

- [1] (4 points each) For the following problems, just write down the answers. No work is necessary. Right or wrong.
- (a) If  $A$  is an  $8 \times 7$  matrix and  $\dim N(A) = 5$ , then  $\text{rank}(A) =$ \_\_\_\_\_.
  - (b) If the eigenvalues of a  $3 \times 3$  matrix  $A$  are  $-2$ ,  $1$ , and  $3$ , then the characteristic polynomial of  $A$  is \_\_\_\_\_.
  - (c) If a matrix  $A$  is nonsingular, then  $A$  cannot have an eigenvalue \_\_\_\_\_.
  - (d) If the rank of a  $4 \times 5$  matrix  $A$  is  $3$ , then the rank of  $A^T$  is \_\_\_\_\_.
  - (e) If  $A$  is a  $4 \times 5$  matrix and nullity of  $A^T$  is  $1$ , then the nullity of  $A$  is \_\_\_\_\_.
  - (f) The dimension of the row space of a  $8 \times 10$  matrix is at most \_\_\_\_\_.
  - (g) If  $U$  is an orthogonal matrix and  $x$  is a vector then the length of the vector  $Ux$  is\_\_\_\_\_.
  - (h) If  $\{[3, 2, 2]^T, [-2, a, 0]^T, [b, c, -13]^T\}$  is an orthogonal basis of  $R^3$ , then  $a =$ \_\_\_\_\_,  $b =$ \_\_\_\_\_,  $c =$ \_\_\_\_\_.
  - (i) If  $S$  is an orthogonal st of  $5$  nonzero vectors in  $R^7$ , then the dimension of  $\text{Span}(S)$  is \_\_\_\_\_.
  - (j) The number of solutions of the homogeneous  $3 \times 4$  system of linear equations is either \_\_\_\_\_ or \_\_\_\_\_.

[2] (4 points each, -2 points for wrong answer) True/False: Just write T(rue) or F(alse).

- (a) Any system of linear equations has at least one solution.
- (b) If a square matrix  $A$  has no zero rows, then  $A$  has an inverse.
- (c) If the product  $AB$  of two  $n \times n$  matrices  $A$  and  $B$  has an inverse, then an inverse of  $BA$  also exists.
- (d) If  $A^2$  has an eigenvalue 1, then  $A$  must have an eigenvalue 1.
- (e) Any nonzero square matrix has at least one nonzero eigenvalue.
- (f) The zero vector can never be an eigenvector of any square matrix.
- (g) If the characteristic polynomial of a  $2 \times 2$  matrix  $A$  is  $(X - 1)(X - 2)$ , then  $A^{-1}$  exists.
- (h) Any system of linear equations with more equations than the number of unknowns is always inconsistent.
- (i) If an echelon form  $E$  of a matrix  $A$  does not have (columns of) free variables, then  $E$  must have a zero row.
- (j) Any  $n \times n$  matrix  $A$  has infinitely many  $n \times n$  matrices  $B$  such that  $AB = BA$ .

[3] Let  $V = \text{Span}\{[3 \ 1 \ 1 \ -1]^T, [2 \ 5 \ 1 \ 0]^T, [1 \ 2 \ -2 \ -1]^T\}$ .

(a) (10) Use the Gram-Schmidt method to find an orthogonal basis of  $V$ .

(b) (10) Determine whether or not the vector  $[3 \ 7 \ -1 \ -1]^T$  is in  $V$ . If it is in  $V$ , then express it as a linear combination of the orthogonal basis obtained in (a).

[4] Let

$$A = \begin{vmatrix} 1 & 5 & 1 & 1 & 0 \\ 1 & 5 & 2 & 4 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 2 & 10 & 0 & -4 & 0 \end{vmatrix}$$

(a) (5) Reduce  $A$  to its reduced echelon form.

(b) (5) Find a basis of the range space (= column space) of  $A$ .

(c) (5) Find a basis of the null space of  $A$ .

[5] Let

$$A = \begin{vmatrix} 2 & 1 & 4 \\ -1 & -2 & -1 \\ -1 & -1 & -3 \end{vmatrix}$$

- (a) (10) Find the characteristic polynomial of  $A$ .
- (b) (5) Find all eigenvalues of  $A$ .
- (c) (10) Find all linearly independent eigenvectors of  $A$  belonging to each eigenvalue of  $A$ .

6

- [6] (15) If  $A$  is a  $6 \times 6$ -matrix with four eigenvalues, and the eigenspaces of two of these eigenvalues are two dimensional each, must  $A$  be diagonalizable? Credit only for correct answer with correct explanation.

[7] Let

$$A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

- (a) (10) Compute  $\det A$  using cofactor expansion. Is  $A$  invertible?
- (b) (10) What are  $\det(A^{-1})$  and  $\det(A^2)$ ?  $\det(3A)$  and  $\det(A^T)$ ? Let  $B$  be the matrix obtained from  $A$  by adding 5 times its  $3^{rd}$  row to  $1^{st}$  row. Find  $\det B$ .
- (c) (10) Find the inverse of  $A$ , if it exists.

- [8] (15) Let  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Find all subspaces  $W$  of  $\mathbb{R}^2$  such that if  $x$  lies in  $W$ , so does  $Ax$ .