map i is simply induction. Our aim in this section is to study the local and global lifting in the case of the σ-endo-lift e'. This, or the alternative approach of 5.3, will be used in II.6 for the study of the quasi-endo-lift e, and the basechange lift b.

Our first aim is to study the local lifting e'. Let E_w/F_w be a quadratic extension of p-adic fields.

5.1.1 Proposition. *Suppose that τ_w is the stable basechange b'' lift of an irreducible H_w-module ρ_w. Then for any matching measures $\phi_w dg'_w$ and $'\phi_w dh_w$, we have*

$$\operatorname{tr} I(\tau_w; \phi_w dg'_w \times \sigma) = \operatorname{tr}\{\rho_w\}('\phi_w dh_w).$$

Proof. This is shown in I.4.1 for induced representations. The case of the one-dimensional H_w-module follows from the case of the Steinberg representation, as its character is the difference of the characters of an induced and the Steinberg representation.

Suppose then that ρ_w is a discrete-series H_w-packet (consisting of discrete series H_w-modules). Fix a global totally imaginary extension E/F

whose completion at w is the chosen local quadratic extension. At two finite places $v = u$, u', say u splits and u' does not split in E/F, we choose cuspidal representations ρ_u and $\rho_{u'}$. Let V be a finite set containing w, u, u', and the places which ramify in E/F, but no infinite places.

It is easy to see (using the trace formula) that there is a cuspidal $\mathbf{H}(\mathbb{A})$-module ρ whose components at w, u, u' are the given ones, which is unramified at all finite v outside V, and its components at the v in V are all discrete series. We choose a sequence $\{t_v;\ v \text{ outside } V\}$ so that ρ makes a contribution to the sum in the trace formula for $\mathbf{H}(\mathbb{A})$, which is associated with ${}'\phi$. Then the trace formula identity of II.3 asserts

$$\prod \operatorname{tr} I(\tau_v; \phi_v dg'_v \times \sigma) = \prod \operatorname{tr}\{\rho_v\}({}'\phi_v dh_v) + 2\sum \prod \operatorname{tr} \pi_v(f_v dg_v)$$

$$- \sum n({}'\rho) \prod \operatorname{tr}\{{}'\rho_v\}({}'f_v dh_v).$$

The products extend over the finite places in V. The $\{\rho_v\}$ are the packets of the components of our ρ. But by [F3;II], ρ lifts via the stable basechange map b'' to an automorphic $\mathbf{H}'(\mathbb{A})$-module τ. Rigidity theorem for $\mathbf{G}'(\mathbb{A}) = \mathrm{GL}(3, \mathbb{A})$ (see [JS]) implies that $I(\tau)$ is the only contribution to the terms involving ϕ in Proposition II.3.3. The terms $I(\mu)$ do not appear due to the condition at the split place u. Further, $\{\rho\}$ is the only packet which lifts to $I(\tau)$.

Moreover, since u' is a nonsplit place, and the character of $\{\rho_{u'}\}$ (namely sum of characters of the members in the packet) is nonzero on the elliptic set, we may choose ${}'\phi_{u'}$ supported on the regular $H_{u'}$-elliptic set with $\operatorname{tr}\{\rho_{u'}\}({}'\phi_{u'} dh_{u'}) \neq 0$. Then the matching $\phi_{u'}$ can be chosen so that its stable σ-orbital integrals are 0. Namely we can take $f_{u'} = 0$, and ${}'f_{u'} = 0$. Consequently

$$\prod \operatorname{tr} I(\tau_v; \phi_v dg'_v \times \sigma) = \prod \operatorname{tr}\{\rho_v\}({}'\phi_v dh_v). \tag{5.1.1}$$

We can repeat the same discussion with an automorphic $\mathbf{H}(\mathbb{A})$-module ρ' which is unramified outside V, its components at all finite $v \neq w$ in V are in the packets $\{\rho_v\}$, and at w the component is induced. In this case we obtain the identity (5.1.1), in which the product extends over all finite $v \neq w$ in V. Since there are ${}'\phi_v$ supported on the regular set, with $\operatorname{tr}\{\rho_v\}({}'\phi_v dh_v) \neq 0$, for discrete-spectrum representations ρ_w and nonarchimedean E_w/F_w, the proposition follows.

Moreover, the proposition holds also when E_w/F_w is $\mathbb{C}/\mathbb{R}$, and $\{\rho_w\}$ is unitary. It suffices to consider discrete-series ρ_w, and take $F = \mathbb{Q}$ and an imaginary quadratic E. Repeating the proof of (5.1.1), the proposition follows in this case too. □

Note that in the proof of the Proposition above, besides the identity of trace formulae we have used only the (generalized) fundamental lemma, but we do not need to transfer general test functions. It suffices to work with test functions supported on the regular set. These are easily transfered.

5.1.2 COROLLARY. *The twisted character of the representation $I(\tau_w)$ induced from the stable basechange lift τ_w of an irreducible H_w-module ρ_w is unstable.*

Here unstable means that if δ and δ' are distinct σ-regular (σ-elliptic) σ-conjugacy classes in G'_w which are stably σ-conjugate, then $\chi^\sigma_{I(\tau_w)}(\delta') = -\chi^\sigma_{I(\tau_w)}(\delta)$.

5.1.3 PROPOSITION. *Let E/F be a quadratic extension of local fields. Let Π be a square-integrable σ-invariant representation of* $\mathrm{GL}(3, E)$. *Then its σ-character is not identically zero on the σ-elliptic regular set, and it is a σ-stable function on the σ-elliptic regular set.*

PROOF. The σ-character is not identically zero on the σ-elliptic regular set by the twisted orthonormality relations.

We need to show that the σ-character is a σ-stable function on the σ-elliptic regular set. This is clear for the one-dimensional representations of $\mathrm{GL}(3, E)$, hence for the Steinberg representations. We then assume that Π is cuspidal and E/F is nonarchimedean. Suppose the σ-character of Π is not a σ-stable function on the σ-elliptic regular set. Let ϕ be a σ-pseudo-coefficient. Then its unstable σ-orbital integral is nonzero at some σ-elliptic regular element.

At this stage we choose a quadratic extension E/F of totally imaginary number fields, whose completion at a place v_0 is our local situation. At two further finite places v_1, v_2 of F which remain prime in E, and in the places which ramify in E/F, we choose Π_v which are λ_1-lifts of square-integrable ρ_v on $\mathrm{U}(2, E_v/F_v)$. The σ-orbital integral of a σ-pseudo-coefficient $\phi_v dg'_v$ of such a Π_v is a σ-unstable function. Hence for a test measure $\phi dg' = \otimes_v \phi_v dg'_v$ with such components at the specified v, spherical components at all other finite places, and f_∞ vanishing on the σ-singular set, only

σ-unstable orbital integrals occur in the geometric side of the σ-twisted trace formula for $\mathrm{GL}(3, \mathbb{A}_E)$. As usual, the choice of the spherical and the archimedean components can be used to reduce the geometric sum to a single stable σ-elliptic regular conjugacy class. In the trace formulae identity we may choose $f = 0$ on $\mathbf{U}(3, E/F)(\mathbb{A})$, that is, $f_v = 0$ at the places v where ϕ_v is a twisted coefficient of $\lambda_1(\rho_v)$. The only remaining terms on the nontwisted side of the trace formulae identity are those involving discrete-spectrum ρ on $\mathbf{U}(2, E/F)(\mathbb{A})$. The corresponding terms on the twisted side are the induced $I(\tau)$. We obtain the identity (5.1.1), which implies that the local Π of the proposition is induced $I(\tau)$ and not square-integrable (in fact cuspidal) as assumed. This is a contradiction to the assumption that the σ-character of Π is not a σ-stable function on the σ-elliptic regular set. □

5.1.4 PROPOSITION. *Let E/F be a quadratic extension of local fields. Let Π be a square-integrable σ-invariant representation of* $\mathrm{GL}(3, E)$. *Then it is the endoscopic lift of a square-integrable representation π of* $\mathrm{U}(3, E/F)$. *Thus* $\operatorname{tr}\Pi(\phi dg' \times \sigma) = \operatorname{tr}\pi(f dg)$ *for all matching measures $\phi dg'$ on* $\mathrm{GL}(3, E)$ *and $f dg$ on* $\mathrm{U}(3, E/F)$.

PROOF. If Π is Steinberg, so is π, so we may assume that Π is cuspidal and E/F is nonarchimedean. The σ-character of Π is a σ-stable function on the σ-elliptic regular set. We can view our local extension as the completion of a global totally imaginary one, and use a global cuspidal σ-invariant representation of $\mathrm{GL}(3, \mathbb{A}_E)$ whose component at this place is our Π, at a few additional places, including those which ramify, the component be Steinberg, and all other components at finite places be unramified. The trace formula identity for such a global representation (having fixed almost all components by "generalized linear independence of characters") will contain only π on $\mathbf{U}(3, E/F)(\mathbb{A})$ on the twisted side. Using the (known) lifting for Steinberg representations and at the archimedean places, we get an identity

$$\operatorname{tr}\Pi(\phi dg' \times \sigma) = \sum_{\pi} m(\pi) \operatorname{tr}\pi(f dg)$$

for all matching $\phi dg'$ and $f dg$. Using the orthonormality relations for twisted characters of square-integrable representations we see that the sum reduces to a single term, with $m(\pi) = 1$. □

5.2 Alternative approach

In the proof of Proposition 5.1 we used only the σ-endo-transfer e' of the unit element ϕ^0 in the Hecke algebra of G' to the unit element $'\phi^0$ in the Hecke algebra of H; and the transfer of spherical functions with respect to $e' : {}^LH \to {}^LG'$, which follows from the statement for $(\phi^0, {}'\phi^0)$ by a global method. This is needed only at places where E/F, κ, ω are unramified. At the other places it suffices to transfer functions supported on the regular set, and this is easily done

We shall now give an alternative approach, whose purpose is to show that the character of $I(\tau_w)$ is an unstable function, namely that $\operatorname{tr} I(\tau_w; \phi_w dg'_w \times \sigma)$ depends only on $'\phi_w dh_w$. We shall not use the fundamental lemma for e', and conclude that complete local, and some global, results about the endo-lifting e can be obtained without using any knowledge of the σ-endo-transfer e'. We use the results of Keys [Ke] concerning the reducibility of induced G-modules recorded in I.4.4.

We shall also make use of the following result of [F3;II]. A local module is called *elliptic* if its character is nonzero on the elliptic regular set. Put $C_F = F^\times$ if F is a local field, and $C_F = \mathbb{A}_F/F^\times$ if F is a global field.

5.2.1 PROPOSITION. (1) *If τ is an elliptic* (*resp. discrete-spectrum*) *σ-invariant local* (*resp. global*) *H'-* (*resp.* $\mathbf{H}'(\mathbb{A})$)*-module, then its central character is trivial on C_F.* (2) *Such τ is the basechange lift of a unique elliptic or discrete-spectrum H- or $\mathbf{H}'(\mathbb{A})$-module ρ, either through b' or through b'', but not both.*

A proof of (1) in a more general context is given in [F1;VI].

The second statement here implies, in the global case, that if $I(\tau)$ is the only term on the left side of the trace formula identity II.3.3, then precisely one of the sums involving $'f$ and $'\phi$ on the right is nonzero, and it consists of a single term.

Note that the elliptic (local) ρ are the one-dimensional, Steinberg and cuspidal, and also the components of a reducible tempered induced H-module, which make a packet.

We shall now prove a special case of Proposition 5.1, but without using the fundamental lemma for e' stated in I.2.

5.2.2 PROPOSITION. *Let τ_w be the stable basechange lift of the elliptic*

H_w-module ρ_w. Then $\operatorname{tr} I(\tau_w; \phi_w dg'_w \times \sigma) = 0$ *if $\phi_w dg'_w$ matches $'\phi_w dh_w$ and $'\phi_w dh_w$ is* 0.

Proof. We deal with the one-dimensional case first. Let ρ be a one-dimensional H-module, and τ its basechange lift. Then ρ is a constituent of an induced $'I(\mu\nu^{1/2})$, and τ of $I(\mu\nu^{1/2}, \mu\nu^{-1/2})$. We choose $'\phi_w = 0$, so that $'\phi = 0$, and no term involving $'\phi$ appears in the trace formula identity II.3.3. We choose a sequence $\{t_v\}$ so that our $I(\tau)$ is the only contribution associated with ϕ. The only other possible terms in II.3.3 are of the form $\operatorname{tr} \pi(f dg)$ since we can choose $'f_w = 0$, thus $'f = 0$. The local components of any such π are almost all of the form $I(\mu\nu^{1/2})$. In any case, we conclude that for any $v \neq w$, if $\operatorname{tr} I(\tau_w; \phi_w dg'_w \times \sigma) \neq 0$ then $\operatorname{tr} I(\tau_v; \phi_v dg'_v \times \sigma)$ depends only on $f_v dg_v$. More precisely, there are G_v-modules π_v and complex constants $c(\pi_v)$ with

$$\operatorname{tr} I(\tau_v; \phi_v dg'_v \times \sigma) = \sum c(\pi_v) \operatorname{tr} \pi_v(f_v dg_v)$$

for all matching $\phi_v dg'_v$, $f_v dg_v$. Taking matching functions whose orbital integrals are supported on the conjugacy classes of the $\operatorname{diag}(a, b, \overline{a}^{-1})$, $|a| \neq 1$, the Deligne-Casselman [C1] theorem implies that

$$\operatorname{tr} I(\tau_v)_A(\phi_{vA} da'_v \times \sigma) = \sum c(\pi_v) \operatorname{tr} \pi_{vA}(f_{vA} da_v).$$

Here Π_A, π_A denote the modules of coinvariants of Π, π (see I.4.4) with respect to the upper triangular parabolic subgroup with Levi subgroup A, tensored by $\delta^{-1/2}$, where $\delta(\operatorname{diag}(a, b, \overline{a}^{-1})) = |a|^2$ (resp. $\delta(\operatorname{diag}(a, b, c)) = |a/c|^2$) is the modulus function on G_v (resp. $G'_v = \mathrm{GL}(3, F_v)$), and ϕ_{vA}, f_{vA} are the functions on A, A' defined by

$$f_{vA}(\operatorname{diag}(a, b, \overline{a}^{-1})) = |a| \int_K \int_N f_v(k^{-1} ank)\, dndk,$$

$$\phi_{vA}(\operatorname{diag}(a, b, c)) = |a/c| \int_K \int_N \phi_v(\sigma(k)^{-1} ank)\, dndk.$$

Since the functions f_{vA}, ϕ_{vA} are arbitrary, and the module $I(\tau_v)_A$ of coinvariants consists of a single (increasing) σ-invariant exponent, we conclude from the Harish-Chandra finiteness theorem [BJ], and linear independence of characters on A, that on the right there should be a single π_v with nonvanishing nonunitary π_{vA}, and then π_{vA} should consist of a single exponent which lifts to $I(\tau_v)_A$. Here we used the fact (see I.4.3) that if the irreducible π_v and π'_v have nonunitary characters in π_{vA} and π'_{vA} which are equal, then π_v and π'_v are equivalent. Hence our π_v is a subquotient of $I = I(\mu\nu^{1/2})$. But I is irreducible (see I.4.3), hence $\pi_v = I$, and π_{vA} has two exponents, one increasing and one decaying. This contradiction establishes the proposition when ρ_w is one dimensional, hence also when it is special.

To deal with the cuspidal ρ_w, it suffices to construct a cuspidal ρ with this component, and a component ρ_v which is special. If $'\phi_w = 0$ we conclude as above that $\operatorname{tr} I(\tau_v; \phi_v dg'_v \times \sigma)$ depends only on f_v, where τ_v is the stable base change lift of ρ_v. This contradicts the previous conclusion in the special case, as required.

It is clear that taking $F = \mathbb{Q}$ we obtain the above conclusion also in the archimedean case. $\square$

II.6 The quasi-endo-lifting *e*

6.1 Cancellation

The results of II.5 concerning the σ-endo-lifting e' can be used to simplify the identity I.3.3 of trace formulas. First the terms $\operatorname{tr} I(\tau; \phi dg' \times \sigma)$, where τ is a stable basechange lift of an $\mathbf{H}(\mathbb{A})$-module ρ, are canceled with the terms $\operatorname{tr}\{\rho\}('\phi dh)$. Indeed, if a discrete-spectrum $\{\rho\}$ basechanges to a discrete-spectrum τ, then $n(\rho) = 1$ according to [F3;II]. When $n(\rho) \neq 1$, it is equal to $1/2$, and ρ is of the form $\rho(\theta)$ in the notations of [F3;II], p. 721, (where it is denoted by $\pi(\theta)$). According to Proposition 1 there, $\rho(\theta)$ lifts to an induced $\mathbf{H}'(\mathbb{A})$-module $\tau = I(\theta'\kappa, \theta''\kappa)$, where θ', θ'' are distinct characters of C_E/C_F related to the character θ (of $C_E^1 \times C_E^1$). There is no need to elaborate on this result. We simply note that the $\operatorname{tr}\{\rho\}('\phi dh)$ with $n(\rho) = 1/2$ cancel the $\operatorname{tr} I(\eta; \phi dg' \times \sigma)$ with $\eta = (\kappa\theta', \kappa\theta'', \mu)$ (where $\mu\kappa^2\theta'\theta'' = \omega'$), as these appear with coefficient $1/4$.

There remains $\operatorname{tr} I(\mu, '\phi dh)$, which depends on $'\phi$. The induced representation $I(\mu)$ lifts via e' to the $\mathbf{G}'(\mathbb{A})$-module $I(\mu,\, \mu,\, \omega'/\mu^2)$. If $\omega' \neq \mu^3$ then we obtain a cancellation with the term $\operatorname{tr} I((\mu, \mu', \mu); \phi dg' \times \sigma)$, which also appears with coefficient $-1/8$. If $\omega' = \mu^3$ then we obtain a partial cancellation, which replaces the coefficient $-3/8$ by $-1/4$, in the twisted side of the formula.

6.2 Identity

So far we eliminated all terms which depend on $'\phi$. Let us record those terms which are left. We denote by μ any character of C_E trivial on C_F. Put

$$\Phi_1 = \sum \prod \operatorname{tr} \Pi_v(\phi_v dg'_v \times \sigma), \qquad \Phi_2 = \sum \prod \operatorname{tr} I(\tau_v \otimes \kappa_v; \phi_v dg'_v \times \sigma).$$

In Φ_1 the sum is over all (equivalence classes of) σ-invariant discrete-spectrum $\mathbf{G}'(\mathbb{A})$-modules. In Φ_2 the sum is over the σ-invariant discrete-spectrum $\mathbf{H}'(\mathbb{A})$-modules τ which are obtained by the stable base change map b'', namely $\tau \otimes \kappa$ is obtained by the unstable map b'. Further,

$$\Phi_3 = \sum \prod \operatorname{tr} I((\mu, \mu', \mu''); \phi_v dg'_v \times \sigma) \quad (\text{distinct } \mu, \mu', \mu''),$$

and

$$\Phi_4 = \sum \prod \operatorname{tr} I((\kappa\mu, \mu', \kappa\mu); \phi_v dg'_v \times \sigma),$$

$$\Phi_5 = \sum \prod \operatorname{tr} I((\mu, \mu, \mu); \phi_v dg'_v \times \sigma).$$

On the other hand, we put

$$F_1 = \sum_{\pi} m(\pi) \prod \operatorname{tr} \pi_v(f_v dg_v).$$

The sum is over the equivalence classes π in the discrete spectrum of $\mathbf{G}(\mathbb{A})$. They occur with finite multiplicities $m(\pi)$. Further,

$$F_2 = \sum_{\rho \neq \rho(\theta, '\theta)} \prod \operatorname{tr}\{\rho_v\}('f_v dh_v).$$

The sum ranges over the automorphic discrete-spectrum packets of ρ of $\mathbf{H}(\mathbb{A})$, which are not of the form $\rho(\theta, '\theta)$. In this case $n(\rho) = 1$ (see [F3;II]). Also,

$$F_3 = \sum_{\rho = \rho(\theta, '\theta)} \prod \operatorname{tr}\{\rho_v\}('f_v dh_v).$$

Here the sum ranges over the packets $\rho = \rho(\theta,' \theta)$, where θ, θ' and $\omega/\theta'\theta$ are distinct. In this case $n(\rho) = 1/2$. Finally we put

$$F_4 = \sum_\mu m(\mu\kappa) \prod \operatorname{tr} I(\mu_v\kappa_v, f_v dg_v) + \frac{1}{2} \sum \prod \operatorname{tr}' I(\mu_v,' f_v dh_v),$$

$$F_5 = \sum_\mu m(\mu) \prod \operatorname{tr} I(\mu_v, f_v dg_v) \qquad (\mu^3 = \omega'),$$

$$F_6 = \sum_\mu m(\mu) \prod \operatorname{tr} R(\mu_v) I(\mu_v, f_v dg_v) - \sum_\rho \prod \operatorname{tr}\{\rho_v\}('f_v dh_v).$$

In F_6, the first sum is over all μ with $\mu^3 \neq \omega'$. The second is over the packets $\rho = \rho(\theta, \omega/\theta^2)$, where $\theta^3 \neq \omega$. We deduce from the identity II.3.3 of trace formulas the following

6.2.1 Proposition. *The identity of trace formulas takes the form*

$$\Phi_1 + \frac{1}{2}\Phi_2 + \frac{1}{4}\Phi_3 - \frac{1}{8}\Phi_4 - \frac{1}{4}\Phi_5 = F_1 - \frac{1}{2}F_2 - \frac{1}{4}F_3 + \frac{1}{4}F_4 + \frac{1}{4}F_5 + \frac{1}{4}F_6.$$

To simplify the formula we first note that the normalizing factor m which appears in F_4 and F_5 can be evaluated as a limit. It is equal to -1. The representations $I(\mu_v\kappa_v)$, $I(\mu_v)$ of $G_v = \mathbf{G}(F_v)$ in F_4 and F_5 are irreducible, and Proposition I.1.4 asserts the following. In the notations of F_4 and Φ_4 we have at each v

$$\operatorname{tr} I(\mu_v,' f_v dh_v) = \operatorname{tr} I(\mu_v\kappa_v, f_v dg_v) = \operatorname{tr} I((\kappa_v\mu_v, \mu_v', \kappa_v\mu_v); \phi_v dg_v' \times \sigma).$$

In the case of F_5 and Φ_5 we have

$$\operatorname{tr} I(\mu_v, f_v dg_v) = \operatorname{tr} I((\mu_v, \mu_v, \mu_v); \phi_v dg_v' \times \sigma).$$

Hence $\Phi_4 = -2F_4$ and $\Phi_5 = -F_5$, and these terms are canceled in the comparison of the Proposition.

The $\mathbf{G}(\mathbb{A})$-modules in F_4 and F_5 are irreducible, and their characters are supported on the split set. If f has a component f_v such that the orbital integral $\Phi(f_v dg_v)$ is supported on the elliptic set, we can conclude that F_4, F_5 are equal to 0.

The normalizing factor $m(\mu)$ of F_6 can be shown to be equal to 1, and F_6 can be shown to be equal to 0, but this will not be done here. However, it is clear from Proposition I.4.1 that $\rho = \rho(\theta, \omega/\theta^2)$ with $\theta^3 \neq \omega$ quasi-lifts to $I(\mu)$, where $\mu = \theta \circ N_{E/F}$. In any case the trace identity takes the form

6.2.2 Proposition. *At each v, let $\phi_v dg'_v$, $f_v dg_v$, $'f_v dh_v$ be matching functions. Fix unramified π_v, namely the corresponding Satake parameters t_v. Then*

$$\Phi_1 + \frac{1}{2}\Phi_2 + \frac{1}{4}\Phi_3 = F_1 - \frac{1}{2}F_2 - \frac{1}{4}F_3 + \frac{1}{4}F_6.$$

The terms consist of products over a finite set of places, and at most one of the terms on the left is nonzero, consisting of a single nonzero representation.

We conclude

6.2.3 Theorem. *Every discrete-spectrum automorphic $\mathbf{H}(\mathbb{A})$-module ρ with two elliptic components quasi endo lifts to an automorphic $\mathbf{G}(\mathbb{A})$-module.*

Proof. It is clear from Proposition I.4.1 that if ρ appears in F_3 then there is a nontrivial term in Φ_3, but if ρ appears in F_2 then there is a contribution in Φ_2. So we apply the identity with a function ϕ so that the suitable Φ is nonzero, and such that $'f$ is 0. Indeed, if Π_u is the component at u of the unique term Π on the left, then $\operatorname{tr}\Pi_u(\phi_u dg'_u \times \sigma)$ is nonzero, and depends only on the stable orbital integral of $\phi_u dg'_u$, namely on the stable orbital integral of f_u, which is supported on the nonsplit set. We can take $f_u dg_u$ with $\Phi(f_u dg_u)$ supported on the regular nonsplit set, with vanishing unstable orbital integrals. Namely the orbital integrals of $'f_u dh_u$, and consequently $'f_u dh_u$ itself, can be taken to be identically 0. Hence $'f dh$ is 0, so that $F_2 = F_3 = F_6 = 0$, but the left side is nonzero, hence the right side is nonzero. Hence $F_1 \neq 0$, as required. □

Note that the same proof implies that for every π which appears in F_1 there exists a σ-invariant Π (with σ-stable components), so that π basechange quasi-lifts to Π, and for each such Π there exists a π with this property.

One case of the theorem which is particularly interesting is that of the one-dimensional $\mathbf{H}(\mathbb{A})$-module, which occurs in F_2 and quasi-endo-lifts to $\mathbf{G}(\mathbb{A})$-modules π whose components almost everywhere are nontempered. Such π may have finitely many cuspidal components, hence be cuspidal, and make a counterexample to the generalized Ramanujan hypothesis.

The purpose of chapter III will be to refine Theorem 3.2.3 above to remove the assumptions on the elliptic components, and sharpen the quasi-lifting to complete results on the local and global endo-lifting and on the basechange lifting.

II.7 Unitary symmetric square

Let E/F be a quadratic extension of number fields. Put $\mathbf{H} = \mathrm{SL}(2)$. If π_0 is an automorphic $\mathbf{H}(\mathbb{A})$-module, then for almost all v its component π_{0v} is the irreducible unramified subquotient of the H_v-module $I_0(\mu_v)$ induced from the character

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto \mu_v(a) \quad (a \text{ in } F_v^\times).$$

For almost all v, the component Π_v of an automorphic $\mathrm{PGL}(3,\mathbb{A})$-module Π is similarly associated with the representation $I(\mu_{1v}, \mu_{2v}, \mu_{3v})$ normalizedly induced from the unramified character $(\mu_{1v}, \mu_{2v}, \mu_{3v})$ of the upper triangular subgroup. Here $\mu_{1v}\mu_{2v}\mu_{3v} = 1$. In [F2;I] it is shown that

7.1 Lemma. *Given an irreducible automorphic representation π_0 of $\mathrm{SL}(2,\mathbb{A})$ with π_{0v} in $I_0(\mu_v)$ for all v, there exists an irreducible automorphic representation Π of $\mathrm{PGL}(3,\mathbb{A})$ with Π_v in $I(\mu_v, 1, \mu_v^{-1})$ for almost all v.*

Note that π_{0v} in $I_0(\mu_v)$ is represented by the conjugacy class $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, with $a/b = \mu_v(\boldsymbol{\pi})$ in the dual group ${}^L H = \mathrm{PGL}(2,\mathbb{C})$, and $\Pi_v = I(\mu_{1v}, \mu_{2v}, \mu_{3v})$ by the class of the diagonal matrix $(\mu_{1v}(\boldsymbol{\pi}), \mu_{2v}(\boldsymbol{\pi}), \mu_{3v}(\boldsymbol{\pi}))$ in the dual group $\widehat{M} = \mathrm{SL}(3,\mathbb{C})$ of $M = \mathrm{PGL}(3)$. The lifting of the Lemma is compatible with the three-dimensional symmetric square-representation Sym of $\widehat{H}$ on $\widehat{M}$, which maps (a,b) to $(a/b, 1, b/a)$ (see [F2;I]). Hence we denote Π of the Lemma by $\mathrm{Sym}(\pi_0)$, and name it the symmetric square lift of π_0.

Recall that the connected component $\widehat{G}$ of the dual group ${}^L G$ of the projective unitary group $\mathbf{G} = \mathbf{PU}(3)$ is also $\mathrm{SL}(3,\mathbb{C})$. Given an automorphic $\mathbf{H}(\mathbb{A})$-module π_0, we wish to find an automorphic $\mathbf{G}(\mathbb{A})$-module π, to be called the *unitary symmetric square* $\mathrm{US}(\pi_0)$, whose local components are defined by those of π_0, and the map Sym: ${}^L H \to {}^L G$, for almost all v. Thus, when v splits E/F, G_v is $\mathrm{PGL}(3,F_v)$, and $\mathrm{US}(\pi_{0v})$ is $I(\mu_v, 1, \mu_v^{-1})$ if π_{0v} is $I_0(\mu_v)$. If v stays prime in E, the induced unramified G_v-module $I(\mu_v)$ is parametrized by the conjugacy class of $(\mu_v(\boldsymbol{\pi}), 1, 1) \times \sigma$ in ${}^L G = \widehat{G} \times \langle\sigma\rangle$. In this case, $\pi_{0v} = I_0(\mu_v)$ determines $(\mu_v(\boldsymbol{\pi}), 1)$ in ${}^L H$, hence $(\mu_v(\boldsymbol{\pi}), 1, \mu_v(\boldsymbol{\pi})^{-1}) \times \sigma$ in ${}^L G$, which is conjugate to $((\mu_v \circ N)(\boldsymbol{\pi}), 1, 1) \times \sigma$, and $\mathrm{US}(\pi_{0v})$ is $I(\mu_v \circ N)$. Here N denotes the norm map from E_v to F_v. We now assume the availability of all liftings used below under no restrictions at any component.

7.2 PROPOSITION. *Given an automorphic* $\mathbf{H}(\mathbb{A})$*-module* π_0*, there exists an automorphic* $\mathbf{G}(\mathbb{A})$*-module* $\pi = \mathrm{US}(\pi_0)$ *whose component is* $\mathrm{US}(\pi_{0v})$ *for almost all* v*.*

PROOF. We follow the arrows in the following diagram:

$$\begin{array}{ccc} I_0(\mu)\times I_0(\mu) \text{ or } I_0(\mu\circ N) & \overset{\mathrm{Sym}}{\longrightarrow} & I(\mu,1,\mu^{-1})\times I(\mu,1,\mu^{-1}) \\ \text{on } \mathrm{SL}(2,E) & & \text{or } I(\mu\circ N,1,\mu^{-1}\circ N) \text{ on } \mathrm{PGL}(3,E) \\ BC\uparrow & & \uparrow BC \\ I_0(\mu) \text{ on } \mathrm{SL}(2,F) & \overset{\mathrm{US}}{\longrightarrow} & I(\mu,1,\mu^{-1}) \text{ or } I(\mu\circ N) \text{ on } \mathrm{PU}(3). \end{array}$$

The basechange theory for GL(2) implies the existence of an automorphic $\mathrm{SL}(2,\mathbb{A}_E)$-packet π_0^E whose local components are obtained from those $I_0(\mu_v)$ of π_0 as indicated by the vertical arrow on the left (they are $I_0(\mu_v)\times I_0(\mu_v)$ when v splits, and $I_0(\mu_v\circ N)$ when v stays prime). The Lemma implies the existence of an automorphic $\mathrm{PGL}(3,\mathbb{A}_E)$-module $\mathrm{Sym}(\pi_0^E)$, whose components are as indicted by the top horizontal arrow for almost all v. If $\sigma(g) = J{}^t\overline{g}^{-1}J$ is the automorphism of $\mathrm{GL}(3,E)$ which defines U(3), then it is clear that for almost all v we have that $\mathrm{Sym}(\pi_0^E)_v$ is σ-invariant. Hence $\mathrm{Sym}(\pi_0^E)$ is σ-invariant by the rigidity theorem for $\mathrm{GL}(n)$ of [JS]. The E/F-basechange result for U(3) implies that there exists an automorphic $\mathbf{G}(\mathbb{A})$-module π (G=PU(3)) which quasi-lifts to $\mathrm{Sym}(\pi_0^E)$. But π is the required $\mathrm{US}(\pi_0)$, as it has the desired local components for almost all v. □

It will be interesting — and may have interesting applications — to verify the existence of the local unitary symmetric square lifting by means of character relations between representations of SL(2), and bar-invariant PU(3)-modules. In uncirculated notes I defined a suitable norm map of stable conjugacy classes. Further, I computed the trace formula for PU(3), twisted by the bar-automorphism $g\mapsto \overline{g} = \sigma(\overline{g}) = J{}^tg^{-1}J$; note that the rank is one. The required transfer of orbital integrals of spherical functions is available, see [F2;I]), at a place v of F which splits in E. It is not yet available at inert v. The important case is that of the unit element of the Hecke algebra. But I have not pursued these questions.

III. LIFTINGS AND PACKETS

III.1 Local identity

1.1 Trace formulae

Our aim here is to study the local liftings. Thus we fix a quadratic extension of local nonarchimedean fields. We start with the identity of trace formulae of Proposition I.6.2. We denote by E/F a quadratic extension of number fields such that F has no real places and at the place w of F we obtain that E_w/F_w is our chosen quadratic extension. Denote by V a finite set of places of F including the archimedean and those which ramify in E. The products below range over V. At each v in V we choose matching functions $\phi_v dg'_v$, $f_v dg_v$, $'f_v dh_v$, as in I.2. We fix an unramified G_v-module π_v^0 at each v outside V. The sums below range over the automorphic $\mathbf{G}'(\mathbb{A})$, $\mathbf{G}(\mathbb{A})$ or $\mathbf{H}(\mathbb{A})$-modules with component matching π_v^0 at all v outside V. The main result of II.4 and II.6 asserts the following

1.1.1 Proposition. *The identity of trace formulae takes the form*

$$\Phi_1 + \frac{1}{2}\Phi_2 + \frac{1}{4}\Phi_3 = F_1 - \frac{1}{2}F_2 - \frac{1}{4}F_3 + \frac{1}{4}F_6.$$

The left side depends on a choice of a Haar measure dg' on $\mathbf{G}'(\mathbb{A})$, and the right side on a choice of a Haar measure dg on $\mathbf{G}(\mathbb{A})$, defined using a nondegenerate F-rational differential form of maximal degree on $\mathbf{G}$, which yields such a form on the F-group $\mathbf{G}' = \mathrm{R}_{E/F}\,\mathbf{G}$. These measures are sometimes suppressed to simplify the notations.

By the rigidity theorem for $\mathbf{G}'(\mathbb{A})$ at most one of the terms Φ_i is nonzero, and consists of a single contribution. Here

$$\Phi_1 = \sum_{\Pi} \prod_{v \in V} \operatorname{tr} \Pi_v(\phi_v dg'_v \times \sigma).$$

The sum is over the σ-invariant discrete-spectrum (by which we mean automorphic in the discrete spectrum) $\mathbf{G}'(\mathbb{A})$-modules Π. These are the (σ-invariant) cuspidal or one-dimensional $\mathbf{G}'(\mathbb{A})$-modules. Next

$$\Phi_2 = \sum_{\tau} \prod_{v \in V} \operatorname{tr} I(\tau_v \otimes \kappa_v; \phi_v dg'_v \times \sigma).$$

The sum is over the σ-invariant discrete-spectrum (i.e. cuspidal or one-dimensional) $\mathbf{H}'(\mathbb{A})$-modules τ which are obtained by the stable basechange map b'' in [F3;II]. Further

$$\Phi_3 = \sum \prod_{v \in V} \operatorname{tr} I((\mu, \mu', \mu''); \phi_v dg'_v \times \sigma).$$

Here the sum is over the distinct unordered triples μ, μ', μ'' of characters of C_E/C_F.

On the right,

$$F_1 = \sum_{\pi} m(\pi) \prod_{v \in V} \operatorname{tr} \pi_v(f_v dg_v).$$

The sum is over the equivalence classes of discrete-spectrum (automorphic) $\mathbf{G}(\mathbb{A})$-modules π. They occur with finite multiplicities $m(\pi)$. Next

$$F_2 = \sum_{\rho \neq \rho(\theta, '\theta)} \prod_{v \in V} \operatorname{tr}\{\rho_v\}('f_v dh_v).$$

The sum ranges over the (automorphic) discrete-spectrum packets ρ of $\mathbf{H}(\mathbb{A})$ which are not of the form $\rho(\theta,\ '\theta)$ (see [F3;II]). These packets ρ are cuspidal or one dimensional (see [F3;II]). Also

$$F_3 = \sum_{\rho = \rho(\theta, '\theta)} \prod_{v \in V} \operatorname{tr}\{\rho_v\}('f_v dh_v).$$

The sum ranges over the packets $\rho = \rho(\theta, '\theta)$, where θ, $'\theta$ and $\omega/\theta \cdot {}'\theta$ are distinct. Further

$$F_6 = \sum_{\mu} \prod_{v \in V} \operatorname{tr} R(\mu_v) I(\mu_v; f_v dg_v) - \sum_{\rho} \prod_{v \in V} \operatorname{tr}\{\rho_v\}('f_v dh_v).$$

The first sum is over the characters μ of C_E/C_F with $\mu^3 \neq \omega'$. The second is over the packets $\rho = \rho(\theta, \omega/\theta^2)$, where $\theta^3 \neq \omega$.

1.2 Coinvariants

We shall use the result of [C1], [D6] and I.4 to study the following local identity. Here E/F is an extension of local p-adic fields. Suppose that $\{\rho\}$ is a square-integrable H-module, and $m(\rho, \pi)$, c and c' are complex numbers, where π are (equivalence classes of) unitarizable G-modules, and the sum $\sum_\pi m(\rho,\pi)\,\mathrm{tr}\,\pi(fdg)$ is absolutely convergent. Moreover, suppose that this sum ranges over a countable set S which has the following property. For every open compact subgroup K_1 of G there is a finite set $S(K_1)$ such that $\mathrm{tr}\,\pi(fdg)=0$ for every π in $S-S(K_1)$ and every K_1-biinvariant f. Suppose that for all matching $(\phi dg',\ fdg,\ 'fdh)$ we have

$$c\,\mathrm{tr}\,I(\tau\otimes\kappa;\phi dg'\times\sigma)+c'\,\mathrm{tr}\{\rho\}('fdh)=\sum_{\pi\in S}m(\rho,\pi)\,\mathrm{tr}\,\pi(fdg), \tag{1.2.1}$$

where τ is the stable basechange lift of $\{\rho\}$. In this case we have

1.2.1 Proposition. *(i) The set S consists of* (1) *square-integrable but not Steinberg G-modules, and* (2) *proper submodules of G-modules induced from a unitary character of A.*
(ii) If $\{\rho\}$ is cuspidal then the π of (1) *are cuspidal.*
(iii) If $\{\rho\}$ is Steinberg then precisely one π of (1) *is not cuspidal. It is the Steinberg subquotient of an induced G-module $I(\mu\kappa\nu^{1/2})$.*
(iv) If the $m(\rho,\pi)$ are all positive then the π are all square integrable.

Remark. (a) Then π mentioned in (2) above are not square integrable, since their central exponents do not decay. They exist, and are described in I.4, but we need not use this fact. (b) In (iii), $\nu(x)=|x|$ and μ is a (unitary) character of $E^\times$ trivial on $F^\times$. Our proof implies that if the identity (1.2.1) exists, then $I(\mu\kappa\nu^{1/2})$ is reducible. In this way, we recover a result of Keys [Ke], recorded in I.4. In I.4, we give a complete list of reducible induced G-modules. There we quote the work of Keys [Ke]. Our work here gives an alternative proof that the list describes all reducible induced G-modules.

Proof. Let η be a character of $E^\times$. For every $n\geq 1$ let f_n be a function which is supported on the conjugacy classes of $\mathrm{diag}(\alpha,\beta,\overline{\alpha}^{-1})$ with $|\alpha|=q^n$, with $F(a,f_n)=\eta(\alpha)+\eta(\overline{\alpha}^{-1})$ if $a=\mathrm{diag}(\alpha,1,\overline{\alpha}^{-1})$ with $|\alpha|=q^{-n}$. If $\{\rho\}$ is cuspidal then $\{\rho_N\}$ is zero and so is $I(\tau\otimes\kappa)_N$ as an $A\times\langle\sigma\rangle$-module, that is, $I(\tau\otimes\kappa)_N$ has no σ-invariant irreducible constituents, and so $\mathrm{tr}\,I(\tau\otimes\kappa)_N(f_N\times\sigma)=0$ for all f_N. We omit the Haar measure da from the notations.

If ρ is Steinberg then $I(\tau\otimes\kappa)_N$ has a single σ-invariant exponent, which satisfies

$$\mathrm{tr}[I(\tau\otimes\kappa)_N](\phi_N\times\sigma)=\mathrm{tr}\{\rho\}_N('f_N)$$

for any triple $(\phi dg', fdg, {}'fdh)$ of matching measures, where f is in the span of the f_n, $n \geq 1$. In particular, (1.2.1) takes the form

$$(c' + c)\,\mathrm{tr}\{\rho\}_N({}'f_N) = \sum_\pi m(\rho, \pi)\,\mathrm{tr}\,\pi_N(f_N) \qquad (1.2.2)$$

for fdg as above. It is clear that there exists a compact open subgroup K_1 of G, depending only on the restriction of η to the group $R_E^\times$ of units in $E^\times$, such that f can be chosen to be K_1-biinvariant. Hence the sum in (1.2.2) is finite. Applying linear independence of finitely many characters of the form $n \mapsto z^n$, the proposition follows once we make the following observation. Since G is of rank one, the composition series of an induced representation is at most of length two. Thus if π and π' are irreducible inequivalent G-modules which have equal central exponent, then they are the (only) constituents of a reducible G-module $I(\eta)$ induced from a character η of A with $\eta(a) = \eta(JaJ^{-1})$. Namely the composition series of $I(\eta)_N$ consists of two equal characters, necessarily unitary. Then $\mathrm{tr}\,\pi_N(f_N) = \mathrm{tr}\,\pi'_N(f_N)$, and $m(\rho, \pi)\,\mathrm{tr}\,\pi_N(f_N) + m(\rho, \pi')\,\mathrm{tr}\,\pi'_N(f_N)$ is zero if $m(\rho, \pi) + m(\rho, \pi')$ is zero. If $m(\rho, \pi)$ and $m(\rho, \pi')$ are both positive then their central exponents cannot cancel each other, and (iv) follows. □

REMARK. We have $m(\rho, \pi) = c + c'$ for the noncuspidal (Steinberg) π of (iii).

1.3 Global from local

Given a square-integrable local representation, we wish to create a global cuspidal representation with this component, in order to use the global trace formula in the study of the local lifting. A key tool is the existence of a pseudo-coefficient, constructed by Kazhdan in [K2]. We recall this first.

Let G be a connected reductive p-adic group. Each irreducible representation π is the subquotient of a representation $I(\tau\nu^s)$ induced parabolically and normalizedly from a cuspidal representation τ with unitary central character, of a Levi subgroup M of a parabolic subgroup P, twisted by an unramified character ν^s of M. The data (M, τ) is uniquely determined by π up to conjugation in G.

DEFINITION. Let π be a square-integrable irreducible representation of a connected reductive p-adic group G. A *pseudo-coefficient* of π is a locally constant function on G which transforms under the center of G by the inverse of the central character of π and is compactly supported modulo center, such that $\operatorname{tr}\pi(f dg) = 1$ and $\operatorname{tr}\pi'(f dg) = 0$ for every properly induced representation π' and for every irreducible representation π' which is not a subquotient of any $I(\tau\nu^s)$, any s, determined by π.

In [K2], Kazhdan proves the existence of a pseudo-coefficient of any square-integrable representation. A σ-twisted analogue of these definition and result are as follows. A *twisted pseudo-coefficient* ϕ of a σ-invariant σ-elliptic (its σ-character is not identically zero on the σ-elliptic regular set) representation Π of a connected reductive p-adic group G is a locally constant function on G which transforms under the center of G by the inverse of the central character of Π and is compactly supported modulo center, such that $\operatorname{tr}\Pi(\phi dg) = 1$ and $\operatorname{tr}\Pi'(\phi dg) = 0$ for every properly induced representation Π' and for every irreducible representation Π' which is not a subquotient of any $I(\tau\nu^s)$, any s, with constituent Π. The proof of [K2] extends to show the existence of twisted pseudo-coefficients.

Here is a variant of standard construction.

1.3.1 PROPOSITION. *Let $\{\rho'\}$ be a packet of square-integrable representations of $H = \mathrm{U}(2)$ associated with a local quadratic extension. Then there exists a global quadratic extension E/F which splits at each archimedean place and which is the chosen local quadratic extension at a place w of F, and a global packet $\{\rho\}$ of discrete-spectrum (i.e., containing an automorphic in the discrete spectrum) representations of $\mathbf{H}(\mathbb{A})$ whose component at the place w of F is the packet $\{\rho'\}$, at a place w' the component is cuspidal, at other finitely many finite places w_i of F the component is preassigned square integrable, and at all other finite places the component be fully induced, even unramified at all split (in E) such places and those places of F unramified in E.*

PROOF. Fix a quadratic extension of global fields where F has no real places, such that for some place w of F the completion E_w/F_w is the local quadratic extension of the proposition. Denote by $\mathbf{Z}'$ the center of $\mathrm{R}_{E/F}\,\mathrm{GL}(2)$. Let $\mathbf{H}_1(\mathbb{A})$ be the group of g in $\mathrm{GL}(2,\mathbb{A})$ with determinant in $N_{E/F}\mathbb{A}_E^\times$. Using the relation $\mathbf{Z}'(\mathbb{A})\mathbf{H}_1(\mathbb{A}) = \mathbf{Z}'(\mathbb{A})\mathbf{H}(\mathbb{A})$, it suffices to show the existence of a cuspidal representation of $\mathrm{GL}(2,\mathbb{A})$ with prespecified square-integrable components at the finite places w, w', w_i, which is unramified at all other finite places. Note that the component at w' is cuspidal.

This can easily be done for $\mathrm{GL}(n, \mathbb{A})$, provided the number of w_i is at least $n-1$.

In this case we write the trace formula for a test measure $f dg = \otimes_v f_v dg_v$ where the component $f_{w'} dg_{w'}$ is a (normalized by $\operatorname{tr} \pi_{w'}(f_{w'} dg_{w'}) = 1$) coefficient of the cuspidal $\pi_{w'}$, $f_w dg_w$ and $f_{w_i} dg_{w_i}$ are pseudo-coefficients of discrete-series representations which we choose at these places, and the other $f_v dg_v$ for finite v are taken to be spherical.

We can take the support of some of these $f_v dg_v$ to be sufficiently large so that $\otimes_{v<\infty} f_v dg_v$ has orbital integral nonzero at some rational elliptic regular element γ. We choose the nonarchimedean components so that the orbital integral of f is nonzero at γ, but these components vanish at all other rational conjugacy classes, and on the singular set. Note that the set of characteristic polynomials of rational conjugacy classes (of $\mathrm{GL}(n, F)$ in $\mathrm{GL}(n, \mathbb{A})$) is discrete ($F^n$ in $\mathbb{A}^n$), and the support of f is compact (and it is easy to adjust this "discrete and compact is finite" argument to take the center into account).

As f has $n+1$ elliptic components, the trace formula for f has no weighted orbital integrals. It has no singular orbital integrals by the choice of the archimedean components. The geometric side of the trace formula then reduces to a single nonzero term: $\Phi(\gamma, f dg) \neq 0$.

As the component $f_{w'}$ is cuspidal, the convolution operator $r(f dg)$ on L^2 factorizes through the cuspidal spectrum. Hence the spectral side of the trace formula for f consists only of traces of cuspidal representations. This sum is nonzero, since so is the other, geometric, side: $\sum_{\pi \subset L_0} \operatorname{tr} \pi(f dg) = \Phi(\gamma, f dg) \neq 0$.

If π occurs in the sum, thus $\operatorname{tr} \pi(f dg) \neq 0$, then the component at w' is the chosen cuspidal representation, since $f_{w'}$ is a coefficient thereof: $\operatorname{tr} \pi_{w'}(f_{w'} dg_{w'}) = 1$. Hence π is cuspidal, and its components at w, w_i, are the prechosen discrete-series representations, since the f_w, f_{w_i} are pseudo-coefficients. Since $f_v dg_v$ at the other finite places are spherical, the components of π are unramified, hence fully induced, as required. □

1.4 Local identity

Let E/F be a quadratic extension of local nonarchimedean fields. Let $\{\rho\}$ be a square-integrable $\mathbf{H}(\mathbb{A})$-packet, and τ its stable basechange lift.

1.4.1 Proposition. *For every square-integrable* $\mathbf{G}(\mathbb{A})$*-module* π *there exists a nonnegative integer* $m(\rho,\pi)$ *such that for every triple* $(\phi dg',\ fdg,\ 'fdh)$ *of matching measures we have the identity*

$$\operatorname{tr}\{\rho\}('fdh) + \operatorname{tr} I(\tau\otimes\kappa;\phi dg'\times\sigma) = 2\sum_{\pi} m(\rho,\pi)\operatorname{tr}\pi(fdg). \quad (1.4.1)$$

Proof. Let $\{\rho'\}$ be a cuspidal packet as constructed in Proposition 1.3.1, where E/F is a totally imaginary quadratic extension which localizes at a place w to our local quadratic extension. This $\{\rho'\}$ has the cuspidal packet $\{\rho\}$ as its component at w. At sufficiently many split in E places of F we construct $\{\rho'\}$ to have cuspidal components, as well as a Steinberg component. This last requirement will guarantee that no terms of F_3 and F_6 of Proposition 1.1 will occur, when it is applied with $\{\rho'\}$ making a contribution to the term F_2. There is then a corresponding contribution at Φ_2. Other possible contributions may occur only in F_1. Since the local lifting is available at the split places and where the components are properly induced, in particular unramified, we obtain the identity of the proposition on applying a standard argument of "generalized linear independence of characters".

The fact that only square-integrable π occur on the right of (1.4.1) follows from Proposition 1.2.1. It can be used by a well-known fact about the space of automorphic forms with fixed infinitesimal character and ramification at all finite places, namely that this space is finite dimensional. □

Since $fdg = 0$ implies $'fdh = 0$, the Proposition has the following

Corollary. *Let* ρ *be a square-integrable representation of* $\mathrm{U}(2,E/F)$*, local quadratic field extension* E/F*, and* τ *its stable basechange lift to* $\mathrm{GL}(2,E)$*. Then the* σ*-twisted character of* $I(\tau\otimes\kappa)$ *is stable*: $\operatorname{tr} I(\tau\otimes\kappa;\phi dg'\times\sigma)$ *is zero for any test measure* $\phi dg'$ *whose* σ*-stable orbital integrals are zero, that is, if* $\phi dg'$ *matches* $fdg = 0$*.*

Our next aim will be to show that the sum of (1.4.1) is finite. For that we need a basechange result.

1.4.2 Proposition. *Let* E/F *be a local quadratic field extension,* ρ *a square-integrable representation of* $\mathrm{U}(2,E/F)$*, and* τ *its stable basechange lift to* $\mathrm{GL}(2,E)$*. Then for each square-integrable* π *on* $\mathrm{U}(3,E/F)$ *there is* *a nonnegative integer* $m'(\rho,\pi)$ *such that*

$$\operatorname{tr} I(\tau\otimes\kappa;\phi dg'\times\sigma)=\sum_{\pi} m'(\rho,\pi)\operatorname{tr}\pi(f dg). \tag{1.4.2}$$

The sum is finite, the π *are square integrable.*

Proof. This is essentially the same as that of Proposition 1.2. But instead we use the twisted trace formula. Again we work with a totally imaginary number field F such that the completion of E/F at a place w is our local extension, and with a test function ϕ as follows. At the place w we take $\phi_w dg'_w$ to be a twisted pseudo-coefficient of our $I(\tau\otimes\kappa)$. At sufficiently many places $v\in\{w_i\}$ of F which split in E we take the component to be $(\phi_v dg'_v, \phi_v^0 dg'_v)$, $\phi_v dg'_v$ is a coefficient of a cuspidal representation of $\mathrm{GL}(3,F_v)$, $\phi_v^0 dg'_v$ is an idempotent in the Hecke algebra with $\phi_v dg'_v * \phi_v^0 dg'_v = \phi_v dg'_v$. At all other finite places v we take a spherical function. The choice of the σ-stable component at w guarantees that the twisted trace formula for $\phi dg'=\otimes_v\phi_v dg'_v$ is σ-stable. We can choose the spherical components so that there is a rational σ-regular elliptic element $\delta\in\mathrm{GL}(3,E)$ with $\Phi^{\mathrm{st},\sigma}(\delta,\phi dg')\neq 0$, and then choose the components of ϕ at the archimedean places to vanish on the σ-singular set, and such that $\Phi^{\mathrm{st},\sigma}$ vanishes on any σ-regular stable conjugacy class other than δ. Such δ exists since ϕ_w is a σ-stable function: its σ-orbital integrals are σ-stable. This shows that the geometric side of the twisted trace formula reduces to a single term, $\Phi^{\mathrm{st},\sigma}(\delta,\phi dg')\neq 0$. As $\phi dg'$ has cuspidal components the convolution operator $r(\phi dg'\times\sigma)$ factorizes through the cuspidal spectrum. The spectral side is a sum of $\operatorname{tr}\Pi(\phi dg'\times\sigma)$, Π cuspidal. These Π will be unramified at all places but w, where the component be that of our proposition, and w_i, places which split in E, where the component be cuspidal.

We can now apply the trace formulae identity of Proposition 1.1 with Π as constructed here in the term Φ_1 on the left. This will be the only term on the left, while the terms on the right can only occur in F_1. Applying once again "generalized linear independence of characters", noting that the lifting is known for the places which split, the identity of the proposition follows.

The π which occur are square integrable by the proof of Proposition 1.2. It remains to apply orthogonality relations for characters — this is discussed in detail in the following sections. □

Putting (1.4.1) and (1.4.2) together we obtain

Corollary. *For ρ as in the Proposition, we have*

$$\operatorname{tr}\{\rho\}('f dh) = \sum_{\pi} m''(\rho,\pi) \operatorname{tr}\pi(f dg), \tag{1.4.3}$$

where $m''(\rho,\pi) = 2m(\rho,\pi) - m'(\rho,\pi)$ is an integer, which need not be positive.

Note that the right side of (1.4.3) is not yet known at this stage to be finite, but it is independent of the orbital integrals of fdg on the cubic tori of G.

III.2 Separation

2.1 Transfer

In this section we study a transfer $'D \to {}'D_G$ of distributions which is dual to the transfer $f \to {}'f$ of orbital integrals from $G = \mathrm{U}(3, E/F)$ to $H = \mathrm{U}(2, E/F)$. Here $\tilde{H} = Z_G(\operatorname{diag}(1,-1,1)) = HZ$, and H is viewed as the subgroup of (a_{ij}) in G with $a_{ij} = 0$ if $i + j$ is odd and with $a_{22} = 1$. This study is used to conclude that the sum of (1.4.3) (and of (1.4.2), hence of (1.4.1)) is finite.

Definition. (1) A distribution $'D$ on H is called *stable* if $'D('f)$ depends only on the stable orbital integrals of $'f$.
(2) A function $'f$ on H extends uniquely to a function $'\widetilde{f}$ on $\widetilde{H}$ with $'\widetilde{f}(zh) = \omega^{-1}(z) \cdot {}'\widetilde{f}(h)$ (z in Z, h in $\widetilde{H}$). A distribution $'D$ on H extends to $'\widetilde{D}$ on $\widetilde{H}$ by $'\widetilde{D}(\widetilde{f}) = {}'D(f)$.
(3) Given a stable distribution $'D$ on H, let $'D_G$ be the distribution on G with $'D_G(f) = {}'D('f)$ $(= {}'\widetilde{D}('\widetilde{f}))$, where $'f$ is a function on H matching f.

Remark. (1) The set $W'(T)/W(T)$ embeds as a subset of $C(\mathbf{T}/F)$ via the map

$$w \mapsto \mathbf{w} = \{\tau \mapsto w_\tau = \tau(w)w^{-1}; \tau \in \operatorname{Gal}(\overline{F}/F)\}.$$

(2) The group $W'(T)$ acts on $C(\mathbf{T}/F)$. If w lies in $W'(T)$, and δ in $C(\mathbf{T}/F)$ is represented by $\{g_\tau = \tau(g)g^{-1}\}$ with g in $A(\mathbf{T}/F)$, then

$$\begin{aligned} w(\delta) &= \mathbf{w}^{-1} \cdot \{(wg)_\tau\}(= w\tau(w)^{-1} \cdot \tau(wg)(wg)^{-1}\} \\ &= \{w\tau(g)g^{-1}w^{-1}\} = w\delta w^{-1} \in C(\mathbf{T}/F). \end{aligned}$$

(3) Let d be a locally-integrable conjugacy invariant complex-valued function on G with $d(zg) = \omega(z)d(g)(z \text{ in } Z)$. Note that the regular set G^{reg} of G has the form $G^{\mathrm{reg}} = \cup_{\{T\}} \cup_{g\in G/T} gT^{\mathrm{reg}}g^{-1}$. Here $\{T\}$ indicates a set of representatives T for the conjugacy classes of tori in G. Using the Jacobian $\Delta^2(t) = |\det(1 - \mathrm{Ad}(t))|\mathfrak{g}/\mathfrak{t})|$ we obtain the Weyl integration formula

$$\int_{G/Z} f(g)d(g)dg = \sum_{\{T\}}[W(T)]^{-1} \int_{T/Z} \Delta(t)^2\Phi(t, fdg)d(t)dt.$$

Suppose that t is a regular element of G which lies in T. Then the number of δ in $C(\mathbf{T}/F)$ such that t^δ is conjugate to an element of T is $[W'(T)]/[W(T)]$. If the function d is invariant under stable conjugacy then we have

$$\int_{G/Z} f(g)d(g)dg = \sum_{\{T\}_s}[W'(T)]^{-1} \int_{T/Z} \Delta(t)^2\Phi^{\mathrm{st}}(t, fdg)d(t)dt.$$

Here $\{T\}_s$ is a set of representatives for the stable conjugacy classes of tori in G.

If $'\widetilde{d}$ is a locally-integrable stable function on $\widetilde{H}$ then

$$\int_{\widetilde{H}/Z} {}'\widetilde{f}(h)\cdot {}'\widetilde{d}(h)dh = \sum_{\{T_H\}_s}[W'(T_H)]^{-1} \int_{T_H} \Delta'(t)^2\Phi^{\mathrm{st}}(t, {}'\widetilde{f}dh)\cdot {}'\widetilde{d}(t)dt.$$

The set $\{T_H\}_s$ is a set of representatives for the stable conjugacy classes of tori in H. The symbol $W'(T_H)$ indicates the Weyl group in $A(\mathbf{T}_H/F)$. It consists of two elements.

As in I.2, $\Phi^{\mathrm{st}}(t, {}'\widetilde{f}dh)$ denotes the stable orbital integral of $'\widetilde{f}dh$, and $\Phi^{\mathrm{st}}(t, fdg)$ is that of fdg. In fact the orbital integral Φ is taken over H/T_H or G/T against the measure dh/dt or dg/dt, but we omit dt to simplify the notations. Since T_H and T/Z are isomorphic, we take the corresponding measures dt to equal each other under this isomorphism.

2.2 Orthogonality

Denote by S the torus of G specified in Proposition I.1.3 as T^* in type (0), T_1 in type (1), T_H in type (2). They all lie in HZ. Denote by S_H the corresponding torus of H.

2.2.1 Proposition. *Suppose that* $'\widetilde{D}$ *is a stable distribution on* $\widetilde{H}$ *represented by the locally-integrable (stable) function* $'\widetilde{d}$. *Then the corresponding distribution* $'D_G$ *on* G *is given by a locally-integrable function* $'d_G$ *defined on the regular set of* G *by* $'d_G(t) = 0$ *if* t *lies in a torus of type* (3), *and by*

$$\Delta(t) \cdot {}'d_G(t^\delta) = \sum_w \boldsymbol{\kappa}(w(t))\Delta'(w(t))\kappa(\mathbf{w})\kappa(w(\delta)) \cdot {}'\widetilde{d}(w(t)) \quad (2.2.1)$$

if t *lies in the chosen torus* S *of type* (0), (1) *or* (2), *and* δ *lies in* $C(\mathbf{S}/F)(= B(\mathbf{S}/F))$. *Here* $w(t) = wtw^{-1}$. *The sum ranges over* $W'(S_H)\backslash W'(S)$.

Proof. Fix $i = 0$, 1 or 2, and let S be the distinguished torus of type (i). Let δ be an element of $B(\mathbf{S}/F)$, g a representative of δ in $A(\mathbf{S}/F)$, and $T = S^\delta = g^{-1}Sg$ the associated torus. Let f be a function on the regular set of G such that $\Phi(t, fdg)$ is zero unless a conjugate of t lies in T. Then

$$\begin{aligned}'D_G(f) = {}'\widetilde{D}(\widetilde{f}) &= [W'(S_H)]^{-1} \int_{S/Z} \Delta'(t)^2 \Phi^{\mathrm{st}}(t, {}'\widetilde{f}dh) \cdot {}'d(t)dt \\ &= [W'(S_H)]^{-1} \int_{S/Z} \Delta'(t)[\boldsymbol{\kappa}(t)\Delta(t) \sum_{\delta'} \kappa(\delta')\Phi(t^{\delta'}, fdg)] \cdot {}'d(t)dt.\end{aligned}$$

The sum ranges over all δ' in $B(\mathbf{S}/F)$ such that $S^{\delta'} = T$. Thus δ' is represented by wg (i.e. $\delta' = \{(wg)_\tau = \tau(wg)(wg)^{-1}\}$), where w ranges over $W'(S)/gW(T)g^{-1}$. Since κ is trivial on the image of $B(\mathbf{S}_H/F)$ in $B(\mathbf{S}/F)$, we obtain $[W(T)]^{-1}$ times

$$\int_{S/Z} \Delta(t)\boldsymbol{\kappa}(t)\Delta'(t)\Big[\sum_w \kappa(\mathbf{w}\cdot w(\delta))\Phi((w^{-1}tw)^\delta, fdg)\Big]{}'\widetilde{d}(t)dt$$

$$= \int_{S/Z} \Big[\sum_w \boldsymbol{\kappa}(w(t))\Delta'(w(t))\kappa(\mathbf{w})\kappa(w(\delta)) \cdot {}'\widetilde{d}(w(t))\Big]\Delta(t)\Phi(t^\delta, fdg)dt.$$

Here w ranges over $W'(S_H)\backslash W'(S)$. By definition of $'d_G$ this is equal to

$$= [W(T)]^{-1} \int_{T/Z} \Delta(t)^2\Phi(t, fdg) \cdot {}'d_G(t)dt = \int_{G/Z} f(g) \cdot {}'d_G(g)dg;$$

hence the proposition follows. ∎

DEFINITION. (1) Let d, d' be conjugacy invariant functions on the elliptic set of G. Put

$$\langle d, d' \rangle = \sum_{\{T\}_e} [W(T)]^{-1} \int_{T/Z} \Delta(t)^2 d(t) \overline{d}'(t) dt$$
$$= \sum_{\{T\}_{e,s}} [W'(T)]^{-1} \sum_{\delta \in B(\mathbf{T}/F)} \int_{T/Z} \Delta(t)^2 d(t^\delta) \overline{d}'(t^\delta) dt.$$

Here $\{T\}_e$ (resp. $\{T\}_{e,s}$) is a set of representatives for the (resp. stable) conjugacy classes of elliptic tori T in G.
(2) Let $'d$, $'d'$ be stable conjugacy invariant functions of the elliptic set of H. Put

$$'\langle 'd, 'd' \rangle = \sum_{\{T_H\}_{e,s}} \frac{[B(\mathbf{T}_H/F)]}{[W'(T_H)]} \int_{T_H} \Delta'(t)^2 \cdot 'd(t) \cdot '\overline{d}(t) dt.$$

Here $\{T_H\}_{e,s}$ is a set of representatives for the stable conjugacy classes of elliptic tori in H.

2.2.2 PROPOSITION. *Let $'d$, $'d'$ be stable functions on (the elliptic set of) $\widetilde{H}$, and $'d_G$, $'d'_G$ the associated class functions on (the elliptic set of) G. Then*

$$\langle 'd_G, 'd'_G \rangle = 2 \cdot '\langle 'd, 'd' \rangle.$$

PROOF. By (2.2.1) we have

$$\langle 'd_G, 'd'_G \rangle = \sum_{\{S\}} \sum_{\delta \in C(\mathbf{S}/F)} [W'(S)]^{-1} \int_{S/Z} \sum_{w, w' \in W'(S_H) \backslash W'(S)} \boldsymbol{\kappa}(w(t)) \boldsymbol{\kappa}(w'(t))$$
$$\Delta'(w(t)) \Delta'(w'(t)) \kappa(\mathbf{w}) \kappa(\mathbf{w}') '\widetilde{d}(w(t)) '\overline{\widetilde{d}'}(w'(t)) \kappa(w(\delta)) \kappa(w'(\delta)).$$

Note that κ is a character of order 2. Here S ranges over the set of (conjugacy classes of) distinguished tori in G of type (1) and (2). The group $W'(S_H) \backslash W'(S)$ acts simply transitively on the set of nontrivial characters of $C(\mathbf{S}/F)$. Hence $\sum_\delta \kappa(w(\delta)) \kappa(w'(\delta)) \neq 0$ implies that $\kappa(w(\delta)) = \kappa(w'(\delta))$

for all δ and that $w = w'$. Changing variables we conclude that

$$\begin{aligned}\langle 'd_G, 'd_G\rangle &= \sum_{\{S\}} \frac{[C(\mathbf{S}/F)]}{[W'(S_H)]} \int\limits_{S/Z} \Delta'(t)^2 \cdot '\widetilde{d}(t) \cdot \overline{'\widetilde{d}'}(t)dt \\ &= 2 \sum_{\{T_H\}_e} \frac{[C(\mathbf{T}_H/F)]}{[W'(T_H)]} \int\limits_{T_H} \Delta'(t)^2 \cdot 'd(t) \cdot \overline{'d}'(t)dt \\ &= 2 \cdot '\langle 'd, 'd'\rangle.\end{aligned}$$

Here we used the relation $[C(\mathbf{T}/F)] = 2[C(\mathbf{T}_H/F)]$ for tori T of type (1) or (2). The proposition follows. □

Definition. (1) Let d be a conjugacy invariant function on the elliptic set G_e of G. Define d_H to be the stable function on the elliptic set $\widetilde{H}_e$ of $\widetilde{H}$ with

$$\Delta'(t)d_H(t) = \Delta(t)\boldsymbol{\kappa}(t) \sum_{\delta\in B(\mathbf{S}/F)} \kappa(\delta)d(t^\delta)$$

on the t in S, where S is a distinguished torus of type (1) or (2) in $\widetilde{H}$.

2.2.3 Proposition. (1) *If d is a conjugacy invariant function on G_e and $'d$ is a stable function on H_e, both locally integrable, then $\langle d, 'd_G\rangle = '\langle d_H, 'd\rangle$.*
(2) *The locally-integrable class function d on G_e is stable if and only if $d_H = 0$, and if and only if $\langle d, \chi(\{\rho\})_G\rangle$ vanishes for every square-integrable H-packet $\{\rho\}$. Here $\chi(\{\rho\})$ is the sum of the characters of the* (*one or two*) *irreducible H-modules in $\{\rho\}$.*

Proof. By (2.2.1) the inner product $\langle d, 'd_G\rangle$ is equal to

$$\begin{aligned}&\sum_{\{S\}} \sum_{\delta\in B(\mathbf{S}/F)} [W'(S)]^{-1} \int\limits_{S/Z} \Delta(t)d(t^\delta) \sum_w \overline{\boldsymbol{\kappa}}(w(t))\Delta'(w(t))\kappa(\mathbf{w})\kappa(w(\delta))\overline{'d}(w(t)) \\ &= \sum_{\{S\}}\sum_{\delta}[W'(S)]^{-1} \int\limits_{S/Z} \Delta(t)\Delta'(t)\overline{\boldsymbol{\kappa}}(t)\Big[\sum_w \kappa(\{(wg)_\tau\})d((w^{-1}tw)^\delta)\Big]'\overline{d}(t)dt \\ &= \sum_{\{S\}}[W'(S_H)]^{-1} \int\limits_{S/Z} \Delta(t)\Delta'(t)\overline{\boldsymbol{\kappa}}(t)\Big[\sum_\delta \kappa(\delta)d(t^\delta)\Big]'\overline{d}(t)dt \\ &= \sum_{\{T_H\}_e} [W'(T_H)]^{-1} \int\limits_{T_H} \Delta(t)^2 d_H(t)\overline{'d}(t)dt = '\langle d_H, 'd\rangle,\end{aligned}$$

where w ranges over $W'(S_H)\backslash W'(S)$, and (1) follows. For (2), note that $d_H = 0$ if and only if $d_H(w^{-1}tw) = 0$ for every T, t in T and w in $W'(T)$, and $W'(T)$ acts transitively on the set of nontrivial characters of $C(\mathbf{T}/F)$. Hence d is stable if and only if $d_H = 0$. Now the $\chi(\{\rho\})$ make a basis for the space of stable functions on the elliptic set of H, hence $d_H = 0$ if and only if $'\langle d_H,\, \chi(\{\rho\})\rangle = 0$ for all square-integrable H-packets $\{\rho\}$, as required. □

2.2.4 PROPOSITION. *The sum of* (1.4.3) *is finite.*

PROOF. Numbering the countable set of π in (1.4.3) with $m''(\rho,\pi) \neq 0$ we rewrite (1.4.3) in the form $\operatorname{tr}\{\rho\}('fdh) = \sum_{1\le i\le b} m_i \operatorname{tr}\pi_i(fdg)$, where $1 \le b \le \infty$. The m_i are nonzero integers, and the π_i are square integrable. For each i in the sum let $f_i dg$ be the product of a pseudo-coefficient of π_i with $m_i/|m_i|$. For any finite a $(1 \le a \le b)$ put $f^a dg = \sum^a f_i dg$, where $\sum^a$ indicates the sum over i $(1 \le i \le a)$. Then

$$\begin{aligned} a^2 &\le \left(\sum^a |m_i|\right)^2 = \left(\sum^a m_i \operatorname{tr}\pi_i(f^a dg)\right)^2 = (\operatorname{tr}\{\rho\}('f^a dg))^2 \\ &= \left\langle \chi(\{\rho\})_G, \sum^a \chi_i m_i/|m_i| \right\rangle^2 \le \langle \chi(\{\rho\})_G, \chi(\{\rho\})_G\rangle \left\langle \sum^a \chi_i, \sum^a \chi_i \right\rangle \\ &= 2a \cdot {}'\langle \chi(\{\rho\}), \chi(\{\rho\})\rangle = 2a[\{\rho\}], \end{aligned}$$

where $[\{\rho\}]$ is the number of irreducibles in the H-packet $\{\rho\}$, and χ_i is the character of π_i. Then $a \le 2[\{\rho\}]$, and the proposition follows. □

In fact, we also proved the

COROLLARY. *The sum of* (1.4.3) *extends over at most two* π *if* $[\{\rho\}] = 1$ *and four* π *if* $[\{\rho\}] = 2$. *The coefficient* $m''(\rho,\pi)$ *are bounded by two in absolute value, and they are equal to one in absolute value if there are at least two* π *in the sum.*

2.3 Evaluation

Let E/F be a quadratic extension of nonarchimedean local fields.

Our next aim is to evaluate the integers $m''(\rho,\pi)$ and $m'(\rho,\pi)$ which appear in (1.4.2) and (1.4.3), and describe the π which occur in these sums. Recall ([F3;II]) that a packet $\{\rho\}$ of square-integrable H-modules consists of a single element, unless it is associated with two distinct characters θ, $'\theta$ of E^1. In the last case $\{\rho\}$ is denoted by $\rho(\theta, '\theta)$. It consists of two cuspidal H-modules. In Corollary 2.2.4 it is shown that the sum of (1.4.3) consists of at most $2[\{\rho\}]$ elements.

2.3.1 PROPOSITION. *The sum in* (1.4.3) *consists of* $2[\{\rho\}]$ *terms. The coefficients* $m''(\rho,\pi)$ *are equal to* 1 *or* -1, *and both values occur for each* ρ.

PROOF. Put $\theta_\rho = \chi(\{\rho\})_G$. Put θ_τ for the (twisted) character of $I(\tau \otimes \kappa)$ (of (1.3.2)), viewed as a stable (conjugacy) function on G. Consider the inner product

$$\langle \theta_\rho, \theta_\tau \rangle = \left\langle \sum_\pi m''(\rho,\pi)\chi_\pi, \sum_{\pi'} m'(\rho,\pi')\chi_{\pi'} \right\rangle = \sum_\pi m''(\rho,\pi)m'(\rho,\pi).$$

By (2.2.1), since θ_τ is a stable function $\langle \theta_\rho, \theta_\tau \rangle$ is equal to

$$\sum_{\{S\}}[W(S)]^{-1} \sum_{\delta\in C(\mathbf{S}/F)} \int_{S/Z} (\Delta\overline{\theta}_\tau)(t) \sum_{w\in W'(S_H)\backslash W'(S)}$$

$$\boldsymbol{\kappa}(w(t))\Delta'(w(t))\kappa(\mathbf{w})\kappa(w(\delta))\widetilde{\chi}(\{\rho\})(w(t))dt.$$

Since κ is a nontrivial character of the group $C(\mathbf{S}/F)$, we have

$$\sum_{\delta\in C(\mathbf{S}/F)} \kappa(w(\delta)) = 0.$$

Hence $\langle \theta_\rho, \theta_\tau \rangle = 0$; the point is that θ_τ is stable and θ_ρ is an anti-stable function. Since the $m'(\rho, \pi)$ are nonnegative integers, we conclude that the integers $m''(\rho, \pi)$ do not all have the same sign. In particular, there are at least two π in (1.4.3). Corollary 2.2.4 then implies that $|m''(\rho,\pi)|$ is one (if it is nonzero). Moreover, if $\{\rho'\}$ is also a square-integrable H-packet, then

$$\begin{aligned} 2\cdot'\langle \chi(\{\rho\}), \chi(\{\rho'\}) \rangle &= \langle \theta_\rho, \theta_{\rho'} \rangle \\ &= \left\langle \sum_\pi m''(\rho,\pi)\chi_\pi, \sum_{\pi'} m''(\rho,\pi')\chi_{\pi'} \right\rangle \\ &= \sum_\pi m''(\rho,\pi)m''(\rho',\pi) \end{aligned}$$

by (2.2.2) and the orthonormality relations (of [K2], Theorem K) for characters χ_π of square-integrable G-modules π. Taking $\rho = \rho'$ we conclude that $\sum_\pi m''(\rho,\pi)^2 = 2[\{\rho\}]$, and the proposition follows. □

Corollary. *For each square-integrable H-packet $\{\rho\}$ there exist $2[\{\rho\}]$ inequivalent square-integrable G-modules which we gather in two nonempty disjoint sets $\pi^+(\rho)$ and $\pi^-(\rho)$, such that*

$$\operatorname{tr}\{\rho\}('fdh) = \operatorname{tr}\pi^+(\rho)(fdg) - \operatorname{tr}\pi^-(\rho)(fdg).$$

Here $\operatorname{tr}\pi^+(\rho)(fdg)$ is the sum of $\operatorname{tr}\pi(fdg)$ over the π in the set $\pi^+(\rho)$. In particular, if $\{\rho\}$ consists of a single term, then $\pi^+(\rho)$ and $\pi^-(\rho)$ are irreducible G-modules.

2.4 Stability

We shall now show that if $m'(\rho,\pi) \neq 0$, namely if π contributes to (1.4.2), then it lies either in $\pi^+(\rho)$ or in $\pi^-(\rho)$. We begin with rewriting (1.4.2). For each (irreducible) π^+ in $\pi^+(\rho)$ there is a nonnegative integer $m(\pi^+)$, and for each π^- in $\pi^-(\rho)$ there is such $m(\pi^-)$, with the following property. Put

$$\sum{}^+(fdg) = \sum(2m(\pi^+)+1)\operatorname{tr}\pi^+(fdg) \qquad (\pi^+ \text{ in } \pi^+(\rho)),$$
$$\sum{}^-(fdg) = \sum(2m(\pi^-)+1)\operatorname{tr}\pi^-(fdg) \qquad (\pi^- \text{ in } \pi^-(\rho)),$$

and

$$\sum{}^0(fdg) = \sum 2m(\rho,\pi)\operatorname{tr}\pi(fdg) \qquad (\pi \text{ not in } \pi^+(\rho), \pi^-(\rho)).$$

Then

$$\sum_\pi m'(\rho,\pi)\operatorname{tr}\pi(fdg) = \sum{}^+(fdg) + \sum{}^-(fdg) + \sum{}^0(fdg)$$

(this relation defines $m(\pi^+)$ and $m(\pi^-)$). Also we write χ^+, χ^-, χ^0 for the corresponding (finite) sums of characters:

$$\chi^+ = \sum_{\pi^+\in\pi^+(\rho)} (2m(\pi^+)+1)\chi(\pi^+),$$
$$\chi^- = \sum_{\pi^-\in\pi^-(\rho)} (2m(\pi^-)+1)\chi(\pi^-),$$
$$\chi^0 = \sum_\pi m(\rho,\pi)\chi(\pi) \qquad (\pi \notin \pi^+(\rho)\cup\pi^-(\rho)).$$

2.4.1 LEMMA. *The class function $\chi^+ + \chi^-$ on G is stable.*

PROOF. In view of Proposition 2.2.3 (2) it suffices to show that $\langle\chi^+ + \chi^-, \theta_{\rho'}\rangle$ vanishes for every square-integrable H-packet $\{\rho'\}$. We distinguish between two cases, when $\rho' \neq \rho$ and when $\rho' = \rho$. In the first case we note that if the irreducible π occurs in $\pi^+(\rho)$ or $\pi^-(\rho)$, then it occurs in $I(\tau\otimes\kappa)$ with $m'(\rho, \pi) \neq 0$. But then $m'(\rho', \pi) = 0$ since the characters of $I(\tau\otimes\kappa)$ and $I(\tau'\otimes\kappa)$ are orthogonal (by the twisted analogue of [K2]), and π does not occur in $\pi^+(\rho')$ or $\pi^-(\rho')$. Consequently

$$\langle\chi^+ + \chi^-, \theta_{\rho'}\rangle = \langle\chi^+ + \chi^-, \chi(\pi^+(\rho')) - \chi(\pi^-(\rho'))\rangle = 0.$$

If $\rho' = \rho$, as in the proof of Proposition 2.3 we have that $0 = \langle\theta_\tau, \theta_\rho\rangle$ is

$$\sum_{\pi^+\in\pi^+(\rho)} (2m(\pi^+) + 1) - \sum_{\pi^-\in\pi^-(\rho)} (2m(\pi^-) + 1) = \langle\chi^+ + \chi^-, \theta_\rho\rangle.$$

This completes the proof of the lemma. □

2.4.2 PROPOSITION. *The sum $\sum^0(fdg)$ is 0 for every f on G. Equivalently, $m(\rho,\pi) = 0$ for every π not in $\pi^+(\rho)$ and $\pi^-(\rho)$.*

PROOF. We claim that χ^0 is zero. If not, $\chi = \langle\chi^1 + \chi^0, \chi^1\rangle \cdot \chi^0 - \langle\chi^1 + \chi^0, \chi^0\rangle \cdot \chi^1$ is a nonzero stable function on the elliptic set of G. Note that $\langle\chi^0, \chi^1\rangle = 0$. Choose $\phi'_{v_0} dg'_{v_0}$ on G'_{v_0} such that $\Phi(t, \phi'_{v_0} dg'_{v_0} \times \sigma) = \chi(Nt)$ on the σ-elliptic set of G'_{v_0} and it is zero outside the σ-elliptic set. As usual fix a totally imaginary field F and create a cuspidal σ-invariant representation Π which is unramified outside our place v_0 and two other finite places v_1, v_2, and has the component St_{v_i} at v_i ($i = 1, 2$), and $\operatorname{tr}\Pi_{v_0}(\phi'_{v_0} dg'_{v_0} \times \sigma) \neq 0$. Since Π is cuspidal, by the usual argument of generalized linear independence of characters we get the local identity

$$\operatorname{tr}\Pi_{v_0}(\phi_{v_0} dg'_{v_0} \times \sigma) = \sum_{\pi_{v_0}} m^1(\pi_{v_0}) \operatorname{tr}\pi(f_{v_0} dg_{v_0})$$

for all matching $\phi_{v_0} dg'_{v_0}$ and $f_{v_0} dg_{v_0}$. The local representation $\Pi_0 = \Pi_{v_0}$ is perpendicular to $I(\tau\otimes\kappa)$ since $\langle\chi, \chi^0 + \chi^1\rangle = 0$, and $\chi^0 + \chi^1 = \chi^\sigma_{I(\tau\otimes\kappa)}$. Since $\chi^1 + \chi^0$ is perpendicular to the σ-twisted character $\chi^\sigma_{\Pi'}$ of any σ-invariant representation Π' inequivalent to $I(\tau\otimes\kappa)$, χ is also perpendicular to all $\chi^\sigma_{\Pi'}$, hence $\operatorname{tr}\Pi'(\phi'_{v_0} dg'_{v_0} \times \sigma) = 0$ for all σ-invariant representations Π', contradicting the construction of Π_{v_0} with $\operatorname{tr}\Pi_{v_0}(\phi'_{v_0} dg'_{v_0} \times \sigma) \neq 0$. Hence $\chi = 0$, which implies that $\chi^0 = 0$.

This completes the proof of the proposition. □

Corollary. *For every square-integrable H-packet $\{\rho\}$ we have*

$$\sum_{\pi^+\in\pi^+(\rho)}(2m(\pi^+)+1)=\sum_{\pi^-\in\pi^-(\rho)}(2m(\pi^-)+1).$$

In particular if the packet $\{\rho\}$ consists of one element then $m(\pi^+)=m(\pi^-)$.

In the next section we deal with each H-module ρ separately to show that $m(\pi^+)=m(\pi^-)=0$. Thus we obtain a precise form of (1.4.2) and (1.4.3).

III.3 Specific lifts

3.1 Steinberg

There are several special cases which we now discuss. Let μ be a character of E^1, and μ' the character of $E^\times$ given by $\mu'(a)=\mu(a/\overline{a})$. Let ρ be the Steinberg (namely square-integrable) subrepresentation $\mathrm{St}(\mu)$ of the H-module $'I=I(\mu'\nu^{1/2})$ normalizedly induced from the character $\mathrm{diag}(a,\overline{a}^{-1})\mapsto\mu'(a)|a|^{1/2}$. The image τ of ρ by the stable basechange map of [F3;II] is the Steinberg H'-module $\mathrm{St}(\mu')$, which is a subrepresentation of the induced module $'I'=I(\mu'\nu^{1/2},\mu'\nu^{-1/2})$. As the packet of this square-integrable ρ consists of a single element, we conclude that there exist two tempered irreducible G-modules denoted $\pi^+=\pi^+(\mu)$ and $\pi^-=\pi^-(\mu)$, and a nonnegative integer m, so that

$$\mathrm{tr}\,\rho('fdh)=\mathrm{tr}\,\pi^+(fdg)-\mathrm{tr}\,\pi^-(fdg) \tag{3.1.1}$$

and

$$\mathrm{tr}\,I(\tau\otimes\kappa;\phi dg'\times\sigma)=(2m+1)[\mathrm{tr}\,\pi^+(fdg)+\mathrm{tr}\,\pi^-(fdg)], \tag{3.1.2}$$

for all matching ϕ, f, $'f$.

3.1.1 Proposition. *The integer m is 0, π^- is cuspidal, and π^+ is the unique square-integrable subquotient $\pi^+_{\mu'}$ of the G-module $I(\mu'\kappa\nu^{1/2})$.*

PROOF. On the set of $x = \operatorname{diag}(a, 1, \overline{a}^{-1})$ in G with $|a| < 1$, since $'f_N(x) = \boldsymbol{\kappa}(x) f_N(x)$ and $\boldsymbol{\kappa}(x) = \kappa(a)$, the theorem of (Deligne [D6] and) Casselman [C1] and the relation (3.1.1) imply that

$$\begin{aligned}\kappa(a)\mu'(a)|a|^{1/2} &= \kappa(a)(\Delta'\chi(\{\rho\}))(\operatorname{diag}(a, \overline{a}^{-1}))\\ &= (\Delta\chi(\pi^+))(\operatorname{diag}(a, 1, \overline{a}^{-1})) - (\Delta\chi(\pi^-))(\operatorname{diag}(a, 1, \overline{a}^{-1}))\\ &= (\chi(\pi_N^+))(\operatorname{diag}(a, 1, \overline{a}^{-1})) - (\chi(\pi_N^-))(\operatorname{diag}(a, 1, \overline{a}^{-1})).\end{aligned}$$

Since the composition series of an induced G-module has length at most two, and at most one of its constituents is square integrable, and since $\pi^+(\rho)$ and $\pi^-(\rho)$ consist of square-integrable G-modules, it follows from linear independence of characters on A that (1) $\chi(\pi_N^-) = 0$, hence π^- is cuspidal, and (2) $(\chi(\pi_N^+))(\operatorname{diag}(a, 1, a^{-1})) = \mu'(a)\kappa(a)|a|^{1/2}$.

By Frobenius reciprocity π^+ is a constituent of $I(\mu'\kappa\nu^{1/2})$. Since π^+ is square integrable we conclude that $I(\mu'\kappa\nu^{1/2})$ is reducible, and $\pi^+ = \pi_{\mu'}^+$.

To show that $2m+1 = 1$ (and $m = 0$) we use again the theorem of [C1] to conclude from (3.1.2) that since the A'-module $I(\tau \otimes \kappa)_N$ of N'-coinvariants has a single decreasing σ-invariant component, and so does π^+, they are equal, and the proposition follows. □

3.2 Trivial

Let $1(\mu)$ be the one-dimensional complement of $\mathrm{St}(\mu)$ in $'I$; $1'(\mu)$ its basechange lift, namely the one-dimensional constituent in $'I'$; and $\pi^\times = \pi_{\mu'}^\times$ the nontempered subquotient of $I = I(\mu'\kappa\nu^{1/2})$.

COROLLARY. *For every matching ϕ, f, $'f$, we have*

$$\operatorname{tr}(1(\mu))('f dh) = \operatorname{tr}\pi^\times(f dg) + \operatorname{tr}\pi^-(f dg),$$

$$\operatorname{tr} I(1'(\mu) \otimes \kappa; \phi dg' \otimes \sigma) = \operatorname{tr}\pi^\times(f dg) - \operatorname{tr}\pi^-(f dg).$$

PROOF. Indeed, the composition series of I consists of $\pi^\times, \pi^+$. □

3.3 Twins

The next special case to be studied is that of $[\{\rho\}] = 2$. Then in the notations of [F3;II], $\{\rho\}$ is of the form $\rho(\theta, '\theta)$, associated with an unordered pair θ, $'\theta$ of characters of E^1. Here $\{\rho\}$ consists of two cuspidals when $\theta \neq {'\theta}$. It lifts to the induced H'-module $\tau \otimes \kappa^{-1} = I(\theta'\kappa^{-1}, {'\theta'}\kappa^{-1})$, where $\theta'(x) = \theta(x/\overline{x})$, $'\theta'(x) = {'\theta}(x/\overline{x})$ (x in $E^\times$), via the stable basechange map of [F3;II], and to $I(\theta', {'\theta'}) = \tau$ via the unstable map. The σ-invariant G'-module $I(\tau)$ is $I(\theta', {'\theta'}, \omega'/\theta' \cdot {'\theta'})$. It is also obtained, by the same process, from the H-module $\rho' = \rho(\theta, \omega/\theta \cdot {'\theta})$, and also from the H-module $\rho'' = \rho({'\theta}, \omega/\theta \cdot {'\theta})$. We now assume that θ, $'\theta$, $\omega/\theta \cdot {'\theta}$ are all distinct, so that $\{\rho\}$, $\{\rho'\}$ and $\{\rho''\}$ are disjoint packets consisting of two cuspidals each.

We also write $\rho_1 = \rho$, $\rho_2 = \rho'$, $\rho_3 = \rho''$. If $\tau = I(\theta', {'\theta'})$, we conclude that there are four inequivalent irreducible cuspidal G-modules π_j $(1 \leq j \leq 4)$, and nonnegative integers m_j, so that

$$\operatorname{tr} I(\tau; \phi dg' \times \sigma) = \sum_j (2m_j + 1) \operatorname{tr} \pi_j(f dg).$$

Moreover, there are numbers ε_{ij} $(1 \leq i \leq 3; 1 \leq j \leq 4)$, equal to 1 or -1, such that for any $i = 1, 2, 3$, the set $\{\varepsilon_{ij}\ (1 \leq j \leq 4)\}$ is equal to the set $\{1, -1\}$, and they satisfy

$$\operatorname{tr} \rho_i({'f} dh) = \sum_{j=1}^{4} \varepsilon_{ij} \operatorname{tr} \pi_j(f dg) \qquad (1 \leq i \leq 3).$$

3.4 Proposition. (1) *For each i there are exactly two j with $\varepsilon_{ij} = 1$.*
(2) *The integer m_j is independent of j. Put $m = m_j$.*
(3) *The product $\varepsilon_{1j}\varepsilon_{2j}\varepsilon_{3j}$ is independent of j.*

Proof. Note that (1) asserts that $\pi^+ = \pi^+(\rho)$ and π^- consist of two elements each. To prove (1), note that the orthogonality relations on H imply that if there exists an i for which exactly two ε_{ij} are 1, then this is valid for all i. Thus, if (1) does not hold, then there are two i for which the number of j with $\varepsilon_{ij} = 1$ is (without loss of generality) one (otherwise this number is three, and this case is dealt with in exactly the same way). Hence we may assume that $i = 1$ and 2, and $\varepsilon_{11} = 1$, $\varepsilon_{22} = 1$ (we cannot have $\varepsilon_{21} = \varepsilon_{11} = 1$ since ρ, ρ' are inequivalent). Since the stable character θ_τ is orthogonal to the unstable character θ_{ρ_i} (all i), we conclude that

$$2m_1 + 1 = 2m_2 + 1 + 2m_3 + 1 + 2m_4 + 1$$

and

$$2m_2 + 1 = 2m_1 + 1 + 2m_3 + 1 + 2m_4 + 1.$$

Hence $m_3+m_4+1=0$, contradicting the assumption that m_j are nonnegative. (1) follows.

To establish (2), we first claim that there exists j so that ε_{ij} is independent of i.

If this claim is false, we may assume that ε_{11}, ε_{12}, ε_{21}, ε_{23}, ε_{32}, ε_{34} are equal. But then the characters of $\{\rho'\}$ and $\{\rho''\}$ are not orthogonal. This contradicts the orthogonality relations on H, hence the claim. Up to reordering indices, the claim implies that ε_{11}, ε_{12}, ε_{21}, ε_{23}, ε_{31}, ε_{34}, are equal. As $\langle\theta_\tau,\theta_{\rho_i}\rangle=0$, we conclude that

$$m_1+m_2=m_3+m_4,\quad m_1+m_3=m_2+m_4,\quad m_1+m_4=m_2+m_3.$$

Hence m_j is independent of j, and (2) follows.

Also it follows that $\varepsilon_{1j}\varepsilon_{2j}\varepsilon_{3j}$ is independent of j, hence (3). □

Let ρ be any square-integrable H-module, so that we have

$$\operatorname{tr}\Pi(\phi dg'\times\sigma)=(2m+1)\sum\operatorname{tr}\pi(fdg),$$

where $\Pi=I(\tau\otimes\kappa)$, the sum ranges over $2[\{\rho\}]$ inequivalent square-integrable π, and m is a nonnegative integer.

3.5 Proposition. *We have $m=0$. There exists a unique generic π in the sum. The other $2[\{\rho\}]-1$ G-modules are not generic.*

Our proof is local. It is based on the following theorem of Rodier [Rd], p. 161, (for any split group H) which computes the number of $\boldsymbol{\psi}_H$-Whittaker models of the admissible irreducible representation π_H of H in terms of values of the character $\operatorname{tr}\pi_H$ or χ_{π_H} of π_H at the measures $\psi_{H,n}dh$ which are supported near the origin. In the course of this proof and in section II.4 (only) we denote our Π, π, $\phi dg'$, fdg, G', G by π, π_H, fdg, $f_H dh$, G, H. For clarity, Proposition 3.5.1 and its twisted analogue 3.5.2 are stated in greater generality than used in this work.

3.5.1 Proposition. *The multiplicity* $\dim_{\mathbb{C}}\operatorname{Hom}_H(\operatorname{Ind}_{U_H}^H\boldsymbol{\psi}_H,\pi_H)$ *is equal to*

$$\lim_n|H_n|^{-1}\operatorname{tr}\pi_H(\psi_{H,n}dh)=\lim_n|H_n|^{-1}\int_{H_n}\chi_{\pi_H}(h)\psi_{H,n}(h)dh.$$

The limit here and below stabilizes for large n. We proceed to explain the notations. For simplicity and clarity, instead of working with a general connected reductive (quasi-) split p-adic group G, we take $G = \mathrm{GL}(r, E)$, where E/F is a quadratic extension of p-adic fields of characteristic zero, $p \neq 2$. Let $x \mapsto \overline{x}$ denote the generator of $\mathrm{Gal}(E/F)$. For $g = (g_{ij})$ in G we put $\overline{g} = (\overline{g}_{ij})$ and ${}^t g = (g_{ji})$. Then $\sigma(g) = J^{-1}{}^t\overline{g}^{-1}J$, $J = ((-1)^{i-1}\delta_{i,r+1-j})$, defines an involution σ on G. The group $H = G^\sigma$ of $g \in G$ fixed by σ is a quasi-split unitary group. Let $\boldsymbol{\psi}_H : U_H \to \mathbb{C}^1 (= \{z \in \mathbb{C}; |z| = 1\})$ be a generic (nontrivial on each simple root subgroup) character on the unipotent upper triangular subgroup U_H of H. There is only one orbit of generic $\boldsymbol{\psi}_H$ under the action of the diagonal subgroup of H on U_H by conjugation.

By $\boldsymbol{\psi}_H$-*Whittaker vectors* we mean vectors in the space of the induced representation $\mathrm{Ind}_{U_H}^H(\boldsymbol{\psi}_H)$. They are the functions $\varphi_H : H \to \mathbb{C}$ with $\varphi_H(uhk) = \boldsymbol{\psi}_H(u)\varphi_H(h)$, $u \in U_H$, $h \in H$, $k \in K_{\varphi_H}$, where K_{φ_H} is a compact open subgroup of H depending on φ_H. The group H acts by right translation. The multiplicity $\dim_{\mathbb{C}} \mathrm{Hom}_H(\mathrm{Ind}_{U_H}^H \boldsymbol{\psi}_H, \pi_H)$ of any irreducible admissible representation π_H of H in the space of $\boldsymbol{\psi}_H$-Whittaker vectors is known to be 0 or 1. In the latter case we say that π_H has a $\boldsymbol{\psi}_H$-Whittaker model or that it is $\boldsymbol{\psi}_H$-*generic*. To be definite, define $\boldsymbol{\psi}_H : U_H \to \mathbb{C}^1$ by $\boldsymbol{\psi}_H((u_{ij})) = \psi(\sum_{1 \leq j < r} u_{j,j+1})$, where $\psi : F \to \mathbb{C}^1$ is an additive character such that the ring R of integers of F is the largest subring of F on which ψ is 1. Note that $u_{r-j,r-j+1} = \overline{u}_{j,j+1}$.

Let $\mathfrak{g}_0$ be the ring of $r \times r$ matrices with entries in the ring of integers R_E of E, and $\mathcal{H}_0$ the set of X in $\mathfrak{g}_0$ fixed by the involution $d\sigma$, defined by $d\sigma(X) = -J^{-1}{}^t\overline{X}J$. Fix a generator $\boldsymbol{\pi}$ of the maximal ideal in R. Write $H_n = \exp(\mathcal{H}_n)$, $\mathcal{H}_n = \boldsymbol{\pi}^n\mathcal{H}_0$. For $n \geq 1$ we have $H_n = {}^tU_{H,n}A_{H,n}U_{H,n}$, where $U_{H,n} = U_H \cap H_n$, and $A_{H,n}$ is the group of diagonal matrices in H_n. Define a character $\psi_{H,n} : H \to \mathbb{C}^1$, supported on H_n, by $\psi_{H,n}({}^tbu) = \psi(\sum_{1 \leq j < r} u_{j,j+1}\boldsymbol{\pi}^{-2n})$, at ${}^tb \in {}^tU_{H,n}A_{H,n}$, $u = (u_{ij}) \in U_{H,n}$. Alternatively, by

$$\psi_{H,n}(\exp X) = \mathrm{ch}_{\mathcal{H}_n}(X)\psi(\mathrm{tr}[X\boldsymbol{\pi}^{-2n}\beta_H]),$$

where $\mathrm{ch}_{\mathcal{H}_n}$ indicates the characteristic function of $\mathcal{H}_n = \boldsymbol{\pi}^n\mathcal{H}_0$ in $\mathcal{H}$, and β_H is the $r \times r$ matrix whose nonzero entries are 1 at the places $(j, j-1)$, $1 < j \leq r$.

We need a twisted analogue of Rodier's theorem. It can be described as follows.

Let π be a σ-invariant admissible irreducible representation of G, thus

$\pi \simeq {}^\sigma\pi$, where ${}^\sigma\pi(\sigma(g)) = \pi(g)$. Then there exists an intertwining operator $A : \pi \to {}^\sigma\pi$, with $A\pi(g) = \pi(\sigma(g))A$ for all $g \in G$. Since π is irreducible, by Schur's lemma A^2 is a scalar which we may normalize by $A^2 = 1$. Thus A is unique up to a sign. Denote by G' the semidirect product $G \rtimes \langle\sigma\rangle$. Then π extends to G' by $\pi(\sigma) = A$. If π is generic, namely realizable in the space of Whittaker functions ($\varphi : G \to \mathbb{C}$ with $\varphi(ugk) = \boldsymbol{\psi}(u)\varphi(g)$, $u \in U$, $g \in G$, k in a compact open K_φ depending on φ), then A is normalized by $A\varphi = {}^\sigma\varphi$, ${}^\sigma\varphi(g) = \varphi(\sigma(g))$.

The twisted character χ_π^σ is a complex valued σ-conjugacy invariant function on G (its value on $\{hg\sigma(h)^{-1}\}$ is independent of $h \in G$) which is locally constant on the σ-regular set (g with regular $g\sigma(g)$), locally integrable ([Cl2], Thm 1, p. 153) and defined by $\operatorname{tr}\pi(f\,dg)A = \int_G \chi_\pi^\sigma(g)f(g)dg$ for all test measures $f\,dg$.

Define $\psi_E : E \to \mathbb{C}^1$ by $\psi_E(x) = \psi(x + \overline{x})$. Define a character $\boldsymbol{\psi} : U \to \mathbb{C}^1$ on the unipotent upper triangular subgroup U of G by $\boldsymbol{\psi}((u_{ij})) = \psi_E(\sum_{1\le j<r} u_{j,j+1})$. This one-dimensional representation has the property that $\boldsymbol{\psi}(\sigma(u)) = \boldsymbol{\psi}(u)$ for all u in U. Note that $\boldsymbol{\psi}(u) = \boldsymbol{\psi}_H(u^2)$ at $u \in U_H = U \cap H$. There is only one orbit of generic σ-invariant characters on U under the adjoint action of the group of σ-invariant diagonal elements in G.

Write $G_n = \exp(\mathfrak{g}_n)$, where $\mathfrak{g}_n = \boldsymbol{\pi}^n\mathfrak{g}_0$. For $n \ge 1$ we have $G_n = {}^tU_nA_nU_n$, where $U_n = U \cap G_n$, and A_n is the group of diagonal matrices in G_n. Define a character $\psi_n : G \to \mathbb{C}^1$ supported on G_n by $\psi_n({}^tbu) = \psi_E(\sum_{1\le j<r} u_{j,j+1}\boldsymbol{\pi}^{-2n})$ where ${}^tb \in {}^tU_nA_n$, $u = (u_{ij}) \in U_n$. Alternatively, $\psi_n : G \to \mathbb{C}^1$ is defined by

$$\psi_n(\exp X) = \operatorname{ch}_{\mathfrak{g}_n}(X)\psi_E(\operatorname{tr}[X\boldsymbol{\pi}^{-2n}\beta])$$

where β is the $r \times r$ matrix with entries 1 at the places $(j, j-1)$, $1 < j \le r$, and 0 elsewhere.

The σ-twisted analogue of Rodier's theorem of interest to us is as follows. Let $\operatorname{ch}_{G_n^\sigma}$ denote the characteristic function of $G_n^\sigma = \{g = \sigma g; g \in G_n\}$ in G_n.

3.5.2 PROPOSITION. *For all sufficiently large n the multiplicity*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\operatorname{Ind}_U^G \boldsymbol{\psi}, \pi) = \dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_U^G \boldsymbol{\psi}, \pi)$$

is equal to

$$|G_n^\sigma|^{-1} \operatorname{tr}\pi(\psi_n \operatorname{ch}_{G_n^\sigma}\, dg \times \sigma) = |G_n^\sigma|^{-1} \int_{G_n^\sigma} \chi_\pi^\sigma(g)\psi_n(g)dg.$$

The proof of this is delayed to the next section.

Proof of Proposition 3.5. The identity

$$\operatorname{tr}\pi(fdg \times \sigma) = (2m+1)\sum_{\pi_H} \operatorname{tr}\pi_H(f_H dh).$$

for all matching test measures fdg and $f_H dh$ implies an identity of characters:

$$\chi_\pi^\sigma(\delta) = (2m+1)\sum_{\pi_H} \chi_{\pi_H}(\gamma)$$

for all $\delta \in G = \mathrm{GL}(3,E)$ with regular norm $\gamma \in H = \mathrm{U}(3,E/F)$. Note that $\delta \mapsto \chi_\pi^\sigma(\delta)$ is a stable σ-conjugacy class function on G, while $\gamma \mapsto \sum_{\pi_H} \chi_{\pi_H}(\gamma)$ is a stable conjugacy class function on H. We use Proposition 3.5.2 with $G = \mathrm{GL}(3,E)$ and $H = G^\sigma$. Then $G_n^\sigma = H_n$. On $\delta \in G_n^\sigma$, the norm $N\delta$ of the stable σ-conjugacy class of δ is just the stable conjugacy class of δ^2. Hence $\chi_\pi^\sigma(\delta) = (2m+1)\sum_{\pi_H} \chi_{\pi_H}(\delta^2)$ at $\delta \in G_n^\sigma = H_n$.

If $\delta = \exp X$, $X \in \mathfrak{g}_n^\sigma = \mathcal{H}_n$, then $\psi_E(\operatorname{tr}[X\boldsymbol{\pi}^{-2n}\beta]) = \psi(\operatorname{tr}[2X\boldsymbol{\pi}^{-2n}\beta_H])$. Indeed $\beta = \beta_H$ and $\psi_E(x) = \psi(x+\overline{x})$, thus $\boldsymbol{\psi}_n({}^t bu) = \psi_E((x+y)\boldsymbol{\pi}^{-2n})$ if $u = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. This is $= \psi(2(x+\overline{x})\boldsymbol{\pi}^{-2n})$ if $y = \overline{x}$, while $\psi_{H,n}({}^t bu) = \psi((x+\overline{x})\boldsymbol{\pi}^{-2n})$ at such $u \in U_H$ (thus with $y = \overline{x}$). Hence $\boldsymbol{\psi}_n(\delta) = \boldsymbol{\psi}_{H,n}(\delta^2)$ for $\delta \in G_n^\sigma = H_n$. Also $d(g^2) = dg$ when $p \neq 2$. It follows that

$$1 = \dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_U^G \boldsymbol{\psi}, \pi) = (2m+1)\sum_{\pi_H} \dim_{\mathbb{C}} \operatorname{Hom}_H(\operatorname{Ind}_{U_H}^H \boldsymbol{\psi}_H, \pi_H).$$

Hence $m = 0$ and there is just one generic π_H in the sum ($\dim_{\mathbb{C}} \neq 0$, necessarily $= 1$). □

3.6 Proposition. *In the notations of Proposition* 3.4, $\varepsilon_{1j}\varepsilon_{2j}\varepsilon_{3j} = 1$.

Proof. Again we use the trace formula, and global notations. We study the situation at a place w. We may and do assume that E/F are totally imaginary. At three finite places $v = v_m$ ($\neq w$; $m = 1, 2, 3$) which do not split (and do not ramify) we choose θ_v, ${}'\theta_v$ so that ρ_v, ρ'_v, ρ''_v are cuspidal. Since $\varepsilon_{1jv}\varepsilon_{2jv}\varepsilon_{3jv}$ is independent of j, then for each v there exists $j = j(v)$ so that ε_{ijv} is independent of i. Since ε_{ijv} can attain only two values, and we have three v at our disposal, we can assume that $\varepsilon_{i_1 j_1 v_1} = \varepsilon_{i_2 j_2 v_2}$, where $j_m = j(v_m)$, and both sides are independent of i_1, i_2.

We now construct global characters θ, $'\theta$ with the chosen components at v_1, v_2 and our place w, which are unramified at each place which does not split in E/F (we can take $\theta_v = '\theta_v$ at the v which ramify). It is clear that $\rho_1 = \rho(\theta, '\theta)$, $\rho_2 = \rho(\theta, \omega/\theta \cdot '\theta)$, $\rho_3 = \rho('\theta, \omega/\theta \cdot '\theta)$ are cuspidal and distinct. All three appear in the trace formula together with $I(\tau \otimes \kappa) = I(\theta', '\theta', \omega'/\theta' \cdot '\theta')$, and with coefficients $n(\rho) = \frac{1}{2}$ (see [F3;II]). Namely, we obtain

$$\prod \left[\sum_j \operatorname{tr} \pi_{jv}(f_v dg_v) \right] + \sum_j \prod \left[\sum_j \varepsilon_{ijv} \operatorname{tr} \pi_{jv}(f_v dg_v) \right]$$
$$= 4 \sum m(\pi) \prod \operatorname{tr} \pi_v(f_v dg_v).$$

The product ranges over $v = w$, v_1, v_2. At $v = v_m$ $(m = 1, 2)$ we take $f_v dg_v$ to be a coefficient of π_{jv}, where $j = j(v)$ was chosen above. Then the product Π can be taken only over our place w. Hence, for every j, we have

$$1 + \sum_i \varepsilon_{ijw} \equiv 0 \pmod 4.$$

This holds only if $\varepsilon_{ijw} = 1$ for an odd number of i, and the proposition follows. □

3.7 Sum up twins

To sum up our case (3.3), fix θ, $'\theta$ so that $\rho_1 = \rho(\theta, '\theta)$, $\rho_2 = \rho(\theta, \omega/\theta \cdot '\theta)$ are disjoint cuspidal H-packets. Denote by Π the induced G'-module $I(\theta', '\theta', \omega'/\theta' \cdot '\theta')$.

Corollary. *There exist four cuspidal G-modules π_j $(1 \leq j \leq 4)$, so that π_1 has a Whittaker model but π_j $(j \neq 1)$ do not, so that*

$$\operatorname{tr} \Pi(\phi dg' \times \sigma) = \sum_j \operatorname{tr} \pi_j(f dg),$$

and

$$\operatorname{tr} \rho_i('f dh) = \operatorname{tr} \pi_1(f dg) + \operatorname{tr} \pi_{i+1}(f dg) - \operatorname{tr} \pi_{i'}(f dg) - \operatorname{tr} \pi_{i''}(f dg).$$

The indices i', i'' are so that $\{i+1, i', i''\} = \{2, 3, 4\}$.

We write $\pi^+(\rho_i)$ for $\{\pi_1, \pi_{i+1}\}$, and $\pi^-(\rho_i)$ for $\{\pi_{i'}, \pi_{i''}\}$.

3.8 $\rho(\theta, \omega/\theta^2)$

The next special case of interest is that of the packet associated with $\rho = \rho(\theta, \omega/\theta^2)$, where $\theta^3 \neq \omega$, so that $\{\rho\}$ consists of cuspidals; in fact $\{\rho\}$ consists of a single element, and this is clear also from the comments below. The associated G'-module is the σ-invariant tempered induced $\Pi = I(\theta', \omega'/\theta'^2, \theta')$. It is the basechange lift of the reducible G-module $\pi = I(\theta')$. The representation π is the direct sum of the tempered irreducibles π^+ and π^-. Then we have

$$\operatorname{tr} \Pi(\phi dg' \times \sigma) = \operatorname{tr} \pi(f dg) = \operatorname{tr} \pi^+(f dg) + \operatorname{tr} \pi^-(f dg),$$

and also

$$\operatorname{tr} \rho('f dh) = \operatorname{tr} \pi^+(f dg) - \operatorname{tr} \pi^-(f dg),$$

for a suitable choice of π^+. Namely π^+ has a Whittaker vector, while π^- does not. In particular $2[\{\rho\}] = [\{\pi^+, \pi^-\}] = 2$, so that $\{\rho\}$ consists of a single element, as asserted.

3.9 Packets

With this we have completed the description of all tempered packets $\{\pi\}$ of G. The packets are in bijection with the tempered σ-stable G'-modules Π. If Π is a square-integrable σ-invariant G'-module, then it is σ-stable, and the packet $\{\pi\}$ consists of a single element (this has been shown already in [F3;III(IV)]). If Π is induced from a square-integrable H'-module, and it is σ-stable, then it is of the form $I(\tau \otimes \kappa)$, where τ is the stable basechange lift of a square-integrable packet $\{\rho\}$ of H. The associated G-packet $\{\pi\}$ consists of $2 = 2[\{\rho\}]$ elements, each occurring with multiplicity one. If Π is induced from the diagonal subgroup, and it is not simply the basechange lift of an induced G-module $I(\mu)$ (in which case the packet $\{\pi\}$ consists of the irreducible constituents of $I(\mu)$), then Π is of the form $I(\theta', {}'\theta', \omega'/\theta' \cdot {}'\theta')$, where the three characters are distinct, and trivial on $F^\times$. In this case the packet $\{\pi\}$ consists of $4 = 2[\{\rho\}]$ elements, where $\rho = \rho(\theta, {}'\theta)$.

Using this, and the related character identities between ρ and the difference of members of $\{\pi\}$, we can use the trace formula to describe the discrete spectrum of G.

III.4 Whittaker models and twisted characters

We shall reduce Proposition 3.5.2 to Proposition 3.5.1 for G (not H), so we begin by recalling the main lines in Rodier's proof in the context of G. Fix

$$d = \mathrm{diag}(\boldsymbol{\pi}^{-r+1}, \boldsymbol{\pi}^{-r+3}, \dots, \overline{\boldsymbol{\pi}}^{r-1})$$

(bar over the last $[r/2]$ entries). Put

$$V_n = d^n G_n d^{-n}, \quad \boldsymbol{\psi}_n(v) = \psi_n(d^{-n} v d^n)\,(v \in V_n).$$

Note that $\sigma(d) = d$, $\sigma(G_n) = G_n$, $\sigma(U_n) = U_n$, $\sigma\psi_n = \psi_n$, and that the entries in the jth line ($j \neq 0$) above or below the diagonal of $v = (v_{ij})$ in V_n lie in $\boldsymbol{\pi}^{(1-2j)n} R_E$ (thus $v_{i,i+j} \in \boldsymbol{\pi}^{(1-2j)n} R_E$ if $j > 0$, and also when $j < 0$). Thus $V_n \cap U$ is a σ-invariant strictly increasing sequence of compact and open subgroups of U whose union is U, while $V_n \cap ({}^tUG)$ — where ${}^tUG$ is the lower triangular subgroup of G — is a strictly decreasing sequence of compact open subgroups of G whose intersection is the element I of G. Note that $\boldsymbol{\psi}_n = \boldsymbol{\psi}$ on $V_n \cap U$.

Consider the induced representations $\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n$, and the intertwining operators

$$A_n^m : \mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n \to \mathrm{Ind}_{V_m}^G \boldsymbol{\psi}_m,$$

$$(A_n^m \varphi)(g) = ((|V_m|^{-1} 1_{V_m} \boldsymbol{\psi}_m) * \varphi)(g) = |V_m|^{-1} \int_{V_m} \boldsymbol{\psi}_m(u) \varphi(u^{-1} g) du$$

(g in G, φ in $\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n$, $|V_m|$ denotes the volume of V_m, 1_{V_m} denotes the characteristic function of V_m). For $m \geq n \geq 1$ we have

$$(A_n^m \varphi)(g) = ((|V_m \cap U|^{-1} 1_{V_m \cap U} \boldsymbol{\psi}) * \varphi)(g)$$
$$= |V_m \cap U|^{-1} \int_{V_m \cap U} \boldsymbol{\psi}(u) \varphi(u^{-1} g) du.$$

Hence $A_m^\ell \circ A_n^m = A_n^\ell$ for $\ell \geq m \geq n \geq 1$. So $(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n, A_n^m\ (m \geq n \geq 1))$ is an inductive system of representations of G. Denote by $(I, A_n : \mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n \to I)$ $(n \geq 1)$ its limit.

The intertwining operators $\phi_n : \mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n \to \mathrm{Ind}_U^G \boldsymbol{\psi}$,

$$(\phi_n(\varphi))(g) = (\boldsymbol{\psi} 1_U * \varphi)(g) = \int_U \boldsymbol{\psi}(u) \varphi(u^{-1} g) du,$$

satisfy $\phi_n \circ A_n^m = \phi_n$ if $m \geq n \geq 1$. Hence there exists a unique intertwining operator $\phi : I \to \mathrm{Ind}_U^G \boldsymbol{\psi}$ with $\phi \circ A_n = \phi_n$ for all $n \geq 1$. Proposition 3 of [Rd] asserts that

4.1.1 LEMMA. *The map ϕ is an isomorphism of G-modules.*

4.1.2 LEMMA. *There exists $n_0 \geq 1$ such that $\boldsymbol{\psi}_n * \boldsymbol{\psi}_m * \boldsymbol{\psi}_n = |V_n||V_m \cap V_n|\boldsymbol{\psi}_n$ for all $m \geq n \geq n_0$.*

PROOF. This is Lemma 5 of [Rd]. We review its proof (the first displayed formula in the proof of this Lemma 5, [Rd], p. 159, line -8, should be erased).

There are finitely many representatives u_i in $U \cap V_m$ for the cosets of V_m modulo $V_n \cap V_m$. Denote by $\varepsilon(g)$ the Dirac measure in a point g of G. Consider $(\varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m \cap V_n})(g)$

$$= \int_G \varepsilon(u_i)(gh^{-1})(\boldsymbol{\psi}_n 1_{V_m\cap V_n})(h)dh = \boldsymbol{\psi}_n(u_i^{-1}g) = \boldsymbol{\psi}_m(u_i)^{-1}\boldsymbol{\psi}_m(g).$$

Note here that if the left side is nonzero, then $g \in u_i(V_m \cap V_n) \subset V_m$. Conversely, if $g \in V_m$, then $g \in u_i(V_m \cap V_n)$ for some i. Hence $\boldsymbol{\psi}_m = \sum_i \boldsymbol{\psi}_m(u_i)\varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m\cap V_n}$, thus

$$\boldsymbol{\psi}_n * \boldsymbol{\psi}_m * \boldsymbol{\psi}_n = \sum_i \boldsymbol{\psi}_m(u_i)\boldsymbol{\psi}_n * \varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m\cap V_n} * \boldsymbol{\psi}_n.$$

Since $\boldsymbol{\psi}_n 1_{V_m\cap V_n} * \boldsymbol{\psi}_n = |V_m \cap V_n|\boldsymbol{\psi}_n$, this is

$$= \sum_i \boldsymbol{\psi}_m(u_i)|V_m \cap V_n|\boldsymbol{\psi}_n * \varepsilon(u_i) * \boldsymbol{\psi}_n.$$

But the key Lemma 4 of [Rd] asserts that $\boldsymbol{\psi}_n * \varepsilon(u) * \boldsymbol{\psi}_n \neq 0$ implies that $u \in V_n$. Hence the last sum reduces to a single term, with $u_i = 1$, and we obtain

$$= |V_m \cap V_n|\boldsymbol{\psi}_n * \boldsymbol{\psi}_n = |V_m \cap V_n||V_m|\boldsymbol{\psi}_n.$$

This completes the proof of the lemma. □

4.1.3 LEMMA. *For an inductive system $\{I_n\}$ we have* $\mathrm{Hom}_G(\varinjlim I_n, \pi) = \varprojlim \mathrm{Hom}_G(I_n, \pi)$.

PROOF. See, e.g., Rotman [Rt], Theorem 2.27. Let us verify this in our context as in [Rd]. Our Lemma 2, which is Lemma 5 of [Rd], implies Proposition 4 of [Rd], that $A_m^n \circ A_n^m = |V_m \cap V_n||V_m|^{-1} \cdot \mathrm{id}(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n)$ if $m \geq n \geq n_0$. This implies that A_n^m is injective, A_m^n is surjective, that A_n

and $\phi_n = \phi \circ A_n$ are injective, and by duality that for any G-module π the maps

$$\mathrm{Hom}_G(\mathrm{Ind}_{V_m}^G \boldsymbol{\psi}_m, \pi) \to \mathrm{Hom}_G(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n, \pi), \qquad \varphi \mapsto \varphi \circ A_n^m,$$

$$\mathrm{Hom}_G(\mathrm{Ind}_U^G \boldsymbol{\psi}, \pi) \to \mathrm{Hom}_G(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n, \pi), \qquad \varphi \mapsto \varphi \circ \phi_n,$$

are surjective for $m \geq n \geq n_0$. In particular $\mathrm{Hom}_G(\mathrm{Ind}_U^G \boldsymbol{\psi}, \pi)$ is equal to $\varprojlim \mathrm{Hom}_G(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$. □

As the $\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$ are increasing with n, if they are bounded we get the first equality in

COROLLARY. *We have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n |G_n|^{-1} \operatorname{tr} \pi(\psi_n dg).$$

PROOF. The left side is $= \lim_n \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$. This equals $\lim_n \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_{G_n}^G \psi_n, \pi)$ since $\boldsymbol{\psi}_n(v) = \psi_n(d^{-n} v d^n)$. This equals $\lim_n \dim_{\mathbb{C}} \mathrm{Hom}_{G_n}(\psi_n, \pi|G_n)$ by Frobenius reciprocity. This equals

$$\lim_n |G_n|^{-1} \operatorname{tr} \pi(\psi_n dg)$$

since $|G_n|^{-1}\pi(\psi_n dg)$ is a projection from π to the space of x in π with $\pi(g)x = \psi_n(g)x$ $(g \in G_n)$, whose dimension is $|G_n|^{-1} \operatorname{tr} \pi(\psi_n dg)$. □

4.2 The twisted case

We now reduce Proposition 3.5.2 to Proposition 3.5.1 for G. Note that since $\sigma\boldsymbol{\psi}_n = \boldsymbol{\psi}_n$, the representations $\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n$ are σ-invariant, where σ acts on $\varphi \in \mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n$ by $\varphi \mapsto \sigma\varphi$, $(\sigma\varphi)(g) = \varphi(\sigma g)$. Similarly $\sigma\boldsymbol{\psi} = \boldsymbol{\psi}$ and $\mathrm{Ind}_U^G \boldsymbol{\psi}$ is σ-invariant. We then extend these representations Ind of G to the semidirect product $G' = G \rtimes \langle\sigma\rangle$ by putting $(I(\sigma)\varphi)(g) = \varphi(\sigma(g))$.

Let π be a σ-invariant irreducible admissible representation of G. Thus there exists an intertwining operator $A : \pi \to {}^\sigma\pi$, where ${}^\sigma\pi(g) = \pi(\sigma(g))$, with $A\pi(g) = \pi(\sigma(g))A$. Then A^2 commutes with every $\pi(g)$ $(g \in G)$, hence A^2 is a scalar by Schur's lemma, and can be normalized to be 1.

This determines A up to a sign. We extend π from G to $G' = G \rtimes \langle\sigma\rangle$ by putting $\pi(\sigma) = A$ once A is chosen.

If $\mathrm{Hom}_G(\mathrm{Ind}_U^G \boldsymbol{\psi}, \pi) \neq 0$, its dimension is 1. Choose a generator ℓ : $\mathrm{Ind}_U^G \boldsymbol{\psi} \to \pi$. Define $A : \pi \to \pi$ by $A\ell(f) = \ell(I(\sigma)f)$. Then

$$\mathrm{Hom}_G(\mathrm{Ind}_U^G \boldsymbol{\psi}, \pi) = \mathrm{Hom}_{G'}(\mathrm{Ind}_U^G \boldsymbol{\psi}, \pi).$$

Similarly we have

$$\mathrm{Hom}_G(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n, \pi) = \mathrm{Hom}_{G'}(\mathrm{Ind}_{V_n}^G \boldsymbol{\psi}_n, \pi).$$

The right side in the last equality can be expressed as

$$\mathrm{Hom}_{G'}(\mathrm{Ind}_{G_n}^G \psi_n, \pi) = \mathrm{Hom}_{G_n'}(\psi_n', \pi|G_n') \qquad (G_n' = G_n \rtimes \langle\sigma\rangle).$$

The last equality follows from Frobenius reciprocity, where we extended ψ_n to ψ_n' on G_n'. Thus $\psi_n' = \psi_n^1 + \psi_n^\sigma$, with $\psi_n^i(g \times j) = \delta_{ij}\psi_n(g)$, $i\, j \in \{1, \sigma\}$.

Now $\mathrm{Hom}_{G_n'}(\psi_n', \pi|G_n')$ is isomorphic to the space π_1 of vectors x in π with $\pi(g)x = \psi_n(g)x$ for all g in G_n'. In particular $\pi(g)x = \psi_n(g)x$ for all g in G_n, and $\pi(\sigma)x = x$. Clearly $|G_n'|^{-1}\pi(\psi_n' dg')$ is a projection from the space of π to π_1. It is independent of the choice of the measure dg'. Its trace is then the dimension of the space Hom. We conclude a twisted analogue of the theorem of [Rd]:

4.2.1 Proposition. *We have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{Ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n |G_n'|^{-1} \operatorname{tr} \pi(\psi_n' dg'),$$

where the limit stabilizes for a large n.

Note that G_n' is the semidirect product of G_n and the two-element group $\langle\sigma\rangle$. With the natural measure assigning 1 to each element of the discrete group $\langle\sigma\rangle$, we have $|G_n'| = 2|G_n|$. The right side is then

$$\frac{1}{2}\lim_n |G_n|^{-1} \operatorname{tr} \pi(\psi_n dg) + \frac{1}{2}\lim_n |G_n|^{-1} \operatorname{tr} \pi(\psi_n dg \times \sigma)$$

(as $\psi_n' = \psi_n^1 + \psi_n^\sigma$, $\psi_n^1 = \psi_n$ and $\operatorname{tr} \pi(\psi_n^\sigma dg) = \operatorname{tr} \pi(\psi_n dg \times \sigma)$). By (the nontwisted) Rodier's Theorem 1,

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n |G_n|^{-1} \operatorname{tr} \pi(\psi_n dg),$$

we conclude

4.2.2 Proposition. *We have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\operatorname{Ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n |G_n|^{-1} \operatorname{tr} \pi(\psi_n dg \times \sigma)$$

for all σ-invariant irreducible representations π of G.

Let $\mathrm{ch}_{G_n^\sigma}$ denote the characteristic function of G_n^σ in G_n.

4.2.3 Proposition. *The terms in the limit on the right of the equality of Proposition 2 are equal to*

$$|G_n^\sigma|^{-1} \operatorname{tr} \pi(\psi_n \,\mathrm{ch}_{G_n^\sigma}\, dg \times \sigma) = |G_n^\sigma|^{-1} \int_{G_n^\sigma} \chi_\pi^\sigma(g) \psi_n(g) dg.$$

Proof. Consider the map $G_n^\sigma \times G_n^\sigma \backslash G_n \to G_n$, $(u,k) \mapsto k^{-1}u\sigma(k)$. It is a closed immersion. More generally, given a semisimple element s in a group G, we can consider the map $Z_{G^0}(s) \times Z_{G^0}(s)\backslash G^0 \to G^0$ by $(u,k) \mapsto k^{-1}usks^{-1}$. Our example is: $(s,G) = (\sigma, G_n \times \langle\sigma\rangle)$.

Our map is in fact an analytic isomorphism since G_n is a small neighborhood of the origin, where the exponential $e : \mathfrak{g}_n \to G_n$ is an isomorphism. Indeed, we can transport the situation to the Lie algebra $\mathfrak{g}_n$. Thus we write $k = e^Y$, $u = e^X$, $\sigma(k) = e^{(d\sigma)(Y)}$, $k^{-1}u\sigma(k) = e^{X-Y+(d\sigma)(Y)}$, up to smaller terms. Here $(d\sigma)(Y) = -J^{-1}{}^t\overline{Y}J$. So we just need to show that $(X,Y) \mapsto X - Y + (d\sigma)(Y)$, $Z_{\mathfrak{g}_n}(\sigma) + \mathfrak{g}_n (\operatorname{mod} Z_{\mathfrak{g}_n}(\sigma)) \to \mathfrak{g}_n$, is bijective. But this is obvious since the kernel of $(1 - d\sigma)$ on $\mathfrak{g}_n$ is precisely $Z_{\mathfrak{g}_n}(\sigma) = \{X \in \mathfrak{g}_n; (d\sigma)(X) = X\}$.

Changing variables on the terms on the right of Proposition 2 we get the equality:

$$|G_n|^{-1} \int_{G_n} \chi_\pi^\sigma(g)\psi_n(g)dg$$

$$= |G_n|^{-1} \int_{G_n^\sigma} \int_{G_n^\sigma \backslash G_n} \chi_\pi^\sigma(k^{-1}u\sigma(k))\psi_n(k^{-1}u\sigma(k))dkdu.$$

But $\sigma\psi_n = \psi_n$, ψ_n is a homomorphism (on G_n), G_n is compact, and χ_π^σ is a σ-conjugacy class function, so we end up with

$$= |G_n^\sigma|^{-1} \int_{G_n^\sigma} \chi_\pi^\sigma(u)\psi_n(u)du.$$

The proposition, and Theorem 2, follow. □

4.3 Germs of twisted characters

Harish-Chandra [HC2] showed that χ_π is locally integrable (Thm 1, p. 1) and has a germ expansion near each semisimple element γ (Thm 5, p. 3), of the form:

$$\chi_\pi(\gamma \exp X) = \sum_{\mathcal{O}} c_\gamma(\mathcal{O}, \pi)\widehat{\mu}_{\mathcal{O}}(X).$$

Here $\mathcal{O}$ ranges over the nilpotent orbits in the Lie algebra $\mathfrak{m}$ of the centralizer M of γ in G, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$, $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform with respect to a symmetric nondegenerate G-invariant bilinear form B on $\mathfrak{m}$ and a self-dual measure, and $c_\gamma(\mathcal{O}, \pi)$ are complex numbers. Both $\mu_{\mathcal{O}}$ and $c_\gamma(\mathcal{O}, \pi)$ depend on a choice of a Haar measure $d_{\mathcal{O}}$ on the centralizer $Z_G(X_0)$ of $X_0 \in \mathcal{O}$, but their product does not. The X ranges over a small neighborhood of the origin in $\mathfrak{m}$. We shall be interested only in the case of $\gamma = 1$, and thus omit γ from the notations.

Suppose that G is quasi split over F, and U is the unipotent radical of a Borel subgroup B. Let $\boldsymbol{\psi} : U \to \mathbb{C}^1$ be the nondegenerate character of U (its restriction to each simple root subgroup is nontrivial) specified in Rodier [Rd], p. 153. The number $\dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{Ind}_U^G \boldsymbol{\psi}, \pi)$ of $\boldsymbol{\psi}$-Whittaker functionals on π is known to be zero or one. Let $\mathfrak{g}_0$ be a self dual lattice in the Lie algebra $\mathfrak{g}$ of G. Denote by ch_0 the characteristic function of $\mathfrak{g}_0$ in $\mathfrak{g}$. Rodier [Rd], p. 163, showed that there is a regular nilpotent orbit $\mathcal{O} = \mathcal{O}_{\boldsymbol{\psi}}$ such that $c(\mathcal{O}, \pi)$ is not zero iff $\dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{Ind}_U^G \boldsymbol{\psi}, \pi)$ is one, in fact $\widehat{\mu}_{\mathcal{O}}(\mathrm{ch}_0)c(\mathcal{O}, \pi)$ is one in this case. Alternatively put, normalizing $\mu_{\mathcal{O}}$ by $\widehat{\mu}_{\mathcal{O}}(\mathrm{ch}_0) = 1$, we have $c(\mathcal{O}, \pi) = \dim_{\mathbb{C}} \operatorname{Hom}(\operatorname{Ind}_U^G \boldsymbol{\psi}, \pi)$. This is shown in [Rd] for all p if $G = \mathrm{GL}(n, F)$, and for general quasi-split G for all $p \geq 1 + 2\sum_{\alpha \in S} n_\alpha$, if the longest root is $\sum_{\alpha \in S} n_\alpha \alpha$ in a basis S of the root system. A generalization of Rodier's theorem to degenerate Whittaker models and nonregular nilpotent orbits is given in Moeglin-Waldspurger [MW]. See [MW], I.8, for the normalization of measures. In particular they show that $c(\mathcal{O}, \pi) > 0$ for the nilpotent orbits $\mathcal{O}$ of maximal dimension with $c(\mathcal{O}, \pi) \neq 0$.

Harish-Chandra's results extend to the twisted case. The twisted character is locally integrable (Clozel [Cl2], Thm 1, p. 153), and there exist unique complex numbers $c^\theta(\mathcal{O}, \pi)$ ([Cl2], Thm 3, p. 154) with $\chi_\pi^\theta(\exp X) = \sum_{\mathcal{O}} c^\theta(\mathcal{O}, \pi)\widehat{\mu}_{\mathcal{O}}(X)$. Here $\mathcal{O}$ ranges over the nilpotent orbits in the Lie algebra $\mathfrak{g}^\theta$ of the group G^θ of the $g \in G$ with $g = \theta(g)$. Further, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$ (it is unique up to a constant, not unique as stated in [HC2], Thm 5, and [Cl2], Thm 3); $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform, and X ranges over a small neighborhood of the origin in $\mathfrak{g}^\theta$.

In this section we compute the expression displayed in Proposition 3 using the germ expansion $\chi_\pi^\sigma(\exp X) = \sum_{\mathcal{O}} c^\sigma(\mathcal{O},\pi)\widehat{\mu}_{\mathcal{O}}(X)$. This expansion means that for any test measure fdg supported on a small enough neighborhood of the identity in G we have

$$\int_{\mathfrak{g}^\sigma} f(\exp X)\chi_\pi^\sigma(\exp X)dX$$

$$= \sum_{\mathcal{O}} c^\sigma(\mathcal{O},\pi)\int_{\mathcal{O}}\left[\int_{\mathfrak{g}^\sigma} f(\exp X)\psi(\operatorname{tr}(XZ))dX\right]d\mu_{\mathcal{O}}(Z).$$

Here $\mathcal{O}$ ranges over the nilpotent orbits in $\mathfrak{g}^\sigma$, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$, $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform. The X range over a small neighborhood of the origin in $\mathfrak{g}^\sigma$. Since we are interested only in the case of the unitary group, and to simplify the exposition, we take $G = \mathrm{GL}(n,E)$ and the involution σ whose group of fixed points is the unitary group. In this case there is a unique regular nilpotent orbit $\mathcal{O}_0$.

We normalize the measure $\mu_{\mathcal{O}_0}$ on the orbit $\mathcal{O}_0$ of β in $\mathfrak{g}^\sigma$ by the requirement that $\widehat{\mu}_{\mathcal{O}_0}(\mathrm{ch}_0^\sigma)$ is 1, thus that $\int_{\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma} d\mu_{\mathcal{O}_0}(X) = q^{n\dim(\mathcal{O}_0)}$ for large n. Equivalently a measure on an orbit $\mathcal{O} \simeq G/Z_G(Y)$ ($Y \in \mathcal{O}$) is defined by a measure on its tangent space $m = \mathfrak{g}/Z_{\mathfrak{g}}(Y)$ ([MW], p. 430) at Y, taken to be the self dual measure with respect to the symmetric bilinear nondegenerate F-valued form $B_Y(X,Z) = \operatorname{tr}(Y[X,Z])$ on m.

4.3 Proposition. *If π is a σ-invariant admissible irreducible representation of G and $\mathcal{O}_0$ is the regular nilpotent orbit in $\mathfrak{g}^\sigma$, then the coefficient $c^\sigma(\mathcal{O}_0,\pi)$ in the germ expansion of the σ-twisted character χ_π^σ of π is equal to*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\operatorname{Ind}_U^G \boldsymbol{\psi},\pi) = \dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_U^G \boldsymbol{\psi},\pi).$$

This number is one if π is generic, and zero otherwise.

Proof. We compute the expression displayed in Proposition 3 as in [MW], I.12. It is a sum over the nilpotent orbits $\mathcal{O}$ in $\mathfrak{g}^\sigma$, of $c^\sigma(\mathcal{O},\pi)$ times

$$|G_n^\sigma|^{-1}\widehat{\mu}_{\mathcal{O}}(\psi_n\circ e) = |G_n^\sigma|^{-1}\mu_{\mathcal{O}}(\widehat{\psi_n\circ e})$$

$$= |G_n^\sigma|^{-1} \int_{\mathcal{O}} \widehat{\psi_n \circ e}(X) d\mu_{\mathcal{O}}(X).$$

The Fourier transform (with respect to the character ψ_E) of $\psi_n \circ e$,

$$\widehat{\psi_n \circ e}(Y) = \int_{\mathfrak{g}^\sigma} \psi_n(\exp Z)\overline{\psi}_E(\operatorname{tr} ZY) dZ$$

$$= \int_{\mathfrak{g}_n^\sigma} \psi_E(\operatorname{tr} Z(\boldsymbol{\pi}^{-2n}\beta - Y)) dZ,$$

is the characteristic function of $\boldsymbol{\pi}^{-2n}\beta + \boldsymbol{\pi}^{-n}\mathfrak{g}_0^\sigma = \boldsymbol{\pi}^{-2n}(\beta + \boldsymbol{\pi}^n \mathfrak{g}_0^\sigma)$ multiplied by the volume $|\mathfrak{g}_n^\sigma| = |G_n^\sigma|$ of $\mathfrak{g}_n^\sigma$. Hence we get

$$= \int_{\mathcal{O}\cap(\boldsymbol{\pi}^{-2n}(\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma))} d\mu_{\mathcal{O}}(X) = q^{n\dim(\mathcal{O})} \int_{\mathcal{O}\cap(\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma)} d\mu_{\mathcal{O}}(X).$$

The last equality follows from the homogeneity result of [HC2], Lemma 3.2, p. 18. For sufficiently large n we have that $\beta + \boldsymbol{\pi}^n \mathfrak{g}_0^\sigma$ is contained only in the orbit $\mathcal{O}_0$ of β. Then only the term indexed by $\mathcal{O}_0$ remains in the sum over $\mathcal{O}$, and

$$\int_{\mathcal{O}_0\cap(\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma)} d\mu_{\mathcal{O}_0}(X) = \int_{\beta+\boldsymbol{\pi}^n\mathfrak{g}_0^\sigma} d\mu_{\mathcal{O}_0}(X)$$

equals $q^{-n\dim(\mathcal{O}_0)}$ (cf. [MW], end of proof of Lemme I.12). The proposition follows. □

III.5 Global lifting

5.1 Terms in trace formulae

First we recall Proposition III.1.1.

PROPOSITION. *We have* $F_1 = \Phi_1 + \frac{1}{2}(\Phi_2 + F_2) + \frac{1}{4}(\Phi_3 + F_3)$.

PROOF. We have to show that F_6 is 0, in the notation of (I.1.1). If μ and θ are related by $\mu(z) = \theta(z/\overline{z})$, and $\rho = \rho(\theta, \omega/\theta^2)$, then the G_v-module $I(\mu_v)$ is the direct sum of $\pi_{\mu_v}^+$ and $\pi_{\mu_v}^-$, and by (III.3.8) we have

$$\operatorname{tr}\{\rho_v\}({}'f_v dh_v) = \operatorname{tr} \pi_{\mu_v}^+(f_v dg_v) - \operatorname{tr} \pi_{\mu_v}^-(f_v dg_v).$$

Keys and Shahidi [KeS] show that

$$\operatorname{tr} R(\mu_v) I(\mu_v, f_v dg_v) = (-1, E_v/F_v)[\operatorname{tr} \pi^+_{\mu_v}(f_v dg_v) - \operatorname{tr} \pi^-_{\mu_v}(f_v dg_v)],$$

where the Hilbert symbol $(-1, E_v/F_v)$ is equal to 1 if -1 lies in $N_{E/F} E_v^\times$, and to -1 otherwise. It is 1 for almost all v, and the product of $(-1, E_v/F_v)$ over all v is 1. Hence $F_6 = 0$, as required. □

In view of the local liftings results, this gives an explicit description of the discrete spectrum of $\mathbf{G}(\mathbb{A})$.

To write out the three terms in the expression for the discrete spectrum F_1, we introduce some notations. If Π_v is a tempered σ-stable G'_v-module, we write $\{\pi_v(\Pi_v)\}$ for the associated packet of G_v-modules. We apply this terminology also when Π_v is one dimensional, where $\{\pi_v(\Pi_v)\}$ consists of a single one-dimensional G_v-module; and also when Π_v is the lift of an induced G_v-module $I(\mu_v)$. If $\{\rho_v\}$ is a packet of H_v which lifts by stable basechange to the H'_v-module τ_v, we put $\{\pi_v(\rho_v)\}$ for $\{\pi_v(I(\tau_v \otimes \kappa_v))\}$. It consists of $2[\{\rho_v\}]$ elements; it is the disjoint union of the set $\pi^+(\rho_v)$ and $\pi^-(\rho_v)$, whose cardinalities are equal if E_v is a field; $\pi^-(\rho_v)$ is empty if $E_v = F_v \oplus F_v$. Given ρ_v, we write $\varepsilon(\pi_v) = 1$ for π_v in $\pi^+(\rho_v)$, and $\varepsilon(\pi_v) = -1$ for π_v in $\pi_v^-(\rho_v)$. In particular, if $[\{\rho_v\}] = 2$, we defined in Proposition I.3.4 the sign ε_{ijv} as a coefficient of π_{jv} in $\{\pi_v(\rho_v)\}$, and we put $\varepsilon_i(\pi_{jv}) = \varepsilon_{ijv}$. We have $\{\pi_v(\rho_{1v})\} = \{\pi_v(\rho_{2v})\} = \{\pi_v(\rho_{3v})\}$, and ε_i depends on ρ_i.

Using these notations we can write

$$\Phi_1 = \sum_\Pi \prod \operatorname{tr}\{\pi_v(\Pi_v)\}(f_v dg_v).$$

The sum ranges over all discrete-spectrum automorphic σ-invariant $\mathbf{G}'(\mathbb{A})$-modules Π. Note that we use here the rigidity theorem, and the multiplicity one theorem for the discrete spectrum of $\mathrm{GL}(3, \mathbb{A}_E)$.

The term $\frac{1}{2}(\Phi_2 + F_2)$ is the sum of two terms. The first is

$$\begin{aligned}\frac{1}{2} \sum_{\rho \neq \rho(\theta, '\theta)} \Big\{ &\prod [\operatorname{tr} \pi_v^+(\rho_v)(f_v dg_v) + \operatorname{tr} \pi_v^-(\rho_v)(f_v dg_v)] \\ &+ \prod [\operatorname{tr} \pi_v^+(\rho_v)(f_v dg_v) - \operatorname{tr} \pi_v^-(\rho_v)(f_v dg_v)] \Big\} \\ = \sum_\pi m(\rho, \pi) &\prod \operatorname{tr} \pi_v(f_v dg_v).\end{aligned}$$

The first sum is over the discrete-spectrum automorphic $\mathbf{H}'(\mathbb{A})$-packets ρ which are neither one dimensional, nor of the form $\rho(\theta, {}'\theta)$. The multiplicity $m(\rho, \pi)$ is $[1+\varepsilon(\pi)]/2$, where $\varepsilon(\pi) = \prod \varepsilon(\pi_v)$; it is 0 or 1. The sum over π is taken over all products $\otimes\pi_v$, such that there exists ρ as above, and π_v is in $\{\pi_v(\rho_v)\}$ for all v, and π_v is unramified (so that $\varepsilon(\pi_v) = 1$) for almost all v. Thus $m(\rho, \pi) = 1$ exactly when the number of components π_v in $\pi_v^-(\rho_v)$ is even. Otherwise the product $\otimes\pi_v$ is not automorphic.

The other term in $\frac{1}{2}(\Phi_2 + F_2)$ is

$$\frac{1}{2}\sum_{\mu}\Big\{\varepsilon(\mu', \kappa)\prod[\operatorname{tr}\pi^{\times}_{\mu_v}(f_v dg_v) - \operatorname{tr}\pi^{-}_{\mu_v}(f_v dg_v)]$$
$$+\prod[\operatorname{tr}\pi^{\times}_{\mu_v}(f_v dg_v) + \operatorname{tr}\pi^{-}_{\mu_v}(f_v dg_v)]\Big\} = \sum_{\pi} m(\mu, \pi)\prod \operatorname{tr}\pi_v(f_v dg_v).$$

The first sum is over all characters μ of C_E^1, or equivalently one-dimensional automorphic $\mathbf{H}(\mathbb{A})$-modules. As usual we put $\mu'(z) = \mu(z/\overline{z})$, $z \in \mathbb{A}_E^{\times}$.

The sum over π ranges over the products $\otimes\pi_v$, such that there exists a μ as above, with $\pi_v = \pi^{\times}_{\mu_v}$ for almost all v, and $\pi_v = \pi^{-}_{\mu_v}$ at the other places.

We put $m(\mu, \pi) = \frac{1}{2}[1+\varepsilon(\mu', \kappa)\varepsilon(\pi)]$, where $\varepsilon(\pi)$ is $\prod \varepsilon(\pi_v)$, and $\varepsilon(\pi^{\times}_{\mu_v}) = 1$, $\varepsilon(\pi^{-}_{\mu_v}) = -1$.

The multiplicity $m(\mu, \pi)$ is 0 or 1 if there is an even or odd number of places v where π_v is π_v^-, depending on the value of $\varepsilon(\mu', \kappa)$.

The factor $\varepsilon(\mu', \kappa)$ is 1 or -1, depending on the normalization of the intertwining operator $\Pi(\sigma)$ given by the fact that Π is the realization of the induced representation $I(1'(\mu) \otimes \kappa)$ as an automorphic representation.

Let us explain this. Recall that since ${}^{\sigma}\Pi \simeq \Pi$ (${}^{\sigma}\Pi(g) = \Pi(\sigma(g))$) there is a unique-up-to-a-sign intertwining operator $\Pi(\sigma)$ with $\Pi(\sigma)^2 = 1$ and $\Pi(\sigma)\Pi(g) = \Pi(\sigma(g))\Pi(\sigma)$. There is a natural choice of the sign, namely of $\Pi(\sigma)$, when Π embeds in the space of automorphic forms, is generic or is unramified (and these choices coincide when they apply). In our case of the induced $I = I(1'(\mu') \otimes \kappa)$ there is a natural choice of $I(\sigma)$ obtained by induction from the natural choice of σ on $1'(\mu') \otimes \kappa$. However our $\Pi(\simeq I)$ is a subquotient of the space of automorphic forms. It is neither generic, nor unramified, nor a subspace of the space of automorphic forms. Hence it is not necessarily true that the choice of sign of $\Pi(\sigma)$ obtained by restricting the natural choice of $r(\sigma)$ ($(r(\sigma)\psi')(h) = \psi'(\sigma h)$, see II.2) should coincide with that of $I(\sigma)$, which is compatible with the local choices of the $I_v(\sigma)$. Consequently there is a sign $\varepsilon(\mu',\kappa)$, depending on μ, or μ', and κ, such that

$$\begin{aligned}\operatorname{tr}\Pi(\phi dg'\times\sigma) &= \varepsilon(\mu',\kappa)\prod \operatorname{tr} I(1'(\mu'_v)\otimes\kappa_v;\phi_v dg'_v\times\sigma)\\ &= \varepsilon(\mu',\kappa)\prod[\operatorname{tr}\pi^{\times}_{\mu_v}(f_v dh_v)-\operatorname{tr}\pi^{-}_{\mu_v}(f_v dh_v)].\end{aligned}$$

In Φ_2 we write $\operatorname{tr} I(1'(\mu')\otimes\kappa)(\phi dg'\times\sigma)$ instead of $\operatorname{tr}\Pi(\phi dg'\times\sigma)$ which emphasizes that in the trace formula it is the automorphic realization of Π rather than its realization as an induced representation which occurs (the difference is in the choice of sign of $\Pi(\sigma)$).

Put $\pi^{\times}_{\mu}=\otimes\pi^{\times}_{\mu_v}$. Then it occurs in the discrete spectrum with multiplicity $m(\pi^{\times}_{\mu})=\frac{1}{2}(1+\varepsilon(\mu',\kappa))$. It is automorphic precisely when $\varepsilon(\mu',\kappa)=1$. Now if the value $L(\frac{1}{2},\mu'\kappa)$ of the L-function of $\mu'\kappa$ at the center $\frac{1}{2}$ of the critical strip is nonzero, then ($\varepsilon(\frac{1}{2},\mu'\kappa)=1$ and) $\pi^{\times}_{\mu}$ is residual, hence $m(\pi^{\times}_{\mu})=1$ and $\varepsilon(\mu',\kappa)=1$. Here $\varepsilon(s,\mu'\kappa)$ denotes the ε-factor in the functional equation of $L(s,\mu'\kappa)$. If $L(\frac{1}{2},\mu'\kappa)=0$ then $\varepsilon(\frac{1}{2},\mu'\kappa)$ may take either value 1 or -1.

It was conjectured by Arthur [A3] and Harder [Ha], p. 173, that $\varepsilon(\mu',\kappa)=\varepsilon(\frac{1}{2},\mu'\kappa)$, namely that when $L(\frac{1}{2},\mu'\kappa)=0$, $\pi^{\times}_{\mu}$ is (equivalent to) an automorphic representation, necessarily cuspidal, precisely when $\varepsilon(\frac{1}{2},\mu'\kappa)=1$. A proof of this, at least for $F=\mathbb{Q}$, is based on the theory of theta liftings.

There remains the sum $\frac{1}{4}(\Phi_3+F_3)$. It is equal to

$$\begin{aligned}&\frac{1}{4}\sum_{\rho}\left[\prod\sum_{j=1}^{4}\operatorname{tr}\pi_{jv}(\rho_v)(f_v dg_v)+\sum_{i=1}^{3}\prod\sum_{j=1}^{4}\varepsilon_{ijv}\operatorname{tr}\pi_{jv}(\rho_v)(f_v dg_v)\right]\\ &\quad=\sum_{\pi} m(\rho,\pi)\prod\operatorname{tr}\pi_v(f_v dg_v).\end{aligned}$$

The first sum ranges over the discrete-spectrum automorphic $\mathbf{H}(\mathbb{A})$-packets of the form $\rho=\rho(\theta,{}'\theta)$, where θ, ${}'\theta$, $\omega/\theta\cdot{}'\theta$ are distinct. They are taken modulo the equivalence relation $\rho(\theta,{}'\theta)\sim\rho(\theta,\omega/\theta\cdot{}'\theta)\sim\rho({}'\theta,\omega/\theta\cdot{}'\theta)$. The multiplicity $m(\rho,\pi)=[1+\sum_{i=1}^{3}\varepsilon_i(\pi)]/4$ is equal to 0 or 1. The sum ranges over the products $\otimes\pi_v$, such that there exists ρ as above so that π_v lies in $\{\pi_v(\rho_v)\}$ for all v, and it is unramified at almost all v (namely it is π_{1v}), so that $\varepsilon_i(\pi_v)$ is 1 at almost all v.

5.2 Global theorems

This gives a complete description of the discrete spectrum of $\mathbf{G}(\mathbb{A})$. We introduce some more terminology. The local packets $\{\pi_v\}$ have been defined in all cases, except for $\pi_v = \pi_v^\times$. This is a nontempered G_v-module. We define the packet of $\pi_v^\times$ to consist of $\pi_v^\times$ alone. The *quasi-packet* $\pi(\mu_v)$ of $\pi_v^\times$ will be the set $\{\pi_v^\times, \pi_v^-\}$, consisting of a nontempered and a cuspidal. Thus a packet consists of tempered G_v-modules, or of a single nontempered element; a quasi-packet is defined for global purposes. Given a local packet P_v at all v, so that it contains an unramified member π_v^0 for almost all v, we define the *global packet* P to be the set of products $\otimes\pi_v$ over all v, so that $\pi_v = \pi_v^0$ for almost all v, and $\{\pi_v\} = P_v$ for all v. Given a character μ of C_E^1, we define the *quasi-packet* $\{\pi(\mu)\}$ as in the case of the packets, where P_v is replaced by the quasi-packet $\pi(\mu_v)$ at all v.

A standard argument, based on the absolute convergence of the sums, and the unitarizability of all representations which occur in the trace formula, implies:

5.2.1 Theorem. *The basechange lifting is a one-to-one correspondence from the set of packets and quasi-packets which contain a discrete-spectrum automorphic* $\mathbf{G}(\mathbb{A})$*-module, to the set of* σ*-invariant automorphic* $\mathbf{G}'(\mathbb{A})$*-modules which appear in* Φ_1, Φ_2 *or* Φ_3*. Namely, a discrete-spectrum* $\mathbf{G}(\mathbb{A})$*-module* π *lies in one of the following.* (1) *A packet* $\{\pi(\Pi)\}$ *associated with a discrete-spectrum* σ*-invariant* $\mathbf{G}'(\mathbb{A})$*-module* Π. (2) *A packet* $\{\pi(\rho)\}$ *associated with a discrete-spectrum automorphic* $\mathbf{H}'(\mathbb{A})$*-module* ρ *which is not of the form* $\rho(\theta, \omega/\theta^2)$. (3) *A quasi-packet* $\{\pi(\mu)\}$*, associated with an automorphic one-dimensional* $\mathbf{H}(\mathbb{A})$*-module* $\rho = \mu(\det)$.

The multiplicity of π from a packet $\{\pi(\Pi)\}$ in the discrete spectrum of $\mathbf{G}(\mathbb{A})$ is 1. Namely each member π of $\{\pi(\Pi)\}$ is automorphic, in the discrete spectrum. The multiplicity of a member π of a packet $\{\pi(\rho)\}$ or a quasi-packet $\{\pi(\mu)\}$ in the discrete spectrum of $\mathbf{G}(\mathbb{A})$ is equal to $m(\rho, \pi)$ or $m(\mu, \pi)$, respectively. This number is 1 or 0, but it is not constant over $\{\pi(\rho)\}$ or $\{\pi(\mu)\}$. Namely, in cases (2) and (3) not each member of $\{\pi(\rho)\}$ or $\{\pi(\mu)\}$ is automorphic.

5.2.2 Corollary. (1) *The multiplicity of an automorphic representation in the discrete spectrum of* $\mathbf{G}(\mathbb{A})$ *is* 1.

(2) *If* π *and* π' *are discrete-spectrum* $\mathbf{G}(\mathbb{A})$*-modules whose components* π_v *and* π'_v *are equivalent at almost all* v*, then they lie in the same packet, or quasi-packet.*

The first statement is called *multiplicity one theorem* for the discrete spectrum of $\mathbf{G}(\mathbb{A})$. The second is the *rigidity theorem.* It can be rephrased as asserting that the packets and quasi-packets partition the discrete spectrum.

The automorphic members π of the quasi-packet $\{\pi(\mu)\}$ have components π_v^- at the remaining finite set of places, which do not split in E/F. Each such π is a counter example to the naive Ramanujan Conjecture, which suggests that all components π_v of a cuspidal $\mathbf{G}(\mathbb{A})$-module π are tempered. However, we expect this Conjecture to be valid for the members π of the packets $\{\pi(\Pi)\}$, $\{\pi(\rho)\}$.

5.2.3 Proposition. *Suppose that* π *is a discrete-spectrum* $\mathbf{G}(\mathbb{A})$*-module which has a component of the form* $\pi_w^\times$*. Then almost all components of* π *are of the form* $\pi_v^\times$*, and* π *lies in a quasi-packet* $\{\pi(\mu)\}$*.*

Proof. The representation π defines a member Π of Φ_1, Φ_2 or Φ_3 whose component at w is of the form $I(\tau_w)$, where τ_w is a one-dimensional H'_w-module. But then Π must be of the form $I(\tau)$, where τ is a one-dimensional $\mathbf{H}'(\mathbb{A})$-module, and the claim follows. □

The Theorem has the following obvious

5.2.4 Corollary. *There is a bijection from the set of automorphic discrete-spectrum* $\mathbf{H}(\mathbb{A})$*-packets* ρ *which are not of the form* $\rho(\theta, \omega/\theta^2)$*, to the set of automorphic discrete-spectrum* $\mathbf{G}(\mathbb{A})$*-packets of the form* $\{\pi(\rho)\}$*.*

Also we deduce

5.2.5 Corollary. *Suppose that* π *is a discrete-spectrum* $\mathbf{G}(\mathbb{A})$*-module whose component* π_v *at a place* v *which splits* E/F *is elliptic. Then* π *lies in a packet* $\{\pi(\Pi)\}$*, where* Π *is discrete spectrum.*

Let $'G'$ be the multiplicative group of a division algebra of dimension 9 central over E, which is unramified outside the places u'_j, u''_j of E above the finite places u_j of F $(1 \leq j \leq j_0)$ which split in E, and which is anisotropic precisely at u'_j and u''_j. Suppose σ is an involution of the second kind, namely its restriction to the center $E^\times$ is $\sigma(z) = \overline{z}$. Denote by $'G$ the associated unitary group, namely the group of x in $'G'$ with $x\sigma(x) = 1$. It is not hard to compare the trace formulae in the compact case and deduce from our local lifting that we have

5.2.6 PROPOSITION. *The basechange lifting defines a bijection between the set of automorphic packets of* $'\mathbf{G}(\mathbb{A})$*-modules, and the set of* σ*-invariant automorphic* $'\mathbf{G}'(\mathbb{A})$*-modules.*

The Deligne-Kazhdan correspondence, from the set of automorphic representations of $'\mathbf{G}'(\mathbb{A})$, to the set of discrete-spectrum automorphic representations of $\mathbf{G}'(\mathbb{A})$ with an elliptic component at u_j and u'_j, implies

5.2.7 COROLLARY. *The relation* $'\pi_v \simeq \pi_v$ *for all* $v \neq u$ *defines a bijection between the set of automorphic packets of* $'\mathbf{G}(\mathbb{A})$*-modules* $'\pi$*, and the set of automorphic packets of* $'\mathbf{G}(\mathbb{A})$*-modules of the form* $\pi = \pi(\Pi)$*, whose component at* u *is elliptic.*

Finally we use the local results results of section I.5 and the global classification results of Theorem II.2.1 and its corollaries to describe the cohomology of automorphic forms on $\mathbf{G}(\mathbb{A})$. Thus let F be a totally real number field, E a totally imaginary quadratic extension of F, $\mathbf{G}'$ an inner form of $\mathbf{G}$ which is defined using the multiplicative group $'\mathbf{G}'$ of a division algebra of dimension 9 central over E and an involution of the second kind.

The set S of archimedean places of F is the disjoint union of the set S' where $'\mathbf{G}$ is quasi-split ($\simeq \mathrm{U}(2,1)$), and the set S'' where $'\mathbf{G}$ is anisotropic ($\simeq \mathrm{U}(3)$).

Put $'G_\infty = \Pi_{v\in S}\,'G_v$, $'K_\infty = \Pi_{v\in S}\,'K_v$. Here $'K_v = {}'G_v$ for v in S'', $'K_v \simeq \mathrm{U}(2)\times\mathrm{U}(1)$ for v in S'. Write $'G'_\infty$, $'G''_\infty$, $'K'_\infty$, $'K''_\infty$ for the corresponding products over S' and S''.

Fix an irreducible finite-dimensional $'G_v$-module F_v for all v in S. Put $\widetilde{F} = \otimes \widetilde{F}_v$ (v in S). Then $F_v = F_v(a_v, b_v, c_v)$ for integral $a_v > b_v > c_v$ if v is in S'.

Let $\pi = \otimes\pi_v$ be a discrete-spectrum infinite-dimensional automorphic $'\mathbf{G}(\mathbb{A})$-module. Then π_v is unitary for all v and π_v is infinite dimensional for all v outside S''. Put $\pi_\infty = \otimes\pi_v$ (v in S). If $H^*('\mathfrak{g}_\infty, 'K_\infty; \pi_\infty \otimes \widetilde{F}) \neq 0$, then $\pi_v = F_v$ for all v in S'', and

$$H^*('\mathfrak{g}_\infty, 'K_\infty; \pi_\infty \otimes \widetilde{F}) = \prod_{v\in S'} H^*('\mathfrak{g}_v, 'K_v; \pi_v \otimes \widetilde{F}_v).$$

5.3 Proposition. *Let π be an automorphic discrete-spectrum* $'\mathbf{G}(\mathbb{A})$*-module. Let d be* $\dim['\mathfrak{g}_\infty/'\mathfrak{k}_\infty]$. *If* $H^j('\mathfrak{g}_\infty), 'K_\infty; \pi_\infty \otimes \widetilde{F}) \neq 0$ *for $j \neq d$ then either π is one dimensional or π lies in a quasi-packet $\{\pi(\mu)\}$ of Theorem* 5.2.1*, associated with an automorphic one-dimensional* $\mathbf{H}(\mathbb{A})$*-module* $\rho = \mu \circ \det$. *In the last case we have* (1) $a_v - b_v = 1$ *or* $b_v - c_v = 1$ *for all v in* S', (2) π_v *is of the form* $\pi_v^\times$ *or* π_v^- *for all v outside* S'' (*it is* $\pi_v^\times$ *for almost all v*)*, and* (3) $'\mathbf{G}$ *is quasi-split at each finite place of the totally real field* F (*thus* $'G' = \mathrm{GL}(3, E)$ *is split*).

Proof. If π is infinite dimensional and $H^j \neq 0$ for $j \neq d$, then there is v in S' such that π_v is of the form $\pi_v^\times$. Theorem 5.2.1 then implies that π is of the form $\{\pi(\mu)\}$, and (2) follows. Since π_v is unitary (for v in S'), (1) follows from (2). Finally (3) results from Corollary 5.2.7 of Theorem 5.2.1, which asserts that if $'\mathbf{G}(\mathbb{A})$ has automorphic representations of the form $\{\pi(\mu)\}$ where μ is a character of $\mathbf{H}(\mathbb{A})$, then $'G' = \mathrm{GL}(3, E)$ is the multiplicative group of the split simple algebra of dimension 9 over E. □

The last assertion of the Proposition can be rephrased as follows.

5.4 Corollary. *If* $'\mathbf{G}'$ *is the multiplicative group of a division algebra, then any discrete-spectrum automorphic* $'\mathbf{G}(\mathbb{A})$*-module with cohomology outside the middle dimension is necessarily one dimensional.*

This sharpens results of Kazhdan [K4], section 4, in the case of $n = 3$.

III.6 Concluding remarks

The endoscopic lifting from U(2) to U(3) was first studied simultaneously with basechange from U(3) to $\mathrm{GL}(3, E)$ by means of the twisted trace formula in our unpublished manuscript [F3;III]: "L-packets and liftings for U(3)", Princeton 1982 (reference [Flicker] in [GP], [2] of [A2], and p. -2 in [L6]). It introduced a definition of packets, and reduced a complete description of these packets, including the rigidity and multiplicity one theorems for U(3) — as well as a complete description of the lifting from U(2) to U(3) and U(3) to $\mathrm{GL}(3, E)$ — to important technical assumptions, proven later; see below.

The problem of studying the endoscopic lifting from U(2) to U(3) was raised by R. Langlands [L6]. An attempt at this problem — based on stabilizing the trace formula for U(3) alone — was made in reference [25] of [L6] (= [Rogawski] in [GP]). But as explained in [F3;V], §4.6, p. 562/3, this attempt was conceptually insufficient for that purpose.

In [F3;V], §4.6, p. 562/3, we wrote (updating notations to refer to the present work instead of to [F3;V]) the following four paragraphs:

Theorem II.6.2.3 here (which is Theorem 4.4 of [F3;V]) deals with the quasi-endo-lifting e from U(2) to U(3). The proof is via the theory of basechange, and uses in addition to the rigidity theorem for GL(3) only the local basechange transfer of spherical functions from G to G' (Proposition I.2.1, I.2.2). At the remaining finite number of places we work with a function which vanishes on the (σ-) singular set. These functions are easy to transfer. We do not use the endo-transfer of I.2.3, although this is needed for the local lifting.

One may like to prove Theorem II.6.2.3 (= Theorem 4.4 of [F3;V]), by stabilizing the trace formula for U(3) alone, using only the fundamental lemma I.2.1 and I.2.2, and setting $\phi_u = 0$, namely choosing f_u with vanishing stable orbital integrals, so that the terms Φ are 0. Then, choosing discrete-spectrum ρ, for example in F_2, one would like to assert that by the rigidity theorem for $\mathbf{H}(\mathbb{A})$-packets [F3;II], there will be a single contribution in F_2. But if $F_2 \neq 0$ then $F_1 \neq 0$, and there exists π such that ρ quasi-endo-lifts to π.

This argument — which is the one on which the preprint [Rogawski] of [GP] (= [25] of [L6]) is based — does not work. The reason is that there are infinitely many places where E/F splits. There the dual-group is a direct product of the Weil group with the connected component, so we may work with ${}^LG = \mathrm{GL}(3,\mathbb{C})$. Then the homomorphism e takes $\mathrm{diag}(a,b)$ to $\mathrm{diag}(a,1/ab,b)$ if the central character is trivial. Since only conjugacy classes matter, and $\mathrm{diag}(a,1/ab,b)$ is conjugate to $\mathrm{diag}(a,b,1/ab)$, this conjugacy class in ${}^LG = \mathrm{GL}(3,\mathbb{C})$ is obtained also from the conjugacy class $\mathrm{diag}(a,1/ab)$ in ${}^LH = \mathrm{GL}(2,\mathbb{C})$.

Hence, using the spherical components of f at almost all v it is not possible to deduce that the components of ρ at almost all v are fixed; it is possible to say that at any split v the component ρ_v has only finitely many (3, if $[\{a,1/ab,b\}] = 3$) possibilities. This makes it *a priori* possible for infinitely many ρ, and we need only two, to appear in F_2. But these may cancel each other, so that one cannot deduce $F_2 \neq 0$. What makes our proof of Theorem II.6.2.3 work is the comparison to GL(3).

This observation was the basis for our preprint *L-packets and liftings for*

U(3). Our preprint was followed by our series of papers [FUi] (discussed below), as well as by a seminar of (Langlands and) Rogawski "to study what was proven in" our preprint (as the latter stated), and a book by Rogawski (*Automorphic Forms on Unitary Groups in Three Variables*, Ann. of Math. Study 123, 1990). This latter book reproduced in particular our false "proof" of multiplicity one theorem for U(3) (but not our correct proof).

Indeed, our preprint, written before [GP] became available, reduced the multiplicity one theorem to its case for generic representations. When [GP] was orally announced (Maryland conference, 1982) I have not checked what was the precise statement claimed in [GP]. It turned out to be insufficient for a proof, as we proceed to explain. This incomplete proof found its way to [F3;VI] as the second "proof" of Proposition 3.5.

The second proof of Proposition 3.5 of [F3;VI], on p. 48, is global, but incomplete. The false assertion is on lines 21-22: "Proposition 8.5(iii) (p. 172) and 2.4(i) of [GP] imply that for some π with $m(\pi) \neq 0$ above, we have $m(\pi) = 1$". Indeed, [GP], Prop. 2.4, defines $L^2_{0,1}$ to be the orthocomplement in the space L^2_0 (of cusp forms) of "all hypercusp forms", and claims: "(i) $L^2_{0,1}$ has multiplicity 1". ([GP], 8.5 (iii), asserts that π is in $L^2_{0,1}$). Now the sentence of [F3;VI], p. 48, l. 21-22 assumes that [GP], 2.4(i), means that any irreducible π in $L^2_{0,1}$ occurs in L^2_0 with multiplicity one. But the standard techniques of [GP], 2.4, show only that any irreducible π in $L^2_{0,1}$ occurs in $L^2_{0,1}$ with multiplicity one. *A priori* there can exist π' in L^2_0, isomorphic and orthogonal to $\pi \subset L^2_{0,1}$. In such a case we would have $m(\pi) > 1$.

Such a π' is locally generic (all of its local components are generic), and the question boils down to whether this implies that π' is generic ("has a Whittaker model"). This last claim might follow on using the theory of the theta correspondence, but this has not been done as yet. In summary, a clear form of [GP], 2.4(i) is: "Any irreducible π in $L^2_{0,1}$ occurs in $L^2_{0,1}$ with multiplicity one". In the analogous situation of GSp(2) such a statement is made in [So]. It is not sufficiently strong to be useful for us.

We noticed that the global argument of [F3;VI], p. 48, is incomplete while generalizing it in [F4;II] to the context of the symplectic group, where work of Kudla, Rallis and others on the Siegel-Weil formula is available to show that a locally generic cuspidal representation which is equivalent at almost all places to a generic cuspidal representation is generic.

However, [F3;VI] provides also a correct proof (p. 47) of the multiplicity one theorem for U(3) (Proposition 3.5 there). It is a local proof, based on a twisted analogue of Rodier's theorem on the coefficients associated with the regular orbits in the germ expansion of the character of an admissible representation. Such a proof was first used in the study [FK1] with D. Kazhdan of the metaplectic correspondence. The details were omitted from [F3;VI]. They are given in Proposition III.3.5 here, and in section III.4.

A local proof, based on a twisted analogue of Rodier's result, is also used in [F2;I].

In addition to providing a correct proof of multiplicity one theorem for U(3), our proof shows that in each packet of representations of U(3) which lifts to a generic representation of $GL(3, E)$ there is precisely one generic representation.

The main achievement of our work — present already in our original preprint — is the introduction of a definition of packets and quasi-packets in terms of liftings, from $U(2, E/F)$ and to $GL(3, E)$, in addition to the observation that the endoscopic lifting from U(2) to U(3) could only be studied (by means of the trace formula) simultaneously with basechange from $U(3, E/F)$ to $GL(3, E)$. Further we obtain a complete description of these packets, including multiplicity one and rigidity for packets of U(3). These results appeared in [F3;IV], [F3;V], [F3;VI], stated for all local representations and global representations with two (in fact only one, using the technique of [FK2]) elliptic components.

In [F3;VII] we introduce a new technique of proving the equality of the trace formulae of interest for sufficiently general matching test measures to establish all our liftings for all global representations, without any restriction. In our original preprint [F3;III] we computed all terms in all trace formulae which occur as a preparation for such a comparison. In [F3;VII], which is II.4 here, we use regular spherical functions, whose orbital integrals vanish on split elements unless the values of the roots on these split elements are far from 1.

In the present case of basechange there is a simplifying fact, that there are places which split in E/F. This leads to a cancellation of weighted orbital integrals at the place in question, and to use of the invariance of the trace formulae at this place. An analogous argument uses regular Iwahori biinvariant functions. Such an analogous argument was used in the study of the metaplectic correspondence and the simple algebras correspondence in

[FK2] with Kazhdan, and for cyclic basechange for GL(n) in [F1;VI] — in both cases for cusp forms with at least one cuspidal component. It was also used in the case of cyclic basechange for GL(2) in [F1;IV] and in the study of the symmetric square in [F2;VI] for all automorphic representations, without any restrictions.

The use of regular functions in the trace formula is motivated by Deligne conjecture on the Lefschetz fixed point formula first used in the study of Drinfeld moduli schemes in [FK3]. The virtue of the technique is that we do not need to carry out the elaborate computations of the nonelliptic terms in the trace formula. The use of regular functions annihilates *a priori* the weighted orbital integrals and the integrals of the singular elements in the trace formulae. Nevertheless the generality of our results is not affected.

This explains why our work is considerably shorter than analogous works in the area.

However, our argument applies so far only in cases of rank one (including twisted-rank one). It will be interesting to develop it to higher-rank cases.

As emphasized by Langlands, there is no result at all without the fundamental lemma. In [F3;VIII] we introduce a new technique to prove the fundamental lemma for $\mathrm{U}(3, E/F)$ and its endoscopic group $\mathrm{U}(2, E/F)$. It is based on an intermediate double coset decomposition $H\backslash G/K$ of the double coset $T\backslash G/K$ which describes the orbital integral. It is given in section I.3 here. It is inspired by Weissauer's work on the fundamental lemma for Sp(2) and its endoscopic groups. A similar argument is used in [F4;I] to prove the fundamental lemma for (GL(4),GSp(2)) and from GSp(2) to its endoscopic groups, and in [F2;VII] to prove the fundamental lemma for the symmetric square lifting from SL(2) to PGL(3). This technique is elementary and explicit.

A computation of the orbital integrals in terms of lattices is offered by Kottwitz in [LR], p. 360.

A new computation, due to J.G.M. Mars, also coached in terms of counting lattices, is described in section I.6 here, based on Mars' letter to me.

PART 3. ZETA FUNCTIONS OF SHIMURA VARIETIES OF U(3)

INTRODUCTION

Eichler expressed the Hasse-Weil Zeta function of an elliptic modular curve as a product of L-functions of modular forms in 1954, and, a few years later, Shimura introduced the theory of canonical models and used it to similarly compute the Zeta functions of the quaternionic Shimura curves. Both authors based their work on congruence relations, relating a Hecke correspondence with the Frobenius on the reduction mod p of the curve.

Ihara introduced (1967) a new technique, based on comparison of the number of points on the Shimura variety over various finite fields with the Selberg trace formula. He used this to study forms of higher weight. Deligne [D1] explained Shimura's theory of canonical models in group theoretical terms, and obtained Ramanujan's conjecture for some cusp forms on $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$: their normalized Hecke eigenvalues are algebraic and all of their conjugates have absolute value 1 in $\mathbb{C}^{\times}$, for almost all components.

Langlands [L1-3] developed Ihara's approach to predict the contribution of the tempered automorphic representations to the Zeta function of arbitrary Shimura varieties, introducing in the process the theory of endoscopic groups. He carried out the computations in [L2] for subgroups of the multiplicative groups of nonsplit quaternionic algebras.

Using Arthur's conjectural description [A2-4] of the automorphic nontempered representations, Kottwitz [Ko5] developed Langlands' conjectural description of the Zeta function to include nontempered representations. In [Ko6] he associated Galois representations to automorphic representations which occur in the cohomology of unitary groups associated to division algebras. In this anisotropic case the trace formula simplifies.

In the anisotropic case the unramified terms of the Zeta function are expressed in terms of the trace of the Frobenius on the virtual cohomology $\sum_i (-1)^i H^i(\mathcal{S} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$ with coefficients in a smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathbb{V}$; here $\mathbb{E}$ is the reflex field and $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. The functional equation follows from Poincaré duality. But when the Shimura variety $\mathcal{S}$ is not proper, the duality relates H^i with the cohomology with compact supports $H_c^{2\dim - i}$. For a Shimura curve $\mathcal{S}$ Deligne interpreted Shimura's "parabolic" cohomology of discrete groups as "interior" cohomology $H^i_! = \mathrm{Im}[H^i_c \to H^i]$ (Harder's notations). It satisfies Poincaré duality, and purity ("Weil's conjecture").

For noncompact higher-dimensional $\mathcal{S}$, to have a functional equation one needs cohomology satisfying Poincaré duality, and this depends on a choice of a compactification. The Satake Baily-Borel compactification $\mathcal{S}'$ is algebraic, and the $\overline{\mathbb{Q}}_\ell$-adic intersection cohomology (with middle perversity) $IH^i(\mathcal{S}' \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$ has the required properties. The Eichler-Shimura relations were extended by Matsushima-Murakami to anisotropic symmetric spaces and by Borel to isotropic such spaces, to express the L^2-cohomology $H_{(2)}$ in terms of discrete-spectrum representations of the underlying reductive group. Zucker's conjecture [Zu] translated the intersection cohomology — tensored by $\mathbb{C}$ — to the L^2-cohomology. In fact, for curves H^1 coincides with IH^1 for the natural compactification, and in general there are natural maps $H^i_c \to IH^i \to H^i$. These considerations suggested that for general Shimura varieties, the natural Zeta function is indeed that defined in terms of $IH^*(\mathcal{S}' \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$.

The only known approach to determine the decomposition of the cohomology is that of comparison of the Lefschetz fixed point formula with the Arthur-Selberg trace formula. But in the isotropic case only Grothendieck's fixed point formula for the powers of the Frobenius was known. The lack of Hecke correspondences would not permit separating the Hecke algebra modules in the cohomology (IH, H or H_c). To overcome this difficulty Deligne conjectured that the Lefschetz fixed point formula for a correspondence on a variety over a finite field remains valid — as though the variety was proper — on the $\overline{\mathbb{Q}}_\ell$-adic cohomology H^*_c with compact supports, provided the correspondence is twisted by a sufficiently high power of the Frobenius. It is not valid for H^*.

Deligne's conjecture was used with Kazhdan in [FK3] to decompose the cohomology with compact supports of the Drinfeld moduli scheme of elliptic modules, and relate Galois representations and automorphic representations of GL(n) over function fields of curves over finite fields. It suggested various forms simplifying the trace formula for automorphic representations ([FK2], [F3;VII], [F1;IV], [F1;VI]).

Deligne's conjecture was proven in some cases by Zink [Zi], Pink [P2], Shpiz [Sc], and in general by Fujiwara [Fu], and recently Varshavsky [Va]. We use it here to express the Zeta function of the Shimura varieties $\mathcal{S}$ of the quasi-split semisimple F-rank one unitary group of similitudes $G = \mathrm{GU}(3, E/F)$ associated with a totally imaginary quadratic extension E of a totally real number field F and with any coefficients, in terms of automorphic representations of this group and of its unique proper elliptic endoscopic group, $H = \mathrm{G}(\mathrm{U}(2, E/F) \times \mathrm{U}(1, E/F))$. Of course by the Zeta function we mean the one defined by means of H_c^*.

Thus our main result is the decomposition of the $\overline{\mathbb{Q}}_\ell$-adic cohomology with compact supports of the Shimura variety $\mathcal{S}$ (with coefficients in a finite-dimensional representation of G) as a Hecke $\times$ Galois module. In fact we consider only the semisimplification of this module. In conclusion we associate a Galois representation to any "cohomological" automorphic representation of $G(\mathbb{A})$. Here $\mathbb{A} = \mathbb{A}_F$ denotes the ring of adèles of F, and $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$. Our results are consistent with the conjectures of Langlands and Kottwitz [Ko5]. We make extensive use of the results of [Ko5], expressing the Zeta function in terms of stable trace formulae of GU(3) and its endoscopic group G(U(2) $\times$ U(1)), also for twisted coefficients. We use the fundamental lemma proven in this case in [F3;VIII] and assumed in [Ko5] in general.

In the case of GSp(2), using congruence relations Taylor [Ty] associated Galois representations to automorphic representations of $\mathrm{GSp}(2, \mathbb{A}_\mathbb{Q})$ which occur in the cohomology of the Shimura three-fold, in the case of $F = \mathbb{Q}$. Laumon [Ln] used the Arthur-Selberg trace formula and Deligne's conjecture to get more precise results on such representations again for the case $F = \mathbb{Q}$ where the Shimura variety is a three-fold, and with trivial coefficients. Similar results were obtained by Weissauer [We] (unpublished) using the topological trace formula of Harder and Goresky-MacPherson. A more precise result is obtained in [F4;VII]. It uses the classification of the automorphic representations of PGSp(2) obtained in [F4;II-IV].

Here we use the description of the automorphic representations of the group $\mathrm{GU}(3, E/F)$ by [F3], together with the fundamental lemma [F3;VIII] and Deligne's conjecture [Fu], [Va], to decompose the $\overline{\mathbb{Q}}_\ell$-adic cohomology with compact supports, compare it with the intersection cohomology, and describe all of its constituents. This permits us to compute the Zeta function, in addition to describing the Galois representation associated to each automorphic representation occurring in the cohomology. We work with any discrete-spectrum automorphic representation. There are no local restrictions.

We work with any coefficients, and with any totally real base field F. In the case $F \neq \mathbb{Q}$ the Galois representations which occur are related to the interesting "twisted tensor" representation of the dual group. Using Deligne's "purity" theorem [D4] (and Gabber in the context of IH) we conclude that for all good primes p the Hecke eigenvalues of any discrete-spectrum representation $\pi = \otimes\pi_p$ occurring in the cohomology are algebraic. All conjugates of these algebraic numbers lie on the unit circle for π which basechange lift ([F3;I, VI]) to cuspidal representations on GL(3) or to representations induced from cuspidal representations of a Levi factor of a parabolic subgroup. This is known as the "generalized" Ramanujan conjecture (for GU(3)). Counter examples to the naive Ramanujan conjecture are given by π which basechange lift to representations induced from one-dimensional representations of the maximal parabolic subgroup.

The cases of surfaces (compact if $F \neq \mathbb{Q}$) associated with forms of $\mathrm{U}(3, E/F)$ which are ramified at all real places but one, in particular the quasi-split case $F = \mathbb{Q}$, are discussed in [LR]. We deal with the quasi-split $\mathrm{U}(3, E/F)$, especially where $F \neq \mathbb{Q}$.

1. Statement of results

To describe our results we briefly introduce the objects of study; more detailed account is given in the body of the work. Let F be a totally real number field, E a totally imaginary quadratic extension of F, $G = \mathrm{GU}(3, E/F)$ the quasi-split unitary group of similitudes in three variables associated with E/F whose Borel subgroup is the group of upper triangular matrices. In fact we define the algebraic group G by means of the Hermitian form $J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$. It suffices to specify G as an F-variety by its $\overline{F}$-points and the $\mathrm{Gal}(\overline{F}/F)$ action. Thus put $G(\overline{F}) = \mathrm{GL}(3, \overline{F}) \times \overline{F}^{\times}$, and let $\tau \in \mathrm{Gal}(\overline{F}/F)$ act on (g, λ), $g = (g_{ij}) \in \mathbf{G}(\overline{F})$, $\lambda \in \overline{F}^{\times}$, by $\tau(g, \lambda) = (\tau g_{ij}, \tau\lambda)$ if $\tau|E = 1$, and $\tau(g, \lambda) = (\theta(\tau g_{ij})\lambda, \tau\lambda)$ if $\tau|E \neq 1$, where $\theta(g) = J^t g^{-1} J$ and ${}^t g$ indicates the transpose (g_{ji}) of g.

Denote by $x \mapsto \overline{x}$ the action of the nontrivial element of $\mathrm{Gal}(E/F)$ on $x \in E$. Put $\overline{g} = (\overline{g}_{ij})$ for g in $\mathrm{GL}(3, E)$. Put $\sigma(g) = \theta(\overline{g})$. Thus the group $G(F)$ of F-points on G is

$$\{(g, \lambda) \in \mathrm{GL}(3, E) \times E^{\times};\ {}^t\overline{g} J g = \lambda J\}$$

$$= \{(g, \lambda) \in \mathrm{GL}(3, E) \times E^\times;\ \lambda\sigma(g) = g\}.$$

Applying transpose-bar to ${}^t\overline{g}Jg = \lambda J$ and taking determinants we see that $\lambda \in N_{E/F}E^\times$.

Denote by $\mathrm{R}_{L/M}$ the functor of restriction of scalars from L to M, where L/M is a finite field extension. If V is a variety over L, $\mathrm{R}_{L/M}V$ is a variety over M, and $(\mathrm{R}_{L/M}V)(A) = V(A \otimes_M L)$ for any M-ring A. We use this construction to work with the group $G' = \mathrm{R}_{F/\mathbb{Q}}G$ over $\mathbb{Q}$, whose group of $\mathbb{Q}$-points is $G(F)$.

Write $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_{\mathbb{Q},f}$ for the rings of adèles and finite adèles of $\mathbb{Q}$. Let K_f be an open compact subgroup of $G'(\mathbb{A}_{\mathbb{Q}f})$ of the form $\prod_{p<\infty} K_p$, K_p open compact in $G'(\mathbb{Z}_p)$ for all p with equality for almost all primes p.

Let $h : \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G'$ be an $\mathbb{R}$-homomorphism satisfying the axioms of [D3]. Let $\mathcal{S}_{K_f}$ be the associated Shimura variety, defined over its reflex field $\mathbb{E}$, a CM-field in E.

The finite-dimensional irreducible algebraic representations of the group G are parametrized by their highest weights

$$(a, b, c) : \mathrm{diag}(x, y, z) \mapsto x^a y^b z^c,$$

where a, b, $c \in \mathbb{Z}$ and $a \geq b \geq c$. Those with trivial central character have $a + b + c = 0$. We denote them by $(\xi_{a,b,c}, V_{a,b,c})$.

Half the sum of the positive roots is $(1, 0, -1)$.

For each rational prime ℓ, the representation

$$(\xi_{\mathbf{a},\mathbf{b},\mathbf{c}} = \otimes_{\sigma\in S}\xi_{a_\sigma,b_\sigma,c_\sigma}, V_{\mathbf{a},\mathbf{b},\mathbf{c}} = \otimes_{\sigma\in S}V_{a_\sigma,b_\sigma,c_\sigma})$$

of G' over $\overline{\mathbb{Q}}$ (S is the set of embeddings of F in $\mathbb{R}$) defines a smooth $\overline{\mathbb{Q}}_\ell$-adic sheaf $\mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\ell}$ on $\mathcal{S}_{K_f}$. Denote by $\mathbb{H}_{K_f,\overline{\mathbb{Q}}_\ell}$ the Hecke convolution algebra $C_c^\infty(K_f\backslash G(\mathbb{A}_f)/K_f, \overline{\mathbb{Q}}_\ell)$ of compactly supported $\overline{\mathbb{Q}}_\ell$-valued bi-$K_f$-invariant functions on $G(\mathbb{A}_{\mathbb{Q}f})$. We are concerned with the decomposition of the $\overline{\mathbb{Q}}_\ell$-adic vector space $H_c^i(\mathcal{S}_{K_f} \otimes_\mathbb{E} \overline{\mathbb{Q}}, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\ell})$ as a $\mathbb{H}_{K_f,\overline{\mathbb{Q}}_\ell} \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$-module, or more precisely the virtual bi-module $H_c^* = \oplus(-1)^i H_c^i$, $0 \leq i \leq 2\dim \mathcal{S}_{K_f}$. We consider only the semisimplification of H_c^*, as we only study traces.

Fix a fields isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$.

Write $H_c^*(\pi_f)$ for $\mathrm{Hom}_{\mathbb{H}_{K_f}}(\pi_f, H_c^*(\mathcal{S}_{K_f} \otimes_\mathbb{E} \overline{\mathbb{Q}}, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c}}))$.

THEOREM 1. *The irreducible* $\mathbb{H}_{K_f,\overline{\mathbb{Q}}_\ell} \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$*-modules which occur nontrivially in* $H_c^*(\mathcal{S}_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\ell})$ *are of the form* $\pi_f^{K_f} \otimes H_c^*(\pi_f)$*, where* π_f *is the finite component* $\otimes_{p<\infty}\pi_p$ *of a discrete-spectrum representation* π *of* $G'(\mathbb{A}_{\mathbb{Q},f})$*, and* $\pi_f^{K_f}$ *denotes its subspace of* K_f*-fixed vectors. The archimedean component* $\pi_\infty = \otimes_{\sigma\in S}\pi_\sigma$ *of* π*, where* $S = \mathrm{Emb}(F,\mathbb{R})$ *and* $G'(\mathbb{R}) = \prod_{\sigma\in S} G(F \otimes_{F,\sigma} \mathbb{R})$*, has components* π_σ *whose infinitesimal character is* $(a_\sigma, b_\sigma, c_\sigma) + (1,0,-1)$*.*

Conversely, if π *is any discrete-spectrum representation of* $G'(\mathbb{A}_{\mathbb{Q},f})$ *whose archimedean component* $\pi_\infty = \otimes_{\sigma\in S}\pi_\sigma$ *is such that the infinitesimal character of* π_σ *is* $(a_\sigma, b_\sigma, c_\sigma) + (1,0,-1)$*,* $a_\sigma \geq b_\sigma \geq c_\sigma$ *for each* $\sigma \in S$ *(we call such representations* π **cohomological***), and* $\pi_f^{K_f} \neq \{0\}$*, then the* π_f*-isotypical part* $H_c^*(\pi_f)$ *of* $H_c^*(\mathcal{S}_{K_f} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\ell})$ *is nonzero.*

The main point here is that the π which occur in H_c^* are automorphic, in fact discrete spectrum with the prescribed behavior at ∞ and ramification controlled by K_f. Each cohomological π occurs for some K_f depending on π. The same statement is known for $H_{(2)}^*$ by the "Matsushima-Murakami" theory of Borel, hence for IH^* by Zucker's conjecture.

We proceed to describe the semisimplification of the Galois representation $H_c^*(\pi_f)$ attached to π_f. For this purpose we first need to list the cohomological π. Recall that $G'(\mathbb{Q}) = G(F)$ and $G'(\mathbb{A}_{\mathbb{Q}}) = G(\mathbb{A}_F)$.

The discrete-spectrum automorphic representations π of our unitary group are described in [F3] in terms of packets and quasi-packets, E/F-basechange lifting $b : {}^L G = \mathrm{GL}(3,\mathbb{C}) \rtimes W_F \to {}^L(\mathrm{R}_{E/F}G) = [\mathrm{GL}(3,\mathbb{C}) \times \mathrm{GL}(3,\mathbb{C})] \rtimes W_F$ (diagonal embedding), and endoscopic lifting $e : {}^L H = [\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})] \rtimes W_F \to {}^L G$. Here $\widehat{H}$ is viewed as the centralizer of $\mathrm{diag}(-1,1,-1)$ in $\widehat{G}$. A detailed account of the lifting theorems of [F3;VI] is given in the text below, as are the definitions of [F3;VI] of packets and quasi-packets; [F3;VI] can be replaced by our [F3;I] everywhere below. Quasi-packets refer to nontempered representations. We distinguish five types of cohomological representations π of $G(\mathbb{A}_F) = \mathrm{GU}(3,\mathbb{A}_F)$.

(1) π in a stable packet which basechange lifts to a cuspidal representation of $\mathrm{GL}(3,\mathbb{A}_E)$. The components π_σ $(\sigma \in S)$ are discrete series with infinitesimal characters $(a_\sigma, b_\sigma, c_\sigma)$ $+(1,0,-1)$.

(2) π in an unstable packet which basechange lifts to a representation of $\mathrm{GL}(3,\mathbb{A}_E)$ normalizedly induced from a cuspidal representation $\rho' \otimes \kappa$ of a maximal parabolic subgroup, where $\rho' \otimes \kappa$ is obtained by the unstable

basechange map $b' : \mathrm{GL}(2,\mathbb{C}) \rtimes W_F \to {}^L(\mathrm{R}_{E/F}G) = [\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})] \rtimes W_F$ on the unitary group $\mathrm{U}(2,E/F)(\mathbb{A}_F)$ in two variables associated with E/F from a single cuspidal packet ρ of $\mathrm{U}(2,E/F)(\mathbb{A}_F)$. This π is the endoscopic lift of ρ.

(3) π in an unstable packet which basechange lifts to a representation of $\mathrm{GL}(3,\mathbb{A}_E)$ normalizedly induced from the Borel subgroup. It is the endoscopic lift of three inequivalent cuspidal packets ρ_i, $i = 1, 2, 3$.

(4) π is the endoscopic lift of a one-dimensional representation μ of $\mathrm{U}(2,E/F)(\mathbb{A}_F)$. It is an unstable quasi-packet (almost all of its components are nontempered $\pi_v^\times$; the remaining finite number of components are cuspidal π_v^-). It lifts to a representation of $\mathrm{GL}(3,\mathbb{A}_E)$ induced from a one-dimensional representation of a maximal parabolic subgroup.

(5) π is one dimensional. Here $(a_v, b_v, c_v) = (0,0,0)$.

A global (quasi-)packet is the restricted product of local (quasi-)packets, which are sets of one or two irreducibles in the nonarchimedean case, pointed by the property of being unramified (the local (quasi-) packets contain a single unramified representation at almost all places). The packets (1) and the quasi-packet (5) are stable: each member is automorphic and occurs in the discrete spectrum with multiplicity one. The packets (2), (3) and quasi-packets (4) are not stable, their members occur in the discrete spectrum with multiplicity one or zero, according to a formula of [F3;VI] recalled below.

We now describe the (semisimple, by our convention) representation $H_c^*(\pi_f)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ associated to the finite component π_f of the π listed above such that $\pi_\infty = \otimes_{\sigma \in S} \pi_\sigma$ has nonzero Lie algebra cohomology. The Chebotarev's density theorem asserts that the Frobenius elements $\mathrm{Fr}_\wp$ for almost all primes $\wp$ of $\mathbb{E}$ make a dense subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$. Hence it suffices to specify the conjugacy class of $\rho(\mathrm{Fr}_\wp)$ for almost all $\wp$. This makes sense since $H_c^*(\pi_f)$ is unramified at almost all $\wp$, trivial on the inertia subgroup $I_\wp$ of the decomposition group $D_\wp = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{E}_\wp)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ at $\wp$, and D_p/I_p is (topologically) generated by $\mathrm{Fr}_\wp$. The conjugacy class $H_c^*(\pi_f)(\mathrm{Fr}_\wp)$ is determined by its trace. Being semisimple, it is determined by $H_c^*(\pi_f)(\mathrm{Fr}_\wp^j)$ for all sufficiently large j. Note that $\dim \mathcal{S}_{K_f} = 2[F:\mathbb{Q}]$. We consider only p which are unramified in E, thus the residual cardinality q_u of F_u at any place u of F over p is p^{n_u}, $n_u = [F_u : \mathbb{Q}_p]$. Further we use only p with $K_f = K_p K^p$, where $K_p = H'(\mathbb{Z}_p)$ is the standard maximal compact, thus $\mathcal{S}_{K_f}$ has good reduction at p.

Part of the data defining the Shimura variety is the $\mathbb{R}$-homomorphism $h : \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G' = \mathrm{R}_{F/\mathbb{Q}}G$. Over $\mathbb{C}$ the one-parameter subgroup $\mu : \mathbb{C}^\times \to G'(\mathbb{C})$, $\mu(z) = h(z,1)$ factorizes through any maximal $\mathbb{C}$-torus $T'(\mathbb{C}) \subset G'(\mathbb{C})$. The $G'(\mathbb{C})$-conjugacy class of μ defines then a Weyl group $W_{\mathbb{C}}$-orbit $\mu = \prod_\tau \mu_\tau$ in $X_*(T') = X^*(\widehat{T}')$. The dual torus $\widehat{T}' = \prod_\sigma \widehat{T}$ in $\widehat{G}' = \prod_\sigma \widehat{G}$, $\sigma \in \mathrm{Emb}(F,\mathbb{R})$, can be taken to be the diagonal subgroup, and $X^*(\widehat{T}) = \mathbb{Z}^3$. Further, τ ranges over a CM-type Σ. Thus Σ is a subset of $\mathrm{Emb}(E,\mathbb{C})$ with empty $\Sigma \cap c\Sigma$ and $\Sigma \cup c\Sigma = \mathrm{Emb}(E,\mathbb{C})$, where c denotes complex conjugation. We choose μ_τ to be the character $(1,0,0) : \mathrm{diag}(a,b,c) \mapsto a$ of $\widehat{T}$. Then $\mu_{c\tau} = (1,1,0)$. Thus the $G(\mathbb{C})$-orbit of the coweight μ_τ determines a $W_{\mathbb{C}}$-orbit of a character — which we again denote by μ_τ — of $\widehat{T}$, which is the highest weight of the standard representation $r^0_\tau = \mathrm{st}$ of $\mathrm{GL}(3,\mathbb{C})$, while $\mu_{c\tau} = (1,1,0)$ is that of $r^0_{c\tau} = \Lambda^2(\mathrm{st})(= \det \otimes \mathrm{st}^\vee)$. Put $r^0_\mu = \otimes_{\tau\in\Sigma} r^0_\tau$. It is a representation of $\widehat{G}'$.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathrm{Emb}(E,\overline{\mathbb{Q}})$. The stabilizer of μ, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$, defines the reflex field $\mathbb{E}$. It is a CM-field contained in E. We work only with primes p unramified in E. Thus for each prime $\wp$ of $\mathbb{E}$ over p, the decomposition subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{E}_\wp)$ at $\wp$ acts on r^0_μ via its quotient $\langle \mathrm{Fr}_\wp \rangle$ by the inertia subgroup. The Frobenius $\mathrm{Fr}_\wp = \mathrm{Fr}_p^{n_\wp}$ at $\wp$ is the $n_\wp = [\mathbb{E}_\wp : \mathbb{Q}_p]$-th power of Fr_p.

An irreducible admissible representation π_p of $G(F \otimes \mathbb{Q}_p) = G'(\mathbb{Q}_p) = \prod_{u|p} G(F_u)$ has the form $\otimes_u \pi_u$. Suppose it is unramified. If u splits in E, thus $E \otimes_F F_u = F_u \oplus F_u$, then π_u has the form $\pi(\mu_{1u}, \mu_{2u}, \mu_{3u})$, a subquotient of the normalizedly induced representation $I(\mu_{1u}, \mu_{2u}, \mu_{3u})$ of $G(F_u) = \mathrm{GL}(3, F_u)$, where μ_{iu} are unramified characters of $F_u^\times$. If u stays prime in E, thus $E_u = E \otimes_F F_u$ is a field, π_u has the form $\pi(\mu_u) \subset I(\mu_u)$. Write μ_{mu} for the value $\mu_{mu}(\boldsymbol{\pi}_u)$ at any uniformizing parameter $\boldsymbol{\pi}_u$ of $F_u^\times$ (and $E_u^\times$). Put $t_u = t(\pi_u) = \mathrm{diag}(\mu_{1u}, \mu_{2u}, \mu_{3u})$ in the split case and $t(\pi_u) = \mathrm{diag}(\mu_u, 1, 1) \times \mathrm{Fr}_u$ if E_u is a field. In the latter case we also write $\mu_{1u} = \mu_u^{1/2}$, $\mu_{2u} = 1$, $\mu_{3u} = \mu_u^{-1/2}$, and $t_u = (t(\pi_u)^2)^{1/2} = \mathrm{diag}(\mu_u^{1/2}, 1, \mu_u^{-1/2})$. Note that $\mathrm{tr}[t_u^j] = \mu_{1u}^j + \mu_{2u}^j + \mu_{3u}^j$.

The representation π_p is parametrized by the conjugacy class of $\mathbf{t}(\pi_p) = \mathbf{t}_p \times \mathrm{Fr}_p$ in the unramified dual group ${}^L G'_p = \widehat{G}^{[F:\mathbb{Q}]} \rtimes \langle \mathrm{Fr}_p \rangle$. Here $\mathbf{t}_p$ is the $[F:\mathbb{Q}]$-tuple $(\mathbf{t}_u; u|p)$ of diagonal matrices in $\widehat{G} = \mathrm{GL}(3,\mathbb{C})$, where each $\mathbf{t}_u = (t_{u1}, \dots, t_{un_u})$ is any $n_u = [F_u : \mathbb{Q}_p]$-tuple with $\prod_i t_{ui} = t_u$. The Frobenius Fr_p acts on each $\mathbf{t}_u$ by permutation to the left: $\mathrm{Fr}_p(\mathbf{t}_u) =$

$(t_{u2},\dots,t_{un_u},\theta(t_{u1}))$. Here $\theta=\mathrm{id}$ if $E_u=F_u\oplus F_u$ and $\theta(t_u)=J^{-1t}t_u^{-1}J$ if E_u is a field. Each π_u is parametrized by the conjugacy class of $\mathbf{t}(\pi_u)=\mathbf{t}_u\times\mathrm{Fr}_p$ in the unramified dual group ${}^LG'_u=\widehat{G}^{n_u}\times\langle\mathrm{Fr}_p\rangle$, or alternatively by the conjugacy class of $t_u\times\mathrm{Fr}_u$ in ${}^LG_u=\widehat{G}\times\langle\mathrm{Fr}_u\rangle$, where $\mathrm{Fr}_u=\mathrm{Fr}_p^{n_u}$.

Our determination of the Galois representation attached to π_f is in terms of the traces of the representation r^0_μ of the dual group ${}^LG'_{\mathbb{E}}=\widehat{G}'\rtimes W_{\mathbb{E}}$ at the positive powers of the $n_\wp$th powers of the classes $\mathbf{t}(\pi_p)=(\mathbf{t}(\pi_u);u|p)$ parametrizing the unramified components $\pi_p=\otimes_{u|p}\pi_u$. The representation $H^*_c(\pi_f)$ is determined by $\mathrm{tr}[\mathrm{Fr}^j_\wp|H^*_c(\pi_f)]$ for every integer $j\geq 0$, prime p unramified in E, and $\mathbb{E}$-prime $\wp$ dividing p, as follows.

THEOREM 2. *Let π_f be an irreducible representation of $G(\mathbb{A}_f)$ so that $H^*_c(\pi_f)\neq 0$. Then there are representations π_σ of $G(\mathbb{R})$ $(\sigma\in S)$ with*

$$H^*(\mathfrak{g},K_\sigma;\pi_\sigma\otimes V_{(a_\sigma,b_\sigma,c_\sigma)})\neq 0,$$

thus with infinitesimal characters $(a_\sigma,b_\sigma,c_\sigma)+(1,0,-1)$, such that $\pi=\pi_f\otimes(\otimes_\sigma\pi_\sigma)$ is in the discrete spectrum.

(1) *Suppose that π (is cuspidal and) basechange lifts to a cuspidal representation of $\mathrm{GL}(3,\mathbb{A})$. Then the trace $\mathrm{tr}[\mathrm{Fr}^j_\wp|H^*_c(\pi_f)]$ is the product of $q_\wp^{\frac{j}{2}\dim\mathcal{S}_{K_f}}$ and*

$$\mathrm{tr}\,r^0_\mu[(\mathbf{t}(\pi_p)\times\mathrm{Fr}_p)^{jn_\wp}]=\prod\mathrm{tr}\,r^0_u[(\mathbf{t}(\pi_u)\times\mathrm{Fr}_p)^{jn_\wp}]=\prod\big(\mathrm{tr}[t_u^{\frac{jn_\wp}{j_u}}]\big)^{j_u}.$$

Here $j_u=(jn_\wp,n_u)$, and all products range over the places u of F over p.

(2) *Suppose that π basechange lifts to a representation normalizedly induced from a cuspidal representation of the maximal parabolic subgroup. Then π is the endoscopic lift of a cuspidal representation $\tilde\rho$ not of the form $\rho(\theta,{}'\theta)\times{}''\theta$ of $H(\mathbb{A})=\mathrm{U}(2,\mathbb{A})\times\mathrm{U}(1,\mathbb{A})$. Its real component is $\otimes_\sigma\tilde\rho_\sigma$, where $\tilde\rho_\sigma$ is $\rho^+_\sigma=\rho(a_\sigma,b_\sigma)\times\rho(c_\sigma)$, $\rho_\sigma=\rho(a_\sigma,c_\sigma)\times\rho(b_\sigma)$ or $\rho^-_\sigma=\rho(b_\sigma,c_\sigma)\times\rho(a_\sigma)$, and $\rho(a):z\mapsto z^a$.*

The finite part $\tilde\rho_f$ defines a sign $\langle\tilde\rho_f,\pi_f\rangle=\prod_{v<\infty}\langle\tilde\rho_v,\pi_v\rangle\in\{\pm1\}$ on π_f. Put $\varepsilon(\{\rho_\sigma\})=-1$, $\varepsilon(\{\rho^\pm_\sigma\})=1$ $(\sigma\in S)$. Then

$$\mathrm{tr}[\mathrm{Fr}^j_\wp|H^*_c(\pi_f)]=\frac{1}{2}q_\wp^{\frac{j}{2}\dim\mathcal{S}_{K_f}}(\mathrm{tr}\,r^0_\mu[(\mathbf{t}(\pi_p)\times\mathrm{Fr}_p)^{jn_\wp}]+B)$$

where B is the product of $\langle\tilde\rho_f,\pi_f\rangle$, $\prod_{\sigma\in S}\varepsilon(\{\tilde\rho_\sigma\})$, and

$$\mathrm{tr}\,r^0_\mu[\mathrm{us}(\mathbf{t}(\pi_p)\times\mathrm{Fr}_p)^{jn_\wp}]=\prod_{u|p}\mathrm{tr}\,r^0_u[\mathrm{us}_u(\mathbf{t}(\pi_u)\times\mathrm{Fr}_p)^{jn_\wp}]$$

$$= \prod_{u|p} \left(\operatorname{tr}[s^{\frac{n_u}{j_u}} t_u^{\frac{jn_\wp}{j_u}}] \right)^{j_u}.$$

Here $\mathrm{us}_u = (s, \dots, s) \in Z(\widehat{H}') = Z(\widehat{H})^{n_u}$ *and* $s = \operatorname{diag}(-1, 1, -1)$.

(3) *Suppose that* π *basechange lifts to a representation normalizedly induced from a character of the Borel subgroup. Namely* π *is the endoscopic lift of precisely the three cuspidal representations* $\rho_1 = \rho(\theta, {}'\theta) \times {}''\theta$, $\rho_2 = \rho(\theta, {}''\theta) \times {}'\theta$, $\rho_3 = \rho({}'\theta, {}''\theta) \times \theta$ *of* $H(\mathbb{A}) = \mathrm{U}(2, \mathbb{A}) \times \mathrm{U}(1, \mathbb{A})$. *Its real component is* $\otimes_\sigma \tilde{\rho}_\sigma$, *where* $\tilde{\rho}_\sigma$ *is* $\rho_\sigma^+ = \rho(a_\sigma, b_\sigma) \times \rho(c_\sigma)$, $\rho_\sigma = \rho(a_\sigma, c_\sigma) \times \rho(b_\sigma)$ *or* $\rho_\sigma^- = \rho(b_\sigma, c_\sigma) \times \rho(a_\sigma)$, *and* $\rho(a) : z \mapsto z^a$.

The finite parts $\rho_{i,f}$ *define signs* $\langle \rho_{i,f}, \pi_f \rangle = \prod_{v<\infty} \langle \rho_{i,v}, \pi_v \rangle \in \{\pm 1\}$ *on* π_f. *Put* $\varepsilon(\{\rho_\sigma\}) = -1$, $\varepsilon(\{\rho_\sigma^\pm\}) = 1$ $(\sigma \in S)$. *Then*

$$\operatorname{tr}[\mathrm{Fr}_\wp^j \,|H_c^*(\pi_f)] = \frac{1}{4} q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}} (\operatorname{tr} r_\mu^0[(\mathbf{t}(\pi_p) \times \mathrm{Fr}_p)^{jn_\wp}] + B_1 + B_2 + B_3)$$

where B_i *is the product of* $\langle \rho_{i,f}, \pi_f \rangle$, $\prod_{\sigma \in S} \varepsilon(\{\rho_{i,\sigma}\})$ *and*

$$\operatorname{tr} r_\mu^0[\mathrm{us}(e(\mathbf{t}(\rho_{i,p})) \times \mathrm{Fr}_p)^{jn_\wp}] = \prod_{u|p} \operatorname{tr} r_u^0[\mathrm{us}_u(e(\mathbf{t}(\rho_{i,u})) \times \mathrm{Fr}_p)^{jn_\wp}]$$

$$= \prod_{u|p} \left[(-1)^{\frac{n_u}{j_u}} \mu_{1(i),u}^{\frac{jn_\wp}{j_u}} + \mu_{2(i),u}^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}} \mu_{3(i),u}^{\frac{jn_\wp}{j_u}} \right]^{j_u}.$$

In cases (1), (2), (3), *the Hecke eigenvalues* μ_{1u}, μ_{2u}, μ_{3u} *are algebraic. Each of their conjugates has complex absolute value one. Moreover,* π_f *contributes to the* L^2*-cohomology only in degree* $[F:\mathbb{Q}]$. *In case* (1) *we have* $\dim_{\overline{\mathbb{Q}}_\ell} H_c^*(\pi_f) = 3^{[F:\mathbb{Q}]}$. *In cases* (2) *and* (3) *the dimension is smaller and computable.*

(4) *Suppose that* π *basechange lifts to a representation normalizedly induced from a one-dimensional representation of the maximal parabolic subgroup. Then* π *is the endoscopic lift of a character* μ *of* $H(\mathbb{A})$. *The components* π_v $(v < \infty)$ *are nontempered* $\pi_v^\times$, *or cuspidal* π_v^-, *we put* $\langle \mu_v, \pi_v \rangle = 1$ *or* -1 *respectively, and* $\langle \mu, \pi \rangle = \prod_{v<\infty} \langle \mu_v, \pi_v \rangle$. *Then* $\operatorname{tr}[\mathrm{Fr}_\wp^j \,|H_c^*(\pi_f)]$ *is the product of*

$$\frac{(-1)^{[F:\mathbb{Q}]}}{2} q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}}$$

and

$$\varepsilon(\mu', \kappa) \operatorname{tr} r_\mu^0[(\mathbf{t}(\pi_p) \times \mathrm{Fr}_p)^{jn_\wp}] + \langle \mu_f, \pi_f \rangle \operatorname{tr} r_\mu^0[\mathrm{us}(\mathbf{t}(\pi_p) \times \mathrm{Fr}_p)^{jn_\wp}]$$

$$= \varepsilon(\mu', \kappa) \prod_{u|p} \left[(\mu_u q_u^{1/2})^{\frac{jn_\wp}{j_u}} + \rho_u^{\frac{jn_\wp}{j_u}} + (\mu_u q_u^{-1/2})^{\frac{jn_\wp}{j_u}} \right]^{j_u}$$

$$+ \langle \mu_f, \pi_f \rangle \prod_{u|p} \left[(-1)^{\frac{n_u}{j_u}} (\mu_u q_u^{1/2})^{\frac{jn_\wp}{j_u}} + \rho_u^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}} (\mu_u q_u^{-1/2})^{\frac{jn_\wp}{j_u}} \right]^{j_u}$$

for a suitable sign $\varepsilon(\mu', \kappa)$. *The numbers* μ_u *and* ρ_u *are algebraic and all their conjugates lie on the unit circle in* $\mathbb{C}$, *but the Hecke eigenvalues* $\mu_u q_u^{\pm 1/2}$ *are not units.*

(5) *Let* π *be a one-dimensional representation. Then* $\operatorname{tr}[\mathrm{Fr}_\wp^j | H_c^*(\pi_f)]$ *is* $q_\wp^{\frac{j}{2} \dim \mathcal{S}_{K_f}}$ *times*

$$\operatorname{tr} r_\mu^0[(\mathbf{t}(\pi_p) \times \mathrm{Fr}_p)^{jn_\wp}] = \prod_{u|p} \operatorname{tr} r_u^0[(\mathbf{t}(\pi_u) \times \mathrm{Fr}_p)^{jn_\wp}]$$

$$= \prod_{u|p} \left[(\xi_u q_u)^{\frac{jn_\wp}{j_u}} + (\xi_u)^{\frac{jn_\wp}{j_u}} + (\xi_u q_u^{-1})^{\frac{jn_\wp}{j_u}} \right]^{j_u}.$$

In stating Theorem 2 we implicitly made a choice of a square root of p.

For unitary groups defined using division algebras endoscopy does not show and Kottwitz [Ko6] used the trace formula in this anisotropic case to associate Galois representations $H^*(\pi)$ to some automorphic π and obtain some of their properties. However, in this case the classification of automorphic representations and their packets is not yet known.

2. The Zeta function

The Zeta function Z of the Shimura variety is a product over the rational primes p of local factors Z_p each of which is a product of local factors $Z_\wp$ over the primes $\wp$ of the reflex field $\mathbb{E}$ which divide p. Write $q = q_\wp$ for the cardinality of the residue field $\mathbb{F} = R_\wp/\wp R_\wp$ ($R_\wp$ denotes the ring of integers of $\mathbb{E}_\wp$). We work only with "good" p, thus $K_f = K_p K_f^p$, $K_p = G'(\mathbb{Z}_p)$, $\mathcal{S}_{K_f}$ is defined over $R_\wp$ and has good reduction mod $\wp$.

A general form of the Zeta function is for a correspondence, namely for a K_f-biinvariant $\overline{\mathbb{Q}}_\ell$-valued function f^p on $G(\mathbb{A}_f^p)$, ($\mathbb{A}$ is $\mathbb{A}_F$ and we fix a field isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$), and with coefficients in the smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\ell}$ constructed from an absolutely irreducible algebraic finite-dimensional representation $V_{\mathbf{a},\mathbf{b},\mathbf{c}} = \otimes_{\sigma\in S} V_{a_\sigma,b_\sigma,c_\sigma}$ of G' over a number field L, each $V_{a_\sigma,b_\sigma,c_\sigma}$ with highest weight $(a_\sigma, b_\sigma, c_\sigma)$, $a_\sigma \geq b_\sigma \geq c_\sigma$.

The standard form of the Zeta function is stated for $f^p = 1_{G(\mathbb{A}_f^p)}$, and for the trivial coefficient system ($(a_\sigma, b_\sigma, c_\sigma) = (0,0,0)$ for all σ). In this case the coefficients of the Zeta function store the number of points of the Shimura variety over finite residue fields. In this case the correspondence and the coefficients are usually omitted from the notations. Thus the Zeta function Z_p, or rather its natural logarithm $\ln Z_p$, is the sum over $\wp|p$ of

$$\ln Z_\wp(s, \mathcal{S}_{K_f}, f^p, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\ell})_c$$

$$= \sum_{j=1}^{\infty} \frac{1}{jq_\wp^{js}} \sum_{i=0}^{2\dim \mathcal{S}_{K_f}} (-1)^i \operatorname{tr}[\mathrm{Fr}_\wp^j \circ f^p; H_c^i(\mathcal{S}_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\ell})].$$

The subscript c on the left emphasizes that we work with H_c rather than H or IH; we drop it from now on. One can add a superscript i on the left to isolate the contribution from H_c^i.

Our results decompose the alternating sum of the traces on the cohomology for a correspondence f^p. Then we obtain an expression for $\ln Z_p$ which is the sum of 4 terms (we combine the two stable terms, of cuspidal and one-dimensional representations), depending on the type of representation.

Recall that r_μ is the representation of ${}^LG'_{\mathbb{Q}_p} = \widehat{G}' \rtimes W_{\mathbb{Q}_p}$ induced from the representation r_μ^0 the subgroup ${}^LG'_{\mathbb{E}_\wp} = \widehat{G}' \rtimes W_{\mathbb{E}_\wp}$ of index $n_\wp = [\mathbb{E}_\wp : \mathbb{Q}_p]$. The class $\mathbf{t}(\pi_p) = \mathbf{t}_p \times \mathrm{Fr}_p$ is such that $\operatorname{tr} r_\mu[(\mathbf{t}_p \times \mathrm{Fr}_p)^j]$ is zero unless j is a multiple of $n_\wp$, and $\operatorname{tr} r_\mu[(\mathbf{t}_p \times \mathrm{Fr}_p)^{jn_\wp}] = n_\wp \operatorname{tr} r_\mu^0[(\mathbf{t}_p \times \mathrm{Fr}_p)^{jn_\wp}]$.

THEOREM 3. *The logarithm of the function $Z_p(s, \mathcal{S}_{K_f}, f^p, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\ell})$ is the sum of the following terms. All components at infinity π_σ ($\sigma \in S$) of all π below have infinitesimal character $(a_\sigma, b_\sigma, c_\sigma) + (1, 0, -1)$.*

The first term is the sum over all irreducibles π in the stable packets $\{\pi\}$ (those which basechange lift to discrete-spectrum representations) of the product of $\operatorname{tr}\{\pi_f^p\}(f^p)$ and the value at $s' = s - \frac{1}{2}\dim \mathcal{S}_{K_f}$ of

$$\ln L_p(s', r, \pi) = \sum_{j\geq 1} \frac{1}{jp^{js'}} \operatorname{tr}[r_\mu(\mathbf{t}(\pi_p)^j)] = \sum_{j\geq 1} \frac{1}{jq_\wp^{js'}} \operatorname{tr}[r_\mu^0(\mathbf{t}(\pi_p)^{jn_\wp})].$$

The second term is the sum over the irreducibles π in the unstable packets $\{\pi\}$ which basechange lift to representations induced from cuspidal representations of the maximal compact subgroup, of

$$\frac{1}{2}\operatorname{tr}\{\pi_f^p\}(f^p)\left[\ln L_p(s',r,\pi)+\langle\tilde\rho_f,\pi_f\rangle\prod_{\sigma\in S}\varepsilon(\{\tilde\rho_\sigma\})\cdot\ln L_p(s',r\circ\mathrm{us},\pi)\right].$$

Here

$$\ln L_p(s',r\circ\mathrm{us},\pi)=\sum_{j\geq 1}\frac{1}{jq_\wp^{js'}}\operatorname{tr}r_\mu^0[\mathrm{us}(\mathbf{t}(\pi_p))^{jn_\wp}]$$

$$=\sum_{j\geq 1}\frac{1}{jq_\wp^{js'}}\prod_{u|p}\operatorname{tr}r_u^0[\mathrm{us}_u(\mathbf{t}(\pi_u))^{jn_\wp}]=\sum_{j\geq 1}\frac{1}{jq_\wp^{js'}}\prod_{u|p}(\operatorname{tr}[s^{\frac{n_u}{j_u}}t_u^{\frac{jn_\wp}{j_u}}])^{j_u}.$$

The third term is the sum over the irreducibles π in the unstable packets $\{\pi\}$ which basechange lift to representations induced from the Borel subgroup, namely is a lift of the ρ_i specified in Theorem 2(3), of $\frac{1}{4}\operatorname{tr}\{\pi_f^p\}(f^p)$ times

$$\ln L_p(s',r,\pi)+\sum_{\{1\leq i\leq 3\}}\langle\rho_{i,f},\pi_f\rangle\prod_{\sigma\in S}\varepsilon(\{\rho_{i,\sigma}\})\cdot\ln L(s',r\circ\mathrm{us},\rho_i).$$

Here

$$\ln L_p(s',r\circ\mathrm{us},\rho_i)=\sum_{j\geq 1}\frac{1}{jq_\wp^{js'}}\operatorname{tr}r_\mu^0[\mathrm{us}(e(\mathbf{t}(\rho_{i,p})))^{jn_\wp}]$$

$$=\sum_{j\geq 1}\frac{1}{jq_\wp^{js'}}\prod_{u|p}\operatorname{tr}r_u^0[\mathrm{us}_u(e(\mathbf{t}(\rho_{i,u})))^{jn_\wp}].$$

The fourth term is the sum over the irreducibles π in the unstable packets $\{\pi\}$ which basechange lift to representations induced from one-dimensional representations μ of the maximal compact subgroup, of

$$\frac{(-1)^{[F:\mathbb{Q}]}}{2}\operatorname{tr}\{\pi_f^p\}(f^p)\big[\varepsilon(\mu',\kappa)\ln L_p(s',r,\pi)+\langle\tilde\mu_f,\pi_f\rangle\ln L_p(s',r\circ\mathrm{us},\pi)\big].$$

In the case of Shimura varieties associated with subgroups of GL(2), a similar statement is obtained in Langlands [L2]. In general, our result is predicted by Langlands [L1-3] and more precisely by Kottwitz [Ko5].

I. PRELIMINARIES

I.1 The Shimura variety

Let G be a connected reductive group over the field $\mathbb{Q}$ of rational numbers. Suppose that there exists a homomorphism $h : \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G$ of algebraic groups over the field $\mathbb{R}$ of real numbers which satisfies the conditions (2.1.1.1-3) of Deligne [D3]. The $G(\mathbb{R})$-conjugacy class $X_\infty = \mathrm{Int}(G(\mathbb{R}))(h)$ of h is isomorphic to $G(\mathbb{R})/K_\infty$, where K_∞ is the fixer of h in $G(\mathbb{R})$. Then X_∞ carries a natural structure of an Hermitian symmetric domain. Let K_f be an open compact subgroup of $G(\mathbb{A}_{\mathbb{Q}f})$, where $\mathbb{A}_{\mathbb{Q}f}$ is the ring of adèles of $\mathbb{Q}$ without the real component, sufficiently small so that the set

$$\mathcal{S}_{K_f}(\mathbb{C}) = G(\mathbb{Q})\backslash[X_\infty \times (G(\mathbb{A}_{\mathbb{Q}f})/K_f)] = G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K_\infty K_f$$

has a structure of a smooth complex variety (manifold).

The group $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ obtained from the multiplicative group $\mathbb{G}_m$ on restricting scalars from the field $\mathbb{C}$ of complex numbers to $\mathbb{R}$ is defined over $\mathbb{R}$. Its group $(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{R})$ of real points can be realized as $\{(z,\overline{z}); z \in \mathbb{C}^\times\}$ in $(\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$. The $G(\mathbb{C})$-conjugacy class $\mathrm{Int}(G(\mathbb{C}))\mu_h$ of the $\mathbb{C}$-homomorphism $\mu_h : \mathbb{G}_m \to G$, $z \mapsto h(z,1)$, is acted upon by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

In fact, let C_k denote the set of conjugacy classes of homomorphisms $\mu : \mathbb{G}_m \to G$ over a field k. The embedding $\overline{\mathbb{Q}} \to \mathbb{C}$ induces an $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-equivariant map $C_{\overline{\mathbb{Q}}} \to C_{\mathbb{C}}$. This map is bijective. Indeed, choose a maximal torus $\overline{T}$ of G defined over $\overline{\mathbb{Q}}$. Then $\mathrm{Hom}_{\overline{\mathbb{Q}}}(\mathbb{G}_m, \overline{T})/W \to C_{\overline{\mathbb{Q}}}$ is a bijection, where W is the Weyl group of $\overline{T}$ in $G(\overline{\mathbb{Q}})$. Similarly, $\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_m, \overline{T})/W \to C_{\mathbb{C}}$ is a bijection. Since $\mathrm{Hom}_{\overline{\mathbb{Q}}}(\mathbb{G}_m, \overline{T}) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_m, \overline{T})$ is an $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-equivariant bijection, so is $C_{\overline{\mathbb{Q}}} \to C_{\mathbb{C}}$. The conjugacy class of μ_h over $\mathbb{C}$ is then a point in $\mathrm{Hom}_{\overline{\mathbb{Q}}}(\mathbb{G}_m, \overline{T})/W$. The subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which fixes it has the form $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$, where $\mathbb{E}$ is a number field, named the *reflex field*. It is contained in any field $\mathbb{E}_1$ over which G splits, since $\overline{T}$ can be chosen to split over $\mathbb{E}_1$.

There is a smooth variety over $\mathbb{E}$ determined by the structure of its special points (see [D3]), named the canonical model $\mathcal{S}_{K_f}$ of the Shimura variety associated with (G, X_∞, K_f), whose set of complex points is $\mathcal{S}_{K_f}(\mathbb{C})$ displayed above.

Let L be a number field, and let ξ be an absolutely irreducible finite dimensional representation of G on an L-vector space V_ξ. Denote by p the natural projection $G(\mathbb{A}_\mathbb{Q})/K_\infty K_f \to \mathcal{S}_{K_f}(\mathbb{C})$. The sheaf $\mathbb{V} : U \mapsto V_\xi(L) \underset{\xi, G(\mathbb{Q})}{\times} p^{-1}U$ of L-vector spaces over $\mathcal{S}_{K_f}(\mathbb{C})$ is locally free of rank $\dim_L V_\xi$. For any finite place λ of L the local system $\mathbb{V} \otimes_L L_\lambda : U \to V_\xi(L_\lambda) \underset{\xi, G(\mathbb{Q})}{\times} p^{-1}U$ defines a smooth L_λ-sheaf $\mathbb{V}_\lambda$ on $\mathcal{S}_{K_f}$ over $\mathbb{E}$.

The Satake Baily-Borel compactification $\mathcal{S}'_{K_f}$ of $\mathcal{S}_{K_f}$ has a canonical model over $\mathbb{E}$ as does $\mathcal{S}_{K_f}$. The Hecke convolution algebra $\mathbb{H}_{K_f,L}$ of compactly supported K_f-biinvariant L-valued functions on $G(\mathbb{A}_{\mathbb{Q}f})$ is generated by the characteristic functions of the double cosets $K_f \cdot g \cdot K_f$ in $G(\mathbb{A}_{\mathbb{Q}f})$. It acts on the cohomology spaces $H^i(\mathcal{S}_{K_f}(\mathbb{C}), \mathbb{V})$, the cohomology with compact supports $H^i_c(\mathcal{S}_{K_f}(\mathbb{C}), \mathbb{V})$, and on the intersection cohomology L-spaces $IH^i(\mathcal{S}'_{K_f}(\mathbb{C}), \mathbb{V})$. These modules are related by maps: $H^i_c \to IH^i \to H^i$. The action is compatible with the isomorphism $H^i_c(\mathcal{S}_{K_f}(\mathbb{C}), \mathbb{V}) \otimes_L L_\lambda \simeq H^i_c(\mathcal{S}_{K_f} \otimes_\mathbb{E} \overline{\mathbb{Q}}, \mathbb{V}_\lambda)$, (same for H^i and for $IH^i(\mathcal{S}')$), but the last étale cohomology spaces have in addition an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$, which commutes with the action of the Hecke algebra ($X \otimes_\mathbb{E} \overline{\mathbb{Q}}$ abbreviates $X \times_{\mathrm{Spec}\,\mathbb{E}} \mathrm{Spec}\,\overline{\mathbb{Q}}$).

I.2 Decomposition of cohomology

Of interest is the decomposition of the finite-dimensional L_λ-vector spaces IH^i, H^i and H^i_c as $\mathbb{H}_{K_f,L_\lambda} \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$-modules. They vanish unless $0 \le i \le 2 \dim \mathcal{S}_{K_f}$. Thus

$$(1; H_c) \qquad H^i_c(\mathcal{S}_{K_f} \otimes_\mathbb{E} \overline{\mathbb{Q}}, \mathbb{V}_\lambda) = \oplus\, \pi^{K_f}_{f,L_\lambda} \otimes H^i_c(\pi^{K_f}_{f,L_\lambda}).$$

The (finite) sum ranges over inequivalent irreducible $\mathbb{H}_{K_f,L_\lambda}$-modules $\pi^{K_f}_{f,L_\lambda}$, and $H^i_c(\pi^{K_f}_{f,L_\lambda})$ are finite-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ over L_λ. Similar decomposition holds for (H^i and) $IH^i(\mathcal{S}')$; we denote it by (1;IH).

In the case of IH, the Zucker conjecture [Zu], proved by Looijenga and Saper-Stern, asserts that the intersection cohomology of $\mathcal{S}'_{K_f}$ is isomorphic to the L^2-cohomology of $\mathcal{S}_{K_f}$ with coefficients in the sheaf $\mathbb{V}_\mathbb{C} : U \mapsto V_\xi(\mathbb{C}) \underset{\xi, G(\mathbb{Q})}{\times} p^{-1}(U)$ of $\mathbb{C}$-vector spaces: for a fixed embedding of L_λ in $\mathbb{C}$, we have an isomorphism of $\mathbb{H}_{K_f,L_\lambda} \otimes_{L_\lambda} \mathbb{C} = \mathbb{H}_{K_f}$-modules

$$IH^i(\mathcal{S}'_{K_f} \otimes_\mathbb{E} \overline{\mathbb{Q}}, \mathbb{V}_\lambda) \otimes_{L_\lambda} \mathbb{C} \xrightarrow{\sim} H^i_{(2)}(\mathcal{S}_{K_f}(\mathbb{C}), \mathbb{V}_\mathbb{C}).$$

The L^2-cohomology $H^i_{(2)}(\mathcal{S}_{K_f}(\mathbb{C}), \mathbb{V}_\mathbb{C})$, has a ("Matsushima-Murakami") decomposition (see Borel-Casselman [BC]) in terms of discrete-spectrum automorphic representations. Thus

$$H^i_{(2)}(\mathcal{S}_{K_f}(\mathbb{C}), \mathbb{V}_\mathbb{C}) = \bigoplus_\pi m(\pi) \pi_f^{K_f} \otimes H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\xi(\mathbb{C})).$$

Here π ranges over the equivalence classes of the (irreducible) automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ in the discrete spectrum

$$L^2_d = L^2_d(G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q}), \mathbb{C}).$$

The integer $m(\pi)$ denotes the multiplicity of π in L^2_d.

Write $\pi = \pi_f \otimes \pi_\infty$ as a product of irreducible representations π_f of $G(\mathbb{A}_{\mathbb{Q}f})$ and π_∞ of $G(\mathbb{R})$, according to $\mathbb{A}_\mathbb{Q} = \mathbb{A}_{\mathbb{Q}f}\mathbb{R}$, and $\pi_f^{K_f}$ for the space of K_f-fixed vectors in π_f. Then $\pi_f^{K_f}$ is a finite-dimensional complex space on which $\mathbb{H}_{K_f} = \mathbb{H}_{K_f,L} \otimes_L \mathbb{C}$ acts irreducibly. The representation π_∞ is viewed as a $(\mathfrak{g}, K_\infty)$-module, where $\mathfrak{g}$ denotes the Lie algebra of $G(\mathbb{R})$, and

$$H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi_\mathbb{C}) = H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\xi(\mathbb{C})), \quad \xi_\mathbb{C} = \xi \otimes_L \mathbb{C},$$

denotes the Lie-algebra cohomology of π_∞ twisted by the finite-dimensional representation $\xi_\mathbb{C}$ of $G(\mathbb{R})$. Then the finite-dimensional complex space $H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \xi_\mathbb{C})$ vanishes unless the central character ω_{π_∞} and the infinitesimal character $\inf(\pi_\infty)$ are equal to those $\omega_{\check{\xi}_\mathbb{C}}$, $\inf(\check{\xi}_\mathbb{C})$ of the contragredient $\check{\xi}_\mathbb{C}$ of $\xi_\mathbb{C}$; see Borel-Wallach [BW].

There are only finitely many equivalence classes of π in L^2_d with central and infinitesimal character equal to given ones, and a nonzero K_f-fixed vector ($\pi_f^{K_f} \neq 0$). The multiplicities $m(\pi)$ are finite. Hence $H^i_{(2)}(\mathcal{S}_{K_f}(\mathbb{C}), \mathbb{V}_\mathbb{C})$ is finite dimensional. The Zucker isomorphism then implies that the decomposition (1;IH) ranges over the finite set of equivalence classes of irreducible π in L^2_d with $\pi_f^{K_f} \neq 0$ and π_∞ with central and infinitesimal characters equal to those of $\check{\xi}_{\mathbb{C}}$. Further, $\pi_{f,L_\lambda}^{K_f}$ of (1;IH) is an irreducible $\mathbb{H}_{K_f,L_\lambda}$-module with $\pi_{f,L_\lambda}^{K_f} \otimes_{L_\lambda} \mathbb{C} = \pi_f^{K_f}$ for such a $\pi = \pi_f \otimes \pi_\infty$ in the discrete spectrum, and

$$\dim_{\mathbb{C}} IH^i(\pi_f^{K_f}) = \sum_{\pi_\infty} m(\pi_f \otimes \pi_\infty) \dim_{\mathbb{C}} H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes \check{\xi}_{\mathbb{C}}).$$

Each $\pi = \pi_f \otimes \pi_\infty$ in the discrete spectrum such that the central and infinitesimal characters of π_∞ coincide with those of $\check{\xi}_{\mathbb{C}}$ (where ξ is an absolutely irreducible representation of G on a finite-dimensional vector space over L) has the property that for some open compact subgroup $K_f \subset G(\mathbb{A}_{\mathbb{Q}f})$ for which $\pi_f^{K_f} \neq \{0\}$, there is an L-model $\pi_{f,L}^{K_f}$ of $\pi_f^{K_f}$.

It is also known that the cuspidal cohomology in H_c^i, that is, its part which is indexed by the cuspidal π, makes an orthogonal direct summand in $H_c^i \otimes_{L_\lambda} \mathbb{C}$, and also in $IH^i \otimes_{L_\lambda} \mathbb{C}$ (and $H^i \otimes_{L_\lambda} \mathbb{C}$). When we study the π_f-isotypic component of $H_c^i \otimes_{L_\lambda} \mathbb{C}$ for the finite component π_f of a cuspidal representation π, we shall then be able to view it as such a component of IH^i.

Our aim is then to recall the classification of automorphic representations of $\mathrm{U}(3, E/F)$ given in [F3;VI], in particular list the possible $\pi = \pi_f \otimes \pi_\infty$ in the cuspidal and discrete spectrum. This means listing the possible π_f, then the π_∞ which make $\pi_f \otimes \pi_\infty$ occur in the cuspidal or discrete spectrum. Further we list the cohomological π_∞, those for which $H^i(\mathfrak{g}, K; \pi_\infty \otimes \xi_{\mathbb{C}})$ is nonzero, and describe these spaces. In particular we can then compute the dimension of the contribution of π_f to IH^*. Then we describe the trace of $\mathrm{Fr}_\wp$ acting on the Galois representation $H_c^*(\pi_f)$ attached to π_f in terms of the Satake parameters of π_p, in fact any sufficiently large power of $\mathrm{Fr}_\wp$. This determines uniquely the Galois representation $H_c^*(\pi_f)$, of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$, and in particular its dimension. The displayed formula of "Matsushima-Murakami" type will be used to estimate the absolute values of the eigenvalues of the action of the Frobenius on $H_c^*(\pi_f)$.

I.3 Galois representations

The decomposition (1;IH) then defines a map $\pi_f \mapsto IH^i(\pi_f)$ from the set of irreducible representations π_f of $G(\mathbb{A}_{\mathbb{Q}f})$ for which there exists an irreducible representation π_∞ of $G(\mathbb{A}_\mathbb{Q})$ with central and infinitesimal characters equal to those of $\check{\xi}_\mathbb{C}$ such that $\pi_\infty \otimes \pi_f$ is in the discrete spectrum, to the set of finite-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$. We wish to determine the representation $IH^i(\pi_f)$ associated with π_f, namely its restriction to the decomposition groups at almost all primes. As we use Deligne's conjecture, we shall determine $H_c^*(\pi_f)$ instead.

Let p be a rational prime. Assume that G is unramified at p, thus it is quasi-split over $\mathbb{Q}_p$ and splits over an unramified extension of $\mathbb{Q}_p$. Assume that K_f is unramified at p, thus it is of the form $K_f^p K_p$ where K_f^p is a compact open subgroup of $G(\mathbb{A}_{\mathbb{Q}f}^p)$ and $K_p = G(\mathbb{Z}_p)$. Then $\mathbb{E}$ is unramified at p. Let $\wp$ be a place of $\mathbb{E}$ lying over p and λ a place of L such that p is a unit in L_λ. Let $f = f^p f_{K_p}$ be a function in the Hecke algebra $\mathbb{H}_{K_f,L}$, where f^p is a function on $G(\mathbb{A}_{\mathbb{Q}f}^p)$ and f_{K_p} is the quotient of the characteristic function of K_p in $G(\mathbb{Q}_p)$ by the volume of K_p. Denote by $\mathrm{Fr}_\wp$ a geometric Frobenius element of the decomposition group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{E}_\wp)$.

Choose models of $\mathcal{S}_{K_f}$ and of $\mathcal{S}'_{K_f}$ over the ring of integers of $\mathbb{E}$. For almost all primes p of $\mathbb{Q}$, for each prime $\wp$ of $\mathbb{E}$ over p, the representation $H_c^i(\mathcal{S}_{K_f} \otimes_\mathbb{E} \overline{\mathbb{Q}}, \mathbb{V}_\lambda)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ is unramified at $\wp$, thus its restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{E}_\wp)$ factorizes through the quotient $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{E}_\wp) \simeq \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ which is (topologically) generated by $\mathrm{Fr}_\wp$; here $\mathbb{Q}_p^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbb{Q}_p$ in the algebraic closure $\overline{\mathbb{Q}}_p$, $\mathbb{F}$ is the residue field of $\mathbb{E}_\wp$ and $\overline{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$. Denote the cardinality of $\mathbb{F}$ by $q_\wp$; it is a power of p. As a $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$-module $H_c^i(\mathcal{S}_{K_f} \otimes_\mathbb{E} \overline{\mathbb{Q}}, \mathbb{V}_\lambda)$ is isomorphic to $H_c^i(\mathcal{S}_{K_f} \otimes_\mathbb{F} \overline{\mathbb{F}}, \mathbb{V}_\lambda)$.

Deligne's conjecture proven by Zink [Zi] for surfaces, by Pink [P2] and Shpiz [Sc] for varieties X (such as $\mathcal{S}_{K_f}$) which have a smooth compactification $\overline{X}$ which differs from X by a divisor with normal crossings, and unconditionally by Fujiwara [Fu], implies that for each correspondence f^p there exists an integer $j_0 \geq 0$ such that for any $j \geq j_0$ the trace of $f^p \cdot \mathrm{Fr}_\wp^j$ on

$$\bigoplus_{i=0}^{2\dim \mathcal{S}_{K_f}} (-1)^i \; H_c^i(\mathcal{S}_{K_f} \otimes_\mathbb{F} \overline{\mathbb{F}}, \mathbb{V}_\lambda)$$

has contributions only from the variety $\mathcal{S}_{K_f}$ and not from any boundary component of $\mathcal{S}'_{K_f}$. The trace is the same in this case as if the scheme $\mathcal{S}_{K_f} \otimes_\mathbb{F} \overline{\mathbb{F}}$ were proper over $\overline{\mathbb{F}}$, and it is given by the usual expression of the Lefschetz fixed point formula. This is the reason why we work with H_c^i in this paper, and not with $IH^i(\mathcal{S}')$.

II. AUTOMORPHIC REPRESENTATIONS

II.1 Stabilization and the test function

Kottwitz computed the trace of $f^p \cdot \mathrm{Fr}_\wp^j$ on this alternating sum (see [Ko7], and [Ko5], chapter III, for $\xi = 1$) at least in the case considered here. The result, stated in [Ko5], (3.1) as a conjecture, is a certain sum

$$\sum_{\gamma_0} \sum_{(\gamma,\delta)} c(\gamma_0;\gamma,\delta) \cdot O(\gamma, f^p) \cdot TO(\delta,\phi_j) \cdot \mathrm{tr}\,\xi(\gamma_0),$$

rewritten in [Ko5], (4.2) in the form

$$\tau(G) \sum_{\gamma_0} \sum_{\kappa} \sum_{(\gamma,\delta)} \langle \alpha(\gamma_0;\gamma,\delta),\kappa\rangle \cdot e(\gamma,\delta)$$

$$\cdot \;\; O(\gamma, f_{\mathbb{C}}^p) \cdot TO(\delta,\phi_j) \cdot \frac{\mathrm{tr}\,\xi_{\mathbb{C}}(\gamma_0)}{|I(\infty)(\mathbb{R})/A_G(\mathbb{R})^0|},$$

where O and TO are orbital and twisted orbital integrals and ϕ_j is a spherical ($K_p = G(\mathbb{Z}_p)$-biinvariant) function on $G(\mathbb{Q}_p)$. Theorem 7.2 of [Ko5] expresses this as a sum

$$\sum \iota(G,H)\,\mathrm{STF}_e^{\mathrm{reg}}(f_{H,\wp}^{j,s,\xi})$$

over a set of representatives for the isomorphism classes of the elliptic endoscopic triples $(H, s, \eta_0 : \widehat{H} \to \widehat{G})$ for G. The $\mathrm{STF}_e^{\mathrm{reg}}(f_{H,\wp}^{j,s,\xi})$ indicates the (G,H)-regular $\mathbb{Q}$-elliptic part of the stable trace formula for a function $f_{H,\wp}^{j,s,\xi}$ on $H(\mathbb{A}_{\mathbb{Q}})$. The function $f_{H,\wp}^{j,s,\xi}$, denoted simply by h in [Ko5], is constructed in [Ko5], section 7 assuming the "fundamental lemma" and "matching orbital integrals", both known in the case considered here by [F3;VIII].

Thus $f_{H,\wp}^{j,s,\xi}$ is the product of the functions: f_H^p on $H(\mathbb{A}_{\mathbb{Q}f}^p)$ which is obtained from $f_{\mathbb{C}}^p$ by matching of orbital integrals, $f_{H,\wp}^{j,s}$ on $H(\mathbb{Q}_p)$ which is a spherical function obtained by the fundamental lemma from the spherical function ϕ_j, and $f_{H,\infty}^{s,\xi}$ on $H(\mathbb{R})$ which is constructed from pseudo-coefficients of discrete-series representations of $H(\mathbb{R})$ which lift to discrete-series representations of $G(\mathbb{R})$ whose central and infinitesimal characters coincide with those of $\check{\xi}_{\mathbb{C}}$. We denote by $f_{H,\wp}^{j,s,\xi} = f_H^p f_{H,\wp}^{j,s} f_{H\infty}^{s,\xi}$ Kottwitz's function $h = h^p h_p h_\infty$, so that functions on the adèle groups are denoted by f, and the notation does not conflict with that of $h : \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G$.

Note also that the factor $\langle \alpha^p(\gamma_0;\gamma), s\rangle$ is missing on the right side of [Ko5], (7.1). Here

$$\alpha^p = \prod_{v\neq p,\infty} \alpha_v, \quad \text{where} \quad \alpha_v(\gamma_0;\gamma_v) \in X^*(Z(\widehat{I}_0)^{\Gamma(v)}/Z(\widehat{I}_0)^{\Gamma(v),0} Z(\widehat{G}^{\Gamma(v)}))$$

is defined in [Ko5], p. 166, bottom paragraph.

We need to compare the elliptic regular part $\mathrm{STF}_e^{\mathrm{reg}}(f_{H,\wp}^{j,s,\xi})$ of the stable trace formula with the spectral side. To simplify matters we shall work only with a special class of test functions $f^p = \otimes_{v\neq p,\infty} f_v$ for which the complicated parts of the trace formulae vanish. Thus we choose a place v_0 where G is quasi-split, and a maximal split torus A of G over $\mathbb{Q}_{v_0}$, and require that the component f_{v_0} of f^p be in the span of the functions on $G(\mathbb{Q}_{v_0})$ which are biinvariant under an Iwahori subgroup I_{v_0} and supported on a double coset $I_{v_0} a I_{v_0}$, where $a \in A(\mathbb{Q}_{v_0})$ has $|\alpha(a)| \neq 1$ for all roots α of A. The orbital integrals of such a function f_{v_0} vanish on the singular set, and the matching functions f_{Hv_0} on $H(\mathbb{Q}_{v_0})$ have the same property. This would permit us to deal only with regular conjugacy classes in the elliptic part of the stable trace formulae $\mathrm{STF}_e^{\mathrm{reg}}(f_{H,\wp}^{j,s,\xi})$, and would restrict no applicability.

We need a description of the automorphic representations of $G(\mathbb{A}_F)$. It is given next.

II.2 Functorial overview of basechange for U(3)

Let E/F be a quadratic extension of local or global fields. Let $\mathbf{G}$ denote the quasi-split unitary group $\mathbf{U}(3, E/F)$ in three variables over F which splits over E. It is an outer form of GL(3). In [F3;VI] we *determine the* *admissible and automorphic representations of this group* by means of the trace formula and the theory of liftings. We now state the results of [F3;VI].

To be definite, we define the algebraic group $\mathbf{G}$ by means of the form $J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$, as a (representable) functor. For any F-algebra A put $A_E = A \otimes_F E$ and $G(A) = \{g \in \mathrm{GL}(3, A_E);\ {}^t\overline{g}Jg = J\}$. Here ${}^tg$ is the transpose (g_{ji}) of $g = (g_{ij})$ and $x \mapsto \overline{x}$ denotes the nontrivial automorphism of A_E over A.

Put $\sigma(g) = \theta(\overline{g})$. Thus the group $G = \mathbf{G}(F)$ of F-points on $\mathbf{G}$ is

$$\{g \in \mathbf{G}(E);\ {}^t\overline{g}Jg = J\} = \{g \in \mathrm{GL}(3, E);\ \sigma(g) = g\}.$$

Similarly we write $\mathrm{U}(n, E/F)$ for the group $\mathbf{U}(n, E/F)(F)$ of F-points on $\mathbf{U}(n, E/F)$.

When F is the field $\mathbb{R}$ of real numbers, the group $\mathbf{G}(\mathbb{R})$ of $\mathbb{R}$-points on $\mathbf{G}$ is usually denoted by $\mathrm{U}(2, 1; \mathbb{C}/\mathbb{R})$, and the notation $\mathrm{U}(3; \mathbb{C}/\mathbb{R})$ is reserved for its anisotropic inner form. We too shall use the $\mathbb{R}$-notations in the $\mathbb{R}$-case (but only in this case).

If v is a place of the global field F which splits in E, thus $E_v = F_v \otimes_F E = F_v \oplus F_v$ is not a field, $\mathbf{G}(F_v) = \mathrm{GL}(3, F_v)$.

The work of [F3;VI] is based on basechange lifting to $\mathbf{U}(3, E/F)(E) = \mathrm{GL}(3, E)$. This last group is defined as an algebraic group over F by applying the functor of restriction of scalars $\mathbf{G}' = \mathrm{R}_{E/F}\mathbf{G}$ to the algebraic group $\mathbf{G}$. Then for each F-algebra A,

$$\mathbf{G}'(A) = \{(g, g') \in \mathrm{GL}(3, A_E) \times \mathrm{GL}(3, A_E);\ (g, g') = (\theta(\overline{g}'), \theta(\overline{g}))\}.$$

Thus $\mathbf{G}'(\overline{F}) = \mathrm{GL}(3, \overline{F}) \times \mathrm{GL}(3, \overline{F})$, and $\tau \in \mathrm{Gal}(\overline{F}/F)$ acts as $\tau(x, y) = (\tau x, \tau y)$ if $\tau|E = 1$, and $\tau(x, y) = \iota\theta(\tau x, \tau y)$ if $\tau|E \neq 1$. Here $\theta(x, y) = (\theta(x), \theta(y))$ and $\iota(x, y) = (y, x)$. In particular $\mathbf{G}'(E) = \mathrm{GL}(3, E) \times \mathrm{GL}(3, E)$ while $G' = \mathbf{G}'(F) = \{(x, \sigma x); x \in \mathrm{GL}(3, E)\}$.

A main aim of [F3;VI] is to *determine the admissible representations* Π *of* $\mathrm{GL}(3, E)$ *and the automorphic representations* Π *of* $\mathrm{GL}(3, \mathbb{A}_E)$ *which are* σ*-invariant*: ${}^\sigma\Pi \simeq \Pi$, where ${}^\sigma\pi(g) = \pi(\sigma(g))$, and again $\sigma(g) = \theta(\overline{g})$ and $\theta(g) = J\,{}^tg^{-1}J$. In other words, we are interested in the representations $\Pi'(x, \sigma x) = \Pi(x)$ of $\mathbf{G}'(F)$ or $\mathbf{G}'(\mathbb{A})$ — admissible or automorphic — which are ι-invariant: ${}^\iota\Pi' \simeq \Pi'$, where ${}^\iota\Pi'(x, \sigma x) = \Pi'(\sigma x, x)$.

The lifting, part of Langlands' principle of functoriality, is defined by means of an L-group homomorphism $b : {}^LG \to {}^LG'$. One is interested in this and related L-group homomorphisms not in the abstract but since via the Satake transform they specify an explicit lifting relation of unramified representations, crucial for stating the global lifting, from which the local lifting is deduced. To state the results of [F3;VI] it suffices to specify the lifting of unramified representations. For this reason we reduce the discussion of functoriality here to a minimum. Thus the L-group ${}^LG$ (see [Bo2]) is the semidirect product of the connected component, $\widehat{G} = \mathrm{GL}(3, \mathbb{C})$, with a group which we take here to be the relative Weil group $W_{E/F}$. We could have equally worked with the absolute Weil group W_F and its subgroup W_E. Note that $W_F/W_E \simeq W_{E/F}/W_{E/E} \simeq \mathrm{Gal}(E/F)$, $W_{E/F} = W_F/W_E^c$, and $W_{E/E} = W_E/W_E^c = W_E^{\mathrm{ab}}$ is the abelianized W_E. Here W_E^c is the closure of the commutator subgroup of W_E (see [D1], [Tt]). Now the relative Weil group is an extension of $\mathrm{Gal}(E/F)$ by $W_{E/E} = C_E, = E^\times$ (locally) or $\mathbb{A}_E^\times/E^\times$ (globally). Thus

$$W_{E/F} = \langle z \in C_E, \sigma; \sigma^2 \in C_F - N_{E/F}C_E, \sigma z = \overline{z}\sigma\rangle$$

and we have an exact sequence

$$1 \to W_{E/E} = C_E \to W_{E/F} \to \mathrm{Gal}(E/F) \to 1.$$

Here $W_{E/F}$ acts on $\widehat{G}$ via its quotient $\mathrm{Gal}(E/F) = \langle\sigma\rangle$, $\sigma : g \mapsto \theta(g) = J^t g^{-1} J$. Further, ${}^LG'$ is $\widehat{G}' \rtimes W_{E/F}$, $\widehat{G}' = \mathrm{GL}(3, \mathbb{C}) \times \mathrm{GL}(3, \mathbb{C})$, where $W_{E/F}$ acts via its quotient $\mathrm{Gal}(E/F)$ by $\sigma = \iota\theta$, $\theta(x, y) = (\theta(x), \theta(y))$, $\iota(x, y) = (y, x)$.

The basechange map $b : {}^LG \to {}^LG'$ is $x \times w \mapsto (x, x) \times w$. In fact $\mathbf{G}$ is an ι-twisted endoscopic group of $\mathbf{G}'$ (see Kottwitz-Shelstad [KS]) with respect to the twisting ι. Namely $\widehat{G}$ is the centralizer $Z_{\widehat{G}'}(\iota) = \{g \in \widehat{G}'; \iota(g) = g\}$ of the involution ι in $\widehat{G}'$. Note that $\mathbf{G}$ is an elliptic ι-endoscopic group, which means that $\widehat{G}$ is not contained in any parabolic subgroup of $\widehat{G}'$.

The F-group $\mathbf{G}'$ has another elliptic ι-endoscopic group $\mathbf{H}$, whose dual group ${}^LH$ has connected component $\widehat{H} = Z_{\widehat{G}'}((s, 1)\iota)$, where $s = \mathrm{diag}(-1, 1, -1)$. Then $\widehat{H}$ consists of the (x, y) with

$$(x, y) = (s, 1)\iota \cdot (x, y) \cdot [(s, 1)\iota]^{-1} = (s, 1)(y, x)(s, 1) = (sys, x),$$

thus $y = x$ and $x = sys = sxs$. In other words $\widehat{H}$ is $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$, embedded in $\widehat{G} = \mathrm{GL}(3, \mathbb{C})$ as (a_{ij}), $a_{ij} = 0$ if $i + j$ is odd, a_{22} is the

GL$(1,\mathbb{C})$-factor, while $[a_{11},\, a_{13};\, a_{31},\, a_{33}]$ is the GL$(2,\mathbb{C})$-factor. Now ${}^LH$ is isomorphic to a subgroup ${}^LH_1$ of ${}^LG'$, and the factor $W_{E/F}$, acting on $\widehat{G}'$ by $\sigma = \iota\theta$, induces on $\widehat{H}_1$ the action $\sigma(x,x) = (\theta x, \theta x)$, namely $W_{E/F}$ acts on $\widehat{H}_1$ via its quotient Gal(E/F) and $\sigma(x)$ is $\theta(x)$. If we write $x = (a,b)$ with a in GL$(2,\mathbb{C})$ and b in GL$(1,\mathbb{C})$, $\sigma(a,b)$ is $(w\,{}^ta^{-1}w, b^{-1})$, where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We prefer to work with $\mathbf{H} = \mathbf{U}(2,E/F) \times \mathbf{U}(1,E/F)$, whose dual group ${}^LH$ is the semidirect product of $\widehat{H} = \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ $(\subset \widehat{G})$ and $W_{E/F}$ which acts via its quotient Gal(E/F) by $\sigma : x \mapsto \varepsilon\theta(x)\varepsilon$, $\varepsilon = \mathrm{diag}(1,-1,-1)$. We denote by $e' : {}^LH \to {}^LG'$ the map $\widehat{H} \hookrightarrow \widehat{G}'$ by $x \mapsto (x,x)$, and $\sigma \mapsto (\theta(\varepsilon),\varepsilon)\sigma$, $z \mapsto z$ $(\in W_{E/F})$. Here $\mathbf{U}(1,E/F)$ is the unitary group in a single variable: its group of F-points is $E^1 = \{x \in E^\times; x\overline{x} = 1\} = \{z/\overline{z}; z \in E^\times\}$. The quasi-split unitary group $\mathbf{U}(2,E/F)$ in two variables has F-points consisting of the a in GL$(2,E)$ with $a = \varepsilon w\,{}^t\overline{a}^{-1}w\varepsilon$.

The homomorphism $e' : {}^LH \to {}^LG'$ factorizes through the embedding $i : {}^LH' \to {}^LG'$, where $\mathbf{H}'$ is the endoscopic group (not elliptic and not ι-endoscopic) of $\mathbf{G}'$ with $\widehat{H}' = Z_{\widehat{G}'}((s,s))$. Thus $\widehat{H}' = \widehat{H} \times \widehat{H} \subset \widehat{G}'$, Gal$(E/F)$ permutes the two factors, and

$$\mathbf{H}' = \mathrm{R}_{E/F}\mathbf{U}(2,E/F) \times \mathrm{R}_{E/F}\mathbf{U}(1,E/F),$$

so that $H' = \mathbf{H}'(F) = \mathrm{GL}(2,E) \times \mathrm{GL}(1,E)$. The map $b'' : {}^LH \to {}^LH'$ is the basechange homomorphism, $b'' : x \mapsto (x,x)$ for $x \in \widehat{H}$, $z \mapsto z$, $\sigma \mapsto (\theta(\varepsilon),\varepsilon)\sigma$ on $W_{E/F}$. Thus $e' = i \circ b''$.

The lifting of representations implied by b is the basechange lifting, described below. On the U$(1,E/F)$ factor it is $\mu \mapsto \mu'$, where $\mu'(x) = \mu(x/\overline{x})$, $x \in E^\times$, is a character of GL$(1,E)$ which is σ-invariant. Thus $\mu' = {}^\sigma\mu'$ where ${}^\sigma\mu'(x) = \mu'(\overline{x}^{-1})$.

The lifting implied by the embedding $i : {}^LH' \to {}^LG'$ is simply normalized induction, taking a representation (ρ',μ') of GL$(2,E) \times$ GL$(1,E)$ to the normalizedly induced representation $I(\rho',\mu')$ from the parabolic subgroup of type $(2,1)$. In particular, if ρ' is irreducible with central character $\omega_{\rho'}$ and $\Pi = I(\rho',\mu')$ has central character ω', then $\omega' = \omega_{\rho'} \cdot \mu'$, and so $\mu' = \omega'/\omega_{\rho'}$ is uniquely determined by ω' and $\omega_{\rho'}$. The relation $\mu' = \omega'/\omega_{\rho'}$ implies that μ' is 1 on $F^\times$, as this is true for ω', $\omega_{\rho'}$. Since we fix the central character ω' $(= {}^\sigma\omega')$, we shall talk about the lifting $i : \rho' \to \Pi$, meaning that $\Pi = I(\rho', \omega'/\omega_{\rho'})$.

Similarly if e' maps a representation (ρ, μ) of $H = \mathrm{U}(2, E/F) \times \mathrm{U}(1, E/F)$ to $\Pi = I(\rho', \mu')$ where $(\rho', \mu') = b((\rho, \mu))$, then $\omega_\Pi(x) = \omega_\rho(x/\overline{x})\mu(x/\overline{x})$, and so μ is uniquely determined by the central character $\omega' = \omega_\Pi$ of Π and ω_ρ of ρ. Hence we talk about the lifting $e' : \rho \mapsto \Pi$, meaning that $\Pi = I(b(\rho),\ \omega'/\omega'_\rho)$, where $\omega'_\rho(x) = \omega_\rho(x/\overline{x})$ and $b(\rho)$ is the basechange of ρ.

The (elliptic ι-endoscopic) F-group $\mathbf{G}$ (of $\mathbf{G}'$) has a single proper elliptic endoscopic group $\mathbf{H}$. Here $\widehat{H} = Z_{\widehat{G}}(s)$ and $W_{E/F}$ acts via its quotient $\mathrm{Gal}(E/F)$ by $\sigma(x) = \varepsilon\theta(x)\varepsilon^{-1}$, $x \in \widehat{H}$. Thus to define ${}^L H \to {}^L G$ to extend $\widehat{H} \hookrightarrow \widehat{G}$ and $\sigma \mapsto \varepsilon \times \sigma$ to include the factor $W_{E/F}$, we need to map $z \in C_E = W_{E/E} = \ker(W_{E/F} \to \mathrm{Gal}(E/F)) = E^\times$ or $\mathbb{A}_E^\times/E^\times$, to $\mathrm{diag}(\kappa(z), 1, \kappa(z)) \times z$, where $\kappa : C_E/N_{E/F}C_E \to \mathbb{C}^\times$ is a homomorphism whose restriction to C_F is nontrivial (namely of order two). Indeed, $\sigma^2 \in C_F - N_{E/F}C_E$, and $\sigma^2 \mapsto \varepsilon\theta(\varepsilon) \times \sigma^2$, where $\varepsilon\theta(\varepsilon) = \mathrm{diag}(-1, 1, -1) = s$. We denote this homomorphism by $e : {}^L H \to {}^L G$ and name it the "*endoscopic map*". The group $\mathbf{H}$ is $\mathbf{U}(2, E/F) \times \mathbf{U}(1, E/F)$. If a representation $\rho \times \mu$ of $H = \mathbf{H}(F)$ or $\mathbf{H}(\mathbb{A})$ e-lifts to a representation π of $G = \mathbf{G}(F)$ or $\mathbf{G}(\mathbb{A})$, then $\omega_\pi = \kappa\omega_\rho\mu$, where the central characters ω_π, ω_ρ, μ are all characters of E^1 (or $\mathbb{A}_E^1/E^1$ globally). Note that $\kappa(z/\overline{z}) = \kappa^2(z)$. We fix $\omega = \omega_\pi$, hence $\mu = \omega_\pi/\omega_\rho\kappa$ is determined by κ and by the central character ω_ρ of ρ, and so it suffices to talk on the endoscopic lifting $\rho \mapsto \pi$, meaning $(\rho, \omega/\omega_\rho\kappa) \mapsto \pi$.

The homomorphism e factorizes via $i : {}^L H' \to {}^L G'$ and the unstable basechange map $b' : {}^L H \to {}^L H'$, $x \mapsto (x, x)$ for $x \in \widehat{H}$, $\sigma \mapsto (\varepsilon\theta(\varepsilon), 1)\sigma$, $z \mapsto (\kappa(z)_1, \kappa(z)_1)z$ for $z \in C_E$. Here $\kappa(z)_1$ indicates $\mathrm{diag}(\kappa(z), 1, \kappa(z))$. The basechange map on the factor $\mathbf{U}(1, E/F)$ is $\mu \mapsto \mu'$, $\mu'(z) = \mu(z/\overline{z})$, and $b : {}^L U(1) \to {}^L U(1)'$ is $x \mapsto (x, x)$, $b|W_{E/F}$ is the identity.

Let us summarize our L-group homomorphisms in a diagram:

$$\begin{array}{ccccccc}
{}^L G = \mathrm{GL}(3, \mathbb{C}) \rtimes W_{E/F} & \xrightarrow{b} & {}^L G' & & & & \\
e \uparrow & & i \uparrow & \nwarrow^{e'} & & & \\
{}^L H = \mathrm{GL}(2, \mathbb{C}) \rtimes W_{E/F} & \underset{b'}{\to} & {}^L H' & \underset{b''}{\leftarrow} & {}^L H & = \mathrm{GL}(2, \mathbb{C}) \rtimes W_{E/F}\,. &
\end{array}$$

Here

$${}^L G' = [\mathrm{GL}(3, \mathbb{C}) \times \mathrm{GL}(3, \mathbb{C})] \rtimes W_{E/F} \ {}^L H' = [\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})] \rtimes W_{E/F}.$$

Implicit is a choice of a character ω' on C_E and ω on C_E^1 related by $\omega'(z) = \omega(z/\overline{z})$.

The definition of the endoscopic map e and the unstable basechange map b' depend on a choice of a character $\kappa : C_E/N_{E/F}C_E \to \mathbb{C}^1$ whose restriction to C_F is nontrivial.

An L-groups homomorphism $\lambda : {}^LG \to {}^LG'$ defines — via the Satake transform — a lifting of unramified representations. It leads to a definition of a norm map N relating stable (σ-) conjugacy classes in G' to stable conjugacy classes in G based on the map $\delta \mapsto \delta\sigma(\delta)$, $G' \to G'$. In the local case it also leads to a suitable definition of matching of compactly supported smooth (locally constant in the p-adic case) complex valued functions on G and G'. Functions f on G and ϕ on G' are matching if a suitable (determined by λ) linear combination of their (σ-) orbital integrals over a stable conjugacy class, is related to the analogous object for the other group, via the norm map. Symbolically: "$\Phi^{\kappa}_{\phi}(\delta\sigma) = \Phi^{\mathrm{st}}_{f}(N\delta)$". The precise definition is given in [F3;VI] (in brief, the stable orbital integrals of f match the σ-twisted stable orbital integrals of ϕ, the orbital integrals of $'\phi$ match the σ-twisted unstable orbital integrals of ϕ, and the unstable orbital integrals of f match the stable orbital integrals of f_H). We state the names of the related functions according to the diagram of the L-groups above:

$$\begin{array}{ccccc} f & \xleftarrow{b} & \phi & & \\ e\downarrow & & & \searrow^{e'} & \\ f_H & & & & '\phi \end{array}$$

In fact we fix characters ω', ω on the centers $Z' = E^\times$ of $G' = \mathrm{GL}(3,E)$, $Z = E^1$ of $G = \mathrm{U}(3,E/F)$, related by $\omega'(z) = \omega(z/\overline{z})$, $z \in Z' = E^\times$, and consider ϕ on G' with $\phi(zg) = \omega'(z)^{-1}\phi(g)$, $z \in Z' = E^\times$, smooth and compactly supported $\operatorname{mod} Z'$, f on G with $f(zg) = \omega(z)^{-1}f(g)$, $z \in Z = E^1$, smooth and compactly supported $\operatorname{mod} Z$, but according to our conventions $f_H \in C_c^\infty(H)$ and $'\phi \in C_c^\infty(H)$ are compactly supported, where now $H = \mathrm{U}(2,E/F)$.

The representation theoretic results of [F3;VI] can be schematically put in a diagram:

$$\begin{array}{cccccccc} \pi & \overset{b}{\leftrightarrow} & \Pi & I(\rho'\otimes\kappa) & & I(\rho') & & \\ e\uparrow & & & \uparrow i & & i\uparrow & \nwarrow^{e'} & \\ \rho & \xrightarrow{b'} & & \rho'\otimes\kappa & & \rho' & \underset{b''}{\leftarrow} & \rho \end{array}$$

Here we make use of our results ([F3;VI]) in the case of basechange from $\mathrm{U}(2,E/F)$ to $\mathrm{GL}(2,E)$, namely that $b''(\rho) = \rho'$ iff $b'(\rho) = \rho'\otimes\kappa$, in the bottom row of the diagram. We describe these liftings in the next section, and in particular the structure of packets of representations on $G = \mathrm{U}(3, E/F)$. Both are defined in terms of character relations.

Nothing will be gained from working with the group of unitary similitudes

$$\mathrm{GU}(3, E/F) = \{(g, \boldsymbol{\lambda}) \in \mathrm{GL}(3, E) \times E^\times; gJ^t\overline{g} = \boldsymbol{\lambda}J\},$$

as it is the product $E^\times \cdot \mathrm{U}(3, E/F)$, where $E^\times$ indicates the diagonal scalar matrices, and $E^\times \cap \mathrm{U}(3, E/F)$ is E^1, the group of $x = z/\overline{z}$, $z \in E^\times$. Indeed, the transpose of $gJ^t\overline{g} = \boldsymbol{\lambda}J$ is $\overline{g}J^tg = \boldsymbol{\lambda}J$, hence $\boldsymbol{\lambda} = \boldsymbol{\lambda}(g) \in F^\times$, and taking determinants we get $x\overline{x} = \boldsymbol{\lambda}^3$ where $x = \det g$. Hence $\boldsymbol{\lambda} \in N_{E/F}E^\times \subset F^\times$, say $\boldsymbol{\lambda} = (u\overline{u})^{-1}$, $u \in E^\times$, then $ug \in \mathrm{U}(3, E/F)$.

Since an irreducible representation has a central character, working with admissible or automorphic representations of $\mathrm{U}(3, E/F)$ is the same as working with such a representation of $\mathrm{GU}(3, E/F)$: just extend the central character from the center $Z = \mathbf{Z}(F) = E^1$ (locally, or $\mathbf{Z}(\mathbb{A}) = \mathbb{A}^1$ globally) of $G = \mathbf{G}(F)$ (or $\mathbf{G}(\mathbb{A})$), to the center $E^\times$ (or $\mathbb{A}_E^\times$) of the group of similitudes. Consequently we shall talk on representations of U(3) as representations of GU(3) and vice versa, using the fixed central character. In our case the central character of the archimedean component π_∞ of the discrete-spectrum representations π occurring in the cohomology is determined by the sheaf of coefficients in the cohomology.

II.3 Local results on basechange for U(3)

We begin with the *local results* of [F3;VI]. Let E/F be a quadratic extension of nonarchimedean local fields of characteristic 0, put $G' = \mathrm{GL}(3, E)$, and denote by G or $\mathrm{U}(3, E/F)$ the group of F-points on the quasi-split unitary group in three variables over F which splits over E. We realize G as the group of g in G' with $\sigma(g) = g$, where $\sigma(g) = \theta(\overline{g})$, $\theta(g) = J^tg^{-1}J$, $\overline{g} = (\overline{g_{ij}})$ and ${}^tg = (g_{ji})$ if $g = (g_{ij})$, and

$$J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}.$$

Similarly, we realize the group of F-points on the quasi-split unitary group H, or $\mathrm{U}(2, E/F)$, in two variables over E/F as the group of h in $H' =$

GL$(2,E)$ with $\sigma(h)=\varepsilon\theta(\overline{h})\varepsilon$, $\theta(h)=w^t h^{-1} w$, $\varepsilon=\operatorname{diag}(1,-1)$ and

$$w=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Let N denote the norm map from E to F, and E^1 the unitary group $\mathrm{U}(1,E/F)$, consisting of $x\in E^\times$ with $Nx=1$.

Let ϕ, f, f_H denote complex valued locally constant functions on G', G, H. The function f_H is compactly supported. The functions ϕ, f transform under the centers $Z'\simeq E^\times$, $Z\simeq E^1$ of G', G by characters ω'^{-1}, ω^{-1} which are matching ($\omega'(z)=\omega(z/\overline{z})$, $z\in E^\times$), and are compactly supported modulo the center. The spaces of such functions are denoted by $C_c^\infty(G',\omega'^{-1})$, $C_c^\infty(G,\omega^{-1})$, $C_c^\infty(H)$. Assume they are *matching*. Thus the "stable" orbital integrals "$\Phi^{\mathrm{st}}(N\delta,fdg)$" of fdg match the twisted "stable" orbital integrals "$\Phi^{\sigma,\mathrm{st}}(\delta,\phi dg')$" of $\phi dg'$, and the unstable orbital integrals of fdg match the stable orbital integrals of $f_H dh$. These notions are defined in [F3;VI]; dg is a Haar measure on G, dg' on G', dh on H.

By a *G-module* π, or a representation π of G, we mean an admissible representation of G. If such a π is irreducible it has a central character by Schur's lemma. We work only with π which has the central character ω, thus $\pi(zg)=\omega(z)\pi(g)$ for all $g\in G$, $z\in Z$. By a representation we usually mean an irreducible one. For fdg as above the operator $\pi(fdg)$ has finite rank, hence it has trace $\operatorname{tr}\pi(fdg)\in\mathbb{C}$. We denote by χ_π the Harish-Chandra character of π. It is a complex valued function on G which is conjugacy invariant and locally constant on the regular set, with central character ω. Moreover it is locally integrable with $\operatorname{tr}\pi(fdg)=\int\chi_\pi(g)f(g)dg$ (g in G) for all measures dg on G and f in $C_c^\infty(G,\omega^{-1})$.

Definition. A G'-module Π is called *σ-invariant* if ${}^\sigma\Pi\simeq\Pi$, where ${}^\sigma\Pi(g)=\Pi(\sigma(g))$.

For such Π there is an intertwining operator $A:\Pi\to{}^\sigma\Pi$, thus $A\Pi(g)=\Pi(\sigma g)A$ for all $g\in G$. Assume that Π is irreducible. Then Schur's lemma implies that A^2 is a (complex) scalar. We normalize it to be 1. This determines A up to a sign. Extend Π to $G'\rtimes\langle\sigma\rangle$ by $\Pi(\sigma)=A$.

The twisted character $g\mapsto\chi_\Pi^\sigma(g)=\chi_\Pi(g\times\sigma)$ of such Π is a function on G' which depends on the σ-conjugacy classes and is locally constant on the σ-regular set. Further it is locally integrable and satisfies, for all measures

ϕdg,

$$\operatorname{tr}\Pi(\phi dg\times\sigma)=\int\chi^\sigma_\Pi(g)\phi(g)dg\qquad(g\text{ in }G').$$

DEFINITION. A σ-invariant G'-module Π is called *σ-stable* if its twisted character χ^σ_Π depends only on the stable σ-conjugacy classes in G, namely $\operatorname{tr}\Pi(\phi dg'\times\sigma)$ depends only on fdg. It is called *σ-unstable* if $\chi^\sigma_\Pi(\delta)=-\chi^\sigma_\Pi(\delta')$ whenever δ, δ' are σ-regular σ-stably conjugate elements which are not σ-conjugate, equivalently, $\operatorname{tr}\Pi(\phi dg'\times\sigma)$ depends only on $'\phi dh$.

An element of G' is called σ-elliptic if its norm in G is elliptic, namely lies in an anisotropic torus. It is called σ-regular if its norm is regular, namely its centralizer is a torus.

A σ-invariant G'-module Π is called *σ-elliptic* if its σ-character χ^σ_Π is not identically zero on the σ-elliptic σ-regular set.

We first deal with the σ-unstable σ-invariant representations.

UNSTABLE REPRESENTATIONS. *Every σ-invariant irreducible representation Π is σ-stable or σ-unstable. All σ-unstable σ-elliptic Π are of the form $I(\rho')$, normalizedly induced from the maximal parabolic subgroup; on the 2×2 factor the H'-module ρ' is obtained by the stable basechange map b'' from an elliptic representation ρ of H. We have*

$$\operatorname{tr}I(\rho';\phi dg'\times\sigma)=\operatorname{tr}\rho('\phi dh)$$

for all matching measures $'\phi dh$ and $\phi dg'$.

Our preliminary basechange result is

LOCAL BASECHANGE. *Let Π be a σ-stable irreducible tempered representation of G'. For every tempered G-module π there exist nonnegative integers $m'(\pi)=m'(\pi,\Pi)$ which are zero except for finitely many π, so that for all matching $\phi dg'$, fdg we have*

$$\operatorname{tr}\Pi(\phi dg'\times\sigma)=\sum_\pi m'(\pi)\operatorname{tr}\pi(fdg).\qquad(*)$$

This relation defines a partition of the set of (equivalence classes of) tempered irreducible G-modules into disjoint finite sets: for each π there is a unique Π for which $m'(\pi)\neq 0$.

Definition. (1) We call the finite set of π which appear in the sum on the right of $(*)$ a *packet*. Denote it by $\{\pi\}$, or $\{\pi(\Pi)\}$. It consists of tempered G-modules.
(2) Π is called the *basechange lift* of (each element π in) the packet $\{\pi(\Pi)\}$.

To refine the identity $(*)$ we prove that the multiplicities $m'(\pi)$ are equal to 1, and count the π which appear in the sum. The result depends on the σ-stable Π. First we note that:

List of the σ-stable Π. *The irreducible σ-stable Π are the σ-invariant Π which are square-integrable, one-dimensional, or induced $I(\rho' \otimes \kappa)$ from a maximal parabolic subgroup, where on the 2×2 factor the H'-module $\rho' \otimes \kappa$ is the tensor product of an H'-module ρ' obtained by the stable basechange map b'' in our diagram, and the fixed character κ of C_E/NC_E which is nontrivial on C_F.*

In the local case $C_E = E^\times$ and N is the norm from E to F. Namely $\rho' \otimes \kappa$ is obtained by the unstable map b' in our diagram, from a packet $\{\rho\}$ of H-modules (defined in [F3;VI]). The main local results of [F3;VI] are as follows:

Local Results. (1) *If Π is square integrable then it is σ-stable and the packet $\{\pi(\Pi)\}$ consists of a single square-integrable G-module π. If Π is of the form $I(\rho' \otimes \kappa)$, and ρ' is the stable basechange lift of a square-integrable H-packet $\{\rho\}$, then Π is σ-stable and the cardinality of $\{\pi(\Pi)\}$ is twice that of $\{\rho\}$.*

Remark. In the last case we denote $\{\pi(\Pi)\}$ also by $\{\pi(\rho)\}$, and say that $\{\rho\}$ *endo-lifts* to $\{\pi(\rho)\} = \{\pi(I(\rho \otimes \kappa))\}$.

Let $\{\rho\}$ be a square-integrable H-packet. It consists of one or two elements.

Local Results. (2) *If $\{\rho\}$ consists of a single element then $\{\pi\}$ consists of two elements, π^+ and π^-, and we have the character relation*

$$\operatorname{tr} \rho(f_H dh) = \operatorname{tr} \pi^+(fdg) - \operatorname{tr} \pi^-(fdg)$$

for all matching measures $f_H dh$, fdg.

If $\{\rho\}$ consists of two elements, then there are four members in $\{\pi(\rho)\}$, and three distinct square-integrable H-packets $\{\rho_i\}$ $(i = 1, 2, 3)$ with $\{\pi(\rho_i)\}$

$=\{\pi(\rho)\}$. *With this indexing, the four members of* $\{\pi_i\}$ *can be indexed so that we have the relations*

$$\operatorname{tr}\{\rho_i\}(f_H dh) = \operatorname{tr}\pi_1(fdg) + \operatorname{tr}\pi_{i+1}(fdg) - \operatorname{tr}\pi_{i'}(fdg) - \operatorname{tr}\pi_{i''}(fdg) \quad (**)$$

for all matching fdg, $f_H dh$. *Here* i', i'' *are so that* $\{i+1, i', i''\} = \{2,3,4\}$. *A single element in the packet has a Whittaker model. It is* π^+ *if* $[\{\rho\}] = 1$, *and* π_1 *if* $[\{\rho\}] = 2$.

REMARK. The proof that a packet contains no more than one generic member is given only in the case of odd residual characteristic. It depends on a twisted analogue of Rodier [F3;IX].

In the case of the Steinberg (or "special") H-module $s(\mu)$, which is the complement of the one-dimensional representation $1(\mu) : g \mapsto \mu(\det g)$ in the suitable induced representation of H, we denote their stable basechange lifts by $s'(\mu')$ and $1'(\mu')$. Here μ is a character of $C_E^1 = E^1$ (norm-one subgroup in $E^\times$), and $\mu'(a) = \mu(a/\overline{a})$ is a character of $C_E = E^\times$.

LOCAL RESULTS. (3) *The packet* $\{\pi(s(\mu))\}$ *consists of a cuspidal* $\pi^- = \pi_\mu^-$, *and the square-integrable subrepresentation* $\pi^+ = \pi_\mu^+$ *of the induced* G*-module* $I = I(\mu'\kappa\nu^{1/2})$. *Here* I *is reducible of length two, and its nontempered quotient is denoted by* $\pi^\times = \pi_\mu^\times$. *The character relations are*

$$\begin{aligned}
\operatorname{tr}(s(\mu))(f_H dh) &= \operatorname{tr}\pi^+(fdg) - \operatorname{tr}\pi^-(fdg),\\
\operatorname{tr}(1(\mu))(f_H dh) &= \operatorname{tr}\pi^\times(fdg) + \operatorname{tr}\pi^-(fdg),\\
\operatorname{tr} I(s'(\mu')\otimes\kappa; \phi dg' \times \sigma) &= \operatorname{tr}\pi^+(fdg) + \operatorname{tr}\pi^-(fdg),\\
\operatorname{tr} I(1'(\mu')\otimes\kappa; \phi dg' \times \sigma) &= \operatorname{tr}\pi^\times(fdg) - \operatorname{tr}\pi^-(fdg).
\end{aligned}$$

As the basechange character relations for induced modules are easy, we obtained the character relations for all (not necessarily tempered) σ-stable G'-modules.

If π is a nontempered irreducible G-module then its packet $\{\pi\}$ is defined to consist of π alone. For example, the packet of $\pi^\times$ consists only of $\pi^\times$. Also we make the following:

DEFINITION. Let μ be a character of $C_E^1 = E^1$. The *quasi-packet* $\{\pi(\mu)\}$ of the nontempered subquotient $\pi^\times = \pi_\mu^\times$ of $I(\mu'\kappa\nu^{1/2})$ consists of $\pi^\times$ and the cuspidal $\pi^- = \pi_\mu^-$.

Note that $\pi^\times$ is unramified when E/F and μ are unramified.

Thus a packet consists of tempered G-modules, or of a single nontempered element. A quasi-packet consists of a nontempered $\pi^\times$ and a cuspidal π^-. The packet of π^- consists of π^- and π^+, where π^+ is the square-integrable constituent of $I(\mu'\kappa\nu^{1/2})$. These local definitions are made for global purposes.

II.4 Global results on basechange for U(3)

We shall now state the *global results* of [F3;VI].

Let E/F be a quadratic extension of number fields, $\mathbb{A}_E$ and $\mathbb{A} = \mathbb{A}_F$ their rings of adèles, $\mathbb{A}_E^\times$ and $\mathbb{A}^\times$ their groups of idèles, N the norm map from E to F, $\mathbb{A}_E^1$ the group of E-idèles with norm 1, $C_E = \mathbb{A}_E^\times/E^\times$ the idèle class group of E, ω a character of $C_E^1 = \mathbb{A}_E^1/E^1$, ω' a character of C_E with $\omega'(z) = \omega(z/\overline{z})$. Denote by $\mathbf{H}$, or $\mathbf{U}(2, E/F)$, and by $\mathbf{G}$, or $\mathbf{U}(3, E/F)$, the quasi-split unitary groups associated to E/F and the forms εw and J as defined in the local case. These are reductive F-groups. We often write G for $\mathbf{G}(F)$, H for $\mathbf{H}(F)$, and $G' = \mathrm{GL}(3, E)$ for $\mathbf{G}'(F) = \mathbf{G}(E)$, where $\mathbf{G}' = \mathrm{R}_{E/F}\mathbf{G}$ is the F-group obtained from $\mathbf{G}$ by restriction of scalars from E to F. Note that the group of E-points $\mathbf{G}'(E)$ is $\mathrm{GL}(3, E) \times \mathrm{GL}(3, E)$.

Denote the places of F by v, and the completion of F at v by F_v. Put $G_v = \mathbf{G}(F_v)$, $G'_v = \mathbf{G}'(F_v) = \mathrm{GL}(3, E_v)$, $H_v = \mathbf{H}(F_v)$. Note that at a place v which splits in E we have that $\mathbf{U}(n, E/F)(F_v)$ is $\mathrm{GL}(n, F_v)$. When v is nonarchimedean denote by R_v the ring of integers of F_v. When v is also unramified in E put $K_v = \mathbf{G}(R_v)$. Also put $K_{Hv} = \mathbf{H}(R_v)$ and $K'_v = \mathbf{G}'(R_v) = \mathrm{GL}(3, R_{E,v})$, where $R_{E,v}$ is the ring of integers of $E_v = E \otimes_F F_v$. When v splits we have $E_v = F_v \oplus F_v$ and $R_{E,v} = R_v \oplus R_v$.

Write $L^2(G, \omega)$ for the space of right-smooth complex-valued functions ϕ on $G\backslash\mathbf{G}(\mathbb{A})$ with $\phi(zg) = \omega(z)\phi(g)$ ($g \in \mathbf{G}(\mathbb{A})$, $z \in \mathbf{Z}(\mathbb{A})$, $\mathbf{Z}$ being the center of $\mathbf{G}$). The group $\mathbf{G}(\mathbb{A})$ acts by right translation: $(r(g)\phi)(h) = \phi(hg)$. The $\mathbf{G}(\mathbb{A})$-module $L^2(G, \omega)$ decomposes as a direct sum of (1) the discrete spectrum $L_d^2(G, \omega)$, defined to be the direct sum of all irreducible subrepresentations, and (2) the continuous spectrum $L_c^2(G, \omega)$, which is described by Langlands' theory of Eisenstein series as a continuous sum.

The $\mathbf{G}(\mathbb{A})$-module $L_d^2(G, \omega)$ further decomposes as a direct sum of the cuspidal spectrum $L_0^2(G, \omega)$, consisting of cusp forms ϕ, and the residual spectrum $L^2_r(G,\omega)$, which is generated by residues of Eisenstein series. Each irreducible constituent of $L^2(G,\omega)$ is called an *automorphic* representation, and we have *discrete-spectrum* representations, *cuspidal*, *residual* and *continuous-spectrum* representations. Each such has central character ω. The discrete-spectrum representations occur in L^2_d with finite multiplicities. Of course, similar definitions apply to the groups $\mathbf{H}$, $\mathbf{G}'$ and $\mathbf{H}'$.

By a $\mathbf{G}(\mathbb{A})$-*module* we mean an admissible representation of $\mathbf{G}(\mathbb{A})$. Any irreducible $\mathbf{G}(\mathbb{A})$-module π is a restricted tensor product $\otimes_v \pi_v$ of admissible irreducible representations π_v of $G_v = \mathbf{G}(F_v)$, which are almost all (at most finitely many exceptions) unramified. A G_v-module π_v is called *unramified* if it has a nonzero K_v-fixed vector. It is a rare property for a $\mathbf{G}(\mathbb{A})$-module to be automorphic.

An L-groups homomorphism ${}^L H \to {}^L G$ defines via the Satake transform a lifting $\rho_v \mapsto \pi_v$ of unramified representations. Given an automorphic representation ρ of $\mathbf{H}(\mathbb{A})$, the L-groups homomorphism ${}^L H \to {}^L G$ defines then unramified π_v at almost all places. We say that ρ *quasi-e-lifts* to π if ρ_v e-lifts to π_v for almost all places v. Here “e” is for “endoscopic” and “b” is for “basechange”.

A preliminary result is an existence result, of π in the following statement.

Quasi-Lifting. *Every automorphic ρ quasi-e-lifts to an automorphic π.*

Every automorphic π quasi-b-lifts to an automorphic σ-invariant Π *on* $\mathrm{GL}(3,\mathbb{A}_E)$.

The same result holds for each of the homomorphisms in our diagram.

To be pedantic, under the identification $\mathrm{GL}(3,E) = G'$, $g \mapsto (g,\sigma g)$, we can introduce $\Pi'(g,\sigma g) = \Pi(g)$. Then ${}^\sigma\Pi = {}^\iota\Pi'$, where $\iota(x,y) = (y,x)$. Thus Π is σ-invariant as a $\mathrm{GL}(3,E)$-module iff Π' is ι-invariant as a G'-module (and similarly globally).

The main global results of [F3;VI] consist of a refinement of the quasi-lifting to lifting in terms of all places. To state the result we need to define and describe packets of discrete-spectrum $\mathbf{G}(\mathbb{A})$-modules. To introduce the definition, recall that we defined above packets of tempered G_v-modules at each v, as well as quasi-packets, which contain a nontempered representation. If v splits then $G_v = \mathrm{GL}(3,F_v)$ and a (quasi-) packet consists of a single irreducible.

Definition. (1) Given a local packet P_v for all v such that P_v contains an unramified member π_v^0 for almost all v, we define the *global packet* P to be the set of products $\otimes\pi_v$ over all v, where π_v lies in P_v for all v, and $\pi_v = \pi_v^0$ for almost all v.
(2) Given a character μ of $C_E^1 = \mathbb{A}_E^1/E^1$, the quasi-packet $\{\pi(\mu)\}$ is defined as in the case of packets, where P_v is replaced by the quasi-packet $\{\pi(\mu_v)\}$ for all v, and π_v^0 is the unramified $\pi_v^\times$ at the v where E/F and μ are unramified.
(3) The $\mathbf{H}(\mathbb{A})$-module $\rho = \otimes\rho_v$ *endo-lifts* to the $\mathbf{G}(\mathbb{A})$-module $\pi = \otimes\pi_v$ if ρ_v endo-lifts to π_v (i.e. $\{\rho_v\}$ endo-lifts to $\{\pi_v\}$) for all v. Similarly, $\pi = \otimes\pi_v$ *basechange* lifts to the $\mathrm{GL}(3, \mathbb{A}_E)$-module $\Pi = \otimes\Pi_v$ if π_v basechange lifts to Π_v for all v.

A complete description of the packets is as follows.

Global Lifting. *The basechange lifting is a one-to-one correspondence from the set of packets and quasi-packets which contain an automorphic* $\mathbf{G}(\mathbb{A})$*-module, to the set of* σ*-invariant automorphic* $\mathrm{GL}(3, \mathbb{A}_E)$*-modules* Π *which are not of the form* $I(\rho')$*. Here* ρ' *is the* $\mathrm{GL}(2, \mathbb{A}_E)$*-module obtained by stable basechange from a discrete-spectrum* $\mathbf{H}(\mathbb{A})$*-packet* $\{\rho\}$*.*

As usual, we write $\{\pi(\rho)\}$ for a packet which basechanges to $\Pi = I(\rho' \otimes \kappa)$, where the $\mathbf{H}'(\mathbb{A})$-module ρ' is the stable basechange lift of the $\mathrm{GL}(2, \mathbb{A}_E)$-packet $\{\rho\}$. We conclude:

Description of packets. *Each irreducible* $\mathbf{G}(\mathbb{A})$*-module* π *in the discrete spectrum lies in one of the following.*
(1) *A packet* $\{\pi(\Pi)\}$ *associated with a discrete-spectrum* σ*-invariant representation* Π *of* $\mathrm{GL}(3, \mathbb{A}_E)$*.*
(2) *A packet* $\{\pi(\rho)\}$ *associated with a cuspidal* $\mathbf{H}(\mathbb{A})$*-module* ρ*.*
(3) *A quasi-packet* $\{\pi(\mu)\}$ *associated with an automorphic one-dimensional* $\mathbf{H}(\mathbb{A})$*-module* $\rho = \mu \circ \det$*.*

Packets of type (1) will be called *stable*, those of type (2) *unstable*, and quasi-packets are unstable too. The terminology is justified by the following result.

Multiplicities. (1) *The multiplicity of a* $\mathbf{G}(\mathbb{A})$*-module* $\pi = \otimes\pi_v$ *from a packet* $\{\pi(\Pi)\}$ *of type* (1) *in the discrete spectrum of* $\mathbf{G}(\mathbb{A})$ *is one. Namely each element* π *of* $\{\pi(\Pi)\}$ *is automorphic, in the discrete spectrum, in fact in the cuspidal spectrum unless* $\dim\pi = 1$*.*

(2) *The multiplicity of* π *from a packet* $\{\pi(\rho)\}$ *or a quasi-packet* $\{\pi(\mu)\}$ *in the discrete spectrum of* $\mathbf{G}(\mathbb{A})$ *is equal to* 1 *or* 0*. It is not constant over* $\{\pi(\rho)\}$ *and* $\{\pi(\mu)\}$.

If π *lies in* $\{\pi(\rho)\}$*, and there is a single* ρ *which endo-lifts to* π*, then the multiplicity is*

$$m(\rho,\pi)=\frac{1}{2}\left(1+\prod_v\langle\rho_v,\pi_v\rangle\right),$$

where $\langle\rho_v,\pi_v\rangle=1$ *if* π_v *lies in* $\pi(\rho_v)^+$*, and* $\langle\rho_v,\pi_v\rangle=-1$ *if* π_v *lies in* $\pi(\rho_v)^-$.

Let π *lie in* $\{\pi(\rho_1)\}=\{\pi(\rho_2)\}=\{\pi(\rho_3)\}$ *where* $\{\rho_1\}$, $\{\rho_2\}$, $\{\rho_3\}$ *are distinct* $\mathbf{H}(\mathbb{A})$*-packets. Then the multiplicity of* π *is* $\frac{1}{4}(1+\sum_{i=1}^3\langle\rho_i,\pi\rangle)$*. The signs* $\langle\rho_i,\pi\rangle=\prod_v\langle\rho_{iv},\pi_v\rangle$ *are defined by* $(**)$*. The* π *of this and the previous paragraph are in fact cuspidal.*

If π *lies in* $\{\pi(\mu)\}$ *the multiplicity is given by*

$$m(\mu,\pi)=\frac{1}{2}\left[1+\varepsilon(\mu',\kappa)\prod_v\langle\mu_v,\pi_v\rangle\right].$$

Here $\varepsilon(\mu',\kappa)$ *is a sign* (1 *or* -1) *depending on* μ (*that is on* $\mu'(x)=\mu(x/\overline{x})$) *and* κ*, and* $\langle\mu_v,\pi_v\rangle=1$ *if* $\pi_v=\pi_{\mu_v}^\times$ *and* $\langle\mu_v,\pi_v\rangle=-1$ *if* $\pi_v=\pi_{\mu_v}^-$.

The sign $\varepsilon(\mu',\kappa)$ is likely to be the value at 1/2 of the ε-factor $\varepsilon(s,\mu'\kappa)$ of the functional equation of the L-function $L(s,\mu'\kappa)$ of $\mu'\kappa$. This is the case when $L(\frac{1}{2},\mu'\kappa)\neq 0$, in which case $\pi_\mu^\times=\prod_v\pi_{\mu_v}^\times$ is residual and $\varepsilon(\frac{1}{2},\mu'\kappa)=1$. When $L(\frac{1}{2},\mu'\kappa)=0$ the automorphic representation $\pi_\mu^\times$ is in the discrete spectrum (necessarily cuspidal) iff $\varepsilon(\mu',\kappa)=1$. An irreducible π in the quasi-packet of $\pi_\mu^\times$ which is in the discrete spectrum (thus $m(\mu,\pi)=1$) with at least one component π_v^- is cuspidal, since π_v^- is cuspidal. Thus with the exception of the residual $\pi_\mu^\times$ (when $L(\frac{1}{2},\mu'\kappa)\neq 0$) and one-dimensional representations, the multiplicity of π in the discrete spectrum is the same as its multiplicity in the cuspidal spectrum. Discrete-spectrum π lie either in the cuspidal or the residual spectrum.

In particular we have the following

Multiplicity One Theorem. *Distinct irreducible constituents in the discrete spectrum of* $L^2(\mathbf{G}(\mathbb{A}),\omega)$ *are inequivalent.*

Rigidity Theorem. *If π and π' are discrete-spectrum $\mathbf{G}(\mathbb{A})$-modules whose components π_v and π'_v are equivalent for almost all v, then they lie in the same packet, or quasi-packet.*

Genericity. *Each G_v- and $\mathbf{G}(\mathbb{A})$-packet contains precisely one generic representation. Quasi-packets do not contain generic representations.*

Corollary. (1) *Suppose that π is a discrete-spectrum $\mathbf{G}(\mathbb{A})$-module which has a component of the form $\pi_w^\times$. Then π lies in a quasi-packet $\{\pi(\mu)\}$, of type* (3). *In particular its components are of the form $\pi_v^\times$ for almost all v, and of the form π_v^- for the remaining finite set* (*of even cardinality iff $\varepsilon(\mu', \kappa)$ is* 1) *of places of F which stay prime in E.*
(2) *If π is a discrete-spectrum $\mathbf{G}(\mathbb{A})$-module with an elliptic component at a place of F which splits in E, or a one-dimensional or Steinberg component at a place of F which stay prime in E, then π lies in a packet $\{\pi(\Pi)\}$, where Π is a discrete-spectrum* $\mathrm{GL}(3, \mathbb{A}_E)$*-module.*

A cuspidal representation in a quasi-packet $\{\pi(\mu)\}$ of type (3) (for example, one which has a component π_v^-) makes a *counter example to the naive Ramanujan conjecture*: almost all of its components are nontempered, namely $\pi_v^\times$. The Ramanujan conjecture for $\mathrm{GL}(n)$ asserts that all local components of a cuspidal representation of $\mathrm{GL}(n, \mathbb{A})$ are tempered. The Ramanujan conjecture for U(3) should say that all local components of a discrete-spectrum representation π of $\mathbf{U}(3, E/F)(\mathbb{A})$ which basechange lifts to a cuspidal representation of GL(3,$\mathbb{A}$) are tempered. This is shown below for π with "cohomological" components at the archimedean places by using the theory of Shimura varieties associated with U(3).

The discrete-spectrum $\mathbf{G}(\mathbb{A})$-modules π with an elliptic component at a nonarchimedean place v of F which splits in E (such π are stable of type (1)) can easily be transferred to discrete-spectrum $'\mathbf{G}(\mathbb{A})$-modules, where $'\mathbf{G}$ is the inner form of $\mathbf{G}$ which is ramified at v. Thus $'\mathbf{G}$ is the unitary F-group associated with the central division algebra of rank three over E which is ramified at the places of E over v of F.

II.5 Spectral side of the stable trace formula

We are now in a position to describe the spectral side of the stable trace formula for a test function $f = \otimes f_v$ on $G(\mathbb{A})$. Thus $\mathrm{STF}_G(f)$ is the sum of four parts: $I(G,1), \ldots, I(G,4)$. The first is

$$I(G,1) = \sum_{\{\pi\}} \prod_v \mathrm{tr}\{\pi_v\}(f_v).$$

The sum ranges over the packets $\{\pi\}$ which basechange lift to cuspidal σ-invariant representations Π of $\mathrm{GL}(3, \mathbb{A}_E)$ as well as over the one-dimensional representations π of $G(\mathbb{A})$.

The second part, $I(G,2)$, of $\mathrm{STF}_G(f)$, is the sum of

$$\frac{1}{2} \prod_v [\mathrm{tr}\, \pi_v^+(f_v) + \mathrm{tr}\, \pi_v^-(f_v)]$$

over all cuspidal representations $\rho \neq \rho(\theta, {}'\theta)$ of

$$\mathrm{U}(2, E/F)(\mathbb{A}) \times \mathrm{U}(1, E/F)(\mathbb{A}).$$

Here $\{\pi\}$ is the e-lift of ρ, thus $e(\rho_v) = \{\pi_v^+, \pi_v^-\}$ for all v; π_v^- is zero if ρ_v is not discrete series or if v splits in E.

The third part, $I(G,3)$, is the sum of

$$\frac{1}{4} \prod_v \mathrm{tr}\{\pi_v\}(f_v)$$

over all unordered triples (μ, μ', μ'') of distinct characters of $\mathbb{A}_E^1/E^1$ with $\mu\mu'\mu'' = \omega$, where $\{\pi\}$ is the lift of $\rho(\mu, \mu')$ on $\mathrm{U}(2)$.

The fourth part, $I(G,4)$, is the sum of

$$\frac{\varepsilon(\mu', \kappa)}{2} \prod_v [\mathrm{tr}\, \pi_v^\times(f_v) - \mathrm{tr}\, \pi_v^-(f_v)]$$

over all one-dimensional representations μ of $\mathrm{U}(2) \times \mathrm{U}(1)$. For each v the pair $\{\pi_v^\times, \pi_v^-\}$ is the quasi-packet $e(\mu_v)$. It consists only of $\pi_v^\times$ (and π_v^- is zero) when v splits.

II.6 Proper endoscopic group

The spectral side of the other trace formula which we need is for a function $f_H = \otimes f_{Hv}$ on $H(\mathbb{A}) = \mathrm{U}(2, E/F)(\mathbb{A}) \times \mathrm{U}(1, E/F)(\mathbb{A})$. It comes multiplied by the coefficient $\frac{1}{2}$, and has the form $I(H,1) + I(H,2) + I(H,3)$, where the three summands are defined by

$$\sum_{\rho \neq \rho(\theta,'\theta)} \prod_v \mathrm{tr}\{\rho_v\}(f_{Hv}) + \frac{1}{2} \sum_{\rho=\rho(\theta,'\theta)} \prod_v \mathrm{tr}\{\rho_v\}(f_{Hv}) + \sum_\mu \prod_v \mathrm{tr}\, \mu_v(f_{Hv}).$$

The first sum, in $I(H,1)$, ranges over the packets of the cuspidal representations of $\mathrm{U}(2, E/F)(\mathbb{A}_F) \times \mathrm{U}(1, E/F)(\mathbb{A}_F)$ not of the form $\rho(\theta,'\theta) \times {}''\theta$. The θ are characters on $\mathbb{A}_E^1/E^1$.

The second sum, in $I(H,2)$, is over the cuspidal packets ρ of the form $\rho(\theta,'\theta) \times {}''\theta$, where $\{\theta,'\theta,''\theta\}$ are distinct characters. The lifting from $\mathrm{U}(2) \times \mathrm{U}(1)$ to $\mathrm{U}(3)$ on this set of packets is 3-to-1. Only $\rho_1 = \rho(\theta,'\theta) \times {}''\theta$, $\rho_2 = \rho(\theta,''\theta) \times {}'\theta$ and $\rho_3 = \rho('\theta,''\theta) \times \theta$ lift to the same packet of $\mathrm{U}(3)$.

The sum of $I(H,3)$ ranges over the one-dimensional representations μ of $\mathrm{U}(2, E/F)(\mathbb{A}_F)$.

At all places v not dividing p or ∞ the component f_{Hv} is matching f_v, so the local factor indexed by v in each of the 3 sums can be replaced by

$$\mathrm{tr}\, \pi_v^+(f_v) - \mathrm{tr}\, \pi_v^-(f_v),$$
$$\mathrm{tr}\, \pi_v^\times(f_v) + \mathrm{tr}\, \pi_v^-(f_v),$$
$$\mathrm{tr}\{\rho_{iv}\}(f_{Hv}) = \sum_{1 \leq j \leq 4} \langle \rho_{iv}, \pi_{jv} \rangle \, \mathrm{tr}\, \pi_{jv}(\rho_v)(f_v).$$

III. LOCAL TERMS

III.1 The reflex field

Our group is $G' = \mathrm{R}_{F/\mathbb{Q}}G$, where G is $\mathrm{GU}(3, E/F)$, F is a totally real field and E is a totally imaginary quadratic extension of F. Thus G' is split over $\mathbb{Q}$, $G'(\mathbb{Q}) = G(F)$ and $G'(\mathbb{R}) = G(\mathbb{R}) \times \cdots \times G(\mathbb{R})$ ($[F:\mathbb{Q}]$ times). The dimension of the corresponding Shimura variety is $2[F:\mathbb{Q}]$. Half the real dimension of the symmetric space $G(\mathbb{R})/K_{G(\mathbb{R})}$ is 2. We proceed to show that the reflex field $\mathbb{E}$ is a CM-field contained in the Galois closure of $E/\mathbb{Q}$.

Since all quasi-split unitary groups of rank one defined using E/F are isomorphic, we choose now the Hermitian form J $(= {}^t\overline{J}$ in $\mathrm{GL}(3, E))$ to be $\mathrm{diag}(1, -1, -1)$. It defines the group $G = \mathrm{GU}(1, 2; E/F)$ of unitary similitudes which is the linear reductive quasi-split algebraic group over F whose value at any F-algebra A is

$$G(A) = \{(g, \lambda) \in \mathrm{GL}(3, A_E) \times A_E^\times; {}^t\overline{g}Jg = \lambda J\}$$

where $A_E = A \otimes_F E$ and $x \mapsto \overline{x}$ is the nontrivial automorphism of A_E over A. Applying transpose-bar to ${}^t\overline{g}Jg = \lambda J$ we see that $\lambda \in A^\times$. Since λ is determined by g, $G(A) \subset \mathrm{GL}(3, A_E)$ and $G(A_E) = \mathrm{GL}(3, A_E) \times A_E^\times$.

A key part of the data which defines the Shimura variety is a $G'(\mathbb{R})$-conjugacy class X_∞ of homomorphisms $h : \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G'$ over $\mathbb{R}$. Over $\mathbb{R}$ the group G' is isomorphic to $\prod_\sigma G_\sigma$, where σ ranges over $\mathrm{Emb}(F, \mathbb{R})$, and

$$G_\sigma = G \otimes_{F,\sigma} \mathbb{R} \quad (= G \times_{\mathrm{Spec}\, F, \sigma} \mathrm{Spec}\, \mathbb{R}) = \mathrm{GU}(1, 2; E \otimes_{F,\sigma} \mathbb{R}/\mathbb{R})$$

is an $\mathbb{R}$-group. Put $h = (h_\sigma)$. Note that $E \otimes_{F,\sigma} \mathbb{R}$ is a quadratic extension of $\mathbb{R}$, but there are two possible isomorphisms to $\mathbb{C}$ over $\mathbb{R}$, determined by the choice of an extension $\tau : E \hookrightarrow \mathbb{C}$ of $\sigma : F \hookrightarrow \mathbb{R}$. Thus if $E = F(\xi)$, $\overline{\xi} = -\xi$ (here bar denotes the automorphism of E/F), $\xi^2 \in F^\times$, $\sigma(\xi^2) < 0$ in $\mathbb{R}$, and $E \otimes_{F,\sigma} \mathbb{R} = \mathbb{R}(\sqrt{\sigma(\xi^2)})$. Given $\tau : E \hookrightarrow \mathbb{C}$, $\tau|F = \sigma$, we have $\tau(\xi) \in \mathbb{C}$, namely a choice of $\sqrt{\sigma(\xi^2)} \mapsto \tau(\xi) \in \mathbb{C}$, that is $\tau_* : E \otimes_{F,\sigma} \mathbb{R} \xrightarrow{\sim} \mathbb{C}$ and $\tau_* : G_\sigma \xrightarrow{\sim} \mathrm{GU}(1, 2; \mathbb{C}/\mathbb{R})$. The embedding $c\tau : E \hookrightarrow \mathbb{C}$, where c denotes complex conjugation in $\mathbb{C}$, defines another isomorphism $c\tau_*$ of G_σ with $\mathrm{GU}(1,2;\mathbb{C}/\mathbb{R})$.

Let Σ be a *CM-type of E/F*. It is a set which consists of one extension $\tau : E \hookrightarrow \mathbb{C}$ of each $\sigma : F \hookrightarrow \mathbb{R}$. Then $\Sigma \cap c\Sigma$ is empty and $\Sigma \cup c\Sigma$ is $\mathrm{Emb}(E,\mathbb{C})$ (if $\Sigma = \{\tau\}$ then $c\Sigma = \{c\tau\}$). For each $\tau \in \Sigma$, $h_\tau = \tau_* \circ h_\sigma$ is an algebraic homomorphism $\mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{GU}(1,2;\mathbb{C}/\mathbb{R})$ which can be diagonalized over $\mathbb{C}$, namely we may assume that h_τ has its image in the diagonal torus T of $\mathrm{GU}(1,2;\mathbb{C}/\mathbb{R})$. We choose $h_\tau(z,\overline{z}) = (\mathrm{diag}(z,\overline{z},\overline{z}), z\overline{z})$. Then $h_{c\tau}(z,\overline{z}) = (\mathrm{diag}(\overline{z},z,z), z\overline{z})$, where $c(z) = \overline{z}$ for $z \in \mathbb{C}^\times$. Over $\mathbb{C}$, $h_\tau : \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{GU}(1,2;\mathbb{C}/\mathbb{R})$ has the form

$$h_{\tau,\mathbb{C}} : \mathbb{C}^\times \times \mathbb{C}^\times \to \mathrm{GL}(3,\mathbb{C}) \times \mathbb{C}^\times, \quad h_{\tau,\mathbb{C}}(z,w) = (\mathrm{diag}(z,w,w), zw).$$

Up to conjugacy by the Weyl group $W_\mathbb{C}$ of $\mathrm{GL}(3,\mathbb{C})$ we have $h_{c\tau,\mathbb{C}}(z,w) = (\mathrm{diag}(z,z,w), zw)$. The restriction $\mu_\tau(z) = h_{\tau,\mathbb{C}}(z,1)$ to the first variable is $z \mapsto (\mathrm{diag}(z,1,1), z)$, and $\mu_{c\tau}(z) = (\mathrm{diag}(z,z,1), z)$. We regard μ_τ and $\mu_{c\tau}$ as representatives of their conjugacy classes.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mu = (\mu_\tau; \tau \in \Sigma)$ since μ is defined over $\mathbb{Q}$. Thus $\varphi \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ maps μ to $\varphi\mu = (\mu_{\varphi\circ\tau})$, where we fix $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and view τ as $E \hookrightarrow \overline{\mathbb{Q}}$. The subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ which fixes μ defines a number field $\mathbb{E}$, called the *reflex field* of μ. This is the same as the reflex field of the CM-type Σ, as the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on μ is determined by its action on Σ.

Let us emphasize that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the $G'(\mathbb{C})$-conjugacy class of $\mu = (\mu_\tau; \tau \in \Sigma)$, or its $\prod_\Sigma W_\mathbb{C}$-conjugacy class if μ is viewed in $\prod_\Sigma T(\mathbb{C})$. In fact the conjugacy classes of μ_τ and $\mu_{c\tau}$ can be distinguished by the determinants of their first components: $\det \mu_\tau(z) = z$, $\det \mu_{c\tau}(z) = z^2$. Then $\mathbb{E}$ is determined equally by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\det \mu = (\det \mu_\tau; \tau \in \Sigma)$.

LEMMA. *The reflex field $\mathbb{E}$ is a totally imaginary quadratic extension of a totally real field $\mathbb{E}^c$ contained in E.*

PROOF. Clearly complex conjugation c does not fix μ, $\det \mu$ or Σ, hence $c \notin \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$. The Galois closure $F' = \cup_\sigma \sigma F$ of F is totally real, and the Galois closure $E' = \cup_\tau \tau E$ (it suffices to take $\tau \in \Sigma$ as $c\tau E = \tau E$ for every $\tau \in \Sigma$) of E is totally imaginary quadratic extension of a totally real Galois extension F'' of $\mathbb{Q}$. Indeed $F'' = F'((\sqrt{\sigma(\xi^2)\sigma'(\xi^2)};\ \sigma \neq \sigma'))$ and $E' = F''(\sqrt{\sigma(\xi^2)}$, any $\sigma)$. Now $\mathbb{E} \subset E'$ since $\mathrm{Gal}(\overline{\mathbb{Q}}/E')$ fixes Σ and μ.

Complex conjugation, c, restricts to the nontrivial element of $\mathrm{Gal}(E'/F'')$ (and of each $\mathrm{Gal}(\tau E/\sigma F)$). The group $\langle c\rangle$ is normal in $\mathrm{Gal}(E'/\mathbb{Q})$ since F'' is Galois over $\mathbb{Q}$. Hence c is a central element. Since $c \notin \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$, c acts on $\mathbb{E}$ nontrivially, and on each conjugate of $\mathbb{E}$ (in E'). Finally, as noted in section III.1, $\mathbb{E}$ is contained in E. □

III.2 The representation of the dual group

The representation (r_μ^0, V_μ) of ${}^L G'_{\mathbb{E}} = \widehat{G}' \rtimes W_{\mathbb{E}}$ associated in [L2] to the conjugacy class $\mathrm{Int}(G'(\mathbb{C}))\mu$ of the weight $\mu = \mu_h$ (see section III.1) is specified by two properties.

(1) The restriction of r_μ^0 to $\widehat{G}'$ is irreducible with extreme weight $-\mu$. Here $\mu = \mu_h \in X^*(\widehat{T}) = X_*(T)$ is a character of a maximal torus $\widehat{T}$ of $\widehat{G}'$, uniquely determined up to the action of the Weyl group.

(2) Let y be a splitting ([Ko3], section 1) of $\widehat{G}'$. Assume that y is fixed by the Weil group $W_{\mathbb{E}}$ of $\mathbb{E}$. Then $W_{\mathbb{E}} \subset {}^L G'_{\mathbb{E}}$ acts trivially on the highest weight space of V_μ attached to y.

If T denotes the diagonal torus in G, T' in G', $\widehat{T}$ in $\widehat{G} = \mathrm{GL}(3,\mathbb{C})$ and $\widehat{T}' = \prod_\sigma \widehat{T}$ in $\widehat{G}' = \prod_\sigma \widehat{G}$, then $\mu_\tau \in X_*(T) = X^*(\widehat{T})$ can be viewed as the character $\mu_\tau = (1,0,0)$ of $\widehat{T}$, mapping $\mathrm{diag}(a,b,c)$ to a. Then $\mu_{c\tau} = (1,1,0)$, and $\mu = \prod \mu_\tau$ ($\tau \in \Sigma$) is $(1,0,0) \times (1,0,0) \times \cdots \times (1,0,0)$. Note that the $G(\mathbb{C})$-orbit of μ_τ determines a $W_{\mathbb{C}}$-orbit of μ_τ in $X^*(\widehat{T})$. The character $\mu_\tau = (1,0,0)$ is the highest weight of the standard representation st of $\mathrm{GL}(3,\mathbb{C})$, which we now denote by r_τ^0, while $\mu_{c\tau} = (1,1,0)$ is that of $r_{c\tau}^0 = \wedge^2(\mathrm{st})$ $(= \det \otimes \mathrm{st}^\vee)$.

A basis for the 3^n-dimensional representation $r_\mu^0 = \otimes_{\tau\in\Sigma} r_\tau^0$ is of the form $\otimes_{\tau\in\Sigma} e^\tau_{\ell(\tau)}$ $(1 \le \ell(\tau) \le 3)$. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via its action on Σ; the stabilizer is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$. Thus we may let the Weil group $W_{\mathbb{E}}$ act on the highest weight vector of μ, via its quotient in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ permuting the factors of r_μ^0, and define $r_\mu = \mathrm{Ind}_{W_{\mathbb{E}}}^{W_{\mathbb{Q}}}(r_\mu^0)$.

An irreducible admissible representation π_p of $G(F \otimes \mathbb{Q}_p) = G'(\mathbb{Q}_p) = \prod_{u|p} G(F_u)$ has the form $\otimes_u \pi_u$. Suppose it is unramified. If u splits in E, thus $E \otimes_F F_u = F_u \oplus F_u$, then π_u has the form $\pi(\mu_{1u}, \mu_{2u}, \mu_{3u})$, a subquotient of the induced representation $I(\mu_{1u}, \mu_{2u}, \mu_{3u})$ of $G(F_u) = \mathrm{GL}(3, F_u)$, where μ_{iu} are unramified characters of $F_u^\times$. If u stays prime in E, thus

$E_u = E \otimes_F F_u$ is a field, π_u has the form $\pi(\mu_u) \subset I(\mu_u)$. Write μ_{mu} for the value $\mu_{mu}(\boldsymbol{\pi}_u)$ at any uniformizing parameter $\boldsymbol{\pi}_u$ of $F_u^\times$ (and $E_u^\times$). Put $t_u = t(\pi_u) = \operatorname{diag}(\mu_{1u}, \mu_{2u}, \mu_{3u})$ if u splits, and $t(\pi_u) = \operatorname{diag}(\mu_u, 1, 1) \times \operatorname{Fr}_u$ if E_u is a field. In the latter case we also write $\mu_{1u} = \mu_u^{1/2}$, $\mu_{2u} = 1$, $\mu_{3u} = \mu_u^{-1/2}$, and $t_u = (t(\pi_u)^2)^{1/2} = \operatorname{diag}(\mu_u^{1/2}, 1, \mu_u^{-1/2})$.

The representation π_p is parametrized by the conjugacy class of $\mathbf{t}_p \times \operatorname{Fr}_p$ in the unramified dual group

$$^L G'_p = \widehat{G}^{[F:\mathbb{Q}]} \rtimes \langle \operatorname{Fr}_p \rangle.$$

Here $\mathbf{t}_p$ is the $[F:\mathbb{Q}]$-tuple $(\mathbf{t}_u; u|p)$ of diagonal matrices in $\widehat{G} = \operatorname{GL}(3,\mathbb{C})$, where each $\mathbf{t}_u = (t_{u1}, \dots, t_{un_u})$ is any $n_u = [F_u:\mathbb{Q}_p]$-tuple with $\prod_i t_{ui} = t_u$. The Frobenius Fr_p acts on each $\mathbf{t}_u$ by permutation to the left: $\operatorname{Fr}_p(\mathbf{t}_u) = (t_{u2}, \dots, t_{un_u}, \theta(t_{u1}))$. Here $\theta = \mathrm{id}$ if $E_u = F_u \oplus F_u$ and $\theta(t_u) = J^{-1\,t}t_u^{-1} J$ if E_u is a field. Each π_u is parametrized by the conjugacy class of $\mathbf{t}_u \times \operatorname{Fr}_p$ in the unramified dual group $^L G'_u = \widehat{G}^{[F_u:\mathbb{Q}_p]} \times \langle \operatorname{Fr}_p \rangle$, or alternatively by the conjugacy class of $t_u \times \operatorname{Fr}_u$ in $^L G_u = \widehat{G} \times \langle \operatorname{Fr}_u \rangle$, where $\operatorname{Fr}_u = \operatorname{Fr}_p^{n_u}$.

Let us compute the trace

$$\operatorname{tr} r_\mu^0[(\mathbf{t}_p \times \operatorname{Fr}_p)^{n_\wp}] = \prod_{u|p} \operatorname{tr} r_u^0[(\mathbf{t}_u \times \operatorname{Fr}_p)^{n_\wp}]$$

where $\wp$ is a place of $\mathbb{E}$ over p and $n_\wp = [\mathbb{E}_\wp : \mathbb{Q}_p]$. By definition of $\mathbb{E}$, $\operatorname{Fr}_\wp = \operatorname{Fr}_p^{n_\wp}$ acts on $r_u^0 = \otimes_{\{\tau \in \Sigma; \tau|F \in u\}} r_\tau^0$. We proceed to describe the action of Fr_p on $\operatorname{Emb}(E, \mathbb{C})$ and $\operatorname{Emb}(F, \mathbb{R})$.

Fixing a $\sigma_0 : F \hookrightarrow \overline{\mathbb{Q}} \cap \mathbb{R}$ $(\subset \mathbb{R})$ and an extension $\tau_0 : E \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$, we identify

$$\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\operatorname{Gal}(\overline{\mathbb{Q}}/E) \quad \text{with} \quad \operatorname{Emb}(E, \overline{\mathbb{Q}}) = \{\tau_1, \dots, \tau_n, c\tau_1, \dots, c\tau_n\}$$

by $\varphi \mapsto \varphi \circ \tau_0$, and

$$\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\operatorname{Gal}(\overline{\mathbb{Q}}/F) \quad \text{with} \quad \operatorname{Emb}(F, \overline{\mathbb{Q}} \cap \mathbb{R}) = \{\sigma_1, \dots, \sigma_n\}.$$

The decomposition group of $\mathbb{Q}$ at p, $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, acts by left multiplication. Suppose p is unramified in E. Then Fr_p acts, and the Fr_p-orbits in $\operatorname{Emb}(F, \mathbb{R})$ are in bijection with the places $u_1, \dots, u_r$ of F over p. If $E_u = E \otimes_F F_u$ is a field, $\{\tau; \tau|F \in u\}$ makes a single Fr_p-orbit, u_E.

If $E_u = F_u \oplus F_u$, it is the disjoint union of two orbits, which we denote by u'_E and $u''_E = cu'_E$. Thus $u'_E = \{\tau_{u1}, \dots, \tau_{un_u}\}$, $\tau_{ui}|F^\times = \sigma_{ui}$ if $u = \{\sigma_{u1}, \dots, \sigma_{un_u}\}$.

The Frobenius Fr_p acts transitively on its orbit $u = \mathrm{Emb}(F_u, \overline{\mathbb{Q}}_p)$ and on u'_E and on $u''_E = cu'_E$ if u splits in E, or on u_E if E_u is a field. The smallest positive power of Fr_p which fixes each $\sigma \in u$, and each τ in u'_E and u''_E when u splits in E, is n_u. When E_u is a field, $\mathrm{Fr}_p^{2n_u}$ fixes each τ in u_E but Fr_p^j, $j < 2n_u$, does not. If E_u is a field then $\mathrm{Fr}_p^{n_u}$ fixes each σ in u, and it interchanges τ and $c\tau$. The positive integer $n_\wp$ is the smallest such that $\mathrm{Fr}_\wp = \mathrm{Fr}_p^{n_\wp}$ stabilizes Σ. Since Fr_p^{2n} fixes each τ, $n_\wp$ divides $2n$.

Now the action of Fr_p on $\widehat{G}'_u = \widehat{G}^{n_u}$ is by

$$\mathrm{Fr}_p(\mathbf{t}_u) = (t_{u2}, \dots, t_{un_u}, \theta(t_{u1})),$$

where $\mathbf{t}_u = (t_{u1}, \dots, t_{un_u})$, and $\theta = \mathrm{id}$ if u splits in E or $\theta(g) = J^{-1}{}^t g^{-1} J$ if E_u is a field. Then $\mathrm{Fr}_p^{n_u}(\mathbf{t}_u)$ is $\theta(\mathbf{t}_u)$ (which is $(\theta(t_{u1}), \dots, \theta(t_{un_u}))$).

We conclude that when $E_u = F_u \oplus F_u$, we have

$$(\mathbf{t}_u \times \mathrm{Fr}_p)^{n_u} = \left(\prod_{1 \le i \le n_u} t_{ui}, \dots, \prod_{1 \le i \le n_u} t_{ui} \right) \times \mathrm{Fr}_p^{n_u},$$

and

$$(\mathbf{t}_u \times \mathrm{Fr}_p)^j = (\dots, t_{u,i} t_{u,i+1} \dots t_{u,i+j-1}, \dots; 1 \le i \le n_u) \times \mathrm{Fr}_p^j.$$

A basis for the 3^{n_u}-dimensional representation $r_u^0 = \otimes_\tau r_\tau^0$, $\tau \in \Sigma$ and $\tau|F \in u$, is given by $\otimes_{\sigma \in u} e^\sigma_{\ell(\sigma)}$, where $e_{\ell(\sigma)}$ lies in the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$ for each σ. To compute the action of Fr_p^j on these vectors it is convenient to enumerate the σ so that the vectors become

$$\otimes_{1 \le i \le n_u} e^i_{\ell(i)} = e^1_{\ell(1)} \otimes e^2_{\ell(2)} \otimes \dots \otimes e^{n_u}_{\ell(n_u)},$$

and Fr_p acts by sending this vector to

$$\otimes_i e^{i-1}_{\ell(i)} = \otimes_i e^i_{\ell(i+1)} = e^1_{\ell(2)} \otimes e^2_{\ell(3)} \otimes \dots \otimes e^{n_u}_{\ell(1)}.$$

Then $\mathrm{Fr}_p^{n_u}$ fixes each vector, and a vector is fixed by Fr_p^j iff it is fixed by $\mathrm{Fr}_p^{j_0}$, $0 \le j_0 < n_u$, $j \equiv j_0 (\mathrm{mod}\, n_u)$. A vector $\otimes_i e^i_{\ell(i)}$ is fixed by Fr_p^j iff it is equal to $\otimes_i e_{\ell(i)}^{i-j} \equiv \otimes_i e_{\ell(i)}^{i-j_0}$, thus $\ell(i)$ depends only on $i \bmod j$ (and $i \bmod n_u$), namely only on $i \bmod j_u$, where $j_u = (j, n_u)$. Then

$$(\mathbf{t}_u \times \mathrm{Fr}_p)^{j_u} = \left(\dots, \prod_{0 \le k < j_u} t_{u,i+k}, \dots\right) \times \mathrm{Fr}_p^{j_u}.$$

This is

$$(t_{u1}t_{u2}\cdots t_{u,j_u}, t_{u2}t_{u3}\cdots t_{u,j_u+1}, \dots, t_{u,j_u}t_{u,j_u+1}\cdots t_{u,2j_u-1};$$
$$t_{u,j_u+1}\cdots t_{u,2j_u}, \dots) \times \mathrm{Fr}_p^{j_u}.$$

It acts on vectors of the form

$$(e^1_{u,\ell(1)} \otimes e^2_{u,\ell(2)} \otimes \cdots \otimes e^{j_u}_{u,\ell(j_u)}) \otimes (e^1_{u,\ell(1)} \otimes e^2_{u,\ell(2)} \otimes \cdots \otimes e^{j_u}_{u,\ell(j_u)}) \otimes \cdots.$$

The product of the first j_u vectors is repeated n_u/j_u times.

On the vectors with superscript 1 the class $(\mathbf{t}_u \times \mathrm{Fr}_p)^{j_u}$ acts as

$$t_{u,1}t_{u,2}\cdots t_{u,j_u} \cdot t_{u,j_u+1}\cdots t_{u,2j_u}\cdots t_{u,(\frac{n_u}{j_u}-1)j_u+1}\cdots t_{u,\frac{n_u}{j_u}j_u}$$
$$= \prod_{1 \le i \le n_u} t_{u,i} = t_u = \mathrm{diag}(\mu_{1u}, \mu_{2u}, \mu_{3u}),$$

and so $(\mathbf{t}_u \times \mathrm{Fr}_p)^j$ acts as t_u^{j/j_u}. The trace is then $\mu_{1u}^{j/j_u} + \mu_{2u}^{j/j_u} + \mu_{3u}^{j/j_u}$. The same holds for each superscript, so we get the product of j_u such factors. Put $j_u = (jn_\wp, n_u)$. We then have

$$\mathrm{tr}\, r_u^0[(\mathbf{t}_u \times \mathrm{Fr}_p)^{jn_\wp}] = \left(\mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + \mu_{3u}^{\frac{jn_\wp}{j_u}}\right)^{j_u}.$$

When E_u is a field we describe the orbit u_E as $\tau_i = \mathrm{Fr}_p^{i-1}\tau_1$, $1 \le i \le 2n_u$. The representation r_u of $\widehat{G}^{n_u}$ is $\otimes_{\tau \in u_E \cap \Sigma} r_\tau$. Here r_{τ_i} $(1 \le i \le n_u)$ is the standard representation of $\widehat{G} = \mathrm{GL}(3, \mathbb{C})$ on $\mathbb{C}^3$, and $r_{\tau_i}(g) = r_{\tau_{i-n_u}}(\theta(g))$ if $n_u < i \le 2n_u$. The representation r_u extends to $\widehat{G}^{n_u} \rtimes \langle \mathrm{Fr}_p^{m_u} \rangle$ provided $\mathrm{Fr}_p^{m_u}$ stabilizes $u_E^\Sigma = u_E \cap \Sigma$. Since $\mathrm{Fr}_p^{2n_u}$ fixes each element of u_E, we may assume $1 \le m_u | 2n_u$. But $\mathrm{Fr}_p^{n_u}$ maps each $\tau \in \Sigma \cap u_E$ to $c\tau \notin \Sigma \cap u_E$. Hence any multiple of m_u divisible by n_u must also be divisible by $2n_u$. This implies that $\mathrm{ord}_2\, m_u \ge \mathrm{ord}_2\, 2n_u$. Indeed, if $\ell_u n_u = m_u k_u$, and $2|\ell_u$, we may assume $2n_u = m_u k_u$ since m_u divides $2n_u$, and if k_u is even m_u divides n_u. Thus $2n_u = m_u k_u$ for an odd positive k_u.

In each $\mathrm{Fr}_p^{m_u}$-orbit ($m_u \bmod n_u$) there are $k_u = \frac{2n_u}{m_u}$ elements. Indeed, $1 + m_u a \equiv 1 + m_u b (\bmod n_u)$ with $0 \le a,\, b < k_u$, iff n_u divides $(a-b)m_u$, thus $k_u | 2(a-b)$ and so $k_u | (a-b)$ (as k_u is odd). So the distinct elements in such an orbit are $1 + m_u a$, $0 \le a < k_u$. It follows that the number of $\mathrm{Fr}_p^{m_u}$-orbits in $\{1, \dots, n_u\}$ is $m_u/2$.

To compute the trace we consider the $\mathrm{Fr}_p^{m_u}$-fixed vectors in

$$r_u^0 = \otimes_{\tau \in u_E \cap \Sigma} r_\tau^0.$$

As is the case when u splits E/F, each $\mathrm{Fr}_p^{m_u}$-orbit contributes a factor $\mathrm{tr}[t_u \theta(t_u)]$ to the trace. Then $\mathrm{tr}\, r_u^0[(\mathbf{t}_u \times \mathrm{Fr}_p)^{jn_\wp}]$ exists if $m_u = (jn_\wp, 2n_u)$ is divisible by the same power of 2 as $2n_u$, thus $\mathrm{ord}_2\, jn_\wp > \mathrm{ord}_2\, n_u$. Put $j_u = (jn_\wp, n_u)$. Then the trace is equal to

$$\mathrm{tr}\, r_u^0[(\mathbf{t}_u \times \mathrm{Fr}_p)^{jn_\wp}] = (\mathrm{tr}([t_u \theta(t_u)]^{jn_\wp/2j_u}))^{j_u} = (\mu_u^{jn_\wp/2j_u} + 1 + \mu_u^{-jn_\wp/2j_u})^{j_u}.$$

Put $\mu_{1u} = \mu_u^{1/2}$, $\mu_{2u} = 1$, $\mu_{3u} = \mu_u^{-1/2}$, to conform with the notations in the split case.

III.3 Local terms at p

The spherical function $f_{H_\wp}^{s,j}$ is defined by means of L-group homomorphisms ${}^L H' \to {}^L G' \to {}^L G'_{j'}$, where $G'_{j'} = \mathrm{R}_{\mathbb{Q}_{j'}/\mathbb{Q}_p} G'$ and $\mathbb{Q}_{j'}$ denotes the unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ of degree $j' = jn_\wp$. Since the groups H' and G' are products of groups $H'_u = \mathrm{R}_{F_u/\mathbb{Q}_p} H$ and $G'_u = \mathrm{R}_{F_u/\mathbb{Q}_p} G$, it suffices to work with these latter groups. Thus $G'_{j'} = \prod_{u|p} G'_{uj'}$, where $G'_{uj'} = \mathrm{R}_{\mathbb{Q}_{j'}/\mathbb{Q}_p} G'_u$. The function $f_\wp^{s,j}$ will be $\otimes f_u^{s,j}$, for analogously defined $f_u^{s,j}$.

Now

$${}^L G'_{j'} = (\widehat{G}')^{j'} \rtimes \langle \mathrm{Fr}_p \rangle = \prod_{u|p} (\widehat{G}'_u)^{j'} \rtimes \langle \mathrm{Fr}_p \rangle, \qquad \widehat{G}' = \widehat{G}, \qquad \widehat{G}'_u = \widehat{G}^{n_u},$$

and Fr_p acts on

$$\mathbf{x} = (\mathbf{x}_u), \qquad \mathbf{x}_u = (\mathbf{x}_{u1}, \dots, \mathbf{x}_{uj'}), \qquad \mathbf{x}_{ui} \in \widehat{G}'_u = \widehat{G}^{n_u},$$

by

$$\mathrm{Fr}_p(\mathbf{x}) = (\mathrm{Fr}_p(\mathbf{x}_u)), \qquad \mathrm{Fr}_p(\mathbf{x}_u) = (\mathrm{Fr}_p(\mathbf{x}_{u2}), \ldots, \mathrm{Fr}_p(\mathbf{x}_{uj'}), \mathrm{Fr}_p(\mathbf{x}_{u1})).$$

It suffices to work with ${}^L G'_{uj'} = (\widehat{G}'_u)^{j'} \rtimes \langle \mathrm{Fr}_p \rangle$.

Let $\mathrm{us}_1, \ldots, \mathrm{us}_{j'}$ be Fr_p-fixed elements in $Z(\widehat{H}'_u) = Z(\widehat{H})^{n_u}$, thus $\mathrm{us}_i = (s_i, \ldots, s_i)$ with $s_i \in Z(\widehat{H}) = \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times$ and $\mathrm{us}_1 \cdots \mathrm{us}_{j'} = \mathrm{us} = (s, \ldots, s)$, $s = \mathrm{diag}(-1, 1, -1)$. Define

$$\tilde{\eta}_{j'} : {}^L H'_u = \widehat{H}^{n_u} \times \langle \mathrm{Fr}_p \rangle \to {}^L G'_{uj'} = (\widehat{G}'_u)^{j'} \rtimes \langle \mathrm{Fr}_p \rangle$$

by

$$\mathbf{t} \mapsto (\mathbf{t}, \ldots, \mathbf{t}), \qquad \mathrm{Fr}_p \mapsto (\mathrm{us}_1, \mathrm{us}_2, \ldots, \mathrm{us}_{j'}) \times \mathrm{Fr}_p,$$

thus

$$\mathrm{Fr}_p^i \mapsto (\mathrm{us}_1\, \mathrm{us}_2 \cdots \mathrm{us}_i, \mathrm{us}_2 \cdots \mathrm{us}_{i+1}, \ldots, \mathrm{us}_{j'}\, \mathrm{us}_1 \ldots \mathrm{us}_{i-1}) \times \mathrm{Fr}_p^i .$$

The diagonal map $G'_u \to G'_{uj'}$ defines

$${}^L G'_{uj'} \to {}^L G'_u, \quad (\mathbf{t}_1, \ldots, \mathbf{t}_{j'}) \times \mathrm{Fr}_p^i \mapsto \mathbf{t}_1 \cdots \mathbf{t}_{j'} \times \mathrm{Fr}_p^i .$$

The composition $\eta_{j'} : {}^L H'_u \to {}^L G'_u$ gives

$$\mathbf{t} \times \mathrm{Fr}_p^i \mapsto \mathbf{t}^{j'}\, \mathrm{us}^i \times \mathrm{Fr}_p^i .$$

The homomorphism $\tilde{\eta}_{j'}$ defines a dual homomorphism

$$\mathbb{H}(K_{uj'} \backslash G_{uj'} / K_{uj'}) \to \mathbb{H}(K_{Hu} \backslash H_u / K_{Hu})$$

of Hecke algebras. The function $f^{s,j}_{Hu}$ is defined to be the image by the relation

$$\mathrm{tr}\, \pi_u(\tilde{\eta}_{j'}(t))(\phi_{uj'}) = \mathrm{tr}\, \pi_{Hu}(t)(f^{s,j}_{Hu})$$

of the function $\phi_{j'}$ of [Ko5], p. 173, or rather the u-component $\phi_{uj'}$ of $\phi_{j'}$, which is the characteristic function of $K_{uj'} \cdot \mu_{F_{j'}}(p^{-1}) \cdot K_{uj'}$. Put $j_u = (jn_\wp, n_u)$. Theorem 2.1.3 of [Ko3] (see also [Ko5], p. 193) asserts that the product over $u|p$ in F of these traces is the product of $q_\wp^{\frac{j}{2} \dim \mathcal{S}_{K_f}}$, where $q_\wp = p^{[\mathbb{E}_\wp : \mathbb{Q}_p]}$, with the product over $u|p$ of

$$\mathrm{tr}\, r_u^0(\mathrm{us}[\mathbf{t}(\pi_u) \times \mathrm{Fr}_p]^{jn_\wp}) = \left(\mathrm{tr} \left[s^{\frac{n_u}{j_u}} t_u^{\frac{jn_\wp}{j_u}} \right] \right)^{j_u}$$

$$= \left[(-1)^{\frac{n_u}{j_u}} \mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}} \mu_{3u}^{\frac{jn_\wp}{j_u}}\right]^{j_u}.$$

Similarly for $s = I$ we have that the analogous factor (with H replaced by G) is the product with factors

$$\operatorname{tr} r_u^0[(\mathbf{t}(\pi_u) \times \mathrm{Fr}_p)^{jn_\wp}] = \left[\operatorname{tr}\left(t_u^{\frac{jn_\wp}{j_u}}\right)\right]^{j_u} = \left[\mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + \mu_{3u}^{\frac{jn_\wp}{j_u}}\right]^{j_u}.$$

III.4 The eigenvalues at p

We proceed to describe the eigenvalues μ_{iu} ($i = 1$ if E_u is a field, $1 \leq i \leq 3$ if $E_u = F_u \oplus F_u$) for the various terms in the formula, beginning with $\mathrm{STF}_G(f)$, according to the parts which make it. If E_u is a field, $\mathrm{bc}(\pi(\mu_{1u})) = \pi_{G'}(\mu_{1u}, 1, \overline{\mu}_{1u}^{-1})$ where $G' = \mathrm{GL}(3, E_u)$. If $E_u = F_u \oplus F_u$ then $\mathrm{bc}(\pi) = \pi \times \check{\pi}$, and $\pi = \pi(\mu_{1u}, \mu_{2u}, \mu_{3u})$ if π is unramified. We choose the complex numbers μ_{1u} to have $|\mu_{1u}| \geq 1$. Write t_u for $\mathrm{diag}(\mu_{1u}, 1, 1) \times \mathrm{Fr}_u$ or for $\mathrm{diag}(\mu_{1u}, \mu_{2u}, \mu_{3u})$.

The first part of $\mathrm{STF}_G(f)$ describes the stable spectrum. It has two types of terms.

(1) For the packets $\{\pi\}$ which basechange-lift to cuspidal $\Pi \simeq \check{\Pi}$ on $G'(\mathbb{A}_F) = \mathrm{GL}(3, \mathbb{A}_E)$, if $E_u = F_u \oplus F_u$ then the μ_{iu} satisfy $q_u^{-1/2} < |\mu_{iu}| < q_u^{1/2}$, where q_u is the cardinality of the residual field of F_u, since Π is unitary and so its component Π_u is unitarizable. Note that the unramified component Π_u is generic (since Π is), hence fully induced. If E_u is a field then $q_{E_u}^{-1/2} < |\mu_{1u}| < q_{E_u}^{1/2}$, where q_{E_u} is the cardinality of the residual field of E_u.

(2) For a one-dimensional representation π, $\mathrm{bc}(\pi) = \Pi$ is a one-dimensional representation $g \mapsto \chi(\det g)$, where χ is a character of $\mathbb{A}_E^1/E^1$. If u splits in E,

$$t(\pi_u) = \mathrm{diag}(\mu_{1u}, \mu_{2u}, \mu_{3u}) \qquad \text{is} \qquad \mathrm{diag}(\chi_u q_u, \chi_u, \chi_u q_u^{-1}),$$

where $\chi_u = \chi(\boldsymbol{\pi}_u)$ has absolute value 1. If E_u is a field,

$$t(\pi_u) = \mathrm{diag}(\mu_{1u}, 1, 1) \times \mathrm{Fr}_u$$

with $\mu_{1u} = q_{E_u}$.

The second part of $\mathrm{STF}_G(f)$ is a sum of terms indexed by $\{\pi\} = e(\rho\times\mu)$. Then $\mathrm{bc}(\{\pi\}) = I(\rho' \otimes \kappa \times \mu')$ where ρ' is the stable basechange lift of ρ. Here ρ is a cuspidal representation of $\mathrm{U}(2, E/F)(\mathbb{A})$, and $\operatorname{tr} \pi_u^-(f_u) = 0$ as f_u is spherical. The component of ρ at u is unramified and fully induced. If u splits, ρ_u is $I_2(\mu_{1u}, \mu_{2u})$. If E_u is a field, ρ_u is $I(\mu_{1u})$. The component $\pi_u = e(\rho_u \times \mu_u)$ lifts to $I(\mu'_{1u}\kappa_u, \mu'_{2u}\kappa_u, \mu'_u)$, where $\mu'_{iu}(z) = \mu_{iu}(z/\overline{z})$, if E_u is a field, and to $I(\mu_{1u}\kappa_u, \mu_{2u}\kappa_u, \mu_u)$ if u splits in E. Then the components μ_{iu} of t_u satisfy $q_u^{-1/2} < |\mu_{iu}| < q_u^{1/2}$ (replace q_u by q_{E_u} if E_u is a field, and μ_{iu} by μ_{1u}).

The terms in the third part correspond to unordered triples (μ, μ', μ'') of characters of $\mathbb{A}_E^1/E^1$, and the entries of t_u are units in $\mathbb{C}^\times$.

The terms in the fourth part of $\mathrm{STF}_G(f)$ are indexed by the quasi-packets $\{\pi\} = e(\mu \times \mu_1)$, that is by the one-dimensional representations $\mu \times \mu_1$ of $\mathrm{U}(2, E/F)(\mathbb{A}_F) \times \mathrm{U}(1, E/F)(\mathbb{A}_F)$. The unramified member of the quasi-packet $e(\mu_u \times \mu_{1u}) = \{\pi_u^\times, \pi_u^-\}$ is $\pi_u^\times$, and $t(\pi_u^\times)$ is

$$\mathrm{diag}(\mu_u q_u^{1/2}, \mu_{1u}, \mu_u q_u^{-1/2})$$

if u splits and $\mathrm{diag}(\mu_u q_{E_u}^{1/2}, 1, 1) \times \mathrm{Fr}_u$ if E_u is a field, and $|\mu_u| = 1$ in $\mathbb{C}^\times$.

In summary, as noted in the last section, the factor at p of each of the summands in $\mathrm{STF}_G(f)$ has the form

$$q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}} \operatorname{tr} r_\mu^0[(\mathbf{t}(\pi_p) \times \mathrm{Fr}_p)^{jn_\wp}] = q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}} \prod_{u|p} (\operatorname{tr}[t_u \times \mathrm{Fr}_p]^{jn_\wp})$$

$$= q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}} \prod_{u|p} \left(\mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + \mu_{3u}^{\frac{jn_\wp}{j_u}} \right)^{j_u}.$$

Here $j_u = (n_u, jn_\wp)$ and $n_\wp = [\mathbb{E}_\wp : \mathbb{Q}_p]$, and $|n_\wp|_2 < |n_u|_2$ for each u where E_u is a field.

Remark. As p splits in F into a product of primes u with $F_u/\mathbb{Q}_p$ unramified with $[F:\mathbb{Q}] = \sum_{u|p}[F_u : \mathbb{Q}_p]$, and the dimension of the symmetric space $G(\mathbb{R})/K_{G(\mathbb{R})}$ is 2, we note that

$$\dim \mathcal{S}_{K_f} = 2[F:\mathbb{Q}] = \sum_{u|p} 2[F_u : \mathbb{Q}_p].$$

III.5 Terms at p for the endoscopic group

The other trace formula which contributes is that of the endoscopic group $\mathrm{U}(2, E/F)(\mathbb{A}_F) \times \mathrm{U}(1, E/F)(\mathbb{A}_F)$ of $G(\mathbb{A}_F)$. The factors at p of the various summands have the form

$$q_{\wp}^{\frac{j}{2}\dim \mathcal{S}_{K_f}} \prod_{u|p} \mathrm{tr}(s[t_u \times \mathrm{Fr}_p]^{jn_\wp})$$

$$= q_{\wp}^{\frac{j}{2}\dim \mathcal{S}_{K_f}} \cdot \prod_{u|p} \left[(-1)^{\frac{n_u}{j_u}} \mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}} \mu_{3u}^{\frac{jn_\wp}{j_u}} \right]^{j_u},$$

where $s = \mathrm{diag}(1, -1, 1)$ is the element in $\widehat{G} = \mathrm{GL}(3, \mathbb{C})$ whose centralizer is $\widehat{H} = \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$. We need to specify the 3-tuples t_u again, according to the three parts of $\mathrm{STF}_H(f_H)$. They correspond to the last three terms of the $\mathrm{STF}_G(f)$ that we listed above.

For the first part, where the summands are indexed by (stable) packets of cuspidal representations $\rho \neq \rho(\theta, '\theta) \times ''\theta$ of

$$\mathrm{U}(2, E/F)(\mathbb{A}_F) \times \mathrm{U}(1, E/F)(\mathbb{A}_F),$$

the t_u is the same as in the second part of $\mathrm{STF}_G(f)$. If $\rho = \rho(\theta, '\theta) \times ''\theta$, they are the same as in the third part. For the one-dimensional representations of $\mathrm{STF}_H(f_H)$, the t_u are as in the 4th part of $\mathrm{STF}_G(f)$.

IV. REAL REPRESENTATIONS

IV.1 Representation of the real GL(2)

Packets of representations of a real group G are parametrized by maps of the Weil group $W_{\mathbb{R}}$ to the L-group ${}^L G$. Recall that $W_{\mathbb{R}} = \langle z, \sigma; z \in \mathbb{C}^\times, \sigma^2 \in \mathbb{R}^\times - N_{\mathbb{C}/\mathbb{R}}\mathbb{C}^\times, \sigma z = \overline{z}\sigma \rangle$ is

$$1 \to W_{\mathbb{C}} \to W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

an extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $W_{\mathbb{C}} = \mathbb{C}^\times$. It can also be viewed as the normalizer $\mathbb{C}^\times \cup \mathbb{C}^\times j$ of $\mathbb{C}^\times$ in $\mathbb{H}^\times$, where $\mathbb{H} = \mathbb{R}\langle 1, i, j, k \rangle$ is the Hamilton quaternions. The norm on $\mathbb{H}$ defines a norm on $W_{\mathbb{R}}$ by restriction ([D2], [Tt]). The discrete-series (packets of) representations of G are parametrized by the homomorphisms $\phi : W_{\mathbb{R}} \to \widehat{G} \times W_{\mathbb{R}}$ whose projection to $W_{\mathbb{R}}$ is the identity and to the connected component $\widehat{G}$ is bounded, and such that $C_\phi Z(\widehat{G})/Z(\widehat{G})$ is finite. Here C_ϕ is the centralizer $Z_{\widehat{G}}(\phi(W_{\mathbb{R}}))$ in $\widehat{G}$ of the image of ϕ.

When $G = \mathrm{GL}(2,\mathbb{R})$ we have $\widehat{G} = \mathrm{GL}(2,\mathbb{R})$, and these maps are ϕ_k ($k \geq 1$), defined by

$$W_{\mathbb{C}} = \mathbb{C}^\times \ni z \mapsto \begin{pmatrix} (z/|z|)^k & 0 \\ 0 & (|z|/z)^k \end{pmatrix} \times z, \qquad \sigma \mapsto \begin{pmatrix} 0 & 1 \\ \iota & 0 \end{pmatrix} \times \sigma.$$

Since $\sigma^2 = -1 \mapsto \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{pmatrix} \times \sigma^2$, ι must be $(-1)^k$. Then $\det \phi_k(\sigma) = (-1)^{k+1}$, and so k must be an odd integer ($= 1, 3, 5, \dots$) to get a discrete-series (packet of) representation of $\mathrm{PGL}(2,\mathbb{R})$. In fact π_1 is the lowest discrete-series representation, and ϕ_0 parametrizes the so called limit of discrete-series representations; it is tempered. Even $k \geq 2$ and $\sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \sigma$ define discrete-series representations of $\mathrm{GL}(2,\mathbb{R})$ with the quadratic nontrivial central character sgn. Packets for $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{R})$ consist of a single discrete-series irreducible representation π_k. Note that $\pi_k \otimes \mathrm{sgn} \simeq \pi_k$. Here $\mathrm{sgn} : \mathrm{GL}(2,\mathbb{R}) \to \{\pm 1\}$, $\mathrm{sgn}(g) = 1$ if $\det g > 0$, $= -1$ if $\det g < 0$.

The π_k $(k > 0)$ have the same central and infinitesimal character as the kth-dimensional nonunitarizable representation

$$\operatorname{Sym}_0^{k-1}\mathbb{C}^2 = |\det g|^{-(k-1)/2}\operatorname{Sym}^{k-1}\mathbb{C}^2$$

into

$$\mathrm{SL}(k,\mathbb{C})^{\pm} = \{g \in \mathrm{GL}(2,\mathbb{C}); \det g \in \{\pm 1\}\}.$$

Note that

$$\det \operatorname{Sym}^{k-1}(g) = \det g^{k(k-1)/2}.$$

The normalizing factor is $|\det \operatorname{Sym}^{k-1}|^{-1/k}$. Then

$$\operatorname{Sym}_0^{k-1}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \operatorname{diag}\left(\operatorname{sgn}(a)^{k-i}\operatorname{sgn}(b)^{i-1}|a|^{k-i-(k-1)/2}|b|^{i-1-(k-1)/2}\right)$$

$(1 \le i \le k)$. In fact both π_k and $\operatorname{Sym}_0^{k-1}\mathbb{C}^2$ are constituents of the normalizedly induced representation $I(\nu^{k/2}, \operatorname{sgn}^{k-1}\nu^{-k/2})$ whose infinitesimal character is $(\frac{k}{2}, -\frac{k}{2})$, where a basis for the lattice of characters of the diagonal torus in SL(2) is taken to be $(1,-1)$.

IV.2 Representations of U(2,1)

Here we record well-known results concerning the representation theories of the groups of this work in the case of the archimedean quadratic extension $\mathbb{C}/\mathbb{R}$. For proofs we refer to [Wh], §7, to [BW], Ch. VI for cohomology, and to [Cl1], [Sd] for character relations. This is used in [F3;VI] to determine all automorphic $\mathbf{G}(\mathbb{A})$-modules with nontrivial cohomology outside of the middle dimension.

We first recall some notations. Denote by σ the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Put $\overline{z} = \sigma(z)$ for z in $\mathbb{C}$, and $\mathbb{C}^1 = \{z/|z|;\ z \text{ in } \mathbb{C}^\times\}$. Put $H' = \mathrm{GL}(2,\mathbb{C}), G' = \mathrm{GL}(3,\mathbb{C})$,

$$H = \mathrm{U}(1,1) = \left\{h \text{ in } H';\ {}^t\overline{h}wh = w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$$

and

$$G = \mathrm{U}(2,1) = \left\{g \text{ in } G';\ {}^t\overline{g}Jg = J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}\right\}.$$

The center Z of G is isomorphic to $\mathbb{C}^1$; so is that of H. Fix an integer $\mathbf{w}$ and a character $\omega(z/|z|) = (z/|z|)^{\mathbf{w}}$ of $\mathbb{C}^1$. Put $\omega'(z) = \omega(z/\overline{z})$. Any representation of any subgroup of G which contains Z will be assumed below to transform under Z by ω.

The diagonal subgroup A_H of H will be identified with the subgroup of the diagonal subgroup A of G consisting of $\operatorname{diag}(z, z', \overline{z}^{-1})$ with $z' = 1$. For any character χ_H of A_H there are complex a, c with $a + c$ in $\mathbb{Z}$ such that

$$\chi_H(\operatorname{diag}(z, \overline{z}^{-1})) = (\text{“}z^a(\overline{z}^{-1})^c = \text{”})|z|^{a-c}(z/|z|)^{a+c}.$$

The character χ_H extends uniquely to a character χ of A whose restriction to Z is ω. In fact $b = \mathbf{w} - a - c$ is integral, and $\chi = \chi(a, b, c)$ is defined by

$$\chi(\operatorname{diag}(z, z', \overline{z}^{-1})) = z'^b|z|^{a-c}(z/|z|)^{a+c}.$$

A character κ of $\mathbb{C}^\times$ which is trivial on the multiplicative group $\mathbb{R}_+^\times$ of positive real numbers but is nontrivial on $\mathbb{R}^\times$ is of the form $\kappa(z) = (z/|z|)^{2k+1}$, where k is integral.

The H-module $I(\chi_H) = I(\chi_H; B_H, H) = \operatorname{Ind}(\delta_H^{1/2}\chi_H; B_H, H)$ normalizedly induced from the character χ_H of A_H extended trivially to the upper triangular subgroup B_H of H, is irreducible unless a, c are equal with $a + c$ an odd integer, or are distinct integers. If $a = c$ and $a + c \in 1 + 2\mathbb{Z}$ then χ_H is unitary and $I(\chi_H)$ is the direct sum of two tempered representations. If a, c are distinct integers the sequence $JH(I(\chi_H))$ of constituents, repeated with their multiplicities, in the composition series of $I(\chi_H)$, consists of (1) an irreducible finite-dimensional H-module $\xi_H = \xi_H(\chi_H) = \xi_H(a, c)$ of dimension $|a - c|$ (and central character $z \mapsto z^{a+c}$), and (2) the two irreducible square-integrable constituents of the packet $\rho = \rho(a, c)$ (of highest weight $|a - c| + 1$) on which the center of the universal enveloping algebra of H acts by the same character as on ξ_H.

The Langlands classification (see [BW], Ch. IV) defines a bijection between the set of packets and the set of $\widehat{H}$-conjugacy classes of homomorphisms from the Weil group $W_\mathbb{R}$ to the dual group ${}^LH = \widehat{H} \rtimes W_\mathbb{R}$ ($W_\mathbb{R}$ acts on the connected component $\widehat{H} = \mathrm{GL}(2, \mathbb{C})$ by $\sigma(h) = w\,{}^th^{-1}w^{-1}$ $(= \frac{1}{\det h}h)$), whose composition with the second projection is the identity. Such homomorphism is called *discrete* if its image is not conjugate by $\widehat{H}$ to a subgroup of $\widehat{B}_H = B_H \rtimes W_\mathbb{R}$. The packet $\rho(a, c) = \rho(c, a)$ corresponds to the homomorphism $y(\chi_H) = y(a, c)$ defined by

$$z \mapsto \begin{pmatrix} (z/|z|)^a & 0 \\ 0 & (z/|z|)^c \end{pmatrix} \times z, \quad \sigma \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \sigma.$$

It is discrete if and only if $a \neq c$. Note that $\sigma^2 \mapsto -I \times \sigma^2$, thus here a, c are odd.

The composition $y(a, b, c)$ of $y(\chi_H \otimes \kappa^{-1}) = y(a - 2k - 1, c - 2k - 1)$ with the endo-lift map $e : {}^LH \to {}^LG$ is the homomorphism $W_{\mathbb{R}} \to {}^LG$ defined by

$$z \mapsto \begin{pmatrix} (z/|z|)^a & & 0 \\ & (z/|z|)^b & \\ 0 & & (z/|z|)^c \end{pmatrix} \times z, \quad \sigma \mapsto J \times \sigma.$$

Since $\sigma^2 \mapsto I \times \sigma^2$, the a, b, c are even. Here $b = \mathbf{w} - a - c$ is determined by a, c, and the central character, thus $\mathbf{w}$. The corresponding G-packet $\pi = \pi(a, b, c)$ depends only on the set $\{a, b, c\}$. It consists of square integrables if and only if a, b, c are distinct.

The irreducible representations of SU(2, 1) (up to equivalence) are described in [Wh], §7. We proceed to summarize these results, but in the standard notations of normalized induction, which are used for example in [Kn], and in our p-adic theory. Thus [Wh], (1) on p. 181, defines the induced representation π_Λ on space of functions transforming by $f(gma) = e^{\Lambda(a)} f(g)$, while [Kn] defines the induced representation I_Λ on space of functions transforming by $f(gma) = e^{(-\Lambda-\rho)(a)} f(g)$. Thus

$$\pi_\Lambda = I_{-\Lambda-\rho}, \qquad \pi_{-\Lambda-\rho} = I_\Lambda,$$

and ρ is half the sum of the positive roots. Note that the convention in representation theory of real groups is that G acts on the left: $(I_\Lambda(h)f)(g) = f(h^{-1}g)$, while in representation theory of p-adic groups the action is by right shifts: $(I(\Lambda)(h)f)(g) = f(gh)$, and f transforms on the left: "$f(mag) = e^{(\Lambda+\rho)(ma)} f(g)$". We write $I(\Lambda)$ for right shift action, which is equivalent to the left shift action I_Λ of e.g. [Kn].

To translate the results of [Wh], §7, to the notations of [Kn], and ours, we simply need to replace Λ of [Wh] by $-\Lambda - \rho$. Explicitly, we choose the basis $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$ of simple roots in the root system Δ of $\mathfrak{g}_{\mathbb{C}} = \mathrm{sl}(3, \mathbb{C})$ relative to the diagonal $\mathfrak{h}$ (note that in the definition of Δ^+ in [Wh], p. 181, h should be H). The basic weights for this order are $\Lambda_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$, $\Lambda_2 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$. [Wh] considers π_Λ only for "*G-integral*" $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$ (thus $k_i \in \mathbb{C}$, $k_1 - k_2 \in \mathbb{Z}$), and $\rho = (1,0,-1) = \alpha_1 + \alpha_2 = \Lambda_1 + \Lambda_2$. Then [Wh], 7.1, asserts that I_Λ is reducible iff $\Lambda \neq 0$ and Λ is *integral* ($k_i \in \mathbb{Z}$), and [Wh], 7.2, asserts that I_Λ is unitarizable iff $\langle \Lambda, \rho \rangle \in i\mathbb{R}$. The normalized notations I_Λ are convenient as the infinitesimal character of $I_{s\Lambda}$ for any element s in the Weyl group $W_\mathbb{C} = S_3$ is the $W_\mathbb{C}$-orbit of Λ. In the unnormalized notations of [Wh], p. 183, l. 13, one has $\chi_\Lambda = \chi_{s(\Lambda+\rho)-\rho}$ instead. The Weyl group $W_\mathbb{C}$ is generated by the reflections $s_i\Lambda = \Lambda - \langle \Lambda, \alpha_i^\vee \rangle \alpha_i$, where $\alpha_i^\vee = 2\alpha_i / \langle \alpha_i, \alpha_i \rangle$ is α_i. Put $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ for the longest element.

For integral $k_i = \langle \Lambda, \alpha_i \rangle < 0$ ($i = 1,2$), [Wh], p. 183, l. -3, shows that I_Λ contains a finite-dimensional representation ξ_Λ. Thus ξ_Λ is a quotient of $I_{w_0\Lambda}$, and has infinitesimal character $w_0\Lambda$ and highest weight $w_0\Lambda - \rho$. Note that $\mathcal{F}$ in midpage 183 and $\mathcal{F}^+$ in 7.6 of [Wh] refer to integral and not G-integral elements. For such Λ the set of discrete-series representations sharing infinitesimal character $(W_\mathbb{C} \cdot \Lambda)$ with ξ_Λ consists of $D^+_{s_1 s_2 \Lambda}$, $D^-_{s_2 s_1 \Lambda}$, $D_{w_0\Lambda}$ ([Wh], 7.6, where "G" should be "$\widehat{G}$"). The holomorphic discrete-series $D^+_{s_2 w_0 \Lambda}$ is defined in [Wh], p. 183, as a subrepresentation of $I_{s_2 w_0 \Lambda}$, and it is a constituent also of $I_{w_0 s_2 w_0 \Lambda} = I_{s_1\Lambda}$ ([Wh], 7.10) but of no other $I_{\Lambda'}$. The antiholomorphic discrete-series $D^-_{s_1 w_0 \Lambda}$ is defined as a sub of $I_{s_1 w_0 \Lambda}$ and it is a constituent of $I_{s_2\Lambda} = I_{w_0 s_1 w_0 \Lambda}$, but of no other $I_{\Lambda'}$. The nonholomorphic discrete-series $D_{w_0\Lambda}$ is defined as a sub of $I_{w_0\Lambda}$ and it is a constituent of $I_{s\Lambda}$ for all $s \in W_\mathbb{C}$, but of no other $I_{\Lambda'}$. It is generic. $\dim \xi_\Lambda = 1$ iff $k_1 = k_2 = 1$.

Let us repeat this with Λ positive: $k_i = \langle \Lambda, \alpha_i \rangle > 0$ ($i = 1,2$) (we replace Λ by $w_0\Lambda$).

ξ_Λ is a quotient of I_Λ;
$D^+_{s_2\Lambda}$ lies (only) in $I_{s_2\Lambda}$, $I_{w_0 s_2 \Lambda}$;
$D^-_{s_1\Lambda}$ lies (only) in $I_{s_1\Lambda}$, $I_{w_0 s_1 \Lambda}$;
D_Λ lies in $I_{s\Lambda}$ for all $s \in W_\mathbb{C}$. It is generic.

The induced I_Λ is reducible and unitarizable iff $\Lambda \neq 0$ and $\langle \Lambda, \rho \rangle = 0$, thus $k_1 + k_2 = 0$, $k_i \neq 0$ in $\mathbb{Z}$, and $\Lambda = k_1(\Lambda_1 - \Lambda_2) = k_1 s_2 \Lambda_2 = -k_1 s_1 \Lambda_1$. The composition series has length two ([Wh], (i) and (ii) on p. 184, and 7.11). We denote them by $\pi^\pm_\Lambda$ (corresponding to $\pi^\pm_{-\Lambda-\rho}$ in [Wh]). These $\pi^\pm_\Lambda$ do not lie in any other $I_{\Lambda'}$ than indicated next.

If $k_1 < 0$ then $\Lambda = -k_1 s_1 \Lambda_1$, π^-_Λ lies in I_Λ and π^+_Λ in $I_{s\Lambda}$ for all $s \in W_\mathbb{C}$.

Thus $\pi^-_{s_1\Lambda}$ lies in $I_{s_1\Lambda}$ and $\pi^+_{s_1\Lambda}$ in $I_{s\Lambda}$ for all $s \in W_{\mathbb{C}}$, where $\Lambda \geq 0$ has $k_2 = 0$, $k_1 > 0$.

If $k_1 > 0$ then $\Lambda = k_1 s_2 \Lambda_2$, π^+_Λ lies in I_Λ and π^-_Λ in $I_{s\Lambda}$ for all $s \in W_{\mathbb{C}}$.

Thus $\pi^+_{s_2\Lambda}$ lies in $I_{s_2\Lambda}$ and $\pi^-_{s_2\Lambda}$ in $I_{s\Lambda}$ for all $s \in W_{\mathbb{C}}$, where $\Lambda \geq 0$ has $k_1 = 0$, $k_2 > 0$.

There are also nontempered unitarizable non one-dimensional representations $J^\pm_k$ $(k \geq -1)$. J^+_k is defined in [Wh], p. 184, as a sub of $I_{-k\Lambda_1-\rho}$, thus a constituent of $I_{-w_0(k\Lambda_1+\rho)} = I_{\Lambda_1+(k+1)\Lambda_2}$, and it is a constituent also of $I_{-s_1(k\Lambda_1+\rho)}$ and $I_{-s_1s_2(k\Lambda_1+\rho)}$ but of no other $I_{\Lambda'}$, unless $k = -1$ where J^+_{-1} is a constituent of $I_{s\Lambda_1}$ for all $s \in W_{\mathbb{C}}$.

Similarly J^-_k is a sub of $I_{-k\Lambda_2-\rho}$ and a constituent of $I_{-w_0(k\Lambda_2+\rho)} = I_{(k+1)\Lambda_1+\Lambda_2}$, and a constituent of $I_{-s_2(k\Lambda_2+\rho)}$, $I_{-s_2s_1(k\Lambda_2+\rho)}$ but of no other $I_{\Lambda'}$, unless $k = -1$ where J^-_{-1} is a constituent of $I_{s\Lambda_2}$ for all $s \in W_{\mathbb{C}}$ (see [Wh], 7.12, where in (1) Λ_2 should be Λ_1).

Let us express this with $\Lambda > 0$.

If $k_1 = 1$, $k_2 = k+1 \geq 0$, $J^+_k = J^+_{s_2\Lambda}$ is a constituent of I_Λ, $I_{w_0\Lambda}$, $I_{s_2\Lambda}$, $I_{s_2s_1\Lambda}$.

If $k_2 = 1$, $k_1 = k+1 \geq 0$, $J^-_k = J^-_{s_1\Lambda}$ is a constituent of I_Λ, $I_{w_0\Lambda}$, $I_{s_1\Lambda}$, $I_{s_1s_2\Lambda}$.

To compare the parameters k_1, k_2 of I_Λ with the (a,b,c) of our induced $I(\chi)$, which is $\mathrm{Ind}(\delta_G^{1/2}\chi; B, G)$, note that $\Lambda(\mathrm{diag}(x, y/x, 1/y)) = x^{k_1}y^{k_2}$ and $\chi(\mathrm{diag}(x, y/x, 1/y)) = x^{a-b}y^{b-c}$. Thus $k_1 = a-b$, $k_2 = b-c$. We then write $I(a,b,c)$ for I_Λ with $k_1 = a-b$, $k_2 = b-c$, extended to U(2,1) with central character $\mathbf{w} = a+b+c$. If ${}^t\overline{g}Jg = J$ and $z = \det g$, then $z\overline{z} = 1$, thus $z = e^{i\theta}$, $-\pi < \theta \leq \pi$, then $x = e^{i\theta/3}$ has that $h = x^{-1}g$ satisfies ${}^t\overline{h}Jh = J$ and $x\overline{x} = 1$, and $\det h = 1$. Note that $I_{s_1\Lambda}$ gives $I(b,a,c)$ and $I_{s_2\Lambda}$ gives $I(a,c,b)$.

Here is a list of all irreducible unitarizable representations with infinitesimal character $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$, integral $k_i \geq 0$, $\Lambda \neq 0$.

$k_1 = k_2 = 1$: ξ_Λ, J^+_0, J^-_0, $D^+_{s_2\Lambda}$, $D^-_{s_1\Lambda}$, D_Λ.

$k_1 > 1$, $k_2 > 1$: ξ_Λ, $D^+_{s_2\Lambda}$, $D^-_{s_1\Lambda}$, D_Λ.

$k_1 > 1$, $k_2 = 1$: ξ_Λ, $J^-_{k_1-1}$, $D^+_{s_2\Lambda}$, $D^-_{s_1\Lambda}$, D_Λ.

$k_1 = 1$, $k_2 > 1$: ξ_Λ, $J^+_{k_2-1}$, $D^+_{s_2\Lambda}$, $D^-_{s_1\Lambda}$, D_Λ.

$k_1 = 0$, $k_2 > 1$: $\pi^+_{k_2s_2\Lambda_2}$, $\pi^-_{k_2s_2\Lambda_2}$.

$k_1 > 1$, $k_2 = 0$: $\pi^+_{k_1s_1\Lambda_1}$, $\pi^-_{k_1s_1\Lambda_1}$.

$k_1 = 0$, $k_2 = 1$: J^-_{-1}, $\pi^+_{s_2\Lambda_2}$, $\pi^-_{s_2\Lambda_2}$.

$k_1 = 1$, $k_2 = 0$: J^+_{-1}, $\pi^+_{s_1\Lambda_1}$, $\pi^-_{s_1\Lambda_1}$.

Here is a list of composition series. $\Lambda \geq 0 \neq \Lambda$.

I_Λ has ξ_Λ, $J^+_{s_2\Lambda}$ (unitarizable iff $k_1 = 1$, $k_2 \geq 0$), $J^-_{s_1\Lambda}$ (unitarizable iff $k_2 = 1$, $k_1 \geq 0$), D_Λ.

$I_{s_1\Lambda}$ has $J^-_{s_1\Lambda}$ (unitarizable iff $k_2 = 1$, $k_1 \geq 0$), $D^-_{s_1\Lambda}$, D_Λ.

$I_{s_2\Lambda}$ has $J^+_{s_2\Lambda}$ (unitarizable iff $k_1 = 1$, $k_2 \geq 0$), $D^+_{s_2\Lambda}$, D_Λ.

$k_1 = 0$, $k_2 = 1$: $I_{s_1\Lambda_2}$ has $J^-_{s_1\Lambda_2}$, $\pi^-_{s_2\Lambda_2}$.

$k_1 = 1$, $k_2 = 0$: $I_{s_2\Lambda_1}$ has $J^+_{s_2\Lambda_1}$, $\pi^+_{s_1\Lambda_1}$.

To fix notations in a manner consistent with the nonarchimedean case, note that if μ is a one-dimensional H-module then there are unique integers $a \geq b \geq c$ with $a + b + c = \mathbf{w}$ and either (i) $a = b + 1, \mu = \xi_H(a, b)$, or (ii) $b = c + 1, \mu = \xi_H(b, c)$. If the central character on the U(1,1)-part is $z \mapsto z^{2k+1}$, case (i) occurs when $\mathbf{w} - 3k \leq 1$, while case (ii) occurs if $\mathbf{w} - 3k \geq 2$.

If, in addition, $a > b > c$, put $\pi^\times_\mu = J^+_{s_2\Lambda}$, $\pi^-_\mu = D^-_{s_1\Lambda}$, and $\pi^+_\mu = D_\Lambda \oplus D^+_{s_2\Lambda}$ in case (i), $\pi^\times_\mu = J^-_{s_1\Lambda}$, $\pi^-_\mu = D^+_{s_2\Lambda}$ and $\pi^+_\mu = D_\Lambda \oplus D^-_{s_1\Lambda}$ in case (ii). D_Λ, hence π^+_μ, is generic in both cases. $\{\pi^\times_\mu, \pi^+_\mu\}$ make the composition series of an induced representation.

The motivation for this choice of notations is the following character identities. Put

$$\rho = \rho(a, c) \otimes \kappa^{-1}, \qquad \rho^- = \rho(b, c) \otimes \kappa^{-1}, \qquad \rho^+ = \rho(a, b) \otimes \kappa^{-1}.$$

Then $\{\rho, \rho^+, \rho^-\}$ is the set of H-packets which lift to the G-packet $\pi = \pi(a, b, c)$ via the endo-lifting e. As noted above, ρ, ρ^+ and ρ^- are distinct if and only if $a > b > c$, equivalently π consists of three square-integrable G-modules. Moreover, every square-integrable H-packet is of the form ρ, ρ^+ or ρ^- for unique $a \geq b \geq c$, $a > c$.

If $a = b = c$ then $\rho = \rho^+ = \rho^-$ is the H-packet which consists of the constituents of $I(\chi_H(a, c) \otimes \kappa^{-1})$, and $\pi = I(\chi(a, b, c))$ is irreducible.

If $a > b = c$ put $\langle \rho, \pi^+ \rangle = 1$, $\langle \rho, \pi^- \rangle = -1$.

If $a = b > c$ put $\langle \rho, \pi^+ \rangle = -1$, $\langle \rho, \pi^- \rangle = 1$.

If $a > b > c$ put $\langle \tilde\rho, D_\Lambda \rangle = 1$ for $\tilde\rho = \rho$, ρ^+, ρ^-, and:

$\langle \rho,\ D^+_{s_2\Lambda} \rangle = -1$, $\langle \rho,\ D^-_{s_1\Lambda} \rangle = -1$;

$\langle \rho^+, D^+_{s_2\Lambda} \rangle = \ \ 1$, $\langle \rho^+, D^-_{s_1\Lambda} \rangle = -1$;

$\langle \rho^-, D^+_{s_2\Lambda} \rangle = -1$, $\langle \rho^-, D^-_{s_1\Lambda} \rangle = \ \ 1$.

16.1 Proposition ([Sd]). *For all matching measures fdg on G and $f_H dh$ on H, we have*

$$\operatorname{tr}\tilde{\rho}(f_H dh) = \sum_{\pi' \in \pi} \langle \tilde{\rho}, \pi' \rangle \operatorname{tr}\pi'(fdg) \quad (\tilde{\rho} = \rho,\ \rho^+ \text{ or } \rho^-).$$

From this and the character relation for induced representations we conclude the following

16.2 Corollary. *For every one-dimensional H-module μ and for all matching measures fdg on G and $f_H dh$ on H we have*

$$\operatorname{tr}\mu(f_H dh) = \operatorname{tr}\pi_\mu^\times(fdg) + \operatorname{tr}\pi_\mu^-(fdg).$$

Let ρ be a tempered H-module, π the endo-lift of ρ (then π is a G-packet), ρ' be the basechange lift of ρ (thus ρ' is a σ-invariant H'-module), and $\pi' = I(\rho')$ be the G'-module normalizedly induced from ρ' (we regard H' as a Levi subgroup of a maximal parabolic subgroup of G'). Then

16.3 Proposition ([Cl1]). *We have* $\operatorname{tr}\pi(fdg) = \operatorname{tr}\pi'(\phi dg' \times \sigma)$ *for all matching fdg on G and $\phi dg'$ on G'.*

From this and the character relation for induced representations we conclude the following

16.4 Corollary. *For all matching measures fdg on G and $\phi dg'$ on G' and every one-dimensional H-module μ we have*

$$\operatorname{tr} I(\mu'; \phi dg' \times \sigma) = \operatorname{tr}\pi_\mu^\times(fdg) - \operatorname{tr}\pi_\mu^-(fdg).$$

Our next aim is to determine the $(\mathfrak{g}, K)$-cohomology of the G-modules described above, where $\mathfrak{g}$ denotes the complexified Lie algebra of G. For that we describe the K-types of these G-modules, following [Wh], §7, and [BW], Ch. VI. Note that $G = \mathrm{U}(2,1)$ can be defined by means of the form

$$J' = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix}$$

whose signature is also (2,1) and it is conjugate to

$$J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix} \qquad \text{by} \qquad \mathbf{B} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} = \mathbf{B}^{-1}$$

of [Wh], p. 181. To ease the comparison with [Wh] we now take G to be defined using J'. In particular we now take A to be the maximal torus of G whose conjugate by $\mathbf{B}$ is the diagonal subgroup of $G(J)$. A character χ of A is again associated with (a, b, c) in $\mathbb{C}^3$ such that $a + c$ and b are integral, and $I(\chi)$ denotes the G-module normalizedly induced from χ extended to the standard Borel subgroup B.

The maximal compact subgroup K of G is isomorphic to $\mathrm{U}(2) \times \mathrm{U}(1)$. It consists of the matrices $\left(\begin{smallmatrix} \alpha u & 0 \\ 0 & \mu \end{smallmatrix}\right)$; u in SU(2); α, μ in $\mathrm{U}(1) = \mathbb{C}^1$. Note that $A \cap K$ consists of $\gamma\,\mathrm{diag}(\alpha,\ \alpha^{-2},\ \alpha)$. The center of K consists of $\gamma\,\mathrm{diag}(\alpha,\ \alpha,\ \alpha^{-2})$.

Let π_K denote the space of K-finite vectors of the admissible G-module π. By Frobenius reciprocity, as a K-module $I(\chi)_K$ is the direct sum of the irreducible K-modules $\mathfrak{h}$, each occurring with multiplicity

$$\dim[\mathrm{Hom}_{A \cap K}(\chi, \mathfrak{h})].$$

The $\mathfrak{h}$ are parametrized by (a', b', c') in $\mathbb{Z}^3$, such that $\dim \mathfrak{h} = a' + 1$, and the central character of $\mathfrak{h}$ is

$$\gamma\,\mathrm{diag}(\mu, \mu, \mu^{-2}) \mapsto \mu^{b'} \gamma^{c'};$$

hence $b' \equiv c' (\mathrm{mod}\, 3)$ and $a' \equiv b' (\mathrm{mod}\, 2)$. In this case we write $\mathfrak{h} = \mathfrak{h}(a', b', c')$. For any integers a, b, c, p, q with p, $q \geq 0$ we also write

$$\mathfrak{h}_{p,q} = \mathfrak{h}(p + q, 3(p - q) - 2(a + c - 2b), a + b + c).$$

16.5 Lemma. *The K-module $I(\chi)_K$, $\chi = \chi(a, b, c)$, is isomorphic to* $\oplus_{p,q \geq 0} \mathfrak{h}_{p,q}$.

Proof. The restriction of $\mathfrak{h} = \mathfrak{h}(a', b', c')$ to the diagonal subgroup

$$D = \{\gamma\,\mathrm{diag}(\beta\alpha, \beta/\alpha, \beta^{-2})\}$$

of K is the direct sum of the characters $\alpha^n \beta^{b'} \gamma^{c'}$ over the integral n with $-a' \le n \le a'$ and $n \equiv a' (\bmod 2)$. Hence the restriction of $\mathfrak{h}$ to $A \cap K$ is the direct sum of the characters $\gamma \operatorname{diag}(\alpha, \alpha^{-2}, \alpha) \mapsto \alpha^{(3n-b')/2} \gamma^{c'}$. On the other hand, the restriction of $\chi = \chi(a, b, c)$ to $A \cap K$ is the character

$$\lambda \operatorname{diag}(\alpha, \alpha^{-2}, \alpha) \mapsto \alpha^{a+c-2b} \lambda^{a+b+c}.$$

If $-a \le n \le a'$ and $n \equiv a'$ (mod 2), there are unique p, $q \ge 0$ with $a' = p+q$, and $n = p - q$. Then $\mathfrak{h}(a', b', c')|(A \cap K)$ contains $\chi(a, b, c)|(A \cap K)$ if and only if there are p, $q \ge 0$ with

$$a' = p + q, \qquad b' = 3(p - q) - 2(a + c - 2b) \qquad c' = a + b + c,$$

as required. $\square$

Definition. For integral a, b, c put $\chi = \chi(a, b, c)$, $\chi^- = \chi(b, a, c)$, $\chi^+ = \chi(a, c, b)$. Also write

$$\mathfrak{h}^-_{p,q} = \mathfrak{h}(p + q, 3(p - q) - 2(b + c - 2a), a + b + c),$$

and

$$\mathfrak{h}^+_{h,q} = \mathfrak{h}(p + q, 3(p - q) - 2(a + b - 2c), a + b + c).$$

Lemma 16.5 implies that (the sums are over p, $q \ge 0$)

$$I(\chi)_K = \oplus \mathfrak{h}_{p,q}, \quad I(\chi^+)_K = \oplus \mathfrak{h}^+_{p,q}, \quad I(\chi^-)_K = \oplus \mathfrak{h}^-_{p,q}.$$

Definition. Write $JH(\pi)$ for the unordered sequence of constituents of the G-module π, repeated with their multiplicities.

If $a > b > c$ then $JH(I(\chi)) = \{\xi, J^+, J^-, D\}$. By [Wh], 7.9, the K-type decomposition of the constituents is of the form $\oplus \mathfrak{h}_{p,q}$. The sums range over:
(1) $p < a - b$, $q < b - c$ for ξ; (2) $p \ge a - b$, $q < b - c$ for J^-; (3) $p < a - b$, $q \ge b - c$ for J^+;
(4) $p \ge a - b$, $q \ge b - c$ for D. D is the unique generic constituent here and in the next two cases.

Next, $JH(I(\chi^-)) = \{J^-, D^-, D\}$. The K-types are of the form $\oplus \mathfrak{h}^-_{p,q}$, with sums over: (1) $p \ge 0$, $a - b \le q < a - c$ for J^-; (2) $p \ge 0$, $q < a - b$ for D^-; (3) $p \ge 0$, $q \ge a - c$ for D.

Finally, $JH(I(\chi^+)) = \{J^+, D^+, D\}$. The K-types are of the form $\oplus\mathfrak{h}^+_{p,q}$, with sums over: (1) $b-c \le p < a-c$, $q \ge 0$ for J^+; (2) $p < b-c$, $q \ge 0$ for D^+; (3) $p \ge a-c$, $q \ge 0$ for D.

Recall that J^- is unitary if and only if $b-c=1$, and J^+ is unitary if and only if $a-b=1$.

If $a > b = c$ (resp. $a = b > c$) then χ^- (resp. χ^+) is unitary, and $I(\chi^-)$ (resp. $I(\chi^+)$) is the direct sum of the unitary G-modules π^+ and π^-. The K-type decomposition is as follows. If $a > b = c$:

$$\pi^+_K = \oplus\mathfrak{h}^+_{p,q} \ (p \ge 0,\ q \ge a-b), \quad \pi^-_K = \oplus\mathfrak{h}^+_{p,q} \ (p \ge 0,\ q < a-b).$$

If $a = b > c$:

$$\pi^+_K = \oplus\mathfrak{h}^-_{p,q} \ (p \ge b-c,\ q \ge 0), \quad \pi^-_K = \oplus\mathfrak{h}^-_{p,q} \ (p < b-c,\ q \ge 0).$$

Moreover, $JH(I(\chi))$ is $\{\pi^\times = J^+,\ \pi^+\}$ if $a > b = c$ (π^+ is generic, π^-, J^+ are not), and $\{\pi^\times = J^-,\ \pi^-\}$ if $a = b > c$ (π^- is generic, π^+, J^- are not). The corresponding K-type decompositions are

$$J^- = \oplus\mathfrak{h}_{p,q} \ (p < a-b,\ q \ge 0), \qquad J^+ = \oplus\mathfrak{h}_{p,q} \ (p \ge 0,\ q < b-c).$$

As noted above, J^+ is unitary if and only if $a-1 = b \ge c$;
J^- is unitary if and only if $a \ge b = c+1$.

Next we define holomorphic and anti-holomorphic vectors, and describe those G-modules which contain such vectors. We have the vector spaces of matrices

$$P^+ = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix} \right\}, \qquad P^- = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

in the complexified Lie algebra $\mathfrak{g} = M(3,\mathbb{C})$. These P^+, P^- are K-modules under the adjoint action of K, clearly isomorphic to $\mathfrak{h}(1,3,0)$ and $\mathfrak{h}(1,-3,0)$.

Definition. A vector in the space π_K of K-finite vectors in a G-module π is called *holomorphic* if it is annihilated by P^-, and *anti-holomorphic* if it is annihilated by P^+.

16.6 Lemma. *If $I(\chi)$ is irreducible then $I(\chi)_K$ contains neither holomorphic nor anti-holomorphic vectors.*

Proof. The K-modules $P^+ = \mathfrak{h}(1,3,0)$ and $P^- = \mathfrak{h}(1,-3,0)$ act by

$$\mathfrak{h}(1,3,0) \otimes \mathfrak{h}(a,b,c) = \mathfrak{h}(a+1,b+3,c) \oplus \mathfrak{h}(a-1,b+3,c)$$

and

$$\mathfrak{h}(1,-3,0) \otimes \mathfrak{h}(a,b,c) = \mathfrak{h}(a+1,b-3,c) \oplus \mathfrak{h}(a-1,b-3,c).$$

Hence the action of P^+ on $I(\chi)_K$ maps $\mathfrak{h}_{p,q}$ to $\mathfrak{h}_{p+1,q} \oplus \mathfrak{h}_{p,q-1}$, and that of P^- maps $\mathfrak{h}_{p,q}$ to $\mathfrak{h}_{p,q+1} \oplus \mathfrak{h}_{p-1,q}$. Consequently if $\mathfrak{h}_{p',q'}$ is annihilated by P^+, then $\oplus\mathfrak{h}_{p,q}$ $(p \geq p', q \leq q')$ is a $(\mathfrak{g}, K)$-submodule of $I(\chi)$, and if P^- annihilates $\mathfrak{h}_{p',q'}$ then $\oplus\mathfrak{h}_{p,q}$ $(p \leq p',\ q \geq q')$ is a $(\mathfrak{g}, K)$-submodule of $I(\chi)$. The lemma follows. □

Definition. Denote by π_K^{hol} the space of holomorphic vectors in π_K, and by π_K^{ah} the space of anti-holomorphic vectors.

The proof above implies also the following

16.7 Lemma. (i) *The irreducible unitary G-modules with holomorphic vectors are*

(1) $\pi = D^+(a,b,c)$, *where* $a > b > c$; *then*

$$\pi_K^{\mathrm{hol}} = \mathfrak{h}(a-b-1, a+b-2c+3, a+b+c);$$

(2) $\pi = J^-(a,b,b-1)$, *with* $a \geq b$; *then*

$$\pi_K^{\mathrm{hol}} = \mathfrak{h}(a-b, a-b+2, a+2b-1);$$

(3) $\pi = \pi^+(a,b,b)$, *with* $a > b$; *then*

$$\pi_K^{\mathrm{hol}} = \mathfrak{h}(a-b-1, a-b+3, a+2b).$$

(ii) *The irreducible unitary G-modules with antiholomorphic vectors are*

(1) $\pi = D^-(a,b,c)$, *where* $a > b > c$; *then*

$$\pi_K^{\mathrm{ah}} = \mathfrak{h}(b-c-1, b+c-2a-3, a+b+c);$$

(2) $\pi = J^+(b+1, b, c)$, *with* $b \geq c$; *then*

$$\pi_K^{\mathrm{ah}} = \mathfrak{h}(b-c, c-b-2, 2b+c+1);$$

(3) $\pi = \pi^-(a, a, c)$, *with* $a > c$; *then*

$$\pi_K^{\mathrm{ah}} = \mathfrak{h}(a-c-1, c-a-3, 2a+c).$$

We could rename the $J^\pm$, but decided to preserve the notations induced from [Wh].

Let $\xi = \xi_{a,b,c}$ be the irreducible finite-dimensional G-module with highest weight

$$\mathrm{diag}(x, y, z) \mapsto x^{a-1} y^b z^{c+1}.$$

It is the unique finite-dimensional quotient of $I(\chi)$, $\chi = \chi(a, b, c)$, $a > b > c$. Let $\widetilde{\xi}$ denote the contragredient of ξ. Let π be an irreducible unitary G-module. Denote by $H^j(\mathfrak{g}, K; \pi \otimes \widetilde{\xi})$ the $(\mathfrak{g}, K)$-cohomology of $\pi \otimes \widetilde{\xi}$. This cohomology vanishes, by [BW], Theorem 5.3, p. 29, unless π and ξ have equal infinitesimal characters, namely π is associated with the triple (a, b, c) of ξ. It follows from the K-type computations above that one has (cf. [BW], Theorem VI.4.11, p. 201) the following

16.8 Proposition. *If* $H^j(\pi \otimes \widetilde{\xi}) \neq 0$ *for some* j *then* π *is one of the following.*

(1) *If* π *is* $D(a, b, c)$, $D^+(a, b, c)$ *or* $D^-(a, b, c)$ (*and* $a > b > c$) *then* $H^j(\pi \otimes \widetilde{\xi})$ *is* $\mathbb{C}$ *if* $j = 2$ *and* 0 *if* $j \neq 2$. *Such* π *have Hodge types* $(1, 1)$, $(2, 0)$, $(0, 2)$, *respectively. Only* D *is generic.*

(2) *If* π *is* $J^+(a, b, c)$ *with* $a - b = 1$ *or* $J^-(a, b, c)$ *with* $b - c = 1$ *then* $H^j(\pi \otimes \widetilde{\xi})$ *is* $\mathbb{C}$ *if* $j = 1$, 3 *and* 0 *if* $j \neq 1$, 3. *Such* π *have Hodge types* $(0, 1)$, $(0, 3)$ *and* $(1, 0)$, $(3, 0)$, *respectively.*

(3) $H^j(\xi \otimes \widetilde{\xi})$ *is* 0 *unless* $j = 0$, 2, 4 *when it is* $\mathbb{C}$. *The Hodge types of* ξ *are* $(0, 0)$, $(1, 1)$, $(2, 2)$.

IV.3 Finite-dimensional representations

The group $G' = \mathrm{R}_{F/\mathbb{Q}} G$, $G = \mathrm{GU}(1, 2; E/F)$, is isomorphic over $\overline{\mathbb{Q}}$, in fact over the Galois closure F' of F, to $\prod_\sigma G_\sigma$, $G_\sigma = \mathrm{GU}(1, 2; \sigma E/\sigma F)$, $\sigma E =$

$E \otimes_{F,\sigma} \sigma F$. Here σ ranges over $S = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, $= \mathrm{Emb}(F, \overline{\mathbb{Q}})$ and so $G' = \{(g_\sigma);\ g_\sigma \in G_\sigma\}$.

An irreducible representation $(\boldsymbol{\xi}, \mathbf{V})$ of G' over $\overline{\mathbb{Q}}$ has the form $(g_\sigma) \mapsto \otimes\xi_\sigma(g_\sigma)$, where ξ_σ is a representation (irreducible and finite dimensional) of G_σ. In fact in our case it has the form $(\xi_{\mathbf{a},\mathbf{b},\mathbf{c}} = \otimes_{\sigma\in S}\xi_{a_\sigma,b_\sigma,c_\sigma}, V_{\mathbf{a},\mathbf{b},\mathbf{c}} = \otimes_{\sigma\in S}V_{a_\sigma,b_\sigma,c_\sigma})$, where $a_\sigma > b_\sigma > c_\sigma$ for all $\sigma \in S$, and $\xi_{a_\sigma,b_\sigma,c_\sigma}$ is as in 16.8.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by $\varphi((g_\sigma)) = ((\varphi g_\sigma)_{\varphi\sigma}) = ((\varphi g_{\varphi^{-1}\sigma})_\sigma)$. The fixed points are the (g_σ) with $g_\sigma = \sigma g_1$, where g_1 ranges over $G(F)$ (the "1" indicates the fixed embedding $F \hookrightarrow \overline{\mathbb{Q}}$). Thus $G'(\mathbb{Q}) = G(F)$ and $G'(\mathbb{R}) = \prod_S G(\mathbb{R})$ with $|S| = [F:\mathbb{Q}]$ since F is totally real; S is also the set of embeddings $F \hookrightarrow \mathbb{R}$.

Now if the representation $\boldsymbol{\xi}$ is defined over $\mathbb{Q}$, it is fixed under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus $\otimes_\sigma\xi_\sigma(g_\sigma) = \otimes_\sigma\xi_{\varphi\sigma}(\varphi g_\sigma)$. The element $g = (g_1, 1, \dots, 1)$ (thus $g_\sigma = 1$ for all $\sigma \neq 1$) is mapped by φ to

$$(1, \dots, 1, \varphi g_1, 1, \dots, 1)$$

(the entry φg_1 is at the place parametrized by φ). Hence $\xi_1(g_1)$ equals $\xi_\varphi(\varphi g_1)$ (both are equal to $\xi(g)(= \xi(\varphi g))$). Hence $\xi_\varphi = {}^\varphi\xi_1(: g_1 \mapsto \xi_1(\varphi^{-1}g_1))$, and the components ξ_φ of ξ are all translates of the same representation ξ_1. For $(g_\sigma) = (\sigma g_1)$ in $G'(\mathbb{Q}) = G(F)$,

$$\boldsymbol{\xi}((g_\sigma)) = \otimes_\sigma\xi_\sigma(\sigma g_1) = \otimes_\sigma\xi_1(g_1) = \xi_1(g_1) \otimes \cdots \otimes \xi_1(g_1)\ ([F:\mathbb{Q}]\ \text{times}).$$

Next we wish to compute the factors at ∞ of each of the terms in $\mathrm{STF}_G(f)$ and $\mathrm{TF}_H(f_H)$. The functions f_∞ ($= h_\infty$ of [Ko5], p. 186) and $f_{H,\infty}$ are products $\otimes f_\sigma$ and $\otimes f_{H\sigma}$ over σ in S. We fixed a $\mathbb{Q}$-rational finite-dimensional representation

$$(\boldsymbol{\xi}, V_{\boldsymbol{\xi}}) = (\xi_{\mathbf{a},\mathbf{b},\mathbf{c}} = \otimes_{\sigma\in S}\xi_{a_\sigma,b_\sigma,c_\sigma}, V_{\mathbf{a},\mathbf{b},\mathbf{c}} = \otimes_{\sigma\in S}V_{a_\sigma,b_\sigma,c_\sigma}), \quad a_\sigma > b_\sigma > c_\sigma$$

for all $\sigma \in S$, of the $\mathbb{Q}$-group G'. The triple $(a_\sigma, b_\sigma, c_\sigma)$ is independent of σ only if $\boldsymbol{\xi}$ is defined over $\mathbb{Q}$. Denote by $\{_\xi\pi_\sigma\}$ the packet of discrete-series representations of $G(\mathbb{R})$ which share infinitesimal character (i.e. $(a_\sigma, b_\sigma, c_\sigma)$) with $\boldsymbol{\xi}$.

For any (ξ, V), the packet $\{_\xi\pi_\sigma\}$ consists of three irreducible representations D, D^+ and D^-. It is the e-lift of the following representations of

$H(\mathbb{R}) = \mathrm{U}(1,1;\mathbb{R}) \times \mathrm{U}(1;\mathbb{R})$:
$\rho^+ \times \rho(c_\sigma)$, where $\rho^+ = \rho(a_\sigma, b_\sigma) \otimes \kappa^{-1}$,
$\rho \times \rho(b_\sigma)$, where $\rho = \rho(a_\sigma, c_\sigma) \otimes \kappa^{-1}$, and
$\rho^- \times \rho(a_\sigma)$, where $\rho^- = \rho(b_\sigma, c_\sigma) \otimes \kappa^{-1}$.

Denote by $h(D')$ a pseudo-coefficient of the representation D'. Then
$h(\rho \otimes \rho(b_\sigma))$ matches $h(D) - h(D^+) - h(D^-)$,
$h(\rho^+ \otimes \rho(c_\sigma))$ matches $h(D) + h(D^+) - h(D^-)$, and
$h(\rho^- \otimes \rho(a_\sigma))$ matches $h(D) - h(D^+) + h(D^-)$.

Following [Ko5], p. 186, we put

$$f_{H,\sigma} = -h(\rho \otimes \rho(b_\sigma)) + h(\rho^+ \otimes \rho(c_\sigma)) + h(\rho^- \otimes \rho(a_\sigma))$$

and $f_{G,\sigma} = \frac{1}{3}[h(D) + h(D^+) + h(D^-)]$. Put $H' = \mathrm{R}_{F/\mathbb{Q}} H$,

$$f_{G',\infty} = {}_\xi f_{G',\infty} = \prod_{\sigma \in S} f_{G,\sigma}, \qquad f_{H',\infty} = {}_\xi f_{H',\infty} = \prod_{\sigma \in S} f_{H,\sigma}.$$

Note that $q(G') = [F:\mathbb{Q}]q(G)$ is half the real dimension of the symmetric space attached to $G'(\mathbb{R})$, and $q(G)$ is that of $G(\mathbb{R})$. Thus $q(G) = 2$ in our case.

Then $\operatorname{tr} D_\Lambda(f_{G,\sigma}) = \frac{1}{3}$, $\operatorname{tr} D_\Lambda^\pm(f_{G,\sigma}) = \frac{1}{3}$, $\operatorname{tr}\{D_\Lambda\}(f_{G,\sigma}) = 1$.

When $a - b = 1$, we have in addition $\operatorname{tr} J^+_{s_2\Lambda}(f_{G,\sigma}) = -\frac{2}{3}$.

When $b - c = 1$, we have $\operatorname{tr} J^-_{s_1\Lambda}(f_{G,\sigma}) = -\frac{2}{3}$.

When $a - b = 1 = b - c$, we have in addition $\operatorname{tr} \xi(f_{G,\sigma}) = 1$.

Note that if π contributes to $I(G,4)$ then its archimedean components π_σ have infinitesimal characters with $a_\sigma - b_\sigma = 1$ or $b_\sigma - c_\sigma = 1$ for all $\sigma \in S$.

There are contributions to $I(G,2)$, $I(G,3)$, $I(G,4)$ precisely when there are corresponding contributions to the corresponding terms in the trace formula of H, as is listed in section II.6.

V. GALOIS REPRESENTATIONS

V.1 Stable case

We shall study the decomposition of the semisimplification of the the étale cohomology

$$H_c^* = H_c^*(\mathcal{S}_{K_f} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\lambda})$$

with compact supports and coefficients in the representation $(\xi_{\mathbf{a},\mathbf{b},\mathbf{c}}, V_{\mathbf{a},\mathbf{b},\mathbf{c}})$, $a_\sigma > b_\sigma > c_\sigma$ for each $\sigma \in S$, as a $C_c[K_f\backslash\mathbb{G}'(\mathbb{A}_f)/K_f] \times \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{E}_\wp)$-module, by means of Deligne's conjecture on the Lefschetz fixed point formula. Its expression as a sum of trace formulae for G' and H at the test functions specified above shows that this module decomposes as a virtual sum of $\pi_f^{K_f} \otimes H_c^*(\pi_f)$, where the π_f range over the finite parts of discrete-spectrum representations $\pi = \pi_f \otimes \pi_\infty$ of $G'(\mathbb{A}_\mathbb{Q}) = G(\mathbb{A})$ and π_∞ are irreducible $(\mathfrak{g}, K)$-modules with central and infinitesimal characters determined by those of $\mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c}}$. Thus we fix such a representation π of $G(\mathbb{A})$ and examine the π_f-isotypic contribution.

We start with a π which occurs in the stable spectrum, namely in $I(G,1)$.

In general, we have trace formulae evaluated at certain test functions. Since π is stable, only the trace formula for G' occurs. The choice of the function ${}_\xi f_{G',\infty}$ guarantees that the components π_σ of the π which occur in the trace formula for G' lie in the packet $\{D,\, D^+,\, D^-\}$, $a_\sigma > b_\sigma > c_\sigma$ determined by ξ, at each archimedean place $\sigma \in S$. Indeed, as π occurs in $I(G,1)$, its components are never the nontempered $\pi_v^\times$, thus not $J^\pm$.

Let us compute $\mathrm{tr}\, H_c^*(\pi_f)(\mathrm{Fr}_\wp^j) = \mathrm{tr}[\mathrm{Fr}_\wp^j \,|\, \mathrm{tr}\, H_c^*(\pi_f)]$ for a place $\wp$ of $\mathbb{E}$ over an unramified place p of $\mathbb{Q}$.

Suppose that $\pi_f^{K_f} \neq 0$. In particular the component at p of π is unramified. It has the form $\otimes_{u|p}\pi_u$, $\pi_u = \pi(\mu_{1u}, \mu_{2u}, \mu_{3u})$ if $E_u = F_u \oplus F_u$ and $\pi_u = \pi(\mu_{1u})$ if E_u is a field.

We use a correspondence f^p, which is a K_f^p-biinvariant compactly supported function on $G(\mathbb{A}_f^p)$. Since there are only finitely many discrete-spectrum representations of $G(\mathbb{A})$ with a given infinitesimal character (determined by ξ) and a nonzero K_f-fixed vector, we can choose f^p to be a projection onto $\{\pi_f^{K_f}\}$. Recall that $j_u = (jn_\wp, n_u)$ and write μ_{mu} for $\mu_{mu}(\boldsymbol{\pi}_u)$, $m = 1, 2, 3$. Then the trace of the action of $\mathrm{Fr}_\wp^j$ on the $\{\pi_f^{K_f}\}$-isotypic component of $H_c^*(\mathcal{S}_{K_f} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \mathbb{V}_\xi)$ is

$$q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}} \cdot \prod_{u|p} \left(\mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + \mu_{3u}^{\frac{jn_\wp}{j_u}} \right)^{j_u}.$$

Thus the $\{\pi_f^{K_f}\}$-isotypic part of $H_c^{2[F:\mathbb{Q}]}$ (namely the $\pi_f^{K_f}$-isotypic part for each member of the packet) is of the form $\{\pi_f^{K_f}\} \otimes H_c^*(\{\pi_f\})$, where $H_c^*(\{\pi_f\})$ is a $3^{[F:\mathbb{Q}]}$-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$. The $3^{\#\{u|p\}}$ nonzero eigenvalues t of the action of $\mathrm{Fr}_\wp$ include $q_\wp^{\frac{1}{2}\dim \mathcal{S}_{K_f}} \prod_{u|p} \mu_{m(u),u}^{n_\wp}$, where $m(u) \in \{1, 2, 3\}$. This we see first for sufficiently large j by Deligne's conjecture, but then for all $j \geq 0$, including $j = 1$, by multiplicativity.

Standard unitarity estimates on $\mathrm{GL}(3, \mathbb{A}_E)$ and the basechange lifting from $\mathrm{U}(3, E/F)$ to $\mathrm{GL}(3, E)$ imply that $|\mu_{iu}|^{\pm 1} < q_u^{1/2}$ at each place u of F which splits in E, and that $|\mu_{1u}|^{\pm 1} < q_u^{1/2} = q_{E_u}$ if u is a place of F which stays prime and is unramified in E. Hence the Hecke eigenvalues are bounded by $\prod_{u|p} q_u^{n_\wp/2} = p^{\frac{n_\wp}{2}\sum_{u|p}[F_u:\mathbb{Q}_p]} = q_\wp^{\frac{1}{2}[F:\mathbb{Q}]} = (\sqrt{q_\wp})^{\frac{1}{2}\dim \mathcal{S}}$.

Deligne's "Weil conjecture" purity theorem asserts that the Frobenius eigenvalues are algebraic numbers and all their conjugates have equal complex absolute values of the form $q_\wp^{i/2}$ $(0 \leq i \leq 2\dim \mathcal{S})$. This is also referred to as "mixed purity". The eigenvalues of $\mathrm{Fr}_\wp$ on IH^i have complex absolute values equal $q_\wp^{i/2}$, by a variant of the purity theorem due to Gabber. We shall use this to show that the absolute values in our case are all equal to $q_\wp^{\frac{1}{2}\dim \mathcal{S}}$.

The cuspidal π define part not only of the cohomology $H_c^i(\mathcal{S}_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$ but also part of the intersection cohomology $IH^i(\mathcal{S}'_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$. By the Zucker isomorphism it defines a contribution to the L^2-cohomology, which is of the form $\pi_f^{K_f} \otimes H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V_\xi(\mathbb{C}))$. We shall compute this $(\mathfrak{g}, K_\infty)$-cohomology space to know for which i there is nonzero contribution corresponding to our π_f. We shall then be able to evaluate the absolute values of the conjugates of the Frobenius eigenvalues using Deligne's "Weil conjecture" theorem.

By Proposition 16.8 the space $H^{i,j}(\mathfrak{g}, K; \pi \otimes \tilde{\xi}_{a,b,c})$ is 0 for $\pi = D$, D^+, D^- (indexed by $a > b > c$) except when $(i,j) = (1,1)$, $(2,0)$, $(0,2)$

(respectively), when this space is $\mathbb{C}$. From the "Matsushima-Murakami" decomposition of section I.2, first for the L^2-cohomology $H_{(2)}$ but then by Zucker's conjecture also for IH^*, and using the Künneth formula, we conclude that $IH^i(\pi_f)$ is zero unless i is equal to $\dim \mathcal{S}_{K_f} = 2[F:\mathbb{Q}]$, and there $\dim IH^{2[F:\mathbb{Q}]}(\pi_f)$ is $3^{[F:\mathbb{Q}]}$ (as there are $[F:\mathbb{Q}]$ real places of F). Since π_f is the finite component of cuspidal representations only, π_f contributes also to the cohomology $H_c^i(\mathcal{S}_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V}_{\mathbf{a},\mathbf{b},\mathbf{c};\lambda})$ only in dimension $i = 2[F:\mathbb{Q}]$, and $\dim H_c^{2[F:\mathbb{Q}]}(\pi_f) = 3^{[F:\mathbb{Q}]}$. This space depends only on the packet of π_f and not on π_f itself.

Deligne's theorem [D4] (in fact its IH-version due to Gabber) asserts that the eigenvalues t of the Frobenius $\mathrm{Fr}_\wp$ acting on the ℓ-adic intersection cohomology IH^i of a variety over a finite field of $q_\wp$ elements are algebraic and "pure", namely all conjugates have the same complex absolute value, of the form $q_\wp^{i/2}$. In our case $i = \dim \mathcal{S}_{K_f} = 2[F:\mathbb{Q}]$, hence the eigenvalues of the Frobenius are algebraic and each of their conjugates is $q_\wp^{[F:\mathbb{Q}]}$ in absolute value. Consequently the eigenvalues μ_{1u}, μ_{2u}, μ_{3u} are algebraic, and all of their conjugates have complex absolute value 1.

Note that we could not use only "mixed-purity" (that the eigenvalues are powers of $q_\wp^{1/2}$ in absolute value) and the unitarity estimates $|\mu_{mu}|^{\pm 1} < q_u^{1/2}$ on the Hecke eigenvalues, since the estimate (less than $(\sqrt{q_\wp})^{\frac{1}{2}\dim \mathcal{S}}$ away from $(\sqrt{q_\wp})^{\dim \mathcal{S}}$) does not define the absolute value ($(\sqrt{q_\wp})^{\dim \mathcal{S}}$) uniquely. This estimate does suffice to show unitarity when $\dim \mathcal{S} = 1$.

In summary, the representation $H_c^*(\pi_f)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ attached to the finite part π_f of a cuspidal π in the stable discrete spectrum, depends only on the packet of π_f, its dimension is $3^{[F:\mathbb{Q}]}$, and it makes the same contribution to H_c^* and to IH^*. Its restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{E}_\wp)$ is unramified, and the trace of $H_c^*(\mathrm{Fr}_\wp)$ on the $\{\pi_f^{K_f}\}$-isotypic part of $H_c^{2[F:\mathbb{Q}]}$ is equal to the trace of $\otimes \nu_u^{-1/2} r_u(\mathbf{t}(\pi_u) \times \mathrm{Fr}_\wp)$. Here $(r_u, (\mathbb{C}^3)^{[F_u:\mathbb{Q}_p]})$ denotes the twisted tensor representation of ${}^L\mathrm{R}_{F_u/\mathbb{Q}_p} G = \widehat{G}^{[F_u:\mathbb{Q}_p]} \rtimes \mathrm{Gal}(F_u/\mathbb{Q}_p)$, $\mathrm{Fr}_\wp$ is $\mathrm{Fr}_p^{[\mathbb{E}_\wp:\mathbb{Q}_p]}$, and ν_u is the character of ${}^L\mathrm{R}_{F_u/\mathbb{Q}_p} G$ which is trivial on the connected component of the identity and whose value at Fr_p is p^{-1}. The eigenvalues of $\mathbf{t}(\pi_u)$ and all of their conjugates lie on the complex unit circle.

V.2 Unstable case

We continue by fixing a cuspidal representation π with π_σ in $\{D, D^+, D^-\}$, determined by $\xi_\sigma = \xi(a_\sigma, b_\sigma, c_\sigma)$, for all σ in S, and with $\pi^{K_f} \neq 0$. But now we assume π occurs in the unstable spectrum, say in $I(G,2)$. We fix a correspondence f^p which projects to the packet $\{\pi_f^p\}$. Since the function f_H^p is chosen to be matching f^p, by [F3;VIII] the contribution to the first part $I(H,1)$ of the stable trace formula of H is precisely that parametrized by a cuspidal representation $\tilde\rho \neq \rho(\theta, {}'\theta) \times {}''\theta$ of $\mathrm{U}(2,\mathbb{A}) \times \mathrm{U}(1,\mathbb{A})$. Its real component is $\otimes_\sigma \tilde\rho_\sigma$, where $\tilde\rho_\sigma$ is $\rho_\sigma^+ = \rho(a_\sigma, b_\sigma) \times \rho(c_\sigma)$, $\rho_\sigma = \rho(a_\sigma, c_\sigma) \times \rho(b_\sigma)$ or $\rho_\sigma^- = \rho(b_\sigma, c_\sigma) \times \rho(a_\sigma)$, and $\rho(a) : z \mapsto z^a$.

Each component π_v of an irreducible $\pi = \otimes \pi_v$ in the packet $\{\pi = \pi(\tilde\rho)\}$ has a sign $\langle \tilde\rho_v, \pi_v \rangle \in \{\pm 1\}$. Thus the sign $\langle \tilde\rho_f, \pi_f \rangle = \prod_{v<\infty} \langle \tilde\rho_v, \pi_v \rangle$ of π_f is $+1$ if the number of its components π_v^- is even, in which case we denote π_f by π_f^+, otherwise the sign $\langle \tilde\rho_f, \pi_f \rangle$ is -1 and we denote π_f by π_f^-. Write $\{\pi_f\}^+$ for the set of π_f^+, and $\{\pi_f\}^-$ for the set of π_f^-.

At the archimedean places $\sigma : F \hookrightarrow \mathbb{R}$ the sign of π_σ in $\{D, D^+, D^-\}$ depends on $\tilde\rho_\sigma$: $\langle \rho, D \rangle = 1$ and $\langle \rho, D^\pm \rangle = -1$; $\langle \rho^+, D \rangle = 1 = \langle \rho^+, D^+ \rangle$ and $\langle \rho^+, D^- \rangle = -1$; and $\langle \rho^-, D \rangle = 1 = \langle \rho^-, D^- \rangle$, $\langle \rho^-, D^+ \rangle = -1$. Then $\langle \tilde\rho, \pi \rangle = \langle \tilde\rho_f, \pi_f \rangle \prod_\sigma \langle \tilde\rho_\sigma, \pi_\sigma \rangle$. An irreducible π in $\{\pi(\tilde\rho)\}$ is automorphic, necessarily cuspidal, when π has sign $\langle \tilde\rho, \pi \rangle$ equal 1.

The contribution of $\{\pi\}$ to $I(G,2)$, for our test function $f = f^p f_\wp^j f_{G,\infty}$, is

$$\frac{1}{2} \prod_{\sigma \in S} \mathrm{tr}\{\pi_\sigma\}(f_{G,\sigma}) \cdot [\mathrm{tr}\{\pi_f\}^+(f^p) + \mathrm{tr}\{\pi_f\}^-(f^p)] \cdot q_\wp^{\frac{j}{2} \dim \mathcal{S}_{K_f}}$$

$$\cdot \prod_{u|p} \left(\mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + \mu_{3u}^{\frac{jn_\wp}{j_u}} \right)^{j_u}.$$

Here and below f^p indicates — as suitable — its product with the unit element of the $G'(\mathbb{Z}_p)$-Hecke algebra of $G'(\mathbb{Q}_p)$.

The contribution to $I(H,1)$ corresponding to $\tilde\rho$ is

$$\frac{1}{2} \prod_{\sigma \in S} \mathrm{tr}\{\tilde\rho_\sigma\}(f_{H,\sigma}) \cdot \mathrm{tr}\{\tilde\rho_f\}(f_H^p) \cdot q_\wp^{\frac{j}{2} \dim \mathcal{S}_{K_f}}$$

$$\cdot \prod_{u|p} \left[(-1)^{\frac{n_u}{j_u}} \mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}} \mu_{3u}^{\frac{jn_\wp}{j_u}} \right]^{j_u}.$$

By choice of f_H^p we have that $\mathrm{tr}\{\tilde\rho_f\}(f_H^p) = \mathrm{tr}\{\pi_f\}^+(f^p) - \mathrm{tr}\{\pi_f\}^-(f^p)$.

The choice of $f_{G,\sigma}$ is such that $\mathrm{tr}\{\pi_\sigma\}(f_{G,\sigma}) = 1$, $\mathrm{tr}\{\rho_\sigma\}(f_{H,\sigma}) = -1$, $\mathrm{tr}\{\rho_\sigma^\pm\}(f_{H,\sigma}) = 1$.

We conclude that for each irreducible π_f under discussion, the $\pi_f^{K_f}$-isotypic part $\pi_f^{K_f} \otimes H_c^*(\pi_f)$ of H_c^* depends only on $\{\pi_f\}^{\langle\tilde\rho_f,\pi_f\rangle}$. Moreover $\mathrm{Fr}_\wp^j$ acts on $H_c^*(\{\pi_f\}^{\langle\tilde\rho_f,\pi_f\rangle})$ with trace $\frac{1}{2} q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}}$ times

$$\prod_{u|p}\left(\mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + \mu_{3u}^{\frac{jn_\wp}{j_u}}\right)^{j_u} + \langle\tilde\rho_f,\pi_f\rangle \cdot \prod_{\sigma\in S} \mathrm{tr}\{\tilde\rho_\sigma\}(f_{H,\sigma})$$
$$\cdot \prod_{u|p}\left[(-1)^{\frac{n_u}{j_u}} \mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}} \mu_{3u}^{\frac{jn_\wp}{j_u}}\right]^{j_u}.$$

For example, when $F = \mathbb{Q}$ and $\tilde\rho_\sigma$ is ρ, the trace of $\mathrm{Fr}_\wp^j$ of $H_c^*(\pi_f)$ is $q_\wp^j \mu_{2u}^{jn_\wp}$ (and $q_\wp = p^2$ as $\mathbb{E} = E$), but if $\tilde\rho_\sigma$ is $\rho^\pm$, the trace is $q_\wp^j(\mu_{1u}^{jn_\wp} + \mu_{3u}^{jn_\wp})$.

We know that the space contributed by π_f to H_c^* is equal to the space contributed by π_f to IH^*, since π is cuspidal. This is compatible with the computation of the dimensions of the contributions to these two cohomologies, using the L^2-decomposition and using the computation of the trace of the Frobenius. Indeed, given π_f, it contributes (by Künneth formula and the computation of the Lie algebra cohomology of D, D^+, D^-) only to IH^i with $i = 2[F:\mathbb{Q}]$. The dimension of its contribution to $IH^{2[F:\mathbb{Q}]}$ is the number of $\otimes_\sigma\pi_\sigma$ such that $\prod_\sigma\langle\tilde\rho_\sigma,\pi_\sigma\rangle$ is $\langle\tilde\rho_f,\pi_f\rangle$, by the "Matsushima-Murakami" formula of section I.2.

For example, if $F = \mathbb{Q}$, $\dim IH^{2[F:\mathbb{Q}]}(\pi_f^+)$ is 1 if $\tilde\rho_\sigma$ is ρ and 2 if $\tilde\rho_\sigma$ is ρ^+ or ρ^-, and $\dim\phi(\pi_f^-)$ is 2 or 1, respectively. If $[F:\mathbb{Q}] = 2$, $\dim IH^{2[F:\mathbb{Q}]}(\pi_f^+)$ is $1\cdot 1 + 2\cdot 2 = 5$ if $\otimes_\sigma\tilde\rho_\sigma$ is $\rho\otimes\rho$, $1\cdot 2 + 2\cdot 1 = 4$ if $\otimes_\sigma\tilde\rho_\sigma$ is $\rho\otimes\rho^\pm$, and $2\cdot 2 + 1\cdot 1 = 5$ if $\otimes_\sigma\tilde\rho_\sigma$ is $\rho^\pm\otimes\rho^\pm$, while $\dim IH^{2[F:\mathbb{Q}]}(\pi_f^-)$ is 4, 5, 4, respectively.

As in the stable case we conclude from Gabber's purity theorem for $IH^{2[F:\mathbb{Q}]}$ and the fact that cuspidal representations make the same contribution to H_c^* and to IH^*, that the Hecke eigenvalues μ_{mu} are algebraic and their conjugates all lie in the unit circle in $\mathbb{C}$. But this follows already from the theory for the group $\mathrm{U}(2,E/F)$, as the π which contribute to $I(G,2)$ are lifts of π_H on H.

We continue by fixing a cuspidal representation π with π_σ in $\{D, D^+, D^-\}$, determined by $\xi_\sigma = \xi(a_\sigma, b_\sigma, c_\sigma)$, for all σ in S and with $\pi^{K_f} \neq 0$. But now we assume π occurs in the unstable spectrum which contributes to $I(H,3)$. We fix a correspondence f^p which projects to the packet $\{\pi_f^p\}$. Since the function f_H^p is chosen to be matching f^p, by [F3;VIII] the contribution to the part $I(H,2)$ of the stable trace formula of H is precisely that parametrized by the cuspidal representations $\rho_1 = \rho(\theta, {}'\theta) \times {}''\theta$, $\rho_2 = \rho(\theta, {}''\theta) \times {}'\theta$, $\rho_3 = \rho({}'\theta, {}''\theta) \times \theta$, of $\mathrm{U}(2,\mathbb{A}) \times \mathrm{U}(1,\mathbb{A})$. The components ρ_{iv} $(v < \infty)$ of ρ_i define signs $\langle \rho_{iv}, \pi_v \rangle$ in $\{\pm 1\}$ on the irreducible π_v in the packet $\{\pi_v\}$, hence signs $\langle \rho_i, \pi_f \rangle = \prod_{v<\infty} \langle \rho_{iv}, \pi_v \rangle$ on the irreducibles π_f in the packet $\{\pi_f\}$. The product is well defined as $\langle \rho_{iv}, \pi_v \rangle$ are 1 when π_v is unramified or v splits. Write $\{\pi_f\}^{a,b}$ for the $\pi_f = \otimes_{v<\infty} \pi_v$ in $\{\pi_f\}$ with $\langle \rho_1, \pi_f \rangle = a$, $\langle \rho_2, \pi_f \rangle = b$. Then $\langle \rho_3, \pi_f \rangle = ab$.

The contribution of $\{\pi\}$ to $I(G,3)$ for our test function $f = f^p f_\wp^j f_{G,\infty}$ is

$$\frac{1}{4} \prod_{\sigma \in S} \mathrm{tr}\{\pi_\sigma\}(f_{G,\sigma}) \cdot \left[\sum_{a,b \in \{\pm 1\}} \mathrm{tr}\{\pi_f\}^{a,b}(f^p) \right] \cdot q_\wp^{\frac{j}{2} \dim \mathcal{S}_{K_f}}$$

$$\cdot \prod_{u|p} \left(\mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + \mu_{3u}^{\frac{jn_\wp}{j_u}} \right)^{j_u}.$$

Here and below f^p indicates — as suitable — its product with the unit element of the $G'(\mathbb{Z}_p)$-Hecke algebra of $G'(\mathbb{Q}_p)$.

The corresponding contribution to $I(H,2)$, attached to ρ_i $(1 \leq i \leq 3)$, is

$$\frac{1}{4} \sum_{1 \leq i \leq 3} \prod_{\sigma \in S} \mathrm{tr}\{\rho_{i,\sigma}\}(f_{H,\sigma}) \cdot \mathrm{tr}\{\rho_{i,f}\}(f_H^p) \cdot q_\wp^{\frac{j}{2} \dim \mathcal{S}_{K_f}}$$

$$\cdot \prod_{u|p} \left[(-1)^{\frac{n_u}{j_u}} \mu_{1(i),u}^{\frac{jn_\wp}{j_u}} + \mu_{2(i),u}^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}} \mu_{3(i),u}^{\frac{jn_\wp}{j_u}} \right]^{j_u}.$$

By choice of f_H^p we have that $\mathrm{tr}\{\rho_{i,f}\}(f_H^p)$ is

$$\sum_{a,b} \langle \rho_{i,f}, \{\pi_f\}^{a,b} \rangle \, \mathrm{tr}\{\pi_f\}^{a,b}(f^p),$$

where

$$\langle \rho_{1,f}, \{\pi_f\}^{a,b} \rangle = a, \quad \langle \rho_{2,f}, \{\pi_f\}^{a,b} \rangle = b, \quad \langle \rho_{3,f}, \{\pi_f\}^{a,b} \rangle = ab.$$

The choice of $f_{G,\sigma}$ is such that $\operatorname{tr}\{\pi_\sigma\}(f_{G,\sigma}) = 1$, $\operatorname{tr}\{\rho_\sigma\}(f_{H,\sigma}) = -1$, $\operatorname{tr}\{\rho_\sigma^\pm\}(f_{H,\sigma}) = 1$.

We conclude that for each irreducible π_f under consideration, $H_c^*(\pi_f) = H_c^*(\pi'_f)$ if $\langle \rho_{i,f}, \pi_f\rangle = \langle \rho_{i,f}, \pi'_f\rangle$ for all i. Then $\operatorname{Fr}_\wp^j$ acts on $H_c^*(\pi_f)$ with trace $\frac{1}{4} q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}}$ times

$$\prod_{u|p}\left(\mu_{1u}^{\frac{jn_\wp}{j_u}} + \mu_{2u}^{\frac{jn_\wp}{j_u}} + \mu_{3u}^{\frac{jn_\wp}{j_u}}\right)^{j_u} + \sum_{i=1,2,3} \langle \rho_{i,f}, \pi_f\rangle \prod_{\sigma\in S} \operatorname{tr}\{\rho_{i,\sigma}\}(f_{H,\sigma})$$
$$\cdot \prod_{u|p}\left[(-1)^{\frac{n_u}{j_u}}\mu_{1(i),u}^{\frac{jn_\wp}{j_u}} + \mu_{2(i),u}^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}}\mu_{3(i),u}^{\frac{jn_\wp}{j_u}}\right]^{j_u}.$$

As for the contribution of π_f to IH^*, each $\pi = \otimes_\sigma \pi_\sigma$ such that

$$m(\pi) = \frac{1}{4}\left[1 + \sum_{1\le i\le 3} \langle \rho_{i,f}, \pi_f\rangle \prod_\sigma \langle \rho_{i,\sigma}, \pi_\sigma\rangle\right]$$

is 1 contributes 1 to the dimension of the π_f-isotypic part $IH^*(\pi_f)$ of IH^*, in fact IH^i with $i = 2[F:\mathbb{Q}]$. Thus this dimension is the number of $\otimes_\sigma \pi_\sigma$ such that $\pi_f \otimes (\otimes_\sigma \pi_\sigma)$ is cuspidal.

For example, suppose that $F = \mathbb{Q}$ and $\rho_{1\sigma} = \rho(a_\sigma, c_\sigma) \times \rho(b_\sigma)$, $\rho_{2\sigma} = \rho(a_\sigma, b_\sigma) \times \rho(c_\sigma)$, $\rho_{3\sigma} = \rho(b_\sigma, c_\sigma) \times \rho(a_\sigma)$, $a_\sigma > b_\sigma > c_\sigma$. If $\langle \rho_{1f}, \pi_f\rangle = 1 = \langle \rho_{2f}, \pi_f\rangle$, $\pi_f \otimes D$ is cuspidal, but $\pi_f \otimes D^\pm$ are not, and $\dim IH^*(\pi_f)$ is 1. If $\langle \rho_{1f}, \pi_f\rangle = 1$ and $\langle \rho_{2f}, \pi_f\rangle = -1$, then π_f cannot be completed to a cuspidal representation. If $\langle \rho_{1f}, \pi_f\rangle = -1$ and $\langle \rho_{2f}, \pi_f\rangle = 1$, then $\pi_f \otimes D^+$ is cuspidal, but $\pi_f \otimes D$ and $\pi_f \otimes D^-$ are not. If $\langle \rho_{1f}, \pi_f\rangle = -1$ and $\langle \rho_{2f}, \pi_f\rangle = -1$, then $\pi_f \otimes D^-$ is cuspidal, but $\pi_f \otimes D$ and $\pi_f \otimes D^+$ are not.

It follows that $IH^i(\pi_f)$ is 0 unless $i = 2[F:\mathbb{Q}] = \dim \mathcal{S}_{K_f}$. As in the stable case we conclude from Gabber's purity for IH that the Hecke eigenvalues μ_{mu} are algebraic and their conjugates all lie in the unit circle in $\mathbb{C}$. But this follows already from the theory for the group $\mathrm{U}(2, E/F)$.

V.3 Nontempered case

We now fix a π with $\pi_f^{K_f} \neq 0$ in a cuspidal packet which is the lift of a character μ of the endoscopic group $\mathrm{U}(2) \times \mathrm{U}(1)$. The choice of $f_{G,\sigma}$, depending on ξ_σ, implies that π_σ lies in $\{D, D^+, D^-, J^+, J^-\}$, determined by $\xi_\sigma = \xi(\Lambda_\sigma)$, $\Lambda_\sigma = (a_\sigma, b_\sigma, c_\sigma)$, for all σ.

If $\mu_\sigma = \xi_H(a_\sigma, a_\sigma - 1) \times \rho(c_\sigma)$ put $\pi_{\mu_\sigma}^\times = J_{s_2\Lambda_\sigma}^+$, $\pi_{\mu_\sigma}^- = D_{s_1\Lambda_\sigma}^-$, $\pi_{\mu_\sigma}^+ = D_{\Lambda_\sigma} \oplus D_{s_2\Lambda_\sigma}^+$. Note that the nonzero Lie algebra cohomology of J^+ is $H^{0,1}$ and $H^{0,3}$, while the nonzero cohomology of $\pi_{\mu_\sigma}^-$ is $H^{0,2}$. If $\mu_\sigma = \rho(a_\sigma) \times \xi_H(b_\sigma, b_\sigma - 1)$ put $\pi_{\mu_\sigma}^\times = J_{s_1\Lambda_\sigma}^-$, $\pi_{\mu_\sigma}^- = D_{s_2\Lambda_\sigma}^+$, $\pi_{\mu_\sigma}^+ = D_{\Lambda_\sigma} \oplus D_{s_1\Lambda_\sigma}^-$. Note that the nonzero cohomology of J^- is $H^{1,0}$ and $H^{3,0}$, while the nonzero cohomology of $\pi_{\mu_\sigma}^-$ is $H^{2,0}$.

For $\pi = \pi_f \otimes (\otimes_\sigma \pi_\sigma)$ in the packet $\{\pi(\mu)\}$ we write $\langle \mu_v, \pi_v \rangle = 1$ if π_v is the nontempered $\pi_v^\times$ and $= -1$ if π_v is the cuspidal π_v^-, and we put $\langle \mu_f, \pi_f \rangle = \prod_{v<\infty} \langle \mu_v, \pi_v \rangle$. We give π_f the superscript $\times$ if $\langle \mu_f, \pi_f \rangle$ is 1, and the superscript $-$ if $\langle \mu_f, \pi_f \rangle$ is -1. We write $\{\pi_f\}^\times$ for the set of $\pi_f^\times$ and $\{\pi_f\}^-$ for the set of π_f^-, coming from the packet $\{\pi(\mu)\}$.

Then π_f can be completed to an irreducible π in the packet $\{\pi(\mu)\}$ on choosing components π_σ for the $\sigma : F \hookrightarrow \mathbb{R}$. Put $\langle \mu_\sigma, \pi_\sigma \rangle = 1$ if $\pi_\sigma = \pi_{\mu_\sigma}^\times$ and $= -1$ if $\pi_\sigma = \pi_{\mu_\sigma}^-$. Then π is in the discrete spectrum precisely when

$$m(\mu, \pi) = \frac{1}{2}\left[1 + \varepsilon(\mu', \kappa)\langle \mu_f, \pi_f \rangle \prod_\sigma \langle \mu_\sigma, \pi_\sigma \rangle\right]$$

is 1. It is cuspidal with the possible exception of $\otimes_v \pi_v^\times$, which is sometimes residual.

Our π occurs in $I(G,4)$. We fix a correspondence f^p which projects to the packet $\{\pi_f^p\}$. Since the function f_H^p is chosen to be matching f^p, by [F3;VIII] the contribution to the part $I(H,3)$ of the stable trace formula of H is precisely that parametrized by the one-dimensional representation μ of $\mathrm{U}(2,\mathbb{A}) \times \mathrm{U}(1,\mathbb{A})$.

The contribution of $\{\pi\}$ to $I(G,4)$ is

$$\frac{\varepsilon(\mu', \kappa)}{2} \prod_{\sigma \in S} [\operatorname{tr} \pi_\sigma^\times(f_{G,\sigma}) - \operatorname{tr} \pi_\sigma^-(f_{G,\sigma})] \cdot [\operatorname{tr}\{\pi_f\}^\times(f^p) - \operatorname{tr}\{\pi_f\}^-(f^p)]$$

$$\cdot q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}} \cdot \prod_{u|p}\left[(\mu_u q_u^{1/2})^{\frac{jn_\wp}{j_u}} + \rho_u^{\frac{jn_\wp}{j_u}} + (\mu_u q_u^{-1/2})^{\frac{jn_\wp}{j_u}}\right]^{j_u}.$$

Here and below f^p indicates — as suitable — its product with the unit element of the $G'(\mathbb{Z}_p)$-Hecke algebra of $G'(\mathbb{Q}_p)$. Here we used the fact that the eigenvalues of $\mu_u \times \rho_u$ are $(\mu_u q_u^{1/2}, \mu_u q_u^{-1/2}, \rho_u)$ at $u|p$ which splits in E. The Langlands class at $u|p$ where E_u is a field is $\operatorname{diag}(\mu_u q_{E_u}^{1/2}, \rho_u, 1) \times \operatorname{Fr}_u$. The $\mu_u = \mu_u(\boldsymbol{\pi}_u)$ and $\rho_u = \rho_u(\boldsymbol{\pi}_u)$ are algebraic whose conjugates have complex absolute value 1.

The corresponding contribution to $I(H,3)$ is

$$\frac{1}{2}\prod_{\sigma\in S} \operatorname{tr}\mu_\sigma(f_{H,\sigma}) \cdot \operatorname{tr}\mu_f(f_H^p) \cdot q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}}$$

$$\cdot \prod_{u|p}\left[(-1)^{\frac{n_u}{j_u}}(\mu_u q_u^{1/2})^{\frac{jn_\wp}{j_u}} + \rho_u^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}}(\mu_u q_u^{-1/2})^{\frac{jn_\wp}{j_u}}\right]^{j_u}.$$

By choice of f_H^p we have that $\operatorname{tr}\mu_f(f_H^p) = \operatorname{tr}\{\pi_f\}^\times(f^p) + \operatorname{tr}\{\pi_f\}^-(f^p)$.

The choice of $f_{G,\sigma}$ is such that $\operatorname{tr}\pi_\sigma^\times(f_{G,\sigma}) = -\frac{2}{3}$, $\operatorname{tr}\pi_\sigma^-(f_{G,\sigma}) = \frac{1}{3}$; $\operatorname{tr}\mu_\sigma(f_{H,\sigma}) = -1$.

We conclude that for each irreducible π_f under consideration, if the $\pi_f^{K_f}$-isotypic part of H_c^* is $\pi_f^{K_f} \otimes H_c^*(\pi_f)$, then $\operatorname{Fr}_\wp^j$ acts on $H_c^*(\pi_f)$ with trace

$$\frac{(-1)^{[F:\mathbb{Q}]}}{2} q_\wp^{\frac{j}{2}\dim \mathcal{S}_{K_f}} \Bigg(\varepsilon(\mu',\kappa)\prod_{u|p}\left[(\mu_u q_u^{1/2})^{\frac{jn_\wp}{j_u}} + \rho_u^{\frac{jn_\wp}{j_u}} + (\mu_u q_u^{-1/2})^{\frac{jn_\wp}{j_u}}\right]^{j_u}$$

$$+ \langle \mu_f, \pi_f\rangle \prod_{u|p}\left[(-1)^{\frac{n_u}{j_u}}(\mu_u q_u^{1/2})^{\frac{jn_\wp}{j_u}} + \rho_u^{\frac{jn_\wp}{j_u}} + (-1)^{\frac{n_u}{j_u}}(\mu_u q_u^{-1/2})^{\frac{jn_\wp}{j_u}}\right]^{j_u}\Bigg).$$

Let us describe also the contribution of π_f to IH^*. By the "Matsushima-Murakami" formula of section I.2 each cuspidal π with $m(\mu,\pi) = 1$ contributes a subspace to $IH^*(\pi_f)$ of dimension 2 to the power $\#\{\sigma; \pi_\sigma = \pi_\sigma^\times\}$ (note that $\{\sigma : F \hookrightarrow \mathbb{R}\}$ is regarded here as an ordered set). Note that if $\pi^\times = \otimes_v \pi_v^\times$ is residual (in particular it has no cuspidal component π_v^-), it should contribute to $IH^*(\mathcal{S}'_{K_f} \otimes_{\mathbb{E}} \overline{\mathbb{Q}}, \mathbb{V})$. It contributes to H_c^* a space of the same dimension by our computation of the eigenvalues.

For example, when $F = \mathbb{Q}$ and $\varepsilon(\mu',\kappa) = 1$, $\pi = \pi_f^\times \otimes \pi_\sigma^\times$ is in the discrete spectrum and $\dim IH^*(\pi_f) = 2$. In fact $IH^*(\pi_f^\times) = H^{0,1} \oplus H^{0,3}$ $(= \mathbb{C}^2)$ if μ_σ has $\pi_{\mu_\sigma}^\times = J_{s_2\Lambda_\sigma}^+$, and $IH^*(\pi_f^\times) = H^{1,0} \oplus H^{3,0}$ if $\pi_{\mu_\sigma}^\times = J_{s_1\Lambda_\sigma}^-$.

Further, $IH^*(\pi_f^-) = H^{0,2}$ $(= \mathbb{C})$ if $\pi_{\mu_\sigma}^- = D_{s_1\Lambda_\sigma}^-$ and $IH^*(\pi_f^-) = H^{2,0}$ $(= \mathbb{C})$ if $\pi_{\mu_\sigma}^- = D_{s_2\Lambda_\sigma}^+$ (the roles of π_f^- and $\pi_f^\times$ interchange if $\varepsilon(\mu', \kappa) = -1$).

However, in this nontempered case the Hecke eigenvalues μ_u, ρ_u are algebraic and their conjugates all lie in the unit circle in $\mathbb{C}$, simply by the theory for the group $\mathrm{U}(1, E/F)$.

Finally we deal with the case of a one-dimensional representation $\pi = \xi_G$, which occurs in $I(G, 1)$. We can choose f^p to factorize through a projection onto this one-dimensional representation $\pi = \xi_G$ such that $\pi_f^{K_f} \neq 0$. Note that the functions $f_{G',\infty} = \otimes_{\sigma \in S} f_{G,\sigma}$ satisfy $\operatorname{tr} \xi_{G,\sigma}(f_{G,\sigma}) = 1$.

The component at p of such π is unramified, and the trace of the action of $\mathrm{Fr}_\wp^j$ on the π_f-isotypic component of $H_c^*(\mathcal{S}_{K_f} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \mathbb{V}_\xi)$ is

$$q_\wp^{\frac{j}{2} \dim \mathcal{S}_{K_f}} \prod_{u|p} \left[(\xi_u q_u)^{\frac{j n_\wp}{j_u}} + (\xi_u)^{\frac{j n_\wp}{j_u}} + (\xi_u q_u^{-1})^{\frac{j n_\wp}{j_u}} \right]^{j_u}.$$

We conclude that the representation $H_c^*(\pi_f)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{E})$ on H_c^* attached to π_f is $3^{[F:\mathbb{Q}]}$-dimensional. Its restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{E}_\wp)$ is unramified. Its trace is equal to the trace of $\otimes_{u|p} \nu_u^{-1/2} r_u(\mathrm{Fr}_u^{n_\wp})$. Here $r_u(\mathrm{Fr}_u^{n_\wp})$ acts on the twisted tensor representation $(r_u, (\mathbb{C}^3)^{[F_u:\mathbb{Q}_p]})$ as $(\mathbf{t}(\xi_u) \times \mathrm{Fr}_u)^{n_\wp}$, $\mathbf{t}(\xi_u) = (t_1, \dots, t_{n_u})$, t_m diagonal with

$$t(\xi_u) = \prod_{1 \leq m \leq n_u} t_m = \operatorname{diag}(\xi_u q_u, \xi_u, \xi_u q_u^{-1}).$$

The contribution to IH^* is as follows. The infinitesimal character of π_σ is $(0, 0, 0)$ for all $\sigma \in S$. The space $H^{ij}(\mathrm{u}(3, \mathbb{C}/\mathbb{R}), \mathrm{SU}(3); \mathbb{C})$ is $\mathbb{C}$ for $(i, j) = (0, 0)$, $(1, 1)$, $(2, 2)$ and $\{0\}$ otherwise. By the "Matsushima-Murakami" formula of section I.2 we have that $\dim IH^*(\pi_f) = 3^{[F:\mathbb{Q}]}$, in fact $IH^*(\pi_f) = \otimes_\sigma (H^{0,0} \oplus H^{1,1} \oplus H^{2,2})$. Moreover, $\pi = \xi_H$ contributes only to the (even) part

$$\bigoplus_{0 \leq m \leq \dim \mathcal{S}_{K_f}} IH^{2m}(\mathcal{S}_{K_f} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \mathbf{1}).$$

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AUTOMORPHIC REPRESENTATIONS OF LOW RANK GROUPS

by Yuval Z. Flicker (The Ohio State University, USA)

The area of automorphic representations is a natural continuation of the 19th and 20th centuries studies in number theory and modular forms. A guiding principle is a reciprocity law relating the infinite-dimensional automorphic representations, with finite-dimensional Galois representations. Simple relations on the Galois side reflect deep relations on the automorphic side, called "liftings". This monograph concentrates on two initial examples: the symmetric square lifting from SL(2) to PGL(3), reflecting the three-dimensional representation of PGL(2) in SL(3); basechange from the unitary group $U(3, E/F)$ to $GL(3, E)$, $[E : F] = 2$.

- It develops the technique of comparison of twisted and stabilized trace formulae. All aspects of the technique are discussed in an elementary way.
- The "Fundamental Lemma", on orbital integrals of spherical functions.
- Comparison of trace formulae is simplified by usage of "regular" functions.
- The "lifting" is stated and proved by means of character relations.

This permits an intrinsic definition of partition of the automorphic representations of SL(2) into packets, and a definition of packets for U(3), a proof of multiplicity one theorem and rigidity theorem for SL(2) and for U(3), a determination of the self-contragredient representations of PGL(3) and those on $GL(3, E)$ fixed by transpose-inverse-bar. In particular, multiplicity one theorem is new and recent.

- Applications to construction of Galois representations by explicit decomposition of the cohomology of Shimura varieties of U(3) using Deligne's (proven) conjecture on the fixed point formula.

This research monograph will benefit an audience of graduate students and researchers in number theory, algebra and representation theory.