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On the symmetric square: unstable local transfer*

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Summary. We prove the "fundamental lemma" for spherical functions with respect to the natural (induction) lifting from $PGL(2)$ to $PGL(3)$ which appears as the unstable counterpart of the stable symmetric-square lifting from $SL(2)$ to $PGL(3)$ (see [IV] for an introduction to this project, and [VI] for the final results). Thus spherical functions on $PGL(2)$ and $PGL(3)$ which correspond to each other by satisfying an elementary representation theoretic relation are shown to have matching orbital integrals. The proof of this local statement is based on an application of the global trace formula.

Let F be a local field. Put

$$G=PGL(3), \qquad H_1=PGL(2), \qquad J=\begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix},$$

and $\sigma\delta=J{}^t\delta^{-1}J$ for δ in $G(F)$. Fix an algebraic closure $\bar{F}$ of F. The elements δ, δ' of $G(F)$ are called (*stably*) *σ-conjugate* if there is g in $G(F)$ (resp. $G(\bar{F})$) with $\delta'=g^{-1}\delta\sigma(g)$. To state our theorems, we first recall the results of [I], §§ 1.2–1.6, concerning these classes. For any δ in $GL(3,F)$, $\delta\sigma(\delta)$ lies in $SL(3,F)$ and depends only on the image of δ in $G(F)$. The eigenvalues of $\delta\sigma(\delta)$ are $\lambda, 1, \lambda^{-1}$ (see [I], § 1.4), with $[F(\lambda):F]\leqq 2$; δ is called *σ-regular* if $\lambda \neq \pm 1$. In this case we write (as in [I], § 1.5) $\gamma_1=N_1\delta$ for the conjugacy class in $H_1(F)$ which corresponds to the conjugacy class with eigenvalues $\lambda, 1, \lambda^{-1}$ in $SO(3,F)$ under the isomorphism $H_1(F)=SO(3,F)$ (i.e., γ_1 is the image in $H_1(F)$ of a conjugacy class in $GL(2,F)$ with eigenvalues a, b with $a/b=\lambda$). It is shown in [I], § 1.5, that the map N_1 is a bijection from the set of stable regular σ-conjugacy classes in $G(F)$ to the set of regular conjugacy classes in $H_1(F)$ (clearly, we say that a conjugacy class γ_1 in $H_1(F)$ is regular if $\lambda=a/b\neq\pm 1$). The set of σ-conjugacy classes in the stable σ-conjugacy class of a σ-regular δ is (shown

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in [I], § 1.5, to be) parametrized by $F^\times/NK^\times$, where K is the field extension $F(\lambda)$ of F, and N is the norm from K to F. Explicitly, if the quotients of the eigenvalues of the regular element γ_1 are λ and λ^{-1}, choose α, β in K with $\lambda=-\alpha/\beta$ (for example with $\beta=1$ if $K=F$, and with $\bar{\beta}=\alpha$ if $K\neq F$); let a be an element of $GL(2, F)$ with eigenvalues α, β; put

$$e=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad \text{and} \quad h_1=\begin{pmatrix}x & 0 & y\\ 0 & 1 & 0\\ z & 0 & t\end{pmatrix} \quad \text{if } h=\begin{pmatrix}x & y\\ z & t\end{pmatrix};$$

then $\delta_u=(uae)_1$ is a complete set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of the δ with $N_1\delta$ equals γ_1, as u varies over $F^\times/NK^\times$ (a set of cardinality one or two). In addition we associate (in [I], § 1.6) to δ a sign $\kappa(\delta)$, as follows: $\kappa(\delta)$ is 1 if the quadratic form x (in F^3)$\mapsto {}^t x\delta J x$ (equivalently $x\mapsto \frac{1}{2}{}^t x[\delta J+{}^t(\delta J)]x$) represents zero, and $\kappa(\delta)=-1$ if this quadratic form is anisotropic. It is clear that $\kappa(\delta)$ depends only on the σ-conjugacy class of δ, but it is not constant on the stable σ-conjugacy class of δ.

Denote by f (resp. f_1) a complex-valued compactly-supported smooth (thus locally-constant if F is non-archimedean) function on $G(F)$ (resp. $H_1(F)$). Fix Haar measures on $G(F)$ and on $H_1(F)$. Write $\Phi(\delta, f)$ for the twisted orbital integral

$$\int f(g\delta\sigma(g^{-1}))\,dg$$

(here g ranges over $G(F)/G_\delta^\sigma(F)$, where $G_\delta^\sigma(F)=\{g \text{ in } G(F);\ g\delta\sigma(g^{-1})=\delta\}$) of f at δ, and put

$$\Phi^{us}(\delta, f)=\sum_{\delta'}\kappa(\delta')\,\Phi(\delta', f);$$

here δ' ranges over a set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of δ. As usual, δ is σ-regular. If γ_1 is regular, we also put

$$\Phi(\gamma_1, f_1)=\int f_1(g^{-1}\gamma_1 g)\,dg$$

(g in $H_{1,\gamma_1}(F)\backslash H_1(F)$, where $H_{1,\gamma_1}(F)$ is the centralizer of γ_1 in $H_1(F)$). Note that when $\gamma_1=N_1\delta$ the groups $G_\delta^\sigma(F)$, $G_{\delta'}^\sigma(F)$, $H_{1,\gamma_1}(F)$ are isomorphic tori, and we transfer Haar measures on them using these isomorphisms.

Definition. The functions f and f_1 are called *matching* if they have matching orbital integrals, namely if $\Delta(\delta)\,\Phi^{us}(\delta, f)=\Delta_1(\gamma_1)\,\Phi(\gamma_1, f_1)$ for all δ with regular $\gamma_1=N_1\delta$.

Here we put $\Delta_1(\gamma_1)=|(a-b)^2/ab|^{1/2}$ if a, b are the eigenvalues of a representative in $GL(2, F)$ of γ_1, and $\Delta(\delta)=|(1-\lambda^2)(1-\lambda^{-2})|^{1/2}$ if $\lambda=a/b$. Thus

$$\Delta_1(\gamma_1)=|(1-\lambda)(1-\lambda^{-1})|^{1/2}, \quad \text{and} \quad \Delta(\delta)/\Delta_1(\gamma_1)=|(1+\lambda)(1+\lambda^{-1})|^{1/2}.$$

Suppose that F is non-archimedean; denote by R its ring of integers. Put $K=G(R)$, $K_1=H_1(R)$. Let $\mathbb{H}$ (resp. $\mathbb{H}_1$) denote the convolution algebra of complex-valued compactly-supported K- (resp. K_1-)biinvariant functions on $G(F)$

(resp. $H_1(F)$). The Haar measures are the same as those used in the definition of the orbital integrals. Denote by f^0 (resp. f_1^0) the unit element in $\mathbb{H}$ (resp. $\mathbb{H}_1$), namely the quotient by the volume $|K|$ (resp. $|K_1|$) of K (resp. K_1) of the characteristic function of K (resp. K_1). We prove below the following

Theorem 1. *The functions f^0 and f_1^0 are matching.*

This result is used in [VI] to complete the study of the symmetric square lifting, for *all* automorphic representations of $H=SL(2)$.

By a G-module π (resp. H_1-module π_1) we mean an admissible representation of $G(F)$ (resp. $H_1(F)$) in a complex space. An irreducible G-module π is called *σ-invariant* if it is equivalent to the G-module ${}^\sigma\pi$, defined by ${}^\sigma\pi(g)=\pi(\sigma g)$. In this case there is an intertwining operator A on the space of π with $\pi(g)A=A\pi(\sigma g)$ for all g. Since $\sigma^2=1$ we have $\pi(g)A^2=A^2\pi(g)$ for all g, and since π is irreducible A^2 is a scalar by Schur's lemma. We choose A with $A^2=1$. This determines A up to a sign, and when π has a Whittaker model, [IV, § 1(1.1.1)] specifies a normalization of A which is compatible with a global normalization. A G-module π is called *unramified* if the space of π contains a non-zero K-fixed vector. The dimension of the space of K-fixed vectors is bounded by one if π is irreducible. If π is σ-invariant and unramified, and $v_0\neq 0$ is a K-fixed vector in the space of π, then Av_0 is a multiple of v_0 (since $\sigma K=K$), namely $Av_0=cv_0$, with $c=\pm 1$. Replace A by cA to have $Av_0=v_0$, and put $\pi(\sigma)=A$. As verified in [IV, § 1(1.1.1)], when π is (irreducible) unramified and has a Whittaker model, both normalizations of the intertwining operator are equal.

For any π and f the convolution operator $\pi(f)=\int\limits_{G(F)} f(g)\,\pi(g)\,dg$ has finite rank. If π is σ-invariant put $\pi(f\times\sigma)=\int\limits_{G(F)} f(g)\,\pi(g)\,\pi(\sigma)\,dg$. Denote by $\operatorname{tr}\pi(f\times\sigma)$ the trace of the operator $\pi(f\times\sigma)$. It depends on the choice of the Haar measure dg, but the *(twisted) character* χ_π of π does not; χ_π is a locally-integrable complex-valued function on $G(F)$ (see [C], [H]) which is σ-conjugacy invariant and locally-constant on the σ-regular set, with $\operatorname{tr}\pi(f\times\sigma)=\int\limits_{G(F)} f(g)\,\chi_\pi(g)\,dg$ for all f.

If f is spherical, namely it lies in $\mathbb{H}$, and π is σ-invariant, then $\pi(f)$ (hence also $\pi(f\times\sigma)$) factorizes through the projection on the subspace of K-fixed vectors in π; thus $\operatorname{tr}\pi(f\times\sigma)\neq 0$ for f in $\mathbb{H}$ implies that π is unramified. Similarly we introduce $\pi_1(f_1)$ and $\operatorname{tr}\pi_1(f_1)$, and conclude that π_1 is unramified if $\operatorname{tr}\pi_1(f_1)\neq 0$ for f_1 in $\mathbb{H}_1$.

A Levi subgroup of a maximal parabolic subgroup P of $G(F)$ is isomorphic to $GL(2,F)$. Hence an $H_1(F)$-module π_1 extends to a P-module trivial on the unipotent radical N of P. Let δ denote the character of P which is trivial on N and whose value at $p=mn$ is $|\det h|$ if m corresponds to h in $GL(2,F)$. Explicitly, if P is the upper triangular parabolic subgroup of type $(2,1)$, and m in M is represented in $GL(3,F)$ by $\begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix}$, then $\delta(m)=|(\det m')/m''^2|$ (m' lies in $GL(2,F)$, m'' in $GL(1,F)$). Denote by $I(\pi_1)$ the G-module $\pi=\operatorname{Ind}(\delta^{1/2}\pi_1;\,P,G)$ unitarily induced from π_1 on P to G. It is clear from [BZ] that when $I(\pi_1)$ is irreducible then it is σ-invariant, and it is unramified if and only if π_1 is unramified.

Definition. The functions f_1 in $\mathbb{H}_1$ and f in $\mathbb{H}$ are called *corresponding* if $\operatorname{tr}\pi_1(f_1)=\operatorname{tr}(I(\pi_1))(f\times\sigma)$ for all unramified H_1-modules π_1, equivalently: for all H_1-modules π_1.

Example. The spherical functions f_1^0 and f^0 are corresponding.

It is shown in [IV, § 2] that Theorem 1 implies

Theorem 2. *If the spherical functions f, f_1 are corresponding, then they are matching.*

However the argument given below establishes Theorem 2 directly, and Theorem 1 will follow as the special case of $f=f^0$, $f_1=f_1^0$.

In [IV, § 2] the following is proven:

Theorem 0. *Suppose that $\pi=I(\pi_1)$ where π_1 is an irreducible $H_1(F)$-module, and δ, δ' are σ-regular stably σ-conjugate but not σ-conjugate elements of $G(F)$. Then $\chi_\pi(\delta')=-\chi_\pi(\delta)$.*

Of course $\delta\neq\delta'$ as in Theorem 0 exist only when $F(\lambda)\neq F$, namely when $N_1\delta$ is elliptic regular. Let χ_{π_1} be the character of π_1; it is a locally-integrable complex-valued conjugacy-invariant function on $H_1(F)$ which is smooth on the regular set and satisfies

$$\operatorname{tr}\pi_1(f_1)=\int_{H_1(F)} f_1(g)\,\chi_{\pi_1}(g)\,dg$$

for all f_1 on $H_1(F)$. It is shown in [IV, § 2] that Theorem 2 implies the following.

Theorem 3. *If $\pi=I(\pi_1)$ then $\kappa(\delta)\,\Delta(\delta)\,\chi_\pi(\delta)=\Delta_1(\gamma_1)\,\chi_{\pi_1}(\gamma_1)$ for all δ with regular $\gamma_1=N_1\delta$.*

In view of Theorem 0, it suffices to prove Theorem 3 only for one σ-conjugacy class within each stable σ-conjugacy class. It is clear that Theorem 3 implies Theorem 2 (see, e.g., proof of Proposition 27.3 in [FK]).

It is shown in [II, § 2] that Theorem 3 is equivalent to the following

Theorem 3'. *For any $H_1(F)$-module π_1 we have $\operatorname{tr}(I(\pi_1))\,(f\times\sigma)=\operatorname{tr}\pi_1(f_1)$ for all pairs f, f_1 of matching functions on $G(F)$ and $H_1(F)$.*

Our plan is to prove Theorem 3 directly only in the easiest case of the trivial representation π_1, and then use the global trace formula to deduce Theorem 2, hence also 1 and 3, 3'. We emphasize that *our method is to compare the representation theoretic sides of the trace formula in order to derive a comparison of orbital integrals.* This is a new type of application of the trace formula.

To simplify our proof we now assume that F has characteristic zero and odd residual characteristic. We shall prove Theorem 2 for any such F. Then Theorem 3 follows for every local F with characteristic zero by [IV, § 2]. Our proof here then establishes Theorem 2 when F has residual characteristic two, and characteristic zero. By virtue of [K'] each of Theorems 2 and 3 holds also when F is local of positive characteristic. For example, Theorem 2 follows at once from a statement which we proceed to state; it is a corollary to [K'], Theorem A. Suppose that G is a group as in [K'], § 1, F is a local field, f is a locally constant measure on $G(F)$, and U is a compact subset of $G(F)$ consisting of regular elements. Clearly there exists a positive integer l such that the function f, and the restriction $\Phi_U(x,f)$ to U of the orbital integral $\Phi(x,f)$ of f, both lie in the Hecke algebra $\mathbb{H}_l(G,F)$ of $K_l(F)$-biinvariant measures on $G(F)$. Theorem A of [K'] asserts that there exists $m \geqq l$, such that for every local field F' which is m-close to F, the morphism $\phi: \mathbb{H}_l(G,F) \to \mathbb{H}_l(G,F')$ (defined in [K']) is an algebra isomorphism. The statement which we require is that $\Phi_{U'}(x',f') = \phi(\Phi_U(x,f))$ for every x in $X^l(F) = K_l(F)\backslash G(F)/K_l(F)$, where $x' = \phi(x)$, $f' = \phi(f)$, and $U' = \phi(U)$. The analogous statement for twisted orbital integrals is equally valid. To deduce Theorem 2 for F of positive characteristic we take F' of characteristic zero.

We begin with the proof of

Proposition 1. *If π_1 is the trivial $H_1(F)$-module, $\pi = I(\pi_1)$, and δ a σ-regular element of $G(F)$ with elliptic regular norm $\gamma_1 = N_1\delta$, then $(\Delta(\delta)/\Delta_1(\gamma_1))\,\chi_\pi(\delta) = \kappa(\delta)$.*

Proof. To compute the character of π we shall express π as an integral operator in a convenient model, and integrate the kernel over the diagonal. Denote by $\mu = \mu_s$ the character $\mu(x) = |x|^{(s+1)/2}$ of $F^\times$. It defines a character $\mu_P = \mu_{s,P}$ of P, trivial on N, by $\mu_P(p) = \mu((\det m')/m''^2)$ if $p = mn$ and $m = \begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix}$ with m' in $GL(2,F)$, m'' in $GL(1,F)$. If $s=0$, then $\mu_P = \delta^{1/2}$. Let W_s be the space of complex-valued smooth functions ψ on $G(F)$ with $\psi(pg) = \mu_P(p)\,\psi(g)$ for all p in P and g in $G(F)$. The group $G(F)$ acts on W_s by right translation: $(\pi_s(g)\psi)\,(h) = \psi(hg)$. By definition, $I(\pi_1)$ is the G-module W_s with $s=0$. The parameter s is introduced for purposes of analytic continuation.

We prefer to work in another model V_s of the G-module W_s. Let V denote the space of column 3-vectors over F. Let V_s be the space of smooth complex-valued functions ϕ on $V - \{0\}$ with $\phi(\lambda v) = \mu(\lambda)^{-3}\phi(v)$. The expression $\mu(\det g)\,\phi({}^t g v)$, which is initially defined for g in $GL(3,F)$, depends only on the image of g in $G(F)$. The group $G(F)$ acts on V_s by $(\tau_s(g)\phi)\,(v) = \mu(\det g)\,\phi({}^t g v)$. Let $v_0 \neq 0$ be a vector of V such that the line $\{\lambda v_0;\ \lambda$ in $F\}$ is fixed under the action of ${}^tP$. Explicitly, we take $v_0 = {}^t(0,0,1)$. It is clear that the map $V_s \to W_s$, $\phi \to \psi = \psi_\phi$, where $\psi(g) = (\tau_s(g)\phi)\,(v_0) = \mu(\det g)\,\phi({}^t g v_0)$, is a G-module isomorphism, with inverse $\psi \to \phi = \phi_\psi$, $\phi(v) = \mu(\det g)^{-1}\psi(g)$ if $v = {}^t g v_0$ (G acts transitively on $V - \{0\}$).

For $v = {}^t(x,y,z)$ in V put $\|v\| = \max(|x|, |y|, |z|)$. Let V^0 be the quotient of the set of v in V with $\|v\| = 1$ by the equivalence relation $v \sim \alpha v$ if α is a unit in R. Denote by $\mathbb{P}V$ the projective space of lines in $V - \{0\}$. If Φ is a function on $V - \{0\}$ with $\Phi(\lambda v) = |\lambda|^{-3}\Phi(v)$ and $dv = dx\,dy\,dz$, then $\Phi(v)\,dv$ is homogeneous of degree zero. Define

$$\int_{\mathbb{P}V} \Phi(v)\,dv \quad \text{to be} \quad \int_{V^0} \Phi(v)\,dv.$$

Clearly we have

$$\int_{\mathbb{P}V} \Phi(v)\,dv = \int_{\mathbb{P}V} \Phi(gv)\,d(gv) = |\det g| \int_{\mathbb{P}V} \Phi(gv)\,dv.$$

Put $\nu(x) = |x|$ and $m = 3(s-1)/2$. Note that $\nu/\mu_s = \mu_{-s}$. Put $\langle v, w\rangle = {}^t v\,J w$. Then $\langle gv, \sigma(g) w\rangle = \langle v, w\rangle$.

Lemma 1. *The operator* $T_s\colon V_s \to V_{-s}$,

$$(T_s\phi)(v) = \int_{\mathbb{P}V} \phi(w)\,|\langle w, v\rangle|^m\,dw,$$

converges when $\operatorname{Re} s > 2/3$ *and satisfies* $T_s\,\tau_s(g) = \tau_{-s}(\sigma g)\,T_s$ *for all* g *in* $G(F)$ *where it converges.*

Proof. We have

$$\begin{aligned}(T_s(\tau_s(g)\phi))(v) &= \int (\tau_s(g)\phi)(w)\ \ |{}^t w\,J\,v|^m\ \ dw = \mu(\det g)\int \phi({}^t g w)\ \ |{}^t w\,J\,v|^m\,dw\\ &= |\det g|^{-1}\,\mu(\det g)\int \phi(w)\ \ |{}^t({}^t g^{-1} w)\,J\,v|^m\,dw\\ &= (\mu/\nu)(\det g)\int \phi(w)\,|{}^t w J\cdot J g^{-1} J v|^m\,dw\\ &= (\mu/\nu)(\det g)\int \phi(w)\ \ |\langle w, \sigma({}^t g)v\rangle|^m\,dw = [(\nu/\mu)(\det \sigma g)]\,[(T_s\phi)(\sigma({}^t g)v)]\\ &= [(\tau_{-s}(\sigma g))(T_s\phi)](v),\end{aligned}$$

as required.

The spaces V_s are isomorphic to the space W of locally-constant complex-valued functions on V^0, and T_s is equivalent to an operator T_s^0 on W. The proof of Lemma 1 implies also

Corollary 1. *The operator* $T_s^0 \circ \tau_s(g^{-1})$ *is an integral operator with kernel*

$$(\mu/\nu)(\det \sigma g)\,|\langle w, \sigma({}^t g^{-1})v\rangle|^m\ (v, w \text{ in } V^0)$$

and trace

$$\operatorname{tr}[T_s^0 \circ \tau_s(g^{-1})] = (\nu/\mu)(\det g)\int_{V^0} |{}^t v g\,J\,v|^m\,dv.$$

Remark. (1) In the domain where the integral converges, it is clear that $\operatorname{tr}[T_s^0 \circ \tau_s(g^{-1})]$ depends only on the σ-conjugacy class of g if (and only if) $s=0$. (2) We evaluate below this integral at $s=0$ in a case where it converges for all s, and no analytic difficulties occur. However, in the context of the Remark following the proof of our proposition, we claim that to compute the trace of the analytic continuation of $T_s^0 \circ \tau_s(g^{-1})$ it suffices to compute this trace for s in the domain of convergence, and then evaluate the resulting expression at the desired s. Indeed, for each compact open σ-invariant subgroup K of G the space W_K of K-biinvariant functions on W is finite dimensional. Denote by $p_K\colon W \to W_K$ the natural projection. Then $T_s^0 \circ \tau_s(g^{-1}) \circ p_K$ acts on W_K, and the trace of the analytic continuation of $T_s^0 \circ \tau_s(g^{-1}) \circ p_K$ is the analytic continuation of the trace of $T_s^0 \circ \tau_s(g^{-1}) \circ p_K$. Since K can be taken to be arbitrarily small the claim follows.

Next we normalize the operator $T = T_s$ so that it acts trivially on the one-dimensional space of K-fixed vectors in V_s. This space is spanned by the function ϕ_0 in V_s with $\phi_0(v) = 1$ for all v in V^0. Fix a local uniformizer π in R. Let q be the cardinality of the quotient field of R. Normalize the valuation $|\cdot|$ by $|\pi| = q^{-1}$. Normalize the measure dx by $\int_{|x| \leq 1} dx = 1$, so that $\int_{|x|=1} dx = 1 - q^{-1}$. In particular, the volume of V^0 is $(1-q^{-3})/(1-q^{-1}) = 1 + q^{-1} + q^{-2}$.

Lemma 2. *We have* $(T\phi_0)(v_0) = (1 - q^{-3(s+1)/2})\,(1 - q^{(1-3s)/2})^{-1}\,\phi_0(v_0)$. *When* $s=0$ *the constant is* $-q^{-1/2}(1 + q^{-1/2} + q^{-1})$.

Proof.

$$\int \phi_0(v)\,|{}^t v\, J\, v_0|^m\, dv = \int_{V^0} |x|^m\, dx\, dy\, dz = (1-q^{-3(s+1)/2}) \int_{|x|\leq 1} |x|^m\, dx \Big/ \int_{|x|=1} dx,$$

as required.

To complete the proof of the proposition we have to compute $\mathrm{tr}[T\circ\tau_s(\delta^{-1})]$, $T=T_s^0$. Put $a=\begin{pmatrix}\alpha & 1\\ \theta & \alpha\end{pmatrix}$ with $\alpha\neq 0$ in F and θ in $F-F^2$ with $|\theta|=1$ or $|\theta|=q^{-1}$. Put

$$\delta=\delta_u=u(u^{-1}ae)_1=\begin{pmatrix}-\alpha & 0 & 1\\ 0 & u & 0\\ -\theta & 0 & \alpha\end{pmatrix},$$

where u ranges over a set of representatives in $F^\times$ for $F^\times/NK^\times$, where $K=F(\theta^{1/2})$. Then $\det\delta=u(\theta-\alpha^2)$. The eigenvalues of $\delta\sigma(\delta)=(-(\det a)^{-1}a^2)_1$ are $\lambda, 1, \lambda^{-1}$ where $\lambda=-(\alpha+\theta^{1/2})/(\alpha-\theta^{1/2})$. We have

$$(1+\lambda)(1+\lambda^{-1})=\left(1-\frac{\alpha+\theta^{1/2}}{\alpha-\theta^{1/2}}\right)\left(1-\frac{\alpha-\theta^{1/2}}{\alpha+\theta^{1/2}}\right)=\frac{-4\theta}{\alpha^2-\theta},$$

hence

$$(v/\mu)(\det\delta)\,\Delta(\delta)/\Delta_1(\gamma_1)=|u(\alpha^2-\theta)|^{(1-s)/2}|4\theta/(\alpha^2-\theta)|^{1/2}=|4u\theta|^{1/2}|u(\alpha^2-\theta)|^{-s/2}.$$

Further,

$$\delta J=\begin{pmatrix}1 & 0 & -\alpha\\ 0 & u & 0\\ \alpha & 0 & -\theta\end{pmatrix},$$

hence ${}^t v\,\delta\, J\, v = x^2+uy^2-\theta z^2$. Consequently

$$\frac{\Delta(\delta)}{\Delta_1(\gamma_1)}\mathrm{tr}[T\circ\tau_s(\delta^{-1})]=|4u\theta|^{1/2}|u(\alpha^2-\theta)|^{-s/2}\int_{V^0}|uy^2+x^2-\theta z^2|^{3(s-1)/2}\,dx\,dy\,dz.$$

We are interested in the value of this expression at $s=0$. When $\kappa(\delta)=1$ the quadratic form $uy^2+x^2-\theta z^2$ represents zero. Then the integral converges only for s with $\mathrm{Re}\, s>2/3$, but not at $s=0$. At $s=0$ the integral can be evaluated by analytic continuation. However when $\kappa(\delta)=-1$ the quadratic form $uy^2+x^2-\theta z^2$ is anisotropic, hence reaches a non-zero minimum (in valuation) on the compact set $\|v\|=1$. Consequently the integral converges for all values of s, and we may restrict our attention to the case of $s=0$. Here the character depends only on the σ-conjugacy class of δ, and we may take $|u|=1$ if $|\theta|=q^{-1}$, and $|u|=q^{-1}$ if $|\theta|=1$. Then $|u\theta|^{1/2}=q^{-1/2}$ and

$$\int_{\|v\|=1}|uy^2+x^2-\theta z^2|^{-3/2}\,dx\,dy\,dz=(1+q^{-1/2}+q^{-1})\int_{|x|=1}dx.$$

We conclude that

$$\frac{\Delta(\delta)}{\Delta_1(\gamma_1)}\mathrm{tr}[\tau_s(\delta)\circ T]=\kappa(\delta)\,(T\phi_0)(v_0)$$

when $\kappa(\delta)=-1$, hence for all σ-regular δ with elliptic $\gamma_1=N_1\delta$, by Theorem 0. Since $\chi_\pi(\delta)=\operatorname{tr}[\tau_s(\delta)\circ T]/(T\phi_0)(v_0)$, the proposition follows.

Remark. It is clear that when $\kappa(\delta)=1$ the proof of Proposition 1 can be completed without using Theorem 0 on computing $\operatorname{tr}[T_s^0\circ\tau_s(\delta^{-1})]$ by analytic continuation, namely first for large $\operatorname{Re}s$ and then on evaluating the resulting expression at $s=0$.

To prove Theorem 2 we have to take corresponding spherical functions f and f_1, and show that $\Delta(\delta)\,\Phi^{us}(\delta,f)=\Delta_1(\gamma_1)\,\Phi(\gamma_1,f_1)$ for all σ-regular δ with $\gamma_1=N_1\delta$. When γ_1 is split (its centralizer in $H_1(F)$ is conjugate to the diagonal torus), then the stable σ-conjugacy class of δ consists of a single σ-conjugacy class, $\kappa(\delta)=1$ and the required relation follows formally from the definition of f, f_1 being corresponding (see [II, § 1]). Hence we have to prove the equality when γ_1 is elliptic regular (the quotient λ of its eigenvalues generates a quadratic extension of F).

The proof of Theorem 2 which is now to follow is global and uses the trace formula. We shall use the notations of [IV, § 1] without much ado. That is, we fix a totally imaginary global field F such that its completion at a place u is our local field, which is now denoted by F_u. Let ϕ_{1u} be a pseudo-coefficient of the Steinberg $H_1(F_u)$-module St_{1u}. Here St_{1u} is the complement of the trivial $H_1(F_u)$-module in the $H_1(F_u)$-module $I_{1u}(v_u^{1/2})$ unitarily induced from the character $\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}\to|a/c|_u^{1/2}$ of the upper triangular subgroup of $H_1(F_u)$. By definition, $\operatorname{tr}\mathrm{St}_{1u}(\phi_{1u})=1$ and $\operatorname{tr}\pi_{1u}(\phi_{1u})=0$ for any tempered irreducible $H_1(F_u)$-module π_{1u} inequivalent to St_{1u}. By [K], Theorem K, the orbital integral $\Phi(\gamma_1,\phi_{1u})$ is zero if the quotients of the eigenvalues of γ_1 lie in $F_u^\times$ but are different from 1 or -1, and $\Phi(\gamma_1,\phi_{1u})$ is equal to -1 if the quotients of the eigenvalues of γ_1 do not lie in $F_u^\times$. Note that $\operatorname{tr}\pi_{1u}(\phi_{1u})=-1$ if π_{1u} is the trivial $H_1(F_u)$-module, and $\operatorname{tr}\pi_{1u}(\phi_{1u})=0$ for any other non-tempered representation.

Let ϕ_u be a function on $G(F_u)$ matching ϕ_{1u} such that $\Phi(\delta,\phi_u)=-\Phi(\delta',\phi_u)$ if δ,δ' are σ-regular stably σ-conjugate but not σ-conjugate elements. The existence of such a function is proven in [I, § 3] by a local elementary proof. If π_{1u} is induced then the (twisted) character χ_{π_u} of $\pi_u=I(\pi_{1u})$ is supported on the σ-split set (see [II], § 1), hence $\operatorname{tr}\pi_u(\phi_u\times\sigma)=0$. By Proposition 1 for σ-elliptic regular δ we have

$$\begin{aligned}\kappa(\delta)\,\Delta(\delta)\,(\chi_{I(\mathrm{St}_{1u})})(\delta)&=\Delta_1(\gamma_1)\,(\chi_{\mathrm{St}_{1u}})(\gamma_1)=\Delta_1(\gamma_1)\,\Phi(\gamma_1,\phi_{1u})\\&=\Delta(\delta)\,\Phi^{us}(\delta,\phi_u)=2\Delta(\delta)\,\Phi(\delta,\phi_u)\,\kappa(\delta),\end{aligned}$$

hence $2\Phi(\delta,\phi_u)=\chi_{I(\mathrm{St}_{1u})}(\delta)$, and by [II, § 3] we conclude that $\operatorname{tr}(I(\mathrm{St}_{1u}))\,(\phi_u\times\sigma)=1$, hence $\operatorname{tr}(I(\pi_{1u}))\,(\phi_u\times\sigma)=-1$ if π_{1u} is the trivial $H_1(F_u)$-module. If π_{1u} is a square-integrable $H_1(F_u)$-module inequivalent to St_{1u} then $I(\pi_{1u})$ and $I(\mathrm{St}_{1u})$ are not relatives in the terminology of [K], hence their characters are orthogonal by [K], Theorem G, as stated in [II, § 3]. Note that although the work of [K] is formulated in the non-twisted case only, the twisted analogue follows by the same proof on noting that the twisted analogue of [K], Theorem 0, is available (in our case it is given in [IV, § 1]). In particular we have $\operatorname{tr}(I(\pi_{1u}))\,(\phi_u\times\sigma)=0$ for all tempered π_{1u} inequivalent to St_{1u}. Moreover, since the residual characteristic of F_u is odd, all σ-invariant elliptic $G(F_u)$-modules not mentioned above are of the form $I(\pi_u(\theta/\bar{\theta}), \chi_u)$ by [IV, § 1] and in the notations of [IV, § 1] (thus χ_u is a quadratic character and θ is a character of the quadratic extension of F_u determined by χ_u and class field theory). Their characters are σ-stable by [IV, § 2], hence $\operatorname{tr}\pi(\phi_u \times \sigma)=0$ for such π. In summary we have

Proposition 2. *There exist matching functions ϕ_u and ϕ_{1u} on $G(F_u)$ and $H_1(F_u)$ with* (1) $\operatorname{tr}(I(\pi_{1u}))\,(\phi_u \times \sigma)=\operatorname{tr}\pi_{1u}(\phi_{1u})$ *for all* π_{1u}; (2) $\operatorname{tr}\pi_{1u}(\phi_{1u})=-1$ *if* π_{1u} *is trivial,* $\operatorname{tr}\pi_{1u}(\phi_{1u})=1$ *if* $\pi_{1u}=\mathrm{St}_{1u}$ *and* $\operatorname{tr}\pi_{1u}(\phi_{1u})=0$ *otherwise;* (3) $\operatorname{tr}\pi_u(\phi_u \times \sigma)=0$ *unless* π_u *is* $I(\mathrm{St}_{1u})$ *or* $I(\pi_{1u})$ *with trivial* π_{1u}; (4) $\Phi(\gamma_1, \phi_{1u})$ *is* -1 *on the regular elliptic set and zero on the regular split set;* (5) $\Phi(\delta, \phi_u)=-\Phi(\delta', \phi_u)$ *if* δ, δ' *are σ-elliptic regular stably σ-conjugate non-σ-conjugate elements of* $G(F)$.

For any place v of F, let μ_v be a character of $F_v^\times$ and π_{1v} the $H_1(F_v)$-module $I(\mu_v)$ unitarily induced from the character $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \to \mu_v(a/c)$ of the upper triangular subgroup. Then the character of π_{1v} is supported on the split set, the character of $I(\pi_{1v})$ is supported on the σ-split set, they are easily computable and comparable (see [II, § 1]), and we have $\operatorname{tr}(I(\pi_{1v})\,(f_v \times \sigma)=\operatorname{tr}\pi_{1v}(f_{1v})$ for all matching f_v and f_{1v}. When v is an archimedean place of F then the completion F_v is $\mathbb{C}$ since F is totally imaginary. Since any unitary non-induced $H_1(\mathbb{C})$-module is one-dimensional, and its character is the difference of the characters of two induced $H_1(\mathbb{C})$-modules, we conclude

Proposition 3. *If $F_v=\mathbb{C}$ and f_v, f_{1v} are matching functions then* $\operatorname{tr}(I(\pi_{1v}))\,(f_v \times \sigma)$ $=\operatorname{tr}\pi_{1v}(f_{1v})$ *for all $H_1(F_v)$-modules π_{1v}.*

This is a trivial case of Theorem 3′, where $F_v=\mathbb{C}$.

Let u_0, u_1 be two finite distinct places of F, different from u, of odd residual characteristic. Denote by $\mathbb{A}$ the ring of adeles of F. Let $f_1=\otimes f_{1v}$ be a function on $H_1(\mathbb{A})$ with $f_{1v}=f_{1v}^0$ for almost all (finite) v, with f_{1v} in the Hecke algebra $\mathbb{H}_{1v}$ for all finite $v \neq u$, u_0, u_1, and with $f_{1u}=\phi_{1u}, f_{1u_0}=\phi_{1u_0}$, and $f_{1u_1}=\phi_{1u_1}$. Let $f=\otimes f_v$ be a function on $G(\mathbb{A})$ such that (1) $f_u=\phi_u, f_{u_0}=\phi_{u_0}$ and $f_{u_1}=\phi_{u_1}$, (2) at each finite $v \neq u$, u_0, u_1 the component f_v lies in $\mathbb{H}_v$ and f_v, f_{1v} are corresponding (in particular, f_v is f_v^0 for almost all v), (3) at each archimedean place v the functions f_v and f_{1v} are matching. Then $\operatorname{tr}\pi_{1v}(f_{1v})=\operatorname{tr}(I(\pi_{1v}))\,(f_v \times \sigma)$ for every $H_1(F_v)$-module π_{1v}, for every place v, and therefore $\operatorname{tr}\pi_1(f_1)=\operatorname{tr}(I(\pi_1))$ $(f \times \sigma)$ for every $H_1(\mathbb{A})$-module π_1. In particular, we obviously have the following

Proposition 4. *We have*

$$\sum_{\pi_1} \operatorname{tr}\pi_1(f_1)=\sum_{\pi_1} \operatorname{tr}(I(\pi_1))\,(f \times \sigma).$$

Here both sums range over all discrete-series (cuspidal or one-dimensional) automorphic $H_1(\mathbb{A})$-modules π_1. Of course the π_1 which contribute a non-zero term have a Steinberg or trivial component at the places u, u_0, u_1, while all of their other finite components are unramified.

Choose a component f_∞ at an archimedean place ∞ to vanish on the set of non-σ-regular elements δ in $G(F_\infty)$. Since at $v=u_0, u_1$ the components f_v

(resp. f_{1v}) have orbital integrals which vanish on the σ-regular-split (resp. regular split) sets, the trace formula for H_1 asserts the following

Proposition 5. *We have*

$$\sum_{\gamma_1} c(\gamma_1)\,\Delta_1(\gamma_1)\,\Phi(\gamma_1,f_1)=\sum_{\pi_1}\operatorname{tr}\pi_1(f_1).$$

Here γ_1 ranges over the set of regular elliptic conjugacy classes in $H_1(F)$, $c(\gamma_1)$ is a volume factor, and

$$\Delta_1(\gamma_1)\,\Phi(\gamma_1,f_1)=\Phi(\gamma_1,f_1)\qquad\text{is the product }\prod_v \Delta_{1v}(\gamma_1)\cdot\Phi(\gamma_1,f_{1v}).$$

Since the stable twisted orbital integrals of f_{u_0} (and f_{u_1}) are zero, the twisted trace formula for G in its stabilized form (see [III, § 3]), asserts the following

Proposition 6. *We have*

$$\sum_{\gamma_1} c(\gamma_1)\,\Delta(\delta)\,\Phi^{us}(\delta,f)=\sum_{\pi_1}\operatorname{tr}(I(\pi_1))\,(f\times\sigma).$$

The sum over γ_1 and the volume factors $c(\gamma_1)$ are the same as above (as noted in [III, § 1]), δ signifies (a representative of) the stable σ-conjugacy class in $G(F)$ with $\gamma_1=N_1\delta$, and $\Delta(\delta)\,\Phi^{us}(\delta,f)=\Phi^{us}(\delta,f)$ the product $\prod_v \Delta_v(\delta)\,\Phi^{us}(\delta,f_v)$.

We briefly sketch the stabilization argument on which the proof of Proposition 6 is based. The sum over σ-conjugacy classes in the twisted trace formula can be expressed as a sum over the set of stable σ-conjugacy classes δ_0 in $G(F)$, of the sums $\Sigma\Phi(\delta,f)$, where here δ ranges over the set $D(\delta_0/F)$ of σ-conjugacy classes in $G(F)$ within the stable σ-conjugacy class of δ_0. Since δ_0 is σ-regular elliptic for our f, the set $D(\delta_0/F)$ is isomorphic to $F^\times/NK^\times$, where $K=F(N_1\delta)$. $D(\delta_0/F)$ has index two in $D(\delta_0/\mathbb{A})\simeq\mathbb{A}^\times/N\mathbb{A}_K^\times$. Hence

$$\sum_{\delta\text{ in }D(\delta_0/F)}\Phi(\delta,f)=\tfrac12\sum_{\delta\text{ in }D(\delta_0/\mathbb{A})}\Phi(\delta,f)+\tfrac12\sum_{\delta\text{ in }D(\delta_0/\mathbb{A})}\kappa(\delta)\,\Phi(\delta,f).$$

Since the stable orbital integral $\Phi^{st}(\delta,f_{u_0})=\Phi(\delta,f_{u_0})+\Phi(\delta',f_{u_0})$ is assumed to be zero, we have that

$$\sum_{\delta\text{ in }D(\delta_0/\mathbb{A})}\Phi(\delta,f)=\prod_v \Phi^{st}(\delta_0,f_v)=\Phi^{st}(\delta_0,f)$$

is zero, implying the desired equality

$$\sum_{\delta\text{ in }D(\delta_0/F)}\Phi(\delta,f)=\tfrac12\,\Phi^{us}(\delta_0,f)$$

for our f, from which Proposition 6 follows.

Combining Propositions 4, 5 and 6 we obtain

Proposition 7. *We have*

$$\sum c(\gamma_1)\,\Delta_1(\gamma_1)\,\Phi(\gamma_1,f_1)=\sum c(\gamma_1)\,\Delta(\delta)\,\Phi^{us}(\delta,f). \tag{7}$$

Lemma. *Both sums in* (7) *are finite.*

Proof. Identifying $PGL(2)$ with the subgroup $SO(3)$ of $GL(3)$, we note that γ_1 is determined by the coefficients in its characteristic polynomial. These coefficients are rational (in F), and f_1 (or f) is compactly supported, whence the sums are finite.

Let γ_{1u}^0 be a regular elliptic element of $H_1(F_u)$. Then there exists an element γ_1^0 of $H_1(F)$ which is elliptic regular in $H_1(F_{u_0})$ and $H_1(F_{u_1})$, and whose orbit in $H_1(F_u)$ is as close to that of γ_{1u}^0 as desired. At each $v \neq u$, u_0, u_1 choose f_{1v} with $\Phi(\gamma_1^0, f_{1v}) \neq 0$ such that $f_{1v} = f_{1v}^0$ for almost all v; this is clearly possible, since γ_1^0 is a rational element, in $H_1(F)$. For our f_1, which depends on γ_1^0, the sum on the left of (7) is finite, and includes γ_1^0. We now replace $f_{1\infty}$ by its product with a smooth function which takes the value one on the orbit of γ_1^0 in $H_1(F_\infty)$ and vanishes outside of a small neighbourhood of this orbit; choosing a suitable replacement we may assume that γ_1^0 is the only class which contributes a non-zero term on the left of (7).

Next we denote by δ^0 the stable σ-conjugacy class in $G(F)$ with $N_1 \delta^0 = \gamma_1^0$. The sum on the right of (7) is also finite. We can replace the component f_∞, as above, by another function with the property that $\Phi^{us}(\delta^0, f_\infty)$ will not change, yet $\Phi^{us}(\delta, f_\infty)$ be zero at each of the finitely many (stable) classes $\delta \neq \delta^0$ which appear on the right of (7). We conclude that $f_{1\infty}$ (and f_∞) can be chosen so that we obtain the following

Proposition 8. *We have*

$$\prod_{v \neq u} \Delta_{1v}(\gamma_1^0)\, \Phi(\gamma_1^0, f_{1v}) = \prod_{v \neq u} \Delta_v(\delta^0)\, \Phi^{us}(\delta^0, f_v). \tag{8}$$

In particular the right side here is non-zero.

Proof. It is clear that we have (8) where the product ranges over all places v. Since γ_1^0 is elliptic regular in $H_1(F_u)$ and f_{1u} is a pseudo-coefficient of the Steinberg $H_1(F_u)$-module, we have $\Phi(\gamma_1^0, f_{1u}) = -1$. Further we have

$$\Delta_{1u}(\gamma_1^0)\, \Phi(\gamma_1^0, f_{1u}) = \Delta_u(\delta^0)\, \Phi^{us}(\delta^0, f_u)$$

since f_{1u} and f_u are matching (by definition of f_u, which uses Proposition 2: here $f_{1u} = \phi_{1u}$ and $f_u = \phi_u$). Hence we can take the product to range only over $v \neq u$, as asserted.

We can now complete the proof of Theorem 2. Let ϕ and ϕ_1 be corresponding elements of $\mathbb{H}_u$ and $\mathbb{H}_{1u}$. We have to show that

$$\Delta_{1u}(\gamma_{1u}^0)\, \Phi(\gamma_{1u}^0, \phi_1) = \Delta_u(\delta_u^0)\, \Phi^{us}(\delta_u^0, \phi)$$

for any regular elliptic γ_{1u}^0 in $H_1(F_u)$ and δ_u^0 with $\gamma_{1u}^0 = N_1 \delta_u^0$. Since ϕ, ϕ_1 are locally constant it suffices to show the following

Proposition 9. *We have*

$$\Delta_{1u}(\gamma_1^0)\, \Phi(\gamma_1^0, \phi_1) = \Delta_u(\delta^0)\, \Phi^{us}(\delta^0, \phi) \tag{9}$$

for γ_1^0 *as above.*

Proof. Let $f_1' = \otimes f_{1v}'$ and $f' = \otimes f_v'$ be the functions obtained from $f_1 = \otimes f_{1v}$ and $f = \otimes f_v$ on replacing the components f_{1u} and f_u by ϕ_1 and ϕ (thus $f_{1v}' = f_{1v}$ and $f_v' = f_v$ for $v \neq u$). Repeating the discussion leading to (8) with f_1', f' instead of f_1, f, we obtain

$$\prod_v \Delta_{1v}(\gamma_1^0)\, \Phi(\gamma_1^0, f_{1v}') = \prod_v \Delta_v(\delta^0)\, \Phi^{us}(\delta^0, f_v').$$

Since both sides of (8) are non-zero, (9) follows, and Theorem 2 is proven.

As explained above, this completes the proof of Theorems 1, 3 and 3′ as well.

Remark. (1) The proof given above can be adapted to establish the analogous unstable twisted transfer of spherical functions from $GL(3,E)$ to $U(2)$, which is stated (but neither used nor proved) in Lemma 3.4 in [U 1]; however we do not discuss this here. (2) The same method applies also in the study of the endo-lifting from $GL(m,E)$ to $GL(n,F)$, where E/F is a cyclic extension of degree n/m; see [F].

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