REGULAR TRACE FORMULA AND BASE CHANGE LIFTING

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Introduction. The Selberg trace formula has become a major tool in the study of automorphic forms on reductive groups. Although its underlying principle, of computing traces of representations by means of orbital integrals, is very simple, the standard expressions for this important formula are rather complicated; this makes applications hard to accomplish. The complexity of the expression for the formula may be due to the choice of truncation made in its proof. It would be advantageous to have a simple expression for the formula, at least for a set of test functions which is sufficiently large for applications. The possibility of its existence was suggested to us by some of Kazhdan's striking work on the trace formula (see, e.g., the density theorem of [K1; Appendix], or the study of lifting in [K2]). Here we derive an asymptotic expression of this nature, in the simplest case of GL(2). For test functions with a component which is sufficiently regular with respect to all other components we obtain a simple, practical form of the trace formula.

More precisely, if F_u is a nonarchimedean local field and $m \geq 1$ is an integer, we say that a locally constant function f_u on $G_u = \mathrm{GL}(2, F_u)$, which is supported on a compact-mod-center, is *m-regular* if it vanishes outside the open closed subset $S_m = \left\{zg^{-1}\begin{pmatrix}\alpha\pi^{-m} & 0\\ 0 & 1\end{pmatrix}g;\ g \text{ in } G_u;\ a,\ z \text{ in } F_u^\times \text{ with } |a| = 1\right\}$ (π denotes a uniformizer in $F_u^\times$) of G_u, and its normalized orbital integral $F(g, f_u) = \Delta(g)\Phi(g, f_u)$ is the characteristic function of S_m in G_u. If F is a global field, u is a nonarchimedean place of F, and $f^u = \otimes_{v \neq u} f_v$ is a product over all places $v \neq u$ of F of smooth compactly supported mod-center functions f_v on G_v, such that f_v is the unit element f_v^0 of the Hecke algebra for almost all v, then we show that there exists $m_0 = m_0(f^u)$ such that for any $m \geq m_0$ and m-regular function $f_u = f_u^{(m)}$, the "regular" test function $f = f_u^{(m)} \otimes f^u$ vanishes on the conjugacy classes in

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Manuscript received 30 March 1987.

*Partially supported by an NSF grant.

American Journal of Mathematics 110 (1988), 739-764.

the adele group $G(\mathbf{A})$ of all elements γ in $G(F)$ whose eigenvalues lie in $F^{\times}$. The trace formula for such regular test functions $f = f_u^{(m)} \otimes f^u$, where the component $f_u^{(m)}$ is sufficiently regular with respect to the other components f_v, $v \neq u$, is called the *"regular trace formula."* The advantage of this regular trace formula is that the only orbital integrals which appear in it are those of regular elliptic conjugacy classes. For our regular test functions the weighted orbital integrals and those of the singular conjugacy classes are equal to zero.

To test the usefulness of the regular trace formula to lifting problems we use it here to prove the well-known cyclic base-change lifting theory for GL(2); the statement of the main theorem of this theory is given in Section II. We compute in Section III a twisted trace formula, as introduced by H. Saito [Sa], for our regular functions, and state the lifting theorem by means of twisted character relations due to T. Shintani [Sh]. However the usage of the regular trace formula—in which we optimize the choice of a test function—permits an easy comparison of the trace formulae, bypassing the "analytic difficulties" which could be created on working with the usual form of these formulae. Moreover, we show that the usage of regular functions, whose support and orbital integrals are easy to control by definition, eliminates the need to compare orbital integrals of spherical functions (this is done in [Sph] for a general group) other than the unit element of the Hecke algebra (this case is due to Kottwitz [Ko]), at least in our case. A special feature of the theory of base change for $\mathrm{GL}(n)$ is that the two groups under comparison admit a-priori rigidity and multiplicity one theorems (see Corollary 6′ here which follows at once from Jacquet-Shalika [JS], Proposition 3.6); consequently here it suffices to use only the most elementary properties of characters and orbital integrals, namely their smooth behaviour on the regular set. In all other cases this is not a-priori available, and additional arguments have to be supplied (see [FK] or [Rig]).

At any rate it is clear that our comparison technique applies in any base-change comparison for groups of rank (and twisted rank) one, since there exists a place which splits in the cyclic extension E/F under consideration, including the comparisons needed to establish the liftings from $U(2, E/F)$ to $\mathrm{GL}(2, E)$ (see [$U(2)$]) and from $U(3, E/F)$ to $\mathrm{GL}(3, E)$ (see [$U(3)$]). In fact our technique applies in the "analytically" more difficult comparisons of the metaplectic correspondence (see [M] and [FK]) and of the symmetric square (see [Sym], where the technique of the present paper is applied). This technique can also be used to simplify the study of the

Drinfeld and the Shimura moduli schemes (see [FK1]), as well as the study of the relative trace formula (see [RTF]). Our approach is likely to generalize to deal with reductive groups of arbitrary rank, but this we do not treat here. For an alternative approach see Langlands [L], who computed the asymptotic behaviour of weighted orbital integral, [GL(3)] and [$U(2)$] who introduced a correction argument to regularize these integrals, and Arthur-Clozel [AC] in the generality of GL(n).

In addition to being inspired by Kazhdan's intrinsic approach, this paper greatly benefitted from his comments.

1. Regular trace formula. Let F be a local field, G the group GL$(2, F)$, Z the center of G, Z_0 a closed subgroup of finite index in Z, ω a unitary character of Z_0. Let f be a complex-valued, compactly-supported modulo Z_0, smooth (that is, locally constant if F is nonarchimedean) function on G with $f(zg) = \omega(z)^{-1}f(g)$ (z in Z_0, g in G). Put $G' = G/Z_0$, and fix a Haar measure dg on G'. An element γ of G is called *regular* if its eigenvalues are distinct. Denote by $Z(\gamma)$ the centralizer of γ in G; $Z(\gamma)$ is a torus if γ is regular. Fix a Haar measure d_γ on $Z(\gamma)/Z_0$ such that if $Z(\gamma)$ is isomorphic to $Z(\gamma')$ then $d_\gamma = d_{\gamma'}$. Write $\Phi(\gamma, fdg)$, or $\Phi(\gamma, f)$ when dg is fixed, for the *orbital integral* $\int_{G/Z(\gamma)} f(g\gamma g^{-1})dg/d_\gamma$ of f at γ. Put $\Delta(\gamma) = |(a - b)^2/ab|$ if γ is regular with eigenvalues a and b. Put $F(\gamma, f) = \Delta(\gamma)\Phi(\gamma, f)$ if γ is regular. Let π be an admissible G-module in a complex space with $\pi(zg) = \omega(z)\pi(g)$ (z in Z_0). The convolution operator $\pi(fdg) = \int_{G'} \pi(g)f(g)dg$ has finite rank; denote its trace by tr $\pi(fdg)$, or by tr $\pi(f)$ if dg is fixed. The distribution $f \to$ tr $\pi(f)$ is locally constant on the regular set of G, namely there is a complex-valued smooth function χ ($= \chi_\pi$ or $\chi(\pi)$) on the regular set of G, called the *character* of π, which satisfies $\chi(zg) = \omega(z)\chi(g)$ and tr $\pi(fdg) = \int_{G'} \chi(g)f(g)dg$ for every f which is supported on the regular set. This χ extends to a locally integrable function on G, but we do not use this fact in the present work. The character χ of an irreducible π determines its equivalence class; $\chi(\gamma)$ depends only on the conjugacy class of γ; χ is independent of the choice of Haar measures used in its definition. A distribution D on the space of the functions f is called *invariant* if it attains the same value at f and at f^x (where $f^x(g) = f(x^{-1}gx)$) for every x in G, equivalently if $D(f * f' - f' * f) = 0$ for every function f. The distributions $f \to \Phi(\gamma, f)$, $f \to$ tr $\pi(f)$, are invariant.

Let F be a local nonarchimedean field, R its ring of integers, π a local uniformizer in R, $\mathbf{q} = \pi^{-1}$, q the cardinality of the residue field $R/(\pi)$, and $|\cdot|$ the valuation on F normalized to have $|\pi| = (|\mathbf{q}|^{-1} =)\, q^{-1}$. $R^\times$ denotes the group of units in R. Let m be a positive integer. Suppose that $Z_0 = Z$, and that ω is *unramified*, namely trivial on $Z(R)$.

Definition. A function f is called *m-regular* if it is supported on the open closed set

$$S_m = \left\{ zg^{-1}\begin{pmatrix} u\mathbf{q}^m & 0 \\ 0 & 1 \end{pmatrix} g;\ z \text{ in } Z,\ g \text{ in } G,\ u \text{ in } R^\times \right\}$$

in G, and the normalized orbital integral $F\left(\begin{pmatrix} u\mathbf{q}^m & 0 \\ 0 & 1 \end{pmatrix}, f\right)$ attains the value one for any u in $R^\times$. A *regular* function is a finite linear combination of m-regular functions ($m \geq 1$). To emphasize that a function f is m-regular we denote it by $f^{(m)}$. Thus $f^{(m)}$ is not uniquely determined, but its orbital integrals are, and it vanishes outside S_m.

Let μ, μ' be characters of $F^\times$. Denote by $I(\mu, \mu')$ the G-module normalizedly induced from the character $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \to \mu(a)\mu'(c)$ of the upper triangular subgroup of G. It is irreducible unless μ/μ' or μ'/μ is equal to ν, where $\nu(x) = |x|$. The composition series of $I(\mu\nu^{1/2}, \mu\nu^{-1/2})$ (and $I(\mu\nu^{-1/2}, \mu\nu^{1/2})$) has length two. It consists of the one-dimensional G-module $\pi(\mu)$, defined by $(\pi(\mu))(g) = \mu(\det g)$, and the Steinberg G-module $st(\mu)$. An irreducible G-module is called *supercuspidal* if it is inequivalent to any $I(\mu, \mu')$, $\pi(\mu)$, $st(\mu)$.

PROPOSITION 1. *If π is irreducible, $f^{(m)}$ is an m-regular function, and* $\operatorname{tr} \pi(f^{(m)}) \neq 0$, *then π is of the form $I(\mu, \mu')$, $\pi(\mu)$ or $st(\mu)$ with unramified μ (and μ'). Put $z = \mu(\mathbf{q})$, $z' = \mu'(\mathbf{q})$, when μ, μ' are unramified. Then*

$$\operatorname{tr}(I(\mu, \mu'))(f^{(m)}) = z^m + z'^m,\ \operatorname{tr}(\pi(\mu))(f^{(m)}) = (q^{1/2}z)^m,$$

$$\operatorname{tr}(st(\mu))(f^{(m)}) = (q^{-1/2}z)^m.$$

Proof. The character of a supercuspidal π is zero on the support of $f^{(m)}$, by virtue of the theorem of [*D*]. The characters of $I(\mu, \mu')$, $\pi(\mu)$, $st(\mu)$ are easy to compute.

Let F be a local nonarchimedean field. Put $K = G(R)$. Suppose that $Z_0 = Z$ and that ω is unramified. Denote by f^0 the unit element of the convolution algebra of K-biinvariant functions f on G. Thus f^0 is supported on ZK, and it is constant on K.

The following notations will be fixed for the rest of this section. Let F

be a global field, $\mathbf{A}$ its ring of adeles, F_v its completion at the place v. Objects defined above with respect to the local field F_v will be given a subscript v; e.g., $G_v = G(F_v)$, Z_v, ω_v, f_v, R_v, K_v, π_v, q_v, etc. Put $Z_0 = \Pi_v Z_{0v}$, assume that $Z_0 Z(F)$ is of finite index e in $Z(\mathbf{A})$, and that Z_{0v} contains $Z(R_v)$ for almost all v. Fix a unitary character ω of $Z_0 Z(F)$ which is trivial on $Z(F)$. Denote the component of ω at v by ω_v. Fix a nonarchimedean place u of F such that $Z_{0u} = Z_u$ and ω_u is unramified. For simplicity replace ω by its product with a global unramified character to assume that $\omega_u = 1$. Fix a function f_v for all $v \neq u$ such that $f_v = f_v^0$ for almost all v.

PROPOSITION 2. *For every sequence $\{f_v;\ v \neq u\}$ with $f_v = f_v^0$ for almost all v, there exists a positive integer m_0, such that if $f_u = f_u^{(m)}$ is an m-regular function on G_u with $m \geq m_0$, and $f = \otimes_v f_v$, then $f(x) = 0$ for any element x in $G(\mathbf{A})$ with eigenvalues in $F^\times$.*

Proof. Denote the eigenvalues of x by a' and a''. If $f(x) \neq 0$ then $f_v(x) \neq 0$ for all v, and there are $C_v \geq 1$ with $C_v = 1$ for almost all v such that

$$(*)_v \qquad C_v^{-1} \leq |a'/a''|_v \leq C_v$$

holds for all $v \neq u$. Since $a = a'/a''$ lies in $F^\times$ we have $\Pi_v |a_v| = 1$. Hence $(*)_u$ holds with $C_u = \Pi_{v \neq u} C_v$. But since $f_u = f_u^{(m)}$ is m-regular and $f_u(x) \neq 0$, we have $|a|_u = q_u^m$ (or q_u^{-m}). The choice of m_0 with $q_u^{m_0} > C_u$ establishes the proposition.

Remark. Let $\mathbf{A}^u$ denote the ring of adeles without component at u. It is clear that $m_0(\{f_v^{g_v};\ v \neq u\}) = m_0(\{f_v;\ v \neq u\})$ for every $g = (g_v)$ in $G(\mathbf{A}^u)$.

Definition. Put $f^u = \otimes_{v \neq u} f_v$. The function $f = f^u \otimes f_u$ on $G(\mathbf{A})$ is called *m-regular* if $m \geq m_0(f^u)$ and f_u is m-regular. It is called *regular* if it lies in the span of the set of m-regular functions.

Let $L(F, \omega)$ be the span of the set of smooth complex-valued functions ψ on $G(F)\backslash G(\mathbf{A})$ with $\psi(zg) = \omega(x)\psi(g)$ (z in $Z_0 Z(F)$) which are eigenfunctions of the Hecke operators for almost all v (equivalently, for a sufficiently large v; see [Av]). By [Av], ψ is slowly increasing on $Z_0 G(F)\backslash G(\mathbf{A})$. $G(\mathbf{A})$ acts on $L(F, \omega)$ by $(r(g)\psi)(h) = \psi(hg)$. The convolution operator $r(fdg) = \int_{G(\mathbf{A})/Z_0} f(g) r(g) dg$ on $L(F, \omega)$ is an integral operator with kernel $\Sigma_\gamma f(g^{-1}\gamma h)$ (γ in $G(F)/Z(F)$); dg is a product Haar measure on $G(\mathbf{A})$. An element γ of $G(F)$ is called *elliptic regular* if $Z(\gamma, \mathbf{A})/Z(\gamma, F)Z_0$ is compact, where $Z(\gamma)$ is the centralizer of γ in G. For any regular function f, we denote the integral of the kernel over the diagonal $g = h$ in $Z_0G(F)\backslash G(\mathbf{A})$ by tr $r(fdg)$; it is equal to

$$\text{(1)} \qquad \operatorname{tr} r(fdg) = e \sum_{\{\gamma\}} |Z(\mathbf{A})Z(\gamma, F)\backslash Z(\gamma, \mathbf{A})| \Phi(\gamma, fdg).$$

The sum ranges over all conjugacy classes $\{\gamma\}$ of elliptic regular elements in $G(F)/Z(F)$; it is finite for each f. To see this, consider the characteristic polynomial of γ, and recall that a discrete subset of a compact set is finite. The index e of $Z_0Z(F)$ in $Z(\mathbf{A})$ appears in (1) since

$$e|Z(\mathbf{A})Z(\gamma, F)\backslash Z(\gamma, \mathbf{A})| = |Z_0Z(\gamma, F)\backslash Z(\gamma, \mathbf{A})|.$$

Our next aim is to define distributions (2), (3), (4) and $(5)_w$ (for each place w of F) in $f = f^u \otimes f_u$, and prove that for every regular function $f = f^u \otimes f_u$ we have (1) = (2) + (3) + (4) + Σ_w $(5)_w$. Moreover, $(5)_w$ is zero for every place w where f_w is spherical. The identity (1) = (2) + (3) + (4) + Σ_w $(5)_w$ for a regular function $f = f^u \otimes f_u$ will be called the *regular trace formula*.

Denote by tr $r_d(fdg)$ the sum

$$\text{(2)} \qquad \operatorname{tr} r_d(fdg) = \sum_{\pi} \operatorname{tr} \pi(fdg)$$

over all cuspidal and one-dimensional constituents of $L(G, \omega)$. It is absolutely convergent.

The Selberg trace formula is an expression for (2), recorded, e.g., on pp. 516/7 of [JL], consisting of several terms denoted in [JL] by (i), . . . , (viii). Since f is regular, Proposition 2 implies that the terms (i), (iv), (v) of [JL] are zero, and (ii) + (iii) of [JL] is our (1). In words, for a regular function the singular and weighted orbital integrals in the trace formula are zero. The term (vi) of [JL] is

$$\text{(3)} \qquad -\frac{1}{4} \sum_{\mu} \operatorname{tr} M(\eta) I(\eta, fdg).$$

The sum ranges over all characters μ of $F^\times\backslash\mathbf{A}^\times$ such that $\mu^2(z) = \omega(z)$ for z in Z_0; η is the character $\eta\left(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\right) = \mu(ab)$ of the diagonal subgroup $A(\mathbf{A})$ of $G(\mathbf{A})$. $M(\eta)$ is an intertwining operator from $I(\eta)$ to $I(\eta)$, easily evaluated to be -1.

The term (vii) of [JL] is an integral over the analytic manifold of characters η of $A(\mathbf{A})/A(F)$ with $\eta(z) = \omega(z)$ for z in Z_0. Each connected component is parametrized by $\eta = \eta_0 \nu^{is}$ (s in $\mathbf{R}$), where η_0 is a representative and $\nu\left(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\right) = |a/b|$. If $\{\eta_0\}$ denotes a set of representatives, the term (vii) of [JL] is

$$(4) \qquad \frac{1}{4\pi} \sum_{\{\eta_0\}} \int_{\mathbf{R}} \frac{m'(\eta)}{m(\eta)} \operatorname{tr} I(\eta, f) |ds|,$$

where $m(\eta) = L(1, \mu^{-1})/L(1, \mu)$ if $\mu(a) = \eta\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right)$.

The term (viii) of [JL] is the sum over all places w of F of the following integrals over the same manifold of characters η:

$$(5)_w \quad \frac{1}{4\pi} \sum_{\{\eta_0\}} \int_{\mathbf{R}} \operatorname{tr}[R^{-1}(\eta_w) R'(\eta_w) I(\eta_w, f_w)] \cdot \prod_{v \neq w} \operatorname{tr} I(\eta_v, f_v) \cdot |ds|.$$

PROPOSITION 3. *If f_w is spherical then $(5)_w$ is zero.*

Proof. Let $I^0(\eta_w)$ be the space of smooth complex-valued functions ψ on K_w with $\psi(ank) = \eta_w(a)\psi(k)$ (a in $A_w \cap K_w$, n in $N_w \cap K_w$, k in K_w). Restriction is an isomorphism from $I(\eta_w)$ to $I^0(\eta_w)$. If f_w is spherical and π_w is a G_w-module, then the image of the operator $\pi_w(f_w)$ consists of K_w-fixed vectors in the space of π_w. The space of K_w-fixed vectors in $I(\eta_w)$ is zero unless η_w is unramified, in which case it is one-dimensional and spanned by the function whose image in $I^0(\eta_w)$ is the characteristic function ψ^0 of K_w. Put $r = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $r\eta_w$ for the character $a \to \eta_w(rar^{-1})$ of A_w. Then $R(\eta_w)$ is an operator from $I^0(\eta_w)$ to $I^0(r\eta_w)$ normalized to act as the identity on the space of K_w-fixed vectors in $I^0(\eta_w)$ when η_w is unramified. In particular, the derivative $R'(\eta_w)$ is zero on the image of $I(\eta_w, f_w)$, when f_w is spherical, and the proposition follows.

In conclusion, we proved the following

THEOREM. *For every regular function $f = f^u \otimes f_u$ we have* (1) = (2) + (3) + (4) + Σ_w $(5)_w$, *and* $(5)_w$ *is zero for every place w where f_w is spherical.*

Definition. The identity (1) = (2) + (3) + (4) + Σ_w $(5)_w$ for a regular function $f = f^u \otimes f_u$ is called the *regular trace formula.*

Let $\{(z_i, z_i^{-1}); i \geq 0\}$ denote the countable set of unordered pairs of complex numbers such that there is π in (2) or $\pi = I(\eta)$ in (3) whose component π_u satisfies tr $\pi_u(f_u^{(m)} dg_u) = z_i^m + z_i^{-m}$ for all $m \geq m_0$. Write

$a_i = b_i + c_i$, where b_i (resp. c_i) is the sum of tr $\pi^u(f^u dg^u)$ (resp. $\frac{1}{4}$ tr $\pi^u(f^u dg^u)$) over all $\pi = \pi^u \otimes \pi_u$ in (2) (resp. $\pi = I(\eta)$ in (3)) whose component π_u satisfies tr $\pi_u(f_u^{(m)} dg_u) = z_i^m + z_i^{-m}$.

Let a_0' (resp. a_0'') be the sum of tr $\pi^u(f^u dg^u)$ over the π in (2) with tr $\pi_u(f_u^{(m)} dg_u)$ equals $q_u^{m/2}$ (resp. $q_u^{-m/2}$); thus π_u is trivial (resp. special). Since π_u is unitary we have that $|z_i| = 1$ or that z_i is real with $q_u^{-1/2} < |z_i| < q_u^{1/2}$. We have

LEMMA. *If $f = f^u \otimes f_u^{(m)}$ is m-regular then* (2) + (3) *is equal to*

$$(2') \qquad \sum_{i \geq 0} a_i(z_i^m + z_i^{-m}) + a_0' q_u^{m/2} + a_0'' q_u^{-m/2},$$

where the sum is absolutely convergent. Moreover, there exists an integrable function $d(z)$ on the unit circle $|z| = 1$ in $\mathbf{C}^\times$ such that (4) + $\Sigma_{w \neq u}$ (5)$_w$ *is equal to*

$$(6) \qquad \int_{|z|=1} d(z)(z^m + z^{-m})|dz|.$$

Proof. Put $z = q_u^{is}$. Then tr $I(\eta_u, f_u^{(m)})$ takes the form $z^m + z^{-m}$. Put $d(z) = d_4(z) + d_5(z)$, where $d_4(z)$ is the sum over all η_0 in (4) and all n in $\mathbf{Z}$, of tr $I(\eta^u, f^u dg^u) m'(\eta)/m(\eta)$, where $\eta = \eta_0 \nu^{is + 2\pi i n/\log q_u}$; $d_5(z)$ is the analogous contribution from $\Sigma_{w \neq u}$ (5)$_w$. Since (2), (3), (4) and Σ (5)$_w$ are absolutely convergent, the latter as double sum-integral, $\Sigma_i |a_i||z_i^m + z_i^{-m}|$ is finite for every m and $d(z)$ is integrable on $|z| = 1$.

II. Base-change lifting. Put $G = \mathrm{GL}(2)$. Let F_v be a local field. Let μ_{iF_v} $(i = 1, 2)$ be continuous homomorphisms from $F_v^\times$ to $\mathbf{C}^\times$. Put $\delta_{F_v}\left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right) = |a/c|_{F_v}$. Let $I((\mu_{iF_v})) = \mathrm{Ind}(\delta_{F_v}^{1/2}(\mu_{iF_v});\ P(F_v),\ G(F_v))$ be the $G(F_v)$-module unitarily induced from the character $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \to \mu_{1F_v}(a)\mu_{2F_v}(c)$ of the upper triangular subgroup $P(F_v)$ of $G(F_v)$. Let $J(F_v)$ be the free abelian group generated by the set of equivalence classes of the irreducible $G(F_v)$-modules. Let $[\pi_{F_v}]$ denote the image in $J(F_v)$ of a $G(F_v)$-module π_{F_v}. Let E_v/F_v be a cyclic extension of F_v of prime degree e. Fix a generator σ of the galois group $\mathrm{Gal}(E_v/F_v)$. Put $\sigma g = (\sigma g_{ij})$ if $g = (g_{ij})$, and ${}^\sigma\pi_{E_v}(g) = \pi_{E_v}(\sigma g)$. Write π_v for π_{F_v} and $\pi_{E,v}$ for π_{E_v}. An irreducible $G(E_v)$-module $\pi_{E,v}$ is called *F_v-invariant* if it is equivalent to ${}^\sigma\pi_{E,v}$. Denote by $J(E_v)^{F_v}$ the subgroup of $J(E_v)$ which is generated by the irreducible F_v-invariant $G(E_v)$-modules. Denote by $N = N_{E_v/F_v}$ the norm map from E_v to F_v. A character μ_v of $F_v^\times$ defines a one-dimensional representation $\pi(\mu_v)$ of

$G(F_v)$ by $(\pi(\mu_v))(g) = \mu_v(\det g)$. Denote by $J(F_v)/E_v$ the quotient of $J(F_v)$ by the equivalence relations $[\pi_v \otimes \epsilon] \equiv [\pi_v]$ and $[I(\mu_{1v}, \epsilon\mu_{2v})] \equiv [I(\mu_{1v}, \mu_{2v}]$ for every character ϵ of $F_v^\times/NE_v^\times$. The base-change lifting is a map from $J(F_v)/E_v$ to $J(E_v)^{F_v}$, described below.

If $\pi(\mu_{E_v})$ is an F_v-invariant $G(E_v)$-module then there exists a character μ_v of $F_v^\times$ with $\mu_{E_v}(x) = \mu_v(Nx)$; although μ_v is not unique, the image of $\pi(\mu_v)$ in $J(F_v)/E_v$ is uniquely determined. If $I((\mu_{iE_v}))$ is F_v-invariant then $\mu_{2E_v}(x) = \mu_{1E_v}(\sigma x)$ and $e = 2$, or there exist characters μ_{iv} of $F_v^\times$ with $\mu_{iE_v}(x) = \mu_{iv}(Nx)$; in the latter case $I((\mu_{iv}))$ is uniquely determined in $J(F_v)/E_v$. Put $\nu_v(x) = |x|_v$. If F_v is nonarchimedean then the $G(F_v)$-module $I((\mu_{iv}))$ is irreducible unless $\mu_{1v}(x)/\mu_{2v}(x)$ is $\nu_v(x)$ or $\nu_v(x)^{-1}$. For any local F_v, the composition series of $\mu_v \otimes I(\nu_v^{1/2}, \nu_v^{-1/2})$ is of length two. It consists of $\pi(\mu_v)$ and of a square-integrable (Steinberg) subquotient $st(\mu_v)$. A *supercuspidal* $G(F_v)$-module is an irreducible $G(F_v)$-module inequivalent to any $I((\mu_{iv}))$, $\pi(\mu_v)$, $st(\mu_v)$. Denote by $J_I(F_v)$ the subgroup of $J(F_v)$ generated by the $I((\mu_{iv}))$, $\pi(\mu_v)$ and $st(\mu_v)$. $J(F_v)$ is the direct sum of $J_I(F_v)$ and the subgroup generated by the supercuspidal $G(F_v)$-modules. $J_I(F_v)/E_v$ is the image of $J_I(F_v)$ in $J(F_v)/E_v$. Denote by $J_I(E_v)^{F_v}$ the direct summand of $J(E_v)^{F_v}$ generated by the $I((\mu_{iv} \circ N))$, $\pi(\mu_v \circ N)$ and $st(\mu_v \circ N)$. The map $L_{I,v}$ from $J_I(F_v)/E_v$ to $J_I(E_v)^{F_v}$ defined by $L_{I,v}(I((\mu_{iv}))) = I((\mu_{iv} \circ N))$, $L_{I,v}(\pi(\mu_v)) = \pi((\mu_v) \circ N))$ and $L_{I,v}(st(\mu_v)) = st(\mu_v \circ N)$, is an isomorphism, called *lifting*. Thus $\pi_{E,v} = L_{I,v}(\pi_v)$ is the lift of π_v, π_v lifts to $\pi_{E,v}$, $\pi_{E,v}$ descends to π_v, π_v is the descent of $\pi_{E,v}$.

We shall now extend the definition of local lifting from $J_I(F_v)/E_v$ to $J(F_v)/E_v$. Let $NZ(E_v)$ be the image by N of the center $Z(E_v)$ of $G(E_v)$; $NZ(E_v)$ is isomorphic to $NE_v^\times$, since $Z(E_v) \simeq E_v^\times$. Fix a unitary character ω_v of $NZ(E_v)$. By f_v we always denote a complex-valued smooth (that is, locally constant if F_v is nonarchimedean) function on $G(F_v)$ with $f_v(zg) = \omega_v(z)^{-1}f_v(g)$ (z in $NZ(E_v)$, g in $G(F_v)$), which is compactly-supported modulo $NZ(E_v)$. Put $G'(F_v) = G(F_v)/NZ(E_v)$. Fix a Haar measure dg on $G'(F_v)$. For any irreducible $G(F_v)$-module π_v with $\pi_v(zg) = \omega_v(z)\pi_v(g)$ put $\pi_v(f_v) = \int_{G'(F_v)} \pi_v(g)f_v(g)dg$. It is an operator of finite rank. Denote its trace by tr $\pi_v(f_v)$. For every γ in $G(F_v)$ fix a Haar measure d_γ on the quotient by $NZ(E_v)$ of the centralizer $G_\gamma(F_v)$, such that if $G_{\gamma'}(F_v)$ is isomorphic to $G_\gamma(F_v)$, then they are assigned the same measure. Write $\Phi(\gamma, f_v)$ for the orbital integral $\int_{G(F_v)/G_\gamma(F_v)} f_v(g\gamma g^{-1})dg/d_\gamma$ of f_v at γ. An element γ is called *regular* if its eigenvalues a, b are distinct; in this case put $\Delta_v(\gamma) = |(a-b)^2/ab|_v^{1/2}$, and $F(\gamma, f_v) = \Delta_v(\gamma)\Phi(\gamma, f_v)$.

Denote by ω_v' the character of $Z(E_v)$ defined by $\omega_v'(z) = \omega_v(Nz)$. Denote by ϕ_v a smooth function on $G(E_v)$ with $\phi_v(zg) = \omega_v'(z)^{-1}\phi_v(g)$ (z in $Z(E_v)$) which is compactly supported modulo $Z(E_v)$. Fix a Haar measure dg on $G'(E_v) = G(E_v)/Z(E_v)$. For any δ in $G(E_v)$ the conjugacy class of $N'\delta = \delta\sigma(\delta)\cdots\sigma^{e-1}(\delta)$ is defined over F_v (since $\sigma(N'\delta) = \delta^{-1}\cdot N'\delta\cdot\delta$), hence contains an element $N\delta$ in $G(F_v)$. The elements δ, δ' are called *σ-conjugate* if $\delta' = g\delta\sigma(g^{-1})$ for some g in $G(E_v)$. The map $\delta \to N\delta$ induces an injection from the set of σ-conjugacy classes in $G(E_v)$ into the set of conjugacy classes in $G(F_v)$. The σ-centralizer $G_\delta^\sigma(E_v)$ of δ in $G(E_v)$ consists of the g with $\delta = g\delta\sigma(g^{-1})$. It is an inner form of $G_{N\delta}(F_v)$, and we choose a Haar measure on $G_\delta^\sigma(E_v)/Z(F_v)$ corresponding to the one fixed above on $G_{N\delta}(F_v)/Z(F_v)$. Since the twisted orbital integral

$$\int \phi_v(g\delta\sigma(g^{-1}))dg \qquad (g \text{ in } G(E_v)/Z(E_v)G_\delta^\sigma(E_v))$$

of ϕ_v at δ depends only on the σ-conjugacy class of δ, it is denoted by $\Phi(N\delta, \phi_v)$. We also write $F(N\delta, \phi_v)$ for $\Delta_v(N\delta)\Phi(N\delta, \phi_v)$.

If $\pi_{E,v}$ is F_v-invariant irreducible $G(E_v)$-module, then there exists an operator A on its space with $A\pi_{E,v}(g) = \pi_{E,v}(\sigma g)A$ for all g. Since $\pi_{E,v}$ is irreducible, A^e is a scalar (by Schur's lemma), which we normalize by $A^e = 1$. If $\pi_{E,v}$ is one-dimensional take $A = 1$. Otherwise $\pi_{E,v}$ can be realized in a space of Whittaker functions W on $G(E_v)$ with respect to an additive character $\psi = \psi' \circ N_{E_v/F_v}$ of E_v, where ψ' is a nontrivial character of F_v. Thus each W satisfies $W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right) = \psi(x)W(g)$ (x in E_v), and $\pi_{E,v}$ acts by $(\pi_{E,v}(g)W)(h) = W(hg)$. We normalize A by the requirement that $(AW)(g) = W(\sigma g)$. Put $\pi_{E,v}(\sigma) = A$, and

$$\pi_{E,v}(\phi_v \times \sigma) = \int \phi_v(g)\,\pi_{E,v}(g)\,\pi_{E,v}(\sigma)\,dg \qquad (g \text{ in } G'(E_v)).$$

Denote the trace of this operator by tr $\pi_{E,v}(\phi_v \times \sigma)$.

Definition. The functions ϕ_v, f_v are called *matching* if $\Phi(N\delta, \phi_v) = \Phi(N\delta, f_v)$ for every δ in $G(E_v)$ such that $N\delta$ is regular, and $\Phi(\gamma, f_v) = 0$ for every γ in $G(F_v)$ which is regular but not a norm.

Remark. It is easy to see that: for every ϕ_v such that $\Phi(\phi_v)$ is supported on the regular set, there exists a matching f_v; and: for every f_v with $\Phi(\gamma, f_v) = 0$ if γ is not regular or not a norm, there exists a matching ϕ_v.

Definition. (1) The irreducible $G(F_v)$-module π_v *lifts* to the irreducible $G(E_v)$-module $\pi_{E,v}$, and we write $\pi_{E,v} = L_v(\pi_v)$, if tr $\pi_{E,v}(\phi_v \times \sigma) =$ tr $\pi_v(f_v)$ for all matching f_v, ϕ_v.

(2) The *character* $\chi(\pi_v)$ of π_v is a complex-valued smooth function on the regular set of $G(F_v)$ with

$$(\chi(\pi_v))(zg) = \omega_v(z)(\chi(\pi_v))(g) \qquad \text{for } z \text{ in } NZ(E_v)$$

and

$$\operatorname{tr} \pi_v(f_v) = \int_{G'(E_v)} (\chi(\pi_v))(g) f_v(g)\, dg$$

for every f_v which is supported on the regular set.

(3) The (*twisted*) *character* $\chi(\pi_{E,v})$ of $\pi_{E,v}$ is a smooth function on the regular set of $G(F_v)$ with

$$(\chi(\pi_{E,v}))(zg) = \omega_v(z)(\chi(\pi_{E,v}))(g)$$

and

$$\operatorname{tr} \pi_{E,v}(\phi_v \times \sigma) = \int_{G'(E_v)} (\chi(\pi_{E,v}))(Ng)\, \phi_v(g)\, dg$$

for every ϕ_v which is supported on the set of g such that Ng is regular in $G(F_v)$.

Remark. (1) It is clear that if $\pi_{E,v} = L_v(\pi_v)$ then $[\pi_{E,v}]$ lies in $J(E_v)^{F_v}$ and it depends only on the image of $[\pi_v]$ in $J(F_v)/E_v$. (2) The characters $\chi(\pi_v)$ and $\chi(\pi_{E,v})$ always exist; they depend only on the conjugacy class of g, and they determine the equivalence class of π_v and $\pi_{E,v}$. They are independent of the choice of Haar measures used in their definition. It is known that $\chi(\pi_v)$ and $\chi(\pi_{E,v}) \circ N$ extend to locally integrable functions on $G(F_v)$, but we do not need this deeper fact in the present work. The distributions tr π_v and tr $\pi_{v,E}$ determine the equivalence class of π_v and $\pi_{E,v}$. (3) The characters of the elements of $J_I(F_v)$ and $J_I(E_v)$ are easy to compute, hence it follows from the Weyl integration formula that if π_v lies in $J_I(F_v)$ and $L_{I,v}(\pi_v) = \pi_{E,v}$, then $L_v(\pi_v) = \pi_{E,v}$, namely that L_v extends the lifting $L_{I,v}$.

If F_v is nonarchimedean, denote by R_v its ring of integers. A character μ_v of $F_v^\times$ is called unramified if it is trivial on $R_v^\times$. Put $K(F_v) = G(R_v)$. A $G(F_v)$-module π_v is called *unramified* if it has a non-zero $K(F_v)$-fixed vector. If $\mu_{E,v}$ is unramified, then $\mu_{E,v} = \mu_v \circ N$ for some unramified μ_v. If π_v

is irreducible, unramified but not one-dimensional, then there are unramified characters μ_{iv} such that $[\pi_v] = [I((\mu_{iv}))]$. Consequently an irreducible unramified $G(E_v)$-module is necessarily invariant; it is the lift of an unramified $G(F_v)$-module.

Let F be a global field and $\mathbf{A}$ its ring of adeles. Denote the completion of F at its place v by F_v. An irreducible $G(\mathbf{A})$-module π is the restricted direct product $\otimes_v \pi_v$ over all places v of F of irreducible $G_v = G(F_v)$-modules π_v; for almost all v the component π_v is unramified, hence lies in $J_I(F_v)$. Similarly, an irreducible $G(\mathbf{A}_E)$-module π_E is the product $\otimes_v \pi_{E,v}$ over all places v of F of irreducible $G(E_v)$-modules $\pi_{E,v}$. If v splits in E then $E_v = E \otimes_F F_v$ is the direct sum of e copies of F_v and $G(E_v) = G_v \times \cdots \times G_v$; $\pi_{E,v}$ is $\otimes_{i=1}^{e} \pi_{iv}$, where π_{iv} are G_v-modules. Gal(E/F) acts by permutation, and $\pi_{E,v}$ is F_v-invariant if and only if $\pi_{iv} = \pi_{1v}$ for all i. Since $NE_v = F_v$ we have $F_v^\times/NE_v^\times = 1$ and $J(F_v)/E_v = J(F_v)$. Lifting, defined by $L_v(\pi_v) = \pi_v \otimes \cdots \otimes \pi_v$, identifies $J(F_v)/E_v$ with $J(E_v)^{F_v}$.

Definition. An irreducible $G(\mathbf{A})$-module $\pi = \otimes \pi_v$ *(quasi)-lifts* to an irreducible $G(\mathbf{A}_E)$-module $\pi_E = \otimes_v \pi_{E,v}$ if π_v lifts to $\pi_{E,v}$ for (almost) all v. Write $L_q(\pi) = \pi_E$ if π quasi-lifts to π_E, and $L(\pi) = \pi_E$ if π lifts to π_E.

Remark. (1) If μ_{F_v} is a character of $\mathbf{A}^\times/F^\times$, then $\pi(\mu_F)$ lifts to $\pi(\mu_F \circ N)$. (2) Denote by ϵ a nontrivial character of $\mathbf{A}^\times$ which is trivial on $F^\times N\mathbf{A}_E^\times$. Then $\pi \otimes \epsilon$ *(quasi)-lifts* to π_E if and only if π does.

Let $L(F)$ be the span of the set of the smooth complex-valued functions ψ on $G(F)\backslash G(\mathbf{A})$ which are eigenfunctions of the Hecke operators for almost all v (see [Av]). Denote by U the group of unipotent upper triangular matrices in G. Let $L_0(F)$ be the space of ψ in $L(F)$ with $\int_{U(F)\backslash U(\mathbf{A})} \psi(ug)du = 0$ for all g in $G(\mathbf{A})$. Then $G(\mathbf{A})$ acts on $L(F)$, and on $L_0(F)$, by right translation. An irreducible constituent of $L(F)$ (resp. $L_0(F)$) is called an *automorphic* (resp. *cuspidal*) $G(\mathbf{A})$-module. An automorphic $G(\mathbf{A})$-module which is not cuspidal is a constituent of an induced $G(\mathbf{A})$-module of the form $I((\mu_i))$, where μ_i are characters of $\mathbf{A}^\times/F^\times$; it lifts to the corresponding constituent of $I((\mu_i \circ N))$. The main theorem of the theory of base-change is the following

Base Change Theorem. (1) *Lifting defines a bijection from the set of orbits under multiplication by ϵ of automorphic $G(\mathbf{A})$-modules, onto the set of F-invariant automorphic $G(\mathbf{A}_E)$-modules, i.e., $L\colon J(\mathbf{A})/E \xrightarrow{\sim} J(\mathbf{A}_E)^F$. $L(\pi)$ is cuspidal if and only if π is cuspidal and inequivalent to $\pi \otimes \epsilon$.*

(2) *If $[E:F] = 2$ then there is a bijection from the set of* Gal(E/F)-*orbits (unordered pairs) $\{\mu_E, \mu_E \circ \sigma\}$ of characters of $\mathbf{A}_E^\times/E^\times$ with $\mu_E \neq$*

$\mu_E \circ \sigma$, *to the set of cuspidal* $G(\mathbf{A})$*-modules* π with $\pi \otimes \epsilon \simeq \pi$; *it is defined by* $L(\pi) = I(\mu_E, \mu_E \circ \sigma)$. *If* $\mu_E = \mu_E \circ \sigma$ *then* $\mu_E = \mu \circ N$ *for some character* μ *of* $\mathbf{A}^\times/F^\times$ *which is uniquely determined up to multiplication by* ϵ, *and* $L(I(\mu, \epsilon^i\mu)) = I(\mu_E, \mu_E)$ $(i = 0,1)$.

(3) *For each cyclic extension* E_v/F_v *of local fields,* L_v *defines a bijection from* $J(F_v)/E_v$ *to* $J(E_v)^{F_v}$. $L(\pi_v)$ *is supercuspidal if and only if* π_v *is supercuspidal and inequivalent to* $\pi_v \otimes \epsilon_v$, *where* ϵ_v *is a nontrivial character of* $F_v^\times/NE_v^\times$.

(4) *If* $[E_v:F_v] = 2$ *then there is a bijection from the set of* $\mathrm{Gal}(E_v/F_v)$*-orbits* $\{\mu_{E,v}, \mu_{E,v} \circ \sigma\}$ *of characters of* $E_v^\times$ *with* $\mu_{E,v} \neq \mu_{E,v} \circ \sigma$, *to the set of* $G(F_v)$*-modules* π_v *with* $\pi_v \otimes \epsilon_v \simeq \pi_v$; *it is defined by* $L_v(\pi_v) = I(\mu_{E,v}, \mu_{E,v} \circ \sigma)$, *and* π_v *is supercuspidal.*

(5) *If a cuspidal* π *quasi-lifts to an automorphic* π_E *then* $L(\pi) = \pi_E$ *and* $L_v(\pi_v) = \pi_{E,v}$ *for all* v.

The main step in the proof is an identity of trace formulae. To state it, we first recall two local results. If F_v is nonarchimedean, denote by f_v^0 (resp. ϕ_v^0) the unit element of the convolution algebra of *spherical*, namely $K(F_v)$ (resp. $K(E_v)$)-biinvariant functions, with the usual properties of f_v (resp. ϕ_v). The first result is that f_v^0 and ϕ_v^0 are matching (see Kottwitz [Ko], or [Sph], Section 3). Moreover, it is shown in [Sph] that: if f_v and ϕ_v are spherical and *corresponding*, by which we mean that $\mathrm{tr}(L_{I,v}(\pi_v))(\phi_v \times \sigma) = \mathrm{tr}\, \pi_v(f_v)$ for all π_v in $J_I(F_v)$ (equivalently: for all unramified π_v in $J(F_v)$), then f_v and ϕ_v are matching; but we do not use this fact in this paper. The second local result is that if v is a place of F which splits in E, then $\phi_v = (f_{1,v}, \ldots, f_{e,v})$ matches $f_v = f_{1,v} * \cdots * f_{e,v}$; see [L], Section 8, or [GL(3)], Section 1.5, Lemma 13. The results of Section 1.5 in [GL(3)] are general and used in several places below. Yet their proof is elementary and self-contained, and independent of the rest of [GL(3)]. Hence the usage of these results does not complicate the present exposition.

Let F be a global field. Fix a character ω of $NZ(\mathbf{A}_E)/NZ(E) \simeq N\mathbf{A}_E^\times/NE^\times$; let ω' be the character of $Z(\mathbf{A}_E)/Z(E) \simeq \mathbf{A}_E^\times/E^\times$ defined by $\omega'(z) = \omega(Nz)$. The local components of ω, ω' are denoted by ω_v, ω_v'. Fix Haar measures on $G'(\mathbf{A}) = G(\mathbf{A})/NZ(\mathbf{A}_E)$ and $G'(\mathbf{A}_E) = G(\mathbf{A}_E)/Z(\mathbf{A}_E)$, for example by means of rational differential forms of highest degree, and in particular Haar measures on the local groups $G'(F_v)$, $G'(E_v)$. The main step in the proof of the base-change theorem is the following

AUXILIARY THEOREM. *Let* U *be a set of places of* F *which contains the archimedean places and those which ramify in* E. *For each* v *outside* U

fix an unramified $G(F_v)$-module π_v^0. Suppose that f_v and ϕ_v are matching for every v in U, and $f_v = f_v^0$ and $\phi_v = \phi_v^0$ for all but finitely many v in U. Then we have the identity

$$(7) \qquad e \sum_{\pi_E} \prod_v \operatorname{tr} \pi_{E,v}(\phi_v \times \sigma) + \sum_{\mu_E} \prod_v \operatorname{tr}(I(\mu_{E,v}, \mu_{E,v} \circ \sigma))(\phi_v \times \sigma) = \sum_{\pi} \prod_v \operatorname{tr} \pi_v(f_v).$$

The products range over all v in U. The first sum ranges over all cuspidal F-invariant $G(\mathbf{A}_E)$-modules π_E with $\pi_E(zg) = \omega'(z)\pi_E(g)$ (z in $Z(\mathbf{A}_E)$) with $\pi_{E,v} = L_{I,v}(\pi_v^0)$ for all v outside U, and the last is over all cuspidal $G(\mathbf{A})$-modules π with $\pi(zg) = \omega(z)\pi(g)$ (z in $NZ(\mathbf{A}_E)$) such that $L_{I,v}(\pi) = L_{I,v}(\pi_v^0)$ for all v outside U. The sum in the middle is zero if $e \neq 2$; if $e = 2$ it ranges over all unordered pairs $\{\mu_E, \mu_E \circ \sigma\}$ of characters μ_E of $\mathbf{A}_E^\times/E^\times$ with $\mu_E \neq \mu_E \circ \sigma$ and $I(\mu_{E,v}, \mu_{E,v} \circ \sigma) = L_{I,v}(\pi_v^0)$ for all v outside U.

III. Twisted trace formula. To prove the auxiliary theorem we use the regular trace formula for $G(\mathbf{A})$ of Section I, and the regular twisted (by σ) trace formula for $G(\mathbf{A}_E)$, which is introduced in this Section. Thus we fix a nonarchimedean place u of F, and fix (matching) functions f_v and ϕ_v for all $v \neq u$, such that $f_v = f_v^0$ and $\phi_v = \phi_v^0$ for almost all v. At the place u we choose matching regular functions. The notion of a regular function f_u is defined in Section I.

Recall that f_u is called *m-regular* if it is supported on the open closed set

$$S_m = \left\{ z g^{-1} \begin{pmatrix} x\mathbf{q}_u^m & 0 \\ 0 & 1 \end{pmatrix} g;\ z \text{ in } NZ(E_u),\ g \text{ in } G(F_u),\ x \text{ in } R_u^\times \right\} \text{ in } G(F_u),$$

and $F\left(\begin{pmatrix} x\mathbf{q}_u^m & 0 \\ 0 & x' \end{pmatrix}, f_u\right) = 1$ for any x, x' in $R_u^\times$. As usual, R_u denotes the ring of integers in F_u, π_u is a uniformizer, $\mathbf{q}_u = \pi_u^{-1}$. If ϕ_u matches f_u then its orbital integral is supported on S_m as a function on $G(F_u)$.

Definition. The function ϕ_u is called an *m-regular* function and is denoted by $\phi_u^{(m)}$, if it matches an m-regular function f_u on $G(F_u)$ and it is supported on the open closed set of δ in $G(E_u)$ with $N\delta$ in S_m.

We now take u to be a place of F which splits in E. Then $E_u = E \otimes_F F_u = F_u \oplus \cdots \oplus F_u$, and $G(E_u) = G(F_u) \oplus \cdots \oplus G(F_u)$ (sum of $e = [E:F]$ factors). Moreover, $N_{E_u/F_u}\colon E_u \to F_u$ is surjective, and $NZ(E_u) = Z(F_u)$. Let K_u' be an open compact subgroup of $G(F_u)$ such that f_u is K_u'-biinvariant; it can be shown, see [Sym], that K_u' can be taken to be an

Iwahori subgroup of $G(F_u)$. We take ϕ_u to be the function $\phi_u(g_1, \cdots, g_e) = f_u(g_1)f_{1u}(g_2) \cdots f_{1u}(g_e)$, where f_{1u} is a function on $G(F_u)$ which is supported on $Z(F_u)K'_u$, with $f_u = f_u * f_{1u}$. Lemma 13 of [GL(3); (1.5.2)] asserts that $\Phi(\gamma, \phi_u) = \Phi(\gamma, f_u)$ for every γ in $G(F_u)$. Moreover, if $\pi_{E,u}$ is an irreducible $G(E_u)$-module and tr $\pi_{E,u}(\phi_u \times \sigma) \neq 0$, then there is an irreducible $G(F_u)$-module π_u such that $\pi_{E,u} = \pi_u \otimes \cdots \otimes \pi_u$ and tr $\pi_{E,u}(\phi_u \times \sigma) =$ tr $\pi_u(f_u)$ (see [GL(3); (1.5.3)]). We deduce that Proposition 1 holds for our regular ϕ_u, and it is clear that the immediate twisted analogue of Proposition 2 holds as well. Namely, there exists a positive integer m_0, depending on $\{\phi_v; v \neq u\}$, such that if ϕ_u is an m-regular function with $m \geq m_0$, $\phi = \otimes\phi_v$ and x is an element in $G(\mathbf{A}_E)$ such that the eigenvalues of Nx lie in $F_{v'}^{\times}$, then $\phi(x) = 0$.

We are now ready to write out the twisted trace formula for $G(E)$ and a *regular* global function $\phi = \otimes_v \phi_v$, namely one whose component ϕ_u at u is in the space of the m-regular functions with $m \geq m_0 = m_0(\{\phi_v; v \neq u\})$. We shall call this the regular twisted trace formula.

Let $L(E, \omega')$ be the space of smooth complex-valued functions ψ on $G(E)\backslash G(\mathbf{A}_E)$ with $\psi(zg) = \omega'(z)\psi(g)$ (z in $Z(\mathbf{A}_E)$) such that ψ is an eigenvector of all Hecke operators for almost all v. Then (see [Av]), ψ is slowly increasing on $Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)$. $G(\mathbf{A}_E)$ acts on $L(E, \omega')$ by $(r_E(g)\psi)(h) = \psi(hg)$, and Gal$(E/F)$ acts by $(r_E(\sigma)\psi)(h) = \psi(\sigma h)$. The convolution operator

$$r_E(\phi \times \sigma) = \int \phi(g) r_E(g) r_E(\sigma) dg \qquad (g \text{ in } Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E))$$

on $L(E, \omega')$ is an integral operator with kernel

$$\sum_{\delta} f(x^{-1}\delta\sigma(y)) \qquad (\delta \text{ in } G(E)/Z(E)).$$

The integral of the kernel over the diagonal $x = y$ in $Z(\mathbf{A}_E)G(E)\backslash G(\mathbf{A}_E)$, which by abuse of notations we denote by tr $r_E(\phi \times \sigma)$ (although this is not the trace of $r_E(\phi \times \sigma)$), is easy to compute. Indeed, $\phi(x^{-1}\delta\sigma(x)) = 0$ unless $N\delta$ is elliptic regular, in which case

$$Z(\mathbf{A}_E)G_\delta^\sigma(E)\backslash Z(\mathbf{A}_E)G_\delta^\sigma(\mathbf{A}_E) \simeq Z(\mathbf{A}_F)G_\delta^\sigma(E)\backslash G_\delta^\sigma(\mathbf{A}_E)$$

is compact and isomorphic to $Z(\mathbf{A}_F)G_{N\delta}(F)\backslash G_{N\delta}(\mathbf{A}_F)$. A standard change of sums and integrals shows that

(1σ) $$\operatorname{tr} r_E(\phi\times\sigma) = \sum_{\{\gamma\}} |Z(\mathbf{A})G_\gamma(F)\backslash G_\gamma(\mathbf{A})|\,\Phi(\gamma, \phi).$$

The sum ranges over all twisted conjugacy classes $\{\delta\}$ of elements in $G(E)/Z(E)$ whose norm $\gamma = N\delta$ is elliptic regular, or equivalently over all conjugacy classes $\{\gamma\}$ of elliptic regular elements in $G(F)/Z(F)$ which are of the form $N\delta$. The sum is finite, as noted in the lines following (1). Since ϕ and f are chosen so that ϕ_v and f_v are matching for all v, the sums here and in (1) range over equal sets, and $\Phi(\gamma, \phi) = \Phi(\gamma, f)$ for every γ in the indexing set. Hence

(8) $$\operatorname{tr} r(f) = e \operatorname{tr} r_E(\phi\times\sigma).$$

It is clear that we take Z_0 of Section I to be $NZ(E_v)$ locally and $NZ(\mathbf{A}_E)$ globally, in our case of base-change with respect to E/F.

Write $\operatorname{tr} r_{E,d}(\phi\times\sigma)$ for the sum

(2σ) $$\operatorname{tr} r_{E,d}(\phi\times\sigma) = \sum_{\pi_E} \operatorname{tr} \pi_E(\phi\times\sigma)$$

over all F-invariant cuspidal and one-dimensional constituents of $L(E, \omega')$. It is absolutely convergent, and by Proposition 1 it can be written in the form (2′), but with new values of a_i, a_0', a_0''.

Following Saito (see [L], pp. 190–195), the contribution to the twisted trace formula of the term analogous to (vi) in [JL] is

(3σ) $$-\frac{1}{4}\sum_{\eta_E} \operatorname{tr}[M(\sigma\eta_E)I(\eta_E, \sigma)I(\eta_E, \phi)].$$

The sum ranges over all characters η_E of $A(\mathbf{A}_E)/A(E)$ with $\eta_E(z) = \omega'(z)$ (z in $Z(\mathbf{A}_E)$) such that $\sigma\eta_E = r\eta_E$ (recall that $(\sigma\eta_E)(a) = \eta_E(\sigma a)$ and $(r\eta_E)(a) = \eta_E(rar^{-1})$, $r = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$). Thus there is a character μ_E of $\mathbf{A}_E^\times/E^\times$ with $\eta_E\left(\left(\begin{smallmatrix} a' & 0 \\ 0 & a''\end{smallmatrix}\right)\right) = \mu_E(a')\mu_E(\sigma a'')$. $M(\sigma\eta_E)$ is an intertwining operator from $I(\sigma\eta_E)$ to $I(\eta_E)$. If $\mu_E \circ \sigma = \mu_E$ then there is a character μ of $\mathbf{A}^\times/F^\times$ such that $\mu_E(z) = \mu(Nz)$. Then $M(\sigma\eta_E) = M(\eta_E)$ is the scalar -1, as noted below (3), and $\operatorname{tr} I(\eta_E, \phi\times\sigma) = \operatorname{tr} I(\eta, f)$ where $\eta_E(z) = \eta(Nz)$. When $\eta_E \neq \eta_E \circ \sigma$, we have $e = 2$, the normalizing factor

$$m(\sigma\eta_E) = L(1, \mu_E/\mu_E \circ \sigma)/L(1, \mu_E \circ \sigma/\mu_E)$$

in the operator $M(\sigma\eta_E) = m(\sigma\eta_E) \otimes_v R(\eta_{E,v})$ is clearly one, and we have

$$\mathrm{tr}[M(\sigma r\eta_E)I(r\eta_E,\sigma)I(r\eta_E,\phi)] = \mathrm{tr}[M(\sigma\eta_E)I(\eta_E,\sigma)\,I(\eta_E,\phi)].$$

Hence the difference between (3) and (3σ) is

$$\frac{-1}{e}\sum_{\mu_E}\mathrm{tr}[I(\mu_E,\mu_E\circ\sigma)](\phi\times\sigma),$$

where the sum ranges over the unordered pairs $(\mu_E, \mu_E \circ \sigma)$ with $\mu_E \circ \sigma \neq \mu_E$. This is the second sum in (7). It is empty if $e \neq 2$.

The contribution to the twisted trace formula from the term analogous to (vii) of [JL] is analogous to (4), but f has to be replaced by $\phi\times\sigma$ and η by η_E. Namely, it is

$$(4\sigma)\qquad \frac{1}{4\pi}\sum_{\{\eta_{E,0}\}}\int_{\mathbf{R}}\frac{m'(\eta_E)}{m(\eta_E)}\,\mathrm{tr}\,I(\eta_E,\phi\times\sigma)|ds|.$$

The sum-integral ranges over the analytic manifold of characters η_E of $A(\mathbf{A}_E)/A(E)$ with $\sigma\eta_E = \eta_E$ and $\eta_E(z) = \omega'(z)$ (z in $Z(\mathbf{A}_E)$). Each connected component is parametrized by $\eta_E = \eta_{E,0}\nu_E^{is}$ (s in $\mathbf{R}$), and $\{\eta_{E,0}\}$ is a set of representatives. It is easy to see that $\Sigma_\eta\, m'(\eta)/m(\eta) = em'(\eta_E)/m(\eta_E)$ (sum over η with $\eta_E = \eta\circ N$), hence that this term is equal (after being multiplied by e) to the corresponding term (4). For our purposes it suffices to note that both terms can be expressed in the form (6), for suitable functions $d(z)$.

The contribution to the twisted trace formula from the term analogous to (viii) of [JL] is the sum over all places w of F of integrals over the manifold of characters η_E of $A(\mathbf{A}_E)/A(E)$ with $\eta_E(z) = \omega'(z)$ for z in $Z(\mathbf{A}_E)$ and $\sigma\eta_E = \eta_E$, or equivalently over the manifold of characters $\eta_E = \eta\circ N$, where η is as in $(5)_w$, of the form

$$(5\sigma)_w\qquad \frac{1}{4\pi}\sum_{\eta_0}\int_{\mathbf{R}}\mathrm{tr}[R^{-1}(\eta_{E,w})R'(\eta_{E,w})I(\eta_{E,w},\phi_w\times\sigma)]\cdot\prod_{v\neq w}\mathrm{tr}\,I(\eta_{E,v},\phi_v\times\sigma)\cdot|ds|.$$

Note that each of $(5)_w$, $(5\sigma)_w$, can be expressed in the form (4′), and that by virtue of Proposition 3 only finitely many terms $(5)_w$, $(5\sigma)_w$ are nonzero.

In summary, we proved the following

THEOREM. *For every regular function* $\phi = \phi_u \otimes \phi^u$ *(where* $\phi^u = \otimes_{v\neq u}\phi_v$*), we have* $(1\sigma) = (2\sigma) + (3\sigma) + (4\sigma) + \Sigma_w (5\sigma)_w$, *and* $(5\sigma)_w$ *is zero for every place* w *where* ϕ_w *is spherical and* E_w/F_w *is unramified.*

The identity $(1\sigma) = (2\sigma) + (3\sigma) + (4\sigma) + \Sigma_w (5\sigma)_w$ for a regular function $\phi = \phi^u \otimes \phi_u$ is called the *regular twisted trace formula.*

We use below the following

PROPOSITION 4. *For $\phi_u = (f_u, f_{1u}, \ldots, f_{1u})$ with $f_u = f_u * f_{1u}$ we have*

$$e \operatorname{tr} [R^{-1}(\eta_{E,u})R'(\eta_{E,u})I(\eta_{E,u}, \phi_u \times \sigma)] = \operatorname{tr}[R^{-1}(\eta_u)R'(\eta_u)I(\eta_u, f_u)]$$

if $\eta_{E,u} = \eta_u \circ N_{E_u/F_u}$.

Proof. This is Lemma 16 of [GL(3); (1.5.5)] (whose proof on p. 47 of [GL(3)] is self-contained and straightforward).

IV. Proof of Auxiliary Theorem. We first prove (7) in the case where the complement $\bar{U}$ of the set U consists of the place u alone. For regular matching f, ϕ we have $(1) = e(1\sigma)$, by (8). Recall that the regular trace formula asserts that $(1) = (2) + (3) + (4) + \Sigma_w (5)_w$, and its twisted version asserts that $(1\sigma) = (2\sigma) + (3\sigma) + (4\sigma) + \Sigma_w (5\sigma)_w$. Our aim is to show that $(2) + (3) = e(2\sigma) + e(3\sigma)$ (when $\bar{U}$ is empty). Proposition 4 implies that for our choice of ϕ_u and f_u we have $(5)_u = e(5\sigma)_u$. Hence the Lemma of Section I shows that $(4) - e(4\sigma) + \Sigma_w ((5)_w - e(5\sigma)_w)$ takes the form (6), while $(2) + (3) - e(2\sigma) - e(3\sigma)$ takes the form (2′). Let $X'(q_u)$ be the union of $|z| = 1$ in $\mathbf{C}$ and the set of real z with $q_u^{-1/2} \leq |z| \leq q_u^{1/2}$. Let $X(q_u)$ be the quotient of $X'(q_u)$ by the equivalence relation $z \sim z^{-1}$. Since all representations which appear in the trace formula are unitary, (7) holds for the set U whose complement $\bar{U}$ consists only of u once we prove the following

PROPOSITION 5. *Let a_i $(i \geq 0)$ be complex numbers; $d(z)$ an integrable function on $|z| = 1$; and z_i $(i \geq 0)$ distinct elements of $X(q_u)$ with $z_i \neq q_u^{1/2}, q_u^{-1/2}$, and $\Sigma_i |a_i||z_i^m + z_i^{-m}| < \infty$ for all integral m. Suppose that*

$$(9)_m \quad \sum_{i \geq 0} a_i(z_i^m + z_i^{-m}) + a_0' q_u^{m/2} + a_0'' q_u^{-m/2} = \int_{|z|=1} d(z)(z^m + z^{-m})|dz|$$

for all $m \geq m_0$. Then $a_i = 0$ for all $i \geq 0$, and $a_0' = a_0'' = 0$.

Proof. Put $q = q_u$, $f_n(z) = z^n + z^{-n}$, $F_n(z) = q^{1/2}f_{n+1}(z) - f_n(z)$; note that $F_m(q^{1/2}) = (q - 1)q^{m/2}$, and that $q^{1/2}(11)_{m+1} - (11)_m$ is equal to

$$(10)\qquad \sum_{i\geq 0} a_i F_m(z_i) + (q-1)a_0' F_m(q^{1/2}) = \int_{|z|=1} d(z)F_m(z)|dz|.$$

Had we replaced q by $1/q$ in the definition of F_n, a_0' would be replaced by a_0'' here. Express the left side of (10) in the form $\Sigma_{i\geq 0} b_i F_m(z_i)$ (distinct z_i in $X(q)$). The sum ranges over all $i \geq 0$ such that $b_i \neq 0$. Our aim is to show that the sum is empty. Suppose that it is not empty.

We first show that $|z_i| = 1$ for all i. If this is false, we may assume that $z_0 > 1$. Let $m' > m_0$ be an odd integer. For every $m \geq 0$ we obtain from (10) the identity

$$\sum_{i\geq 0} b_i F_{m'}(z_i) f_m(z_i^{3m'}) = \int d(z)\, F_{m'}(z)\, f_m(z^{3m'})\,|dz|.$$

Writing t_j for the distinct $z_i^{3m'}$, and t for $z^{3m'}$, we obtain (for all $m \geq 0$)

$$(11)\qquad \sum_{j\geq 0} b_j' f_m(t_j) = \int_{|t|=1} d_1(t)\, f_m(t)\,|dt|.$$

The sum ranges over distinct points of $X(q^{3m'/2})$. Since $F_{m'}(z_0) \neq 0$ and $z_0^{3m'} \neq z_i^{3m'}$ for all z_i, we have that $b_0' = b_0 F_{m'}(z_0) \neq 0$. We may assume that $b_0' = 1$. The absolute convergence of the sum and integral implies that there is $c > 0$ with $|d_1(t)| \leq c$, and for each $\epsilon > 0$ there is $N > 0$ such that $\Sigma_{j>N} |b_j'| < \epsilon$. Let $B = B(q^{1/2})$ be the space spanned over $\mathbf{C}$ by the functions f_m on $X = X(q^{1/2})$, where $m \geq 0$. It is closed under multiplication, contains the scalars, and separates points on X. Moreover, if f lies in B then so does its complex conjugate $\bar{f}$. Hence the Stone-Weierstrass theorem implies the following

LEMMA. *B is dense in the sup norm in the space of complex-valued continuous functions on X.*

This lemma implies that there is f in $B(q^{3m'/2})$ with $f(t_0) = 1$ which is bounded by 2 on $X(q^{3m'/2})$, whose value at $t_1, \ldots, t_N$ and on $|t| = 1$ is very small. Evaluating the linear functional (11) at f (f is a finite linear combination of f_m's), we conclude that $b_0' = 0$ and $b_0 = 0$.

We now know that $|z_i| = 1$ for all i. Let $X = X(1)$ be the quotient of the unit circle $|z| = 1$ by the relation $z \sim 1/z$. In particular, $a_0' = a_0'' = 0$, and $(9)_m$ takes the form

$$(12)\qquad \sum_{i\geq 0} a_i f_m(z_i) = \int d(z) f_m(z)|dz|.$$

z and z_i are in $|z| = 1$. Arguing as above we have $c > 0$ with $|d(z)| \leq c$, and $N > 0$ with $\Sigma_{i>N} |b_i| < \epsilon$. Moreover there is f in $B = B(1)$ with $f(z_0) = 1$, with $|f| \leq 2$ on X, such that outside a small neighborhood of z_0 the value of f is small. Our problem is that (12) holds only when $m \geq m_0$. But this is easy to overcome. Take k larger than the sum of m_0 and the degree of f, such that z_0^k is close to one. Then $|z^k + z^{-k}| \leq 2$ on X, and we can apply (12) with f_m replaced by $g(z) = f(z)(z^k + z^{-k})$, to obtain a contradiction to $b_0 \neq 0$. This establishes the proposition.

We repeat the conclusion of Proposition 5 as

LEMMA 1. *The Auxiliary Theorem holds in the case where the complement $\bar{U}$ of the set U consists of u alone.*

The special case where $d(z) \equiv 0$ and $m_0 = 0$ of Proposition 5 can now be used to deduce by induction on the cardinality of $\bar{U}$ the validity of

LEMMA 2. *The Auxiliary Theorem holds where $\bar{U}$ is any finite set of nonarchimedean places of F which split in E or are unramified in E.*

Proof. If v is unramified in E, the identities $(9)_m$ hold only for the integers m divisible by $e = [E_v : F_v]$. Hence the sum on the right of (7) ranges over the π with $L_{I,v}(\pi_v) = L_{I,v}(\pi_v^0)$ for all v in $\bar{U}$, as required.

It remains to show the following

LEMMA 3. *The Auxiliary Theorem holds where U is any finite set.*

Proof. In this proof we say that unramified irreducible G_v-modules π_v and π_v' are equivalent if $L_{I,v}(\pi_v) = L_{I,v}(\pi_v')$. Consider the set of sequences $\{\pi_v^0(v \notin U)\}$ of equivalence classes of irreducible unramified G_v-modules π_v^0. Define $c(\{\pi_v^0(v \notin U)\})$ to be the result of subtracting the right from the left side of (7), for a fixed choice of matching f_v and ϕ_v (v in U). These complex numbers c are zero for all but countably many sequences, which we now denote by $\{\pi_{i,v}^0(v \notin U)\}$ with $i \geq 0$. Put c_i for $c(\{\pi_{i,v}^0(v \notin U)\})$. Moreover, since all sums in the trace formula are absolutely convergent and we may take $f_v = f_v^0$ for all $v \notin U$, the sum $\Sigma_i |c_i|$ is finite.

Our aim is to show that $c_i = 0$ for all i. Suppose that $c_0 \neq 0$. Then there is $N > 0$ such that $\Sigma_{i>N} |c_i| \cup \frac{1}{2} |c_0|$. There is a finite set V outside U such that for every $i (1 \leq i \leq N)$ there is $v = v(i)$ in V such that $L_{I,v}(\pi_{i,v}^0) \neq L_{I,v}(\pi_{0,v}^0)$. Lemma 2 implies that

$$(13) \qquad \sum_i c_i \prod_{v \notin U \cup V} \operatorname{tr} \pi_{i,v}^0(f_v) = 0,$$

where the sum ranges over the $\{\pi_{i,v}^0 (v \notin U)\}$ with $L_{I,v}(\pi_{i,v}^0) = L_{I,v}(\pi_{0,v}^0)$ for v in V. In particular, the indices i $(1 \leq i \leq N)$ do not appear in (13). Taking $f_v = f_v^0$ for all $v \notin U \cup V$ in (13) we deduce that $|c_0| \leq \Sigma_{i>N} |c_i|$. As this last sum is less than $\frac{1}{2}|c_0|$, we obtain a contradiction to the assumption that $c_0 \neq 0$. The lemma follows, and so does the Auxiliary Theorem.

V. Proof of Base-Change Theorem. From now on we use the Auxiliary Theorem in the case where U is a finite set. The rigidity theorem for GL(2) asserts that there exists at most one cuspidal, or automorphic of the form $I(\mu_E, \mu_E \circ \sigma)$, representation of $G(\mathbf{A}_E)$ whose component at each v outside U is the fixed $L_v(\pi_v^0)$. Hence at most one of the two sums on the left of (7) is nonempty, and it consists of at most one entry (π_E or η_E). On the other hand, if π contributes to the sum on the right of (7), then $\pi \otimes \epsilon$ does too, since $\operatorname{tr}(\pi \otimes \epsilon)(f) = \operatorname{tr} \pi(f)$ for every $f = \otimes f_v$ with f_v matching some ϕ_v for all v. Our next aim is to show that there are no other contributions on the right. We shall show that this is a Corollary to Proposition 3.6 of Jacquet-Shalika [JS], which we now recall.

Let F be a global field, and $\pi = \otimes \pi_v$ an irreducible $\mathrm{GL}(n, \mathbf{A})$-module. There is a finite set U of places of F such that for v outside U the component π_v is unramified; in fact, it is the unique unramified constituent in the composition series of a $\mathrm{GL}(n, F_v)$-module $I(\eta_v) = \mathrm{Ind}(\delta_v^{1/2}\eta_v; P_v, \mathrm{GL}(n, F_v))$ unitarily induced from the unramified character $\eta_v((a_{ij})) = \Pi_{i=1}^n \eta_{v,i}(a_{ii})$ of the upper triangular subgroup $(i \leq j)$ P_v of $\mathrm{GL}(n, F_v)$. If π' is a $\mathrm{GL}(n', \mathbf{A})$-module, irreducible and unramified outside U, introduce the Euler product

$$L(s, U, \pi \otimes \pi') = \prod_{v \notin U} \prod_{i=1}^{n} \prod_{j=1}^{n'} (1 - q_v^{-s} \eta_{v,i}(\boldsymbol{\pi}_v)\eta'_{v,j}(\boldsymbol{\pi}_v))^{-1}.$$

As usual, $\boldsymbol{\pi}_v$ denotes a uniformizer of F_v. Proposition 3.6 of [JS] asserts the following

PROPOSITION 6. *Let π, π' be cuspidal* $\mathrm{GL}(n, \mathbf{A})$- *and* $\mathrm{GL}(n', \mathbf{A})$-*modules with a unitary central character.* (1) *The product $L(s, U, \pi \otimes \pi')$ converges absolutely for* $\operatorname{Re} s > 1$. *Let X be the set of s with* $\operatorname{Re} s = 1$ *such that $\pi \otimes \nu^{s-1}$ is equivalent to the contragredient representation $\tilde{\pi}'$ of π'. Then* (2) *$L(s, U, \pi \otimes \pi')$ extends to a continuous function on the complement of X in* $\operatorname{Re} s \geq 1$. (3) *For s_0 in X the limit* $\lim(s - s_0)L(s, U, \pi \otimes \pi')$, *as $s \to s_0$ in* $\operatorname{Re} s \geq 1$, *exists and equals to a finite nonzero number.*

This admits the following

COROLLARY 6′. *Let π, π' be cuspidal* GL(n, **A**)*-modules such that for almost all v the unordered n-tuples $\{\eta_{v,i}(\pi_v)^{\alpha_v}\ (1 \leq i \leq n)\}$ and $\{\eta'_{v,i}(\pi_v)^{\alpha_v}\ (1 \leq i \leq n)\}$ are equal; here $\alpha_v = [F_v^\times : NE_v^\times]$. Then there is $j\ (0 \leq j < e)$ with $\pi' = \pi \otimes \epsilon^j$.*

Proof. We may assume that π, π' have unitary central characters. Let U be a finite set of places of F such that π_v, π'_v are unramified for each v outside U. It is clear that

$$(14) \qquad \prod_{i=0}^{e-1} L(s, U, \pi \otimes \tilde{\pi}' \otimes \epsilon^i) = \prod_{i=0}^{e-1} L(s, U, \pi \otimes \tilde{\pi} \otimes \epsilon^i),$$

since the local factor at v of each of these products is equal to

$$\prod_{i,i'=1}^{n} (1 - q_v^{-s\alpha_v}\eta_{v,i}(\pi_v)^{\alpha_v}\eta'_{v,i}(\pi_v)^{-\alpha_v})^{-e/\alpha_v}.$$

Proposition 6 asserts that the product on the right of (14) has a pole at $s = 1$. Hence the product on the left has a pole, and $\pi' \simeq \pi \otimes \epsilon^j$ for some j, as required.

In particular, if the right side of (7) is nonzero then it is equal to $\Pi_v \operatorname{tr} \pi_v(f_v)$ if $\pi \otimes \epsilon \simeq \pi$, and to $e\Pi_v \operatorname{tr} \pi_v(f_v)$ if $\pi \otimes \epsilon \not\simeq \pi$, namely the sum on the right of (7) ranges over one or e cuspidal π.

To complete the proof of the Base Change Theorem we prove the following

PROPOSITION 7. *Given a local field F_u and a supercuspidal $G(F_u)$-module π_u, there exists a global field F without real completions whose completion at a place u is our F_u, and for each place $u' \neq u$ there is a cuspidal $G(\mathbf{A})$-module π whose component at u is our π_u, its component at some other finite place $w \neq u, u'$ is $st(\mu_w)$ where μ_w is an unramified character of $F_w^\times$, and its component at each finite place $v \neq u, u', w$ is unramified.*

Proof. It is clear that there is F as required, and that the central character ω_u of π_u can be extended to a unitary character ω of $\mathbf{A}^\times/F^\times$ whose component at each finite place $v \neq u, u'$ is unramified. Recall [K1] that for every irreducible square-integrable G_w-module π'_w there exists a pseudo-coefficient, namely a function f_w with $\operatorname{tr} \pi_w(f_w) = 0$ for every irreducible tempered π_w inequivalent to π'_w, and $\operatorname{tr} \pi'_w(f_w) = 1$. Let μ_w be an unramified character with $\mu_w^2 = \omega_w$. Let $f = \otimes f_v$ be the function whose component f_u is a matrix coefficient of the supercuspidal π_u, its component f_w is a pseudo-coefficient of the Steinberg $st(\mu_w), f_v = f_v^0$ at each finite $v \neq u, u', w$, and each of its components at u' and the archimedean places is supported on a small neighborhood of the identity modulo the center. The trace formula then asserts that

$$\sum_{\pi} \operatorname{tr} \pi(f) = |G(\mathbf{A})/G(F)Z(\mathbf{A})| f(1) \neq 0,$$

hence there is π as required.

Remark. When F is a number field the place u' can be chosen to be archimedean.

To prove the local assertion of the Base Change Theorem, choose a place w of F such that E_w is a field. Then π of Proposition 7 satisfies $\pi \otimes \epsilon \neq \pi$, since $st(\mu_w) \otimes \epsilon_w \neq st(\mu_w)$. Choose a place u' of F which splits in E. Applying the identity (7) in which π occurs on the right we conclude that π quasi-lifts to some cuspidal π_E and (7) asserts

$$\prod \operatorname{tr} \pi_{F,v}(\phi_v \times \sigma) = \prod \operatorname{tr} \pi_v(f_v)$$

for all matching ϕ_v and f_v. The product ranges over the set of u, w and the archimedean places. Since $L_v(\pi_v) = \pi_{E,v}$ where v is equal to w or the archimedean places, we conclude that each supercuspidal π_v lifts to a unique $\pi_{E,v}$. The same type of argument shows that each F_v-invariant supercuspidal $\pi_{E,v}$ is a lift of some supercuspidal π_v. It is clear that $I(\mu_{E,v}, \mu_{E,v} \circ \sigma)$ is a lift of a supercuspidal π_v if $\mu_{E,v} \neq \mu_{E,v} \circ \sigma$, that $\pi \simeq \pi \otimes \epsilon$ if and only if it lifts to $I(\mu_E, \mu_E \circ \sigma)$, and finally that $\pi_v \simeq \pi_v \otimes \epsilon_v$ if and only if it lifts to $I(\pi_{E,v}, \pi_{E,v} \circ \sigma)$. The proof of the Base Change Theorem is complete.

Remark. It is easy to deduce from [K] that the local statements (3), (4) of the Base Change Theorem for a local field of positive characteristic follow from the analogous statements in characteristic zero. We do not use this comment in this work as our proofs above hold for a field of any characteristic.

Concluding Remarks. The initial ideas in the theory of base-change are due to H. Saito [Sa], who introduced the twisted trace formula and proved the Main Theorem in the context of modular forms, and T. Shintani [Sh], who introduced the notion of local lifting by means of twisted character relations. Numerous expositions, generalizations, analogues and applications (see, notably, Langlands [L]) followed. The proof of the Auxiliary Theorem was rather lengthy, and centered on computing all terms in the trace formula and the twisted formula, then struggling with the behaviour of the various terms (especially weighted orbital integrals) to force an equality of the two formulae. Matching orbital integrals of spherical functions also played a key role.

In [GL(3)] I presented (in the context of GL(2) and GL(3)) an argument of correcting the weighted orbital integrals (see (2.3.1) and (2.3.2) there), and showed that their limits at the singular set are equal to the singular terms in the trace formula, which are explicitly computed in (2.4) and (2.5). By virtue of the computations of (2.7.1), this gave in the context of GL(2) a proof simpler than that of [L], in addition to a new insight into the trace formula. Also we note that the results of (1.5) in [GL(3)] which are used here are simple, useful and complete. However, the computations of (2.7.2) are wrong (the same applies in the twisted case (3.4.2)). Consequently most of Section 5 there should be discarded. Thus although the final local Theorem 7 *is* proven on p. 198, there is enough (in (2.7.3), which deals with the comparison with division algebras) to prove the final global Theorem 8 on p. 199 *only* for π with two elliptic components, and not unconditionally as asserted there wrongly. The unconditional global theorem is due to Arthur-Clozel [AC], who (in particular) developed the correction argument (in ref. [1e] of [AC]). In this context we note that in a forthcoming paper (see [Reg]) we give a simple proof of the base change theorem for cuspidal representations of GL(n) which have a supercuspidal component. This new proof does not use the correction argument but ideas such as those of this paper instead: we work with test functions for which the weighted and singular orbital integrals vanish a-priori.

Our paper here presents a new approach—but only in the context of GL(2)—to deal with all π without computing the weighted integrals. The focus is on applications to lifting problems, and not on the trace formula itself. Our approach, which is based on the usage of regular functions, is likely to extend and establish unconditional equalities of trace formulae of reductive groups of arbitrary rank, but we have not checked this as yet. For the moment our method is shown in [Sym] to establish the Symmetric Square trace identity, hence also the trace identity used for Base Change from the unitary group $U(3, E/F)$ to GL$(3, E)$ of [U(3)]. It can be seen that the computations of this paper apply also in the cases of the Metaplectic Correspondence [M] and Base Change from $U(2, E/F)$ to GL$(2, E)$ of

[$U(2)$] to establish the trace formulae comparisons, decimating the length and effort invested there. Regular functions are used in [FK] in the context of GL(n) and its metaplectic group, and in [Sph] for arbitrary reductive group, which is the natural setting for our theory, to show that corresponding spherical functions have stable matching orbital integrals.

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