Automorphic forms on SO(4)

By Yuval Z. Flicker

Department of Mathematics, The Ohio State University
231 W. 18th Ave., Columbus, OH 43210-1174, U.S.A.
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Abstract: We announce results of [F1] on automorphic forms on SO(4). An initial result is the proof by means of the trace formula that the *functorial product* of two automorphic representations π_1 and π_2 of the adèle group $\mathrm{GL}(2, \mathbf{A}_F)$ whose central characters ω_1, ω_2 satisfy $\omega_1\omega_2 = 1$, *exists* as an automorphic representation $\pi_1 \boxtimes \pi_2$ of $\mathrm{PGL}(4, \mathbf{A}_F)$. The product is in the discrete spectrum if π_1 is inequivalent to a twist of the contragredient $\check{\pi}_2$ of π_2, and π_1, π_2 are not monomial from the same quadratic extension. If $\pi_2 = \check{\pi}_1$ then $\pi_1 \boxtimes \pi_2$ is the $\mathrm{PGL}(4, \mathbf{A}_F)$-module normalizedly parabolically induced from the $\mathrm{PGL}(3, \mathbf{A}_F)$-module $\mathrm{Sym}^2(\pi_1)$ on the Levi factor of the parabolic subgroup of type $(3, 1)$. Finer results include the definition of a local product $\pi_{1v} \boxtimes \pi_{2v}$ by means of characters, injectivity of the global product, and a description of its image. Thus the product $(\pi_1, \pi_2) \mapsto \pi_1 \boxtimes \pi_2$ is *injective* in the following sense. If π_1, π_2, π_1^0, π_2^0 are discrete spectrum representations of $\mathrm{GL}(2, \mathbf{A})$ with central characters ω_1, ω_2, ω_1^0, ω_2^0 satisfying $\omega_1\omega_2 = 1 = \omega_1^0\omega_2^0$, and for each place v outside a fixed finite set of places of the global field F there is a character χ_v of $F_v^\times$ such that $\{\pi_{1v}\chi_v, \pi_{2v}\chi_v^{-1}\} = \{\pi_{1v}^0, \pi_{1v}^0\}$, then there exists a character χ of $\mathbf{A}^\times/F^\times$ with $\{\pi_1\chi, \pi_2\chi^{-1}\} = \{\pi_1^0, \pi_2^0\}$. In particular, starting with a pair π_1, π_2 of discrete spectrum representations of $\mathrm{GL}(2, \mathbf{A})$ with $\omega_1\omega_2 = 1$, we cannot get another such pair by interchanging a set of their components π_{1v}, π_{2v} and multiplying π_{1v} by a local character and π_{2v} by its inverse, unless we interchange π_1, π_2 and multiply π_1 by a global character and π_2 by its inverse. The injectivity of $(\pi_1, \pi_2) \mapsto \pi_1 \boxtimes \pi_2$ is a strong rigidity theorem for SO(4). The self contragredient discrete spectrum representations of $\mathrm{PGL}(4, \mathbf{A})$ of the form $\pi_1 \boxtimes \pi_2$ are those not obtained from the lifting from the symplectic group $\mathrm{PGSp}(2, \mathbf{A})$.

Key words: Automorphic representations; orthogonal group; liftings; rigidity.

The main results of [F1] concern the study of the automorphic and admissible representations of the projective symplectic group PGSp(2) of similitudes and their relations with the self contragredient such representations of PGL(4). These results are obtained on stabilizing the twisted – by the transpose-inverse involution – trace formula on PGL(4). Stabilization means writing the twisted orbital integrals which occur in the geometric side of this twisted trace formula in terms of stable orbital integrals of the twisted endoscopic groups. In our case the twisted endoscopic groups are PGSp(2) and the split orthogonal group SO(4). The results of [F1] which concern PGSp(2) are summarized in [F2]. Here we go in a direction perpendicular to that of [F2], and report on the results of [F1] which concern the orthogonal group SO(4). These results are of independent interest, and can be viewed not only as results on the representation theory of SO(4) but also as results on pairs of representations of GL(2), as SO(4) is closely related to a product of two GL(2)'s.

The study of the stabilization of the twisted trace formula in [F1] leads to a theory of lifting of automorphic representations of (PGSp(2) and) SO(4) to PGL(4). The lifting from SO(4), which is isomorphic to a central quotient of a determinant-defined subgroup of GL(2) × GL(2), to PGL(4), can be interpreted as the *functorial product* of two discrete spectrum representations π_1 and π_2 of the adèle group $\mathrm{GL}(2, \mathbf{A}_F)$ whose central characters ω_1, ω_2 satisfy $\omega_1\omega_2 = 1$. Our main result in this language is the study of existence and of properties of this product. Thus we first show that the functorial lift **exists** as an automorphic representation $\pi_1 \boxtimes \pi_2$ of

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$\mathrm{PGL}(4, \mathbf{A}_F)$. The product is in the discrete spectrum if π_1 is inequivalent to a twist of the dual $\check{\pi}_2$ of π_2, and π_1, π_2 are not monomial from the same quadratic extension. If $\pi_2 = \check{\pi}_1$ then $\pi_1 \boxtimes \pi_2$ is the $\mathrm{PGL}(4, \mathbf{A}_F)$-module normalizedly induced from the $\mathrm{PGL}(3, \mathbf{A}_F)$-module $\mathrm{Sym}^2(\pi_1)$ on the Levi factor of the parabolic subgroup of type $(3, 1)$. Here $\mathrm{Sym}^2(\pi_1)$ denotes the symmetric square lifting of π_1 from GL(2) (in fact its restriction to SL(2)) to PGL(3) (or GL(3), with trivial central character). We say "normalizedly induced" for "induced in the normalized way, see, e.g., [BZ]". This initial result is claimed using an amalgam of a converse theorem and trace formulae results in [R], Theorem M, in whose last paragraph the 2nd condition for "iff" follows from the 1st.

The second remarkable result is a rigidity theorem for the automorphic representations of SO(4). It can be stated as asserting that the product $(\pi_1, \pi_2) \mapsto \pi_1 \boxtimes \pi_2$ is **injective**:

If π_1, π_2, π_1^0, π_2^0 are discrete spectrum representations of $\mathrm{GL}(2, \mathbf{A})$ with central characters ω_1, ω_2, ω_1^0, ω_2^0 satisfying $\omega_1\omega_2 = 1 = \omega_1^0\omega_2^0$, and for each place v outside a fixed finite set of places of the global field F there is a character χ_v of $F_v^\times$ such that $\{\pi_{1v}\chi_v, \pi_{2v}\chi_v^{-1}\} = \{\pi_{1v}^0, \pi_{1v}^0\}$, then there exists a character χ of $\mathbf{A}^\times/F^\times$ with $\{\pi_1\chi, \pi_2\chi^{-1}\} = \{\pi_1^0, \pi_2^0\}$.

In particular, starting with a pair π_1, π_2 of discrete spectrum representations of $\mathrm{GL}(2, \mathbf{A})$ with $\omega_1\omega_2 = 1$, we cannot get another such pair by interchanging a set of their components π_{1v}, π_{2v} and multiplying π_{1v} by a local character and π_{2v} by its inverse, unless we interchange π_1, π_2 and multiply π_1 by a global character and π_2 by its inverse.

Both results are obtained together with our study of the automorphic representations of the symplectic group PGSp(2) as related to the self dual representaions of PGL(4). They concern the representations of SO(4). The discrete spectrum images of the liftings from PGSp(2) and from SO(4) to PGL(4) are disjoint. They exhaust the set of self contragredient discrete spectrum representations of PGL(4). This determines the **image** of the lifting.

The study of [F1] of the global lifting from SO(4) to PGL(4) is based on local lifting results. In particular the local product $\pi_{1v}\boxtimes\pi_{2v}$ is defined for all admissible representations π_{iv} of $\mathrm{GL}(2, F_v)$ (with $\omega_{1v}\omega_{2v} = 1$). The definition is in terms of character ([H1, H2]) relations. For the precise statement, and proof that the local product exists, see [F1].

To make this report self contained we include Sections 1 and 2, which overlap with [F2].

1. Homomorphisms of dual groups. Let $\mathbf{G}$ be the projective general linear group PGL(4) = PSL(4) over a number field F. Our initial purpose is to determine the automorphic representations π of $\mathbf{G}(\mathbf{A})$, $\mathbf{A}$ is the ring of adèles of F, which are self-contragredient: $\pi \simeq \check{\pi}$, equivalently θ-invariant: $\pi \simeq {}^\theta\pi$. Here θ, $\theta(g) = J^{-1}{}^tg^{-1}J$, is the involution defined by

$$J = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

${}^tg$ is the transpose of $g \in \mathbf{G}$, and ${}^\theta\pi(g) = \pi(\theta(g))$. According to the principle of functoriality ([Bo]) these automorphic representations are essentially described by representations of the Weil group W_F of F (in fact the hypothetical Langlands group L_F) into the dual group $\widehat{G} = \mathrm{SL}(4, \mathbf{C})$ of $\mathbf{G}$ which are $\hat{\theta}$-invariant, namely representations of W_F into centralizers $Z_{\widehat{G}}(\hat{s}\hat{\theta})$ of $\mathrm{Int}(\hat{s})\hat{\theta}$ in $\widehat{G}$. Here $\hat{\theta}$ is the dual involution $\hat{\theta}(\hat{g}) = J^{-1}{}^t\hat{g}^{-1}J$, and $\hat{s}$ is a semisimple element in $\widehat{G}$. These centralizers are the duals of the *twisted* (by $\hat{s}\hat{\theta}$) *endoscopic groups* ([KS]).

A twisted endoscopic group is called *elliptic* if its dual is not contained in a proper parabolic subgroup of $\widehat{G}$. Representations of nonelliptic endoscopic groups can be reduced by parabolic induction to known ones of smaller rank groups. For our $\widehat{G}$, up to conjugacy the elliptic twisted endoscopic groups have as duals the symplectic group $\widehat{H} = Z_{\widehat{G}}(\hat{\theta}) = \mathrm{Sp}(2, \mathbf{C})$ and the special orthogonal group

$$\widehat{C} = Z_{\widehat{G}}(\hat{s}\hat{\theta}) = \text{"SO}(4, \mathbf{C})\text{"}$$
$$= \left\{ g \in \mathrm{SL}(4, \mathbf{C}); g\hat{s}J{}^tg = \hat{s}J = \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix} \right\},$$

of all $\quad A \otimes B = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}$;

$$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B \right) \in (\mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(2, \mathbf{C}))/\mathbf{C}^\times$$

which satisfy $\det A \cdot \det B = 1$. Here $z \in \mathbf{C}^\times$ embeds as (z, z^{-1}), $\hat{s} = \mathrm{diag}(-1, 1, -1, 1)$ and $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The group $\widehat{H}$ is the dual group of the simple F-group $\mathbf{H} = \mathrm{PSp}(2) = \mathrm{PGSp}(2)$.

The group $\widehat{C}$ is the dual group of the special orthogonal group $\mathbf{C}$ ="SO(4)" of pairs $(g_1, g_2) \in$

$(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathbf{G}_m$ with $\det g_1 = \det g_2$. Here $z\in\mathbf{G}_m$ embeds as the central element (z,z). Also we write $((\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1))'$ for $\mathbf{C}$, where the prime indicates that the two factors in GL(2) have equal determinants, and $(\dots)''$ if their product is 1.

The principle of functoriality suggests that automorphic representations of $\mathbf{H}(\mathbf{A})$ and $\mathbf{C}(\mathbf{A})$ parametrize (or lift to) the θ-invariant automorphic representations of the group $\mathbf{G}(\mathbf{A})$ of $\mathbf{A}$-valued points of $\mathbf{G}$. Our main purpose is to describe this parametrization, in particular define tensor products of two automorphic forms of $\mathrm{GL}(2,\mathbf{A})$ the product of whose central characters is 1, and (in [F2]) describe the automorphic representations of the projective symplectic group of similitudes of rank two, $\mathrm{PGSp}(2,\mathbf{A})$, in terms of θ-invariant representations of $\mathrm{PGL}(4,\mathbf{A})$.

2. Unramified lifting. We proceed to explain how the liftings are defined, first for unramified representations.

An irreducible admissible representation π of an adèle group $\mathbf{G}(\mathbf{A})$ is the restricted tensor product $\otimes\pi_v$ of irreducible admissible representations π_v of the groups $\mathbf{G}(F_v)$ of F_v-points of $\mathbf{G}$, where F_v is the completion of F at the place v of F. Almost all the local components π_v are *unramified*, that is contain a (unique up to a scalar multiple) nonzero K_v-fixed vector. Here K_v is the standard maximal compact subgroup of $\mathbf{G}(F_v)$, namely the group $\mathbf{G}(R_v)$ of R_v-points, R_v being the ring of integers of the nonarchimedean local field F_v; $\mathbf{G}$ is defined over R_v at almost all v. An irreducible unramified $\mathbf{G}(F_v)$-module π_v is the unique unramified irreducible constituent in an unramified principal series representation $I(\eta_v)$, normalizedly induced from an unramified character η_v of the maximal torus $\mathbf{T}(F_v)$ of a Borel subgroup $\mathbf{B}(F_v)$ of $\mathbf{G}(F_v)$ (extended trivially to the unipotent radical $\mathbf{N}(F_v)$ of $\mathbf{B}(F_v)$). The space of $I(\eta_v)$ consists of the smooth functions $\phi:\mathbf{G}(F_v)\to\mathbf{C}$ with $\phi(ank) = (\delta_v^{1/2}\eta_v)(a)\phi(k)$, $k\in K_v$, $n\in\mathbf{N}(F_v)$, $a\in\mathbf{T}(F_v)$, $\delta_v(a) = \det[\mathrm{Ad}(a)|\operatorname{Lie}\mathbf{N}(F_v)]$, and the $\mathbf{G}(F_v)$-action is $(g\cdot\phi)(h) = \phi(hg)$, $g,h\in\mathbf{G}(F_v)$.

The character η_v is unramified: it factors as $\eta_v : \mathbf{T}(F_v)/\mathbf{T}(R_v)\to\mathbf{C}^\times$. As $\mathbf{T}(F_v)/\mathbf{T}(R_v)\simeq X_*(\mathbf{T}) = \mathrm{Hom}(\mathbf{G}_m,\mathbf{T})$, η_v lies in $\mathrm{Hom}(X_*(\mathbf{T}),\mathbf{C}^\times) = \mathrm{Hom}(X^*(\widehat{T}),\mathbf{C}^\times)$, where $\widehat{T}$ is the maximal torus in the Borel subgroup $\widehat{B}$ of $\widehat{G}$, both fixed in the definition of the (complex) dual group $\widehat{G}$ ([Bo], [Ko]). Now $\mathrm{Hom}(X^*(\widehat{T}),\mathbf{C}^\times) = X_*(\widehat{T})\otimes\mathbf{C}^\times = \widehat{T}\subset\widehat{G}$, thus the unramified irreducible $\mathbf{G}(F_v)$-module π_v determines a conjugacy class $t(\pi_v) = t(I(\eta_v))$ (the "*Langlands* or *Satake parameter*") in $\widehat{G}$, represented by the image of η_v in $\widehat{T}$.

3. The lifting $\boldsymbol{\lambda_1}$ from SO(4) to PGL(4). Our results on the lifting λ_1 from $\mathbf{C} = \mathrm{SO}(4) = ((\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1))'$ to $\mathbf{G} = \mathrm{PGL}(4)$ are described next. An unramified (irreducible) representation π_{1v} of $\mathrm{GL}(2,F_v)$ is parametrized by a conjugacy class $t(\pi_{1v})$ in $\mathrm{GL}(2,\mathbf{C})$ (the Langlands parameter; its eigenvalues are called the *Hecke eigenvalues* of the representation). An unramified irreducible $\mathbf{C}(F_v)$-module $\pi_{1v}\times\pi_{2v}$ is parametrized by a class $t(\pi_{1v})\times t(\pi_{2v})$ in $((\mathrm{GL}(2,\mathbf{C})\times\mathrm{GL}(2,\mathbf{C}))/\mathbf{C}^\times)'' = \widehat{C}\subset\widehat{G}$. If π_{iv} is the unramified constituent of $I(\eta_{iv})$, $\eta_{ij} = \eta_{ijv}(\boldsymbol{\pi}_v)$, $\eta_{11}\eta_{12}\eta_{21}\eta_{22} = 1$, $t(\pi_{iv}) = \mathrm{diag}(\eta_{i1},\eta_{i2})$, we define the "lift" $\pi_{1v}\boxtimes\pi_{2v} = \lambda_1(\pi_{1v}\times\pi_{2v})$ of $\pi_{1v}\times\pi_{2v}$ with respect to the dual group homomorphism $\lambda_1:\widehat{C} = \mathrm{SO}(4,\mathbf{C})\hookrightarrow\widehat{G} = \mathrm{SL}(4,\mathbf{C})$ (the natural embedding) to be the unramified irreducible constituent π_v of the $\mathrm{PGL}(4,F_v)$-module $I(\eta_v)$ parametrized by the class $t(\pi_v) = \mathrm{diag}(\eta_{11}\eta_{21},\eta_{11}\eta_{22},\eta_{12}\eta_{21},\eta_{12}\eta_{22})$ in $\widehat{G} = \mathrm{SL}(4,\mathbf{C})$. In different notations, $\lambda_1(I(a_1,a_2)\times I(b_1,b_2)) = I(a_1b_1,a_1b_2,a_2b_1,a_2b_2)$ where $a_i,b_i\in\mathbf{C}^\times$, provided that $a_1a_2b_1b_2 = 1$. The inverse image $\lambda_1^{-1}(I(a_1b_1,a_1b_2,b_1a_2,a_2b_2))$ consists only of $\chi I(a_1,a_2)\times\chi^{-1}I(b_1,b_2)$ and $\chi I(b_1,b_2)\times\chi^{-1}I(a_1,a_2)$ where χ is any character of $F_v^\times$. Thus, λ_1 is two-to-one unless $\pi_{1v} = \check{\pi}_{2v}$ (the contragredient of π_{2v}), where λ_1 is injective on the set of orbits of multiplication by χ in $\mathrm{Hom}(F_v^\times,\mathbf{C}^\times)$.

The rigidity theorem for the discrete spectrum of $\mathrm{GL}(n,\mathbf{A})$ asserts that discrete spectrum representations $\pi_1 = \otimes\pi_{1v}$ and $\pi_2 = \otimes\pi_{2v}$ which have $\pi_{1v}\simeq\pi_{2v}$ at almost all places v of F are equivalent ([JS], [MW]). They are even equal, by the multiplicity one theorem for $\mathrm{GL}(n)$. Representations of $\mathrm{PGL}(n,\mathbf{A})$ (or $\mathrm{PGL}(n,F_v)$) are simply representations of $\mathrm{GL}(n,\mathbf{A})$ (or $\mathrm{GL}(n,F_v)$) with trivial central character (since $H^1(F,\mathbf{G}_m) = \{0\}$), and the rigidity theorem applies then to $\mathrm{PGL}(n)$. Both multiplicity one theorem ([F3, F4, F5], [R]), and the rigidity theorem for packets ([F3, F4, F5]; the latter asserts that $\pi = \otimes\pi_v$ and $\pi' = \otimes\pi'_v$ must lie in the same packet if $\pi_v\simeq\pi'_v$ for almost all v) hold for SL(2). Both fail for $\mathrm{SL}(n)$, $n\geq 3$ ([Bla]).

The rigidity theorem holds for $\mathbf{C} = \mathrm{SO}(4)$; this is the content of the assertion that the lifting λ_1 is injective, made in the final paragraph of the following theorem. The existence is claimed in [R] too. We use the trace formula approach. By an *elliptic representation* we mean one whose character ([H1, H2]) is not identically zero on the set of elliptic elements.

Theorem (SO(4) to PGL(4)). *Let $\pi_1 = \otimes\pi_{1v}$, $\pi_2 = \otimes\pi_{2v}$ be automorphic representations of* $\mathrm{GL}(2,\mathbf{A})$ *whose central characters ω_1, ω_2 are equal, and whose components at two places v_1, v_2 are elliptic. Then there* ***exists*** *an automorphic representation $\pi = \lambda_1(\pi_1 \times \check{\pi}_2)$ of* $\mathrm{PGL}(4,\mathbf{A})$ *with $\pi_v = \lambda_1(\pi_{1v} \times \check{\pi}_{2v})$ for almost all v.*

We have $\lambda_1(\chi_1\pi_1 \times \chi_2\pi_2) = \chi_1\chi_2\lambda_1(\pi_1 \times \pi_2)$ for $\chi_i : \mathbf{A}^\times/F^\times \to \mathbf{C}^\times$ with $(\chi_1\chi_2)^2 = 1$.

If $\pi_1 = \pi_E(\mu_1)$, $\pi_2 = \pi_E(\mu_2)$ are cuspidal monomial representations of $\mathrm{GL}(2,\mathbf{A})$ *associated with characters μ_1, μ_2 of $\mathbf{A}_E^\times/E^\times$ where E is a quadratic extension of F such that the restriction of $\mu_1\mu_2$ to $\mathbf{A}^\times$ is 1, then $\lambda_1(\pi_E(\mu_1) \times \pi_E(\mu_2)) = I_{(2,2)}(\pi_E(\mu_1\overline{\mu}_2), \pi_E(\mu_1\mu_2))$.*

If $\{\pi_1, \pi_2\}$ are cuspidal but not of the form $\{\pi_E(\mu_1), \pi_E(\mu_2)\}$, and $\pi_1 \neq \chi\pi_2$ for any quadratic character χ of $\mathbf{A}^\times/F^\times$, then $\pi_1 \boxtimes \pi_2$ is cuspidal.

If π_1 is the trivial representation $\mathbf{1}_2$ and π_2 is a cuspidal representation of $\mathrm{PGL}(2,\mathbf{A})$*, then $\lambda_1(\mathbf{1}_2 \times \pi_2)$ is the discrete spectrum noncuspidal* $\mathrm{PGL}(4,\mathbf{A})$*-module $J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$. Here $\nu(x) = |x|$, and J is the quotient of the representation $I(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ normalizedly induced from the parabolic subgroup of type $(2,2)$ of* $\mathrm{PGL}(4)$.

Moreover, $\lambda_1(\pi_1 \times \check{\pi}_1)$ is the $\mathrm{PGL}(4,\mathbf{A})$*-module normalizedly induced from the* $\mathrm{PGL}(3,\mathbf{A})$*-module* $\mathrm{Sym}^2(\pi_1)$ *on the Levi factor of the parabolic subgroup of type $(3,1)$.*

The global map λ_1 is ***injective*** *on the set of pairs $\pi_1 \times \check{\pi}_2$ with $\omega_1 = \omega_2$ up to the equivalence $\pi_1 \times \check{\pi}_2 \simeq \chi\pi_1 \times \chi^{-1}\check{\pi}_2$, χ a character of $\mathbf{A}^\times/F^\times$, and $\pi_1 \times \check{\pi}_2 \simeq \check{\pi}_2 \times \pi_1$.*

The **image** of λ_1 in the discrete spectrum is determined in [F1] as the set of $\pi \simeq \check{\pi}$ on $\mathrm{PGL}(4,\mathbf{A})$ not in the image of the lifting from $\mathrm{PGSp}(2,\mathbf{A})$.

The injectivity means that if π_1, π_2, π_1^0, π_2^0 are in the discrete spectrum of $\mathrm{GL}(2,\mathbf{A})$ with central characters ω_1, ω_2, ω_1^0, ω_2^0 satisfying $\omega_1\omega_2 = 1 = \omega_1^0\omega_2^0$, each of which has elliptic components at least at the two places v_1, v_2, and if for each v outside a fixed finite set of places of F there is a character χ_v of $F_v^\times$ such that the set $\{\pi_{1v}\chi_v, \pi_{2v}\chi_v^{-1}\}$ is equal to the set $\{\pi_{1v}^0, \pi_{2v}^0\}$ (up to equivalence of representations), then there is a character χ of $\mathbf{A}^\times/F^\times$ such that the set $\{\pi_1\chi, \pi_2\chi^{-1}\}$ is equal to the set $\{\pi_1^0, \pi_2^0\}$. In particular, starting with a pair π_1, π_2 of discrete spectrum representations of $\mathrm{GL}(2,\mathbf{A})$ with $\omega_1\omega_2 = 1$, we cannot get another such pair by interchanging a set of their components π_{1v}, π_{2v} and multiplying π_{1v} by a local character and π_{2v} by its inverse, unless we interchange π_1, π_2 and multiply π_1 by a global character and π_2 by its inverse.

The constraint on two places which occurs in our result is due to our use of simplifying hypothesis on the test functions for which we apply the trace formula identity. They can be reduced to a single constraint with further work using available techniques. An unconditional result depends on progress in the study of trace formulae comparison techniques.

Our global results are complemented and strengthened by very precise local results. We define λ_1-lifting locally by means of **character relations** of the form: $\lambda_1(\pi_1 \times \check{\pi}_2) = \pi$ if $\mathrm{tr}\,\pi(f \times \theta) = \mathrm{tr}(\pi_1 \times \check{\pi}_2)(f_C)$ for all matching functions f, f_C. This definition is compatible with the one given above for purely induced π_1 and π_2 and unramified representations, and we have $\lambda_1(I_2(\mu, \mu') \times \check{\pi}_2) = I_4(\mu\check{\pi}_2, \mu'\check{\pi}_2)$ (the central character of the $\mathrm{GL}(2,F)$-module π_2 is $\mu\mu'$). The local and global results are closely analogous.

4. Comments on character relations. In addition to just proving the existence of the global lifting, the usage of the trace formula affords deriving sharper results. These concern injectivity of the lifting, determination of the image, and derivation of local results formulated in terms of precise and important character relations – too long to record here – in particular for nongeneric representations.

As an example, when $\pi_2 = \check{\pi}_1$ is the contragredient of π_1, $\lambda_1(\pi_1 \times \check{\pi}_1)$ is $I_{(3,1)}(\mathrm{Sym}^2(\pi_1))$. Indeed, if the local component π_{1v} of π_1 at v is unramified then $t(\pi_{1v}) = \mathrm{diag}(a, b)$ (thus π_{1v} is a constituent of $I_2(a, b)$), $\pi_v = \lambda_1(\pi_{1v} \times \check{\pi}_{1v})$ has $t(\pi_v) = \mathrm{diag}(a/b, 1, 1, b/a)$ (thus π_v is a constituent of $I_4(I_3(a/b, 1, b/a), 1)$, and $I_3(a/b, 1, b/a)$ is the symmetric square lifting of $I_2(a, b)$). We write I_n to emphasize that the representation is of the group $\mathrm{GL}(n)$, and e.g. $I_{(3,1)}(\pi_3, \pi_1)$ to indicate the representation of GL(4) induced from its maximal parabolic subgroup of type (3,1). However, the results of [F3, F4, F5] are stronger, in lifting representations of $\mathrm{SL}(2,\mathbf{A})$ to $\mathrm{PGL}(3,\mathbf{A})$ and thus providing new results such as multiplicity one for SL(2).

Although we do not obtain a new proof of the existence of the symmetric square lift of discrete spectrum representations of $\mathrm{PGL}(2, \mathbf{A})$, we do obtain new character relations, of the θ-twisted character of $I_{(3,1)}(\mathrm{Sym}^2 \pi_2, 1)$ with that of $\pi_2 \times \check{\pi}_2$. The proof of this local result is global. Clearly in this case the lift λ_1 is injective: if $\lambda_1(\pi_1 \times \check{\pi}_2) = \lambda_1(\pi_0 \times \check{\pi}_0)$ $(= I_{(3,1)}(\mathrm{Sym}^2(\pi_0), 1))$ then $\pi_1 = \pi_2 = \pi_0\chi$ for some character χ of $\mathbf{A}^\times/F^\times$.

In particular, if π_1 is the one dimensional representation $g \mapsto \chi(\det g)$ of $\mathrm{GL}(2, \mathbf{A})$, then $\lambda_1(\pi_1 \times \check{\pi}_1) = I_{(3,1)}(\mathbf{1}_3, 1)$ is the representation of $\mathrm{PGL}(4, \mathbf{A})$ normalizedly induced from the trivial representation $\mathbf{1}_3$ of the maximal parabolic subgroup of type (3,1). This is obvious in terms of the Satake parameters. However the character relation defining the local relation $\mathbf{1}_2 \boxtimes \mathbf{1}_2 = I_{(3,1)}(\mathbf{1}_3, 1)$ is highly nontrivial. It is only in this case that an alternative purely local computation of the twisted character is known: see [FZ].

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References

[BZ] Bernstein, J., and Zelevinsky, A.: Induced representations of reductive p-adic groups. I. Ann. Sci. École Norm. Sup., **10**, 441–472 (1977).

[Bla] Blasius, D.: On multiplicities for SL(n). Israel J. Math., **88**, 237–251 (1994).

[Bo] Borel, A.: Automorphic L-functions. Proc. Sympos. Pure Math., 33 II. Amer. Math. Soc., Providence, R.I., pp. 27–63 (1979).

[F1] Flicker, Y.: Lifting automorphic forms of PGSp(2) and SO(4) to PGL(4). (2001). (Preprint).

[F2] Flicker, Y.: Automorphic forms on PGSp(2). Elect. Res. Announc. AMS, 10, 39–50 (2004); http://www.ams.org/era.

[F3] Flicker, Y.: On the symmetric square. IV. Applications of a trace formula. Trans. Amer. Math. Soc., **330**, 125–152 (1992).

[F4] Flicker, Y.: On the symmetric square. V. Unit elements. Pacific J. Math., **175**, 507–526 (1996).

[F5] Flicker, Y.: On the symmetric square. VI. Total global comparison. J. Funct. Analyse, **122**, 255–278 (1994).

[FZ] Flicker, Y., and Zinoviev, D.: Twisted character of a small representation of PGL(4). Moscow Math. J. (2004). (To appear).

[H1] Harish-Chandra: Admissible invariant distributions on reductive p-adic groups. Preface and notes by Stephen DeBacker and Paul J. Sally, Jr. University Lecture Series, no. 16, Americal Mathematicsl Society, Providence, RI, xiv + 97 pp. (1999).

[H2] Harish-Chandra: Admissible invariant distributions on reductive p-adic groups. Lie theories and their applications (Proc. Ann. Sem. Canad. Math. Congr., Queen's Univ., Kingston, Ont., 1977), Queen's Papers in Pure Appl. Math. no. 48, Queen's Univ., Kingston, Ont., pp. 281–347 (1978).

[JS] Jacquet, H., and Shalika, J.: On Euler products and the classification of automorphic forms II. Amer. J. Math., **103**, 777–815 (1981).

[Ko] Kottwitz, R.: Stable trace formula: cuspidal tempered terms. Duke Math. J., **51**, 611–650 (1984).

[KS] Kottwitz, R., and Shelstad, D.: Foundations of twisted endoscopy. Asterisque, **255**, vi + 190 pp. (1999).

[MW] Moeglin, C., and Waldspurger, J.-L.: Le spectre résiduel de GL(n). Ann. Sci. Ecole Norm. Sup., **22**, 605–674 (1989).

[R] Ramakrishnan, D.: Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2). Ann. of Math., **152**, 45–111 (2000).