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Automorphic Forms on Covering Groups of $GL(2)$

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The theory of modular forms of half-integral weight has seen much progress since 1973 when Shimura [14] associated to each cusp form of weight $\frac{1}{2}k$ (odd $k \geqq 3$) and character χ a modular form of weight $k-1$ and character χ^2. In a sequence of papers [1, 3, 4] Gelbart and Piatetski-Shapiro reformulated Shimura's correspondence in terms of group representations and based their work on L-functions (as in [14]), the Weil representation [16] and Whittaker models in the sense of Jacquet-Langlands [7]. In particular, they showed that Weil representations of a two-fold covering group of $GL(2)$ correspond to automorphic representations of $GL(2)$ which have one-dimensional components almost everywhere and that cuspidal non-Weil representations correspond to cuspidal representations.

One of the aims of this paper is to give a comprehensive presentation of the correspondence and to obtain a full description of the genuine automorphic representations of the two-fold covering group. In particular, we shall show that the correspondence is one-to-one and determine its image, and deduce that both multiplicity one and strong multiplicity one theorems are valid for the covering group.

More significantly, we shall show that there is a correspondence from the space of automorphic forms on n-fold covering groups $\bar{G}$ (in the sense of Kubota [8]) of $GL(2)$, where $n \geqq 2$ is an arbitrary integer, to the space of automorphic forms on $GL(2)$. The latter correspondence includes Shimura's in the case of $n=2$; it is one-to-one, and its image is determined (in 5.3 below); it affords a full description of the automorphic representations of $\bar{G}$ (in terms of those of $GL(2)$) and in particular those studied by Kubota [8]. Further, it suggests the existence of an intimate relationship between the automorphic representations of general reductive connected groups and those of their covering groups in the sense of Moore [11]. A special feature of the case of even n is that there are cuspidal representations of $\bar{G}$ which correspond to continuous series representations of

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$GL(2)$; if n is odd, then cuspidal representations of $\bar{G}$ always correspond to cuspidal representations of $GL(2)$. The case of $n=2$ is singled out not only because the Weil representation was defined in this case alone but also because here every special representation (à la [7]) of $GL(2)$ is obtained by the correspondence.

Our work is based on the intrinsic approach of character theory, and its exposition in this context may be considered to be an aim in itself. We follow closely Langlands [9] where this approach was developed for the first time. In particular, we make no use of the auxiliary characterization of automorphic forms by L-functions, and although we identify the Weil representations in the case $n=2$ (for the sake of completeness), they do not play any role in this work. For us the main property of the correspondence is that it preserves characters (see 5.0). This also affords some understanding of the correspondence as a reflection of the relationship between conjugacy classes of the groups $\bar{G}$ and $GL(2)$ (see 0.3).

The paper is arranged as follows. We recall Kubota's definition of $\bar{G}$ in 0.2 and discuss the structure of $\bar{G}$ in 0.3 (and 1.1). Lemma 1.2.3 is a key local result, matching orbital integrals on $\bar{G}$ and on $GL(2)$; Lemma 1.4 deals with spherical functions. The local correspondence is established in 2.1 for the principal series, in 2.2 for the Weil representations (when $n=2$), and in 2.4 for the archimedean places. An explicit expression for the Selberg trace formula for $\bar{G}$ is given in Sect. 3, and it is compared in Sect. 4 to the trace formula for $GL(2)$. A full description of the local correspondence is deduced in 5.2 from the resulting equality of traces (see 4.3) using the results of 2.3 concerning square-integrable representations. A new difficulty here (compared to [9] and [7], Chap. 16) is that $\bar{G}$ has no finite-dimensional (local) genuine representations and hence no square-integrable (special) representations which are easy to deal with. The global correspondence is described in the final subsection 5.3.

Now we shall briefly mention some applications of our results in the case $n=2$ to the classical theory of modular forms of half-integral weight. Our work affords a solution to Shimura's question (A) in his list [14], Sect. 4, of open questions. Indeed, we deduce that every holomorphic modular form of weight $k-1$, level $\frac{1}{2}N$ and character χ^2 can be obtained from a modular form of weight $\frac{1}{2}k$, level N and character χ, and (when restricted to new forms) that Shimura's map is one-to-one. A contribution in this direction had been made by Shintani [15], who defined a map reversing Shimura's in some special cases. Our strong multiplicity one theorem (which can also be proved by using L-functions [4]) implies that the quotient of two cusp forms (of type $(\frac{1}{2}k, N, \chi)$) which are eigenfunctions of almost all Hecke operators $T(p^2)$ (see [14]) and have the same eigenvalues is a constant.

We can also answer all parts of question (C) of [14]. Its first part asserts that the image of the orthogonal complement (in the space of cusp forms of type $(\frac{3}{2}, N, \chi)$) of the space generated by the theta-series $\sum_1^\infty \psi(m)\, m e(m^2 z)$ (with $\psi(-1)=-1$, see [14]), is cuspidal. The assertion follows from the representation theoretic statement that cuspidal non-Weil representations of $\bar{G}$ correspond to cuspidal representations of $GL(2)$. This is a special case of (i) and (ii) of 5.3 below, and as we have mentioned, this already occurs in [4].

The second part of (C) was answered by Serre and Stark [13], who proved that each modular form of weight $\frac{1}{2}$ is generated by translates of the theta-series $\sum_{-\infty}^{\infty} \psi(m)\, e(m^2 z)$, where ψ is an even character. A representation theoretic reformulation of this was given by Gelbart and Piatetski-Shapiro [3]. Since our work establishes the correspondence for the whole (not necessarily cuspidal) spectrum of $\bar{G}$ we obtain this as well. Indeed, a modular form of weight $\frac{1}{2}$ generates the space of an automorphic representation $\bar{\pi}$ of $\bar{G}(\mathbb{A}_Q)$ (here $\mathbb{A}_Q$ denotes the adeles of the rationals Q) whose component at infinity has lowest weight $\frac{1}{2}$, and hence (see 2.4) is an even Weil representation. But Corollary 5.3(b) below now implies that $\bar{\pi}$ is a global Weil representation and, as is well known, its space is generated by theta-series which must be of the above form (since these are the only ones with weight $\frac{1}{2}$).

More generally, our work (as well as [4]) applies to $\bar{G}(\mathbb{A}_F)$, where $\mathbb{A}_F$ are the adeles of an arbitrary number field F, not necessarily Q. On applying the above argument with a totally real number field F instead of Q, the same conclusion holds for Hilbert modular forms whose weight vector contains a component equal to $\frac{1}{2}$. In particular, the weight vector of any such form must consist of $\frac{1}{2}$'s and $\frac{3}{2}$'s only since the components at infinity of a Weil representation are either even or odd local Weil representations whose weights are $\frac{1}{2}$ and $\frac{3}{2}$ (respectively; see 2.4 below and [1], p. 93). Other Hilbert modular forms whose weight vector $(\frac{1}{2}k_1, \frac{1}{2}k_2, \ldots)$ includes $\frac{3}{2}$ (or $\frac{1}{2}k$, odd $k \geqq 5$) correspond to Hilbert cusp forms (of weight vector $(k_1-1, k_2-1, \ldots)$).

The last part of question (C) concerns the "basis problem". This asserts that the space of cusp forms of type $(\frac{1}{2}k, 4N, 1)$ (odd $k \geqq 3$, odd square-free $N>0$), in the orthogonal complement of the span of the theta-series associated with $q(x) = x^2$, is generated by theta-series associated with a positive-definite quadratic form in 3 variables. A representation theoretic analogue of this will be given by a parametrization of automorphic representations of $\bar{G}$ by those of $\mathbb{K}/\mathbb{K}^0$ ($\mathbb{K}$ is a quaternion division algebra with a centre $\mathbb{K}^0$), and such parametrization is given by combining our correspondence with that of Jacquet-Langlands [7], from automorphic forms on $\mathbb{K}$ to those on $GL(2)$, as we now explain.

A cusp form of type $(\frac{1}{2}k, 4N, 1)$ as above generates the space of an automorphic representation of $\bar{G}(\mathbb{A}_Q)$ (which is cuspidal non-Weil) with a trivial central character, and hence can be viewed as a representation of $P\bar{G}$ (which is defined to be the quotient of $\bar{G}$ by its centre Z^2). Each irreducible component is of the form $\bar{\pi} = \bar{\pi}_{\frac{1}{2}k} \otimes (\bigotimes_p \bar{\pi}_p)$, where $\bar{\pi}_{\frac{1}{2}k}$ is the discrete series at infinity with lowest weight $\frac{1}{2}k$, $\bar{\pi}_p$ is special for any prime divisor p of $4N$, and unramified elsewhere. Now $\bar{\pi}$ corresponds to an irreducible automorphic representation $\pi = \pi_{k-1} \otimes (\bigotimes_p \pi_p)$ of $PGL(2, \mathbb{A}_Q)$ with discrete series π_{k-1}, special π_p if p divides $4N$ and unramified π_p otherwise. But π is obtained by the Jacquet-Langlands correspondence from an irreducible automorphic representation π' of $\mathbb{K}/\mathbb{K}^0$, where $\mathbb{K}$ is the quaternion division algebra over Q which ramifies at infinity and at all prime divisors p of $4N$. Note that there exists a quadratic form in 3 variables (which is the restriction of the form in 4 variables associated with $\mathbb{K}$ to the "pure quaternions") whose orthogonal algebra is (essentially) $\mathbb{K}/\mathbb{K}^0$. Hence we obtain a one-to-one correspondence $\pi' \to \bar{\pi}$ of automorphic representations on $\mathbb{K}/\mathbb{K}^0$ and $P\bar{G}$, and the determination of its image is a representations theoretic analogue of the "basis problem". A full description of the image is given by combining our results with those of [7] (i.e., $\bar{\pi} = \bigotimes_v \bar{\pi}_v$ lies in the image if $\bar{\pi}_v$ is square-integrable for any v at which $\mathbb{K}$ ramifies). To obtain the classical version of this problem one has to show that if ϕ generates the space of π', then $\int \Theta\phi$ generates the space of $\bar{\pi}$ for some suitable theta-transform Θ. This is likely to follow as in Shimizu's deduction from [7] of Eichler's result concerning the generation of cusp forms of integral weight and square-free level by theta-series, but it is not our purpose to deal with that here.

Clearly, Shimura's suggestion in his question (*D*), to apply representation theoretic techniques in this context, is completely exploited here. However, we have left Shimura's remaining question (*B*) untouched, and our method does not seem to provide any information concerning the Fourier coefficients of modular forms. Suggestive results were obtained by Shintani [15] ($n=2$) and Patterson ($n=3$; see Deligne [0] for exposition in terms of the Eisenstein series of Sect. 3 below).

Modular forms of classical type were defined by Kubota in [8] when $n \geqq 3$, as forms on a product of "complex" upper-half-planes. Our results can be translated into the classical language in this case just as well. The fact that no natural "Weil representation" has ever been defined for $n \geqq 3$ is related to the fact that no natural "classical" theta-series have ever been introduced in this case. However, it is clear that the language of representations is better fitted to describe results of such generality, and we merely comment here that some representation theoretic corollaries are mentioned in 5.2 and 5.3.

Langlands' idea of using representation theory in the above context, and especially the Selberg trace formula, was passed on to me by Gelbart in Jerusalem, spring of 1978, where I benefited from talking to him and Piatetski-Shapiro. Most of this work was written the follwing winter in Princeton where I had the unlimited and invaluable encouragement of Langlands.

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0.1. Notations

We fix an integer $n \geqq 2$ and a number field F which contains a primitive n-th root of unity; this condition is superfluous if $n=2$, but it implies that F is totally imaginary for $n \geqq 3$. For any place v of F we signify by F_v the completion of F at v and by $|\ |_v$ the normalized valuation on F_v, so that the product formula on F is satisfied. When v is non-archimedean, we let ord_v be the order valuation on F_v, O_v the ring of integers in F_v, $\tilde{\omega}_v$ the local uniformizing parameter at v, $q_v = |\tilde{\omega}_v|_v^{-1}$ the cardinality of the residual field $O_v/\tilde{\omega}_v O_v$ and p its characteristic; here and below we drop the index v when reference to the valuation is made clear by the context. Also we denote by $\mathbb{A}$ the adeles of F, $\mathbb{A}^\times$ its ideles, $F^\times$ and $F_v^\times$ the multiplicative group of F and F_v (resp.), $\underline{\zeta}_n$ the group of n-th roots of unity in F; ζ will always denote an element of $\underline{\zeta}_n$.

We write $G(F_v)$ for $GL(2, F_v)$ and $G(\mathbb{A})$ for $GL(2, \mathbb{A})$. We signify by K_v the standard maximal compact subgroup of $G(F_v)$, namely, $O(2, \mathbb{R})$ if F_v is the field $\mathbb{R}$ of real numbers, $U(2, \mathbb{C})$ if F_v is the field $\mathbb{C}$ of complex numbers, and $GL(2, O_v)$ if v is non-archimedean. We denote by $A(F_v)$ (resp. $A(\mathbb{A})$) the subgroup of diagonal elements with entries from $F_v^\times$ (resp. $\mathbb{A}^\times$), and by $Z(F_v)$ (resp. $Z(\mathbb{A})$) the subgroup of scalar matrices over $F_v^\times$ (resp. $\mathbb{A}^\times$). The subgroup consisting of n-th powers of all elements of an abelian group H will be denoted by H^n; for example, $A^n(F_v)$ denotes the group of diagonal matrices whose entries lie in $F_v^{\times n}$. Finally, we let $N(F_v)$ (resp. $N(\mathbb{A})$) be the subgroup of all $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with x in F_v (resp. $\mathbb{A}$), and note once again that the reference to F_v or $\mathbb{A}$ will be omitted whenever it is made redundant by the context.

0.2. Covering Groups

Since covering groups are the basic objects to be studied here, there is no harm in recalling their definition. By a two-cocycle on a locally compact group G we mean a Borel map β from $G \times G$ onto $\underline{\zeta}_n$ with

$$\beta(gg', g'')\,\beta(g, g') = \beta(g, g'g'')\,\beta(g', g'')$$

and

$$\beta(g, e) = \beta(e, g) = 1 \qquad (g, g', g'' \text{ in } G),$$

where e is the identity of G. It is said to be nontrivial if there is no map s from G to $\underline{\zeta}_n$ such that $\beta(g, g')$ is equal to $s(g)\,s(g')\,s(gg')^{-1}$ for all g, g' in G. By an n-fold covering group $\bar{G}$ of G, we mean the group of all pairs (g, ζ) with g in G (and, as usual, ζ in $\underline{\zeta}_n$) with multiplication given by

$$(g, \zeta)(g', \zeta') = (gg', \zeta\zeta'\beta(g, g')),$$

where β is a nontrivial two-cocycle. Then $\bar{G}$ is a locally compact group which is not algebraic. It is central as a group extension of G by $\underline{\zeta}_n$, namely, on identifying $\underline{\zeta}_n$ with its image under the map $\zeta \to (e, \zeta)$ we have that $\underline{\zeta}_n$ is contained in the centre of $\bar{G}$, and the sequence

$$1 \to \underline{\zeta}_n \to \bar{G} \to G \to 1$$

is exact. We say that $(g, 1)$ is the lift (to $\bar{G}$) of the element g of G; where no confusion may arise, we write g for $(g, 1)$ and $g\zeta$ for $(g, 1)(e, \zeta) = (g, \zeta)$ (but not for $(g\zeta, 1)$).

When G is $G(F_v)$ or $G(\mathbb{A})$, an explicit construction of a nontrivial two-cocycle β was given by Kubota [8]; since we deal with Kubota's cocycle only, we shall recall his results. When G is $G(F_v)$, the cocycle $\beta = \beta_v$ is given in terms of the norm residue (or Artin) symbol $(\ ,\ )$ of degree n on $F = F_v$. The symbol $(\ ,\ )$ is defined in most class field theory textbooks; here we shall merely note that $(\ ,\ )$ is a continuous bilinear map from $F^\times \times F^\times$ onto $\underline{\zeta}_n$, that

$$(a, b)(b, a) = (a, -a) = (a, b)(-b/a, a+b) = 1 \tag{1}$$

for any a and b in $F^\times$, and that $(a, b) = 1$ for all a in $F^\times$ if and only if b is in $F^{\times n}$.

For any $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $S = SL(2, F)$ we write $x(s) = c$ if $c \neq 0$ and $x(s) = d$ if $c = 0$, and we put

$$\begin{aligned} \alpha(s, s') &= (x(s), x(s'))\,(-x(s')/x(s), x(ss')) \\ &= (x(ss')/x(s), x(ss')/x(s')) \qquad (s, s' \text{ in } S) \end{aligned}$$

For any g in G with determinant $\underline{g}$, we write $p(g)$ for the element $\begin{pmatrix} 1 & 0 \\ 0 & \underline{g} \end{pmatrix}^{-1} g$ of S, we write g^z for $\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}^{-1} g \begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}$ if z is in $F^\times$, and if $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, we put $v(z,g)=1$ when $c \neq 0$ and $v(z,g)=(z,d)$ when $c=0$. Now a nontrivial two-cocycle is given by

$$\alpha(g,g')=\alpha(p(g)^{g'}, p(g'))\,v(g', p(g)) \qquad (g, g' \text{ in } G).$$

A more convenient but equivalent two-cocycle β is given by

$$\beta(g,g')=\alpha(g,g')\,s(g)\,s(g')\,s(gg')^{-1},$$

where $s(g)=(c,d/g)$ if $cd \neq 0$ and n does not divide $ord\, c$, and $s(g)=1$ otherwise. When F is $\mathbb{R}$, we set $s(g)=1$ for all g, and $\beta=\alpha$ is defined as before; this case may occur only if $n=2$. If F is $\mathbb{C}$, then s, α and β are identically 1 and $\bar{G}$ is the direct product of G and ζ_n. We shall use below Kubota's result [8] that β is continuous in the non-archimedean case.

We say that $\bar{G}$ splits over a subgroup H of G if the map $g \to (g, s(g))$ from H to $\bar{G}$ is a homomorphism for some s, namely, H lifts as a subgroup of $\bar{G}$, or, equivalently, the inverse image $\bar{H}$ of H under $(g,\zeta) \to g$ is the direct product of H and ζ_n. It is easy to see that $\beta(g,g')=(a,c')$ for $g=\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}$ and $g'=\begin{pmatrix}a' & b'\\ 0 & c'\end{pmatrix}$ in G, and hence that $\bar{G}$ splits over N and over A^n. Moreover, $\bar{G}$ splits over K provided that $|n|_v=1$ ([8], Theorem 2), and β is normalized to obtain the value 1 on $K \times K$ in this case. The last property is fundamental for our study (in 1.4) of spherical functions.

As usual we put these local results together to define an n-fold covering group $\bar{G}(\mathbb{A})$ of $G(\mathbb{A})$. It suffices to define a nontrivial two-cocycle $\beta=\beta_{\mathbb{A}}$ on $G(\mathbb{A})$, and this is given by

$$\beta(g,g')=\prod_v \beta_v(g,g'),$$

where $g=\{g_v\}$ and $g'=\{g'_v\}$ are any elements of $G(\mathbb{A})$, and we put $\beta_v(g,g')=\beta_v(g_v,g'_v)$ with the local two-cocycle β_v defined above. The product is taken over all places v of F, and it makes sense since $\beta_v(g,g')$ is 1 for almost all v.

The centre of $\bar{G}(\mathbb{A})$ (resp. $\bar{G}(F_v)$) is clearly $\bar{Z}^n(\mathbb{A})$ (resp. $\bar{Z}^n(F_v)$); here and below we write $\overline{H^n}$ for the inverse image of H^n in $\bar{G}$, where H is an abelian subgroup of G, and not for the set of n-th powers of elements of $\bar{H}$.

Finally, we note that the reciprocity law for (,), namely, that $\prod_v (a,b)_v=1$ for any a,b in $F^\times$, implies that $\alpha(g,g')=\prod_v \alpha_v(g'_v, g_v)$ is 1 for g, g' in $G(F)$, and hence that the map $g \to (g, s(g)^{-1})$ from $G(F)$ to $\bar{G}(\mathbb{A})$ is an injective homomorphism. Here we write $s(g)$ for $\prod_v s_v(g_v)$. We shall regard $G(F)$ as a subgroup of $\bar{G}(\mathbb{A})$ through this map. The fact that $G(F)$ embeds as a subgroup of $\bar{G}(\mathbb{A})$ is fundamental for developing a theory of automorphic forms on $\bar{G}(\mathbb{A})$.

0.3. From G to $\bar{G}$

There is a natural map from G to $\bar{G}$ which is given by

$$\gamma \to \tilde{\gamma} = (\gamma, 1)^n.$$

For us, its main virtue is the following:

Lemma 1. *The map $\gamma \to \tilde{\gamma}$ preserves conjugacy classes.*

Indeed, we have

$$(g, 1)^{-1}\tilde{\gamma}(g, 1) = ((g, 1)^{-1}(\gamma, 1)(g, 1))^n = (g^{-1}\gamma g, \zeta)^n = (g^{-1}\gamma g, 1)^n = (g^{-1}\gamma g)^{\sim},$$

as required.

The significance of this simple lemma is, as we shall see below, that it affords relating characters of representations on G and $\bar{G}$, since these characters are functions on conjugacy classes.

An element γ of G (or $\bar{G}$) is called regular if its eigenvalues are distinct, and singular otherwise. If γ in G is singular, then $\tilde{\gamma}$ is clearly singular in $\bar{G}$. We have to determine which regular γ map to singular $\tilde{\gamma}$. If such γ has eigenvalues in $F^{\times}$, then it is conjugate to $\begin{pmatrix}\zeta a & 0\\ 0 & a\end{pmatrix}$ for some a in $F^{\times}$. If γ is elliptic, its eigenvalues are of the form $a + b\sqrt{\theta}$ with a in F, b in $F^{\times}$, θ nonsquare in $F^{\times}$, and it suffices to determine such a and b for which $(a + b\sqrt{\theta})^n$ lies in $F^{\times}$. Since $b \neq 0$, we put $x = a/b$ and note that $(x + \sqrt{\theta})^n$ lies in $F^{\times}$ if and only if the polynomial

$$\binom{n}{1} x^{n-1} + \theta \binom{n}{3} x^{n-3} + \theta^2 \binom{n}{5} x^{n-5} + \dots \qquad (*)$$

has a zero in F. Clearly, the zeros of this polynomial are obtained by multiplying by $\sqrt{\theta}$ the zeros of the polynomial

$$\binom{n}{1} x^{n-1} + \binom{n}{3} x^{n-3} + \dots = \tfrac{1}{2}[(x+1)^n - (x-1)^n].$$

Since $x = 1$ is not a zero of the last polynomial, we divide it by $(x-1)^n$ and obtain the polynomial

$$\left(\frac{x+1}{x-1}\right)^n - 1,$$

whose zeros are clearly given by $x = \dfrac{\zeta+1}{\zeta-1}$, where $\zeta \neq 1$ runs through $\underline{\zeta}_n$. It follows that $(*)$ has no zero in F if n is odd, and it has the unique solution $x = 0$ ($\zeta = -1$) if n is even. We have proved:

Lemma 2. *If γ is elliptic regular, then $\tilde{\gamma}$ is elliptic regular unless n is even and γ is conjugate to $\begin{pmatrix}0 & \theta b\\ b & 0\end{pmatrix}$ with some b in $F^{\times}$ and a nonsquare θ in $F^{\times}$.*

I. Local Theory

1.0. Orbital Integrals

The explicit expression for the Selberg trace formula will be given below in terms of orbital integrals. Here we prepare the local analysis required for the global considerations. Our aim is to relate orbital integrals on $\bar{G}$ to those on G.

Throughout this section, F will denote a non-archimedean local field; the results can easily be established also for $F=\mathbb{R}$, and they are already known for $F=\mathbb{C}$ since $\bar{G}(\mathbb{C})$ splits over $G(\mathbb{C})$.

Until further notice we shall denote by $\bar{f}$ any locally constant (smooth if $F=\mathbb{R}$ or $\mathbb{C}$) compactly supported genuine function on $\bar{G}(=\bar{G}(F_v))$ with values in $\mathbb{C}$. A genuine function is one which satisfies

$$\bar{f}((g,\zeta))=\zeta\bar{f}((g,1)) \qquad ((g,\zeta) \text{ in } \bar{G});$$

a function invariant under $\underline{\zeta}_n$ (or a nontrivial subgroup of $\underline{\zeta}_n$) reduces to a function on G (or a subcovering of $\bar{G}$), and hence it is of no interest for us.

An element γ of $\bar{G}$ is called regular if its eigenvalues γ_1 and γ_2 are distinct. We denote by $\bar{G}_\gamma$ the centralizer of a regular element γ in $\bar{G}$, and by dg we signify the invariant form on $\bar{G}_\gamma\backslash\bar{G}$ obtained as the quotient of invariant forms on $\bar{G}$ and $\bar{G}_\gamma$, whose normalizations will be specified when used. We put

$$\Delta(\gamma)=\left|\frac{(\gamma_1-\gamma_2)^2}{\gamma_1\gamma_2}\right|^{\frac{1}{2}},$$

and note that $\Delta(\gamma)$ depends only on the conjugacy class of γ.

For any $\bar{f}$ and γ as above, we define the (normalized) orbital integral by

$$F(\gamma,\bar{f})=\Delta(\gamma)\int_{G_\gamma\backslash\bar{G}}\bar{f}(g^{-1}\gamma g)\,dg;$$

since $\bar{f}$ has compact support the integral always converges. We prefer to deal with the $F(\gamma,\bar{f})$ rather than with the un-normalized integrals (where $\Delta(\gamma)$ is deleted), since they extend by continuity to the singular elements of $\bar{G}$; this will be discussed below.

Finally, we define $F(\gamma,f)$ for regular γ in G and locally constant compactly supported functions f on G in an entirely parallel fashion. The $F(\gamma,f)$ have been studied extensively by Harish-Chandra, Shalika and, in the most useful form for us, by Langlands [9].

1.1. Zero Orbits

Many of the $F(\gamma,\bar{f})$ are 0. Indeed, we have:

Lemma 1. *$F(\gamma,\bar{f})$ is 0 unless $\gamma=\zeta\bar{\delta}$ with δ in G.*

Proof. Given a regular γ in $\bar{G}$ which cannot be put in the form $\zeta\tilde{\delta}$ it suffices to find a g in $\bar{G}$ such that $g^{-1}\gamma g=\gamma\zeta$ with $\zeta\neq 1$. Since $\tilde{f}$ is genuine, we shall then deduce that $F(\gamma,\tilde{f})$ is equal to $\zeta F(\gamma,\tilde{f})$; hence $F(\gamma,\tilde{f})=0$ as required.

If the eigenvalues a and b of γ lie in $F^\times$ we may replace it by its conjugate and assume that $\gamma=\begin{pmatrix} a & o\\ 0 & b\end{pmatrix}$. The claim follows from the equality

$$g^{-1}\gamma g=(\gamma,(a,d)(b,c)) \quad \left(g=\begin{pmatrix} c & 0\\ 0 & d\end{pmatrix}\right).$$

If γ is elliptic regular, we may assume that $\gamma=\begin{pmatrix} a & b\theta\\ b & a\end{pmatrix}$ and denote its determinant by x; here a, b are in F while θ is a nonsquare in $F^\times$. We write $\gamma_1=a+b\sqrt{\theta}$ and $\gamma_2=a-b\sqrt{\theta}$ for the eigenvalues of γ, E for the quadratic extension of F generated by $\sqrt{\theta}$ and $(\ ,\ )_E$ for the n-th norm residue symbol on E. We take an arbitrary $g=\begin{pmatrix} c & d\theta\\ d & c\end{pmatrix}$ with c, d in $F^\times$ and denote its determinant by z and its eigenvalues by g_1 and g_2 in the above order. By the continuity of the two-cocycle, we may assume that all elements occurring in the calculation below are nonzero. But the ζ which is defined by $g^{-1}\gamma g=\gamma\zeta$ is given by

$$\begin{aligned}\beta(\gamma,g)/\beta(g,\gamma)&=\left(\frac{b}{xz},\frac{d}{z}\right)\left(\frac{-dx}{b},\frac{ad+bc}{xz}\right)\left(\frac{b}{x},\frac{d}{xz}\right)\left(\frac{ad+bc}{xz},\frac{-bz}{d}\right)\\ &=\left(\frac{b}{x},\frac{d^2}{xz^2}\right)\left(\frac{d}{z},z\right)\left(\frac{ad+bc}{xz},\frac{b^2z}{d^2x}\right)\\ &=(d,b/x)(d/z,b)\left(a/b+c/d,\frac{\theta-c^2/d^2}{\theta-a^2/b^2}\right)\\ &=(d,b/x)(d/z,b)\left(\gamma_1/b+g_2/d,\frac{g_2/d}{\gamma_1/b}\right)_E=(d,b/x)(d/z,b)(\gamma_1/b,g_2/d)_E\\ &=(\gamma_1,g_2)_E;\end{aligned}$$

here we use the fact that $(\alpha,\beta)_E=(N\alpha,\beta)$ for any α in $E^\times$ and β in $F^\times$, where N denotes the norm from E to F. Since g_2 is arbitrary in $E^\times$, the lemma follows from the nontriviality of $(\ ,\ )_E$.

If γ is a regular element of G, it lies in some torus T, and it is clear that the centralizer G_γ of γ in G is T. If $\tilde{\gamma}$ is a regular element of $\bar{G}$, it lies in the subgroup $\bar{T}^n$ of $\bar{T}$ and its centralizer $\bar{G}_{\tilde{\gamma}}$ in $\bar{G}$ is certainly contained in $\bar{T}$. By virtue of Lemma 0.3.1 we deduce that

Lemma 2. *If $\tilde{\gamma}$ is a regular element of $\bar{G}$ then $\bar{G}_{\tilde{\gamma}}$ is equal to $\bar{T}$.*

Consider the unipotent element $\underline{z}=z\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$ of $\bar{G}$ where z is in $\bar{Z}^n$. Since f has compact support, the unnormalized orbital integral

$$\int_{G_{\underline{z}}\backslash\bar{G}} \tilde{f}(g^{-1}\underline{z}g)\,dg$$

is well-defined, and we denote it by $F(z,\tilde{f})$. Clearly, the centralizer $\bar{G}_{\underline{z}}$ of $\underline{z}$ is $\overline{ZN}$, on which we take the measure $d^\times a\,dn$ where $d^\times a = d^\times \alpha = \frac{d\alpha}{|\alpha|}$ if $a=\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$, and $dn = dx$ if $n=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$. We have:

Lemma 3. *The limit of $F(\gamma,\tilde{f})$, as γ in $\bar{A}^n$ tends to z in $\bar{Z}^n$, is equal to $F(z,\tilde{f})$.*

Proof. The Iwasawa decomposition for $\bar{G}$ is $\bar{G}=\bar{A}N\bar{K}$, and the associated decomposition of dg is $da\,dn\,dk$ where $dn=dx$ as above, $da=d^\times\alpha\, d^\times\beta$ if $a=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$, and dk is suitably normalized (see 6 lines below). We have

$$F(\gamma,\tilde{f}) = \Delta(\gamma)\int_N\int_{\zeta_n\backslash\bar{K}} \tilde{f}(k^{-1}n^{-1}\gamma nk)\,dn\,dk$$
$$= \left|\frac{b}{c}\right|^{\frac{1}{2}}\int_N\int_{\underline{\zeta}_n\backslash\bar{K}} \tilde{f}(k^{-1}\gamma nk)\,dn\,dk \qquad \left(\gamma=\begin{pmatrix}b & 0\\ 0 & c\end{pmatrix}\right),$$

and when γ is sufficiently close to z, this is equal to

$$\int_{F^\times}\int_{\underline{\zeta}_n\backslash\bar{K}} \tilde{f}\left(k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}^{-1}\underline{z}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}k\right)\frac{d\alpha}{|\alpha|^2}\,dk = F(z,\tilde{f});$$

the last equality follows from the fact that

$$dg = dx\,d^\times\beta\,\frac{d\alpha}{|\alpha|^2}\,dk \quad \text{ if } g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}k,$$

and the proof is complete.

It follows from Lemma 1 that the limit of $F(\gamma,\tilde{f})$, where γ in $\bar{A}$ tend to z in $\bar{Z}$ but not in $\bar{Z}^n$, must be 0. In this case we write $F(z,\tilde{f})=0$. This is not always the value of the orbital integral $\int \tilde{f}(g^{-1}\underline{z}g)\,dg$. However, Lemma 3 is valid for z in $\bar{Z}^n$, and it suffices for our needs.

1.2. *Matching Orbits*

We are going to relate the orbital integrals $F(\tilde{\gamma},\tilde{f})$ on $\bar{G}$, which are not necessarily 0, to the orbital integrals $F(\gamma,f)$ on G. We shall first recall the characterization of the $F(\gamma,f)$ from Langlands [9], then establish an analogous characterization for the $F(\tilde{\gamma},\tilde{f})$ in a form suitable for their comparison.

If γ is regular in G, it lies in some torus T and we have $G_\gamma = T$. The measure dg used in the definition of $F(\gamma,f)$ is the quotient of invariant forms ω_G and ω_T on G and T (respectively). We write $F(\gamma,f;\omega_T,\omega_G)$ for $F(\gamma,f)$ when its dependence on ω_T and ω_G has to be made explicit.

We say that the set of complex numbers $F(\gamma)=F(\gamma;\omega_T,\omega_G)$, with regular γ in G, is a *G-family* if there exists an f (as in 1.0) on G with $F(\gamma)=F(\gamma,f)$ for all regular γ.

Lemma 0. *A subset $\{F(\gamma)\}$ of $\mathbb{C}$ is a G-family if and only if* (i)–(iv) *below are satisfied.*

(i) *If $\omega'_T = a\omega_T$ and $\omega'_G = b\omega_G$ with a, b in $F^\times$, then*

$$F(\gamma;\omega'_T,\omega'_G) = |b/a|\,F(\gamma;\omega_T,\omega_G).$$

(ii) *If $\gamma' = g^{-1}\gamma g$ and $T' = g^{-1}Tg$ with g in G, and $\omega_{T'}$ is obtained from ω_T, then*

$$F(\gamma';\omega_{T'},\omega_G) = F(\gamma;\omega_T,\omega_G).$$

(iii) *For each torus T the map $\gamma \to F(\gamma)$ is a locally constant compactly supported function on the set T_{reg} of regular elements γ of T.*

(iv) *For each T and z in Z there exists a neighborhood $N(z)$ of z in T and locally constant functions F′ and F″ on $N(z)$ such that*

$$F(\gamma) = F'(\gamma) - \Delta(\gamma)\,c_T F''(\gamma)$$

for all regular γ in $N(z)$. $F'(z)$ and $F''(z)$ are independent of T; if T is split we set $c_T = 0$; otherwise c_T is a positive constant which depends on (the conjugacy class of) T.

Proof. This is Lemma 4.1 of Langlands [9], where it is also shown that for some choice of measures $F'(\gamma, f)$ is equal to

$$F(z,f) = \int_{G_{\underline{z}}\backslash G} f(g^{-1}\underline{z}g)\,dg \qquad \left(\underline{z} = z\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right),$$

and $F''(\gamma, f)$ to $f(z)$, for γ sufficiently close to z, and hence $F'(z)$ and $F''(z)$ are indeed independent of the torus T. We note that the limit of $F(\gamma, f)$ as γ tends to z is well-defined and equal to $F(z, f)$ (see also the proof of Lemma 2 below). The value of c_T will be determined in 1.3 below.

Similar analysis can be applied in order to characterize the orbital integrals $F(\tilde{\gamma}, \tilde{f})$. Since we want to compare the $F(\tilde{\gamma}, \tilde{f})$ to the $F(\gamma, f)$ of Lemma 0, we normalize the $F(\tilde{\gamma}, \tilde{f})$ by writing

$$\bar{F}(\gamma,\tilde{f}) = F(\tilde{\gamma}\zeta_\gamma^{-1},\tilde{f})$$

for any γ in G for which $\tilde{\gamma}$ is regular. Here we put

$$\zeta_\gamma = \alpha(\gamma,\gamma)\,\alpha(\gamma,\gamma^2)\ldots\alpha(\gamma,\gamma^{n-1}).$$

It is easy verify that $\tilde{\gamma} = (\gamma^n, s(\gamma^n)^{-1}\zeta_\gamma)$, and this in fact motivates our definition of ζ_γ. In particular,

$$\bar{F}(\gamma,\tilde{f}) = \Delta(\tilde{\gamma}) \int_{T\backslash G} \tilde{f}(g^{-1}(\gamma^n, s(\gamma^n)^{-1})g)\,dg$$

if γ lies in the torus T.

It follows from Lemma 0.3.1 that $F(\tilde{\gamma}, \tilde{f})$ depends only on the conjugacy class of γ. If we want this to be true for $\bar{F}(\gamma, \tilde{f})$ we must prove:

Lemma 1. *ζ_γ depends only on the conjugacy class of γ.*

Proof. It suffices to show that $\alpha(\gamma,\gamma^j)$ $(j\geqq 1)$ depends only on the trace T and determinant D of γ. For any $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ with $c\neq 0$ we let $c(g)$ be c. Since $\gamma^2=T\cdot\gamma-D\cdot 1$ we have

$$c(\gamma^{j+1})/c(\gamma)=Tc(\gamma^j)/c(\gamma)-Dc(\gamma^{j-1})/c(\gamma).$$

Arguing inductively we see that this expression is a polynomial in T and D, hence a class function for all j. We deduce that

$$\alpha(\gamma,\gamma^j)=(c(\gamma^{j+1})/c(\gamma),\, c(\gamma^{j+1})/c(\gamma^j))=(c(\gamma^{j+1})/c(\gamma),\, -c(\gamma)/c(\gamma^j))$$

is also a class function, as required.

We have proved the Lemma under the conditions $c(\gamma)\,c(\gamma^j)\,c(\gamma^{j+1})\neq 0$. In the non-archimedean case the lemma follows for all γ by the continuity of the two-cocycle α. The observation that $\alpha(\gamma,\gamma)=(a,d)$ for any γ with eigenvalues a,d in $F^\times$ completes the proof in the real case.

As in the case of G we write $\bar F(\gamma,\tilde f;\omega_T,\omega_G)$ for $\bar F(\gamma,\tilde f)$ in order to specify the dependence on the invariant forms ω_T,ω_G on T and G whose pullbacks to $\bar T$ and $\bar G$ serve to define the integral in question. We say that the set of complex numbers $\bar F(\gamma)=\bar F(\gamma;\omega_T,\omega_G)$ (γ in G with regular $\tilde\gamma$) is a *$\bar G$-family* if there exists a genuine $\tilde f$ (as in 1.0) on $\bar G$ with $\bar F(\gamma)=\bar F(\gamma,\tilde f)$ for all such γ.

Lemma 2. *A subset $\{\bar F(\gamma)\}$ of $\mathbb{C}$ is a $\bar G$-family if and only if* (i)–(iv) *of Lemma* 0 *hold with F replaced by $\bar F$, c_T replaced $\bar c_T=|n|\,c_T$, and* (v) *below is satisfied.*

(v) *We have $\bar F(\zeta\gamma)=\bar F(\gamma)$ for all in ζ_n and*

$$\bar F\left(\begin{pmatrix}\zeta a & 0\\ 0 & b\end{pmatrix}\right)=\bar F\left(\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\right)=\bar F\left(\begin{pmatrix} a & 0\\ 0 & \zeta b\end{pmatrix}\right)\qquad (a,b \textit{ in } F^\times).$$

The proof will be given in the next subsection.

We draw the immediate conclusion from Lemmas 0 and 2.

Lemma 3. *Every $\bar G$-family is a G-family; every G-family satisfying* (v) *of Lemma* 2 *is a $\bar G$-family.*

1.3. Proof of Lemma 1.2.2

Property (i) follows at once from the definition of a $\bar G$-family, (ii) from Lemmas 0.3.1, 1.2.1 and the definition of $\Delta(\tilde\gamma)$, and (v) from the fact that $\bar F(\gamma,\tilde f)$ depends only on the n-th power γ^n of γ. Properties (iv) and (iii) for split tori T are obtained from Lemma 1.1.3 and its proof. To obtain these for nonsplit tori, we may assume that $\tilde f$ satisfies $\tilde f(k^{-1}gk)=\tilde f(g)$ for all k in $\bar K$, since we can replace $\tilde f$ by the function $|K|^{-1}\int_K \tilde f(k^{-1}gk)\,dk$.

We may restrict our attention to $\gamma=\begin{pmatrix} a & b\theta \\ b & a \end{pmatrix}$ in a nonsplit torus T, where θ is a nonsquare in $F^\times$ and *ord* $\theta=0$ or 1. Since $\tilde{f}$ is invariant under conjugation by elements of $\bar{K}$, we have the expansion

$$\bar{F}(\gamma,\tilde{f})=\Delta(\tilde{\gamma})|K/T\cap K|\sum_g [TgK:ZK]\tilde{f}(g^{-1}(\gamma^n,s(\gamma^n)^{-1})g) \tag{1}$$

(see [7], p. 254); here $|S|$ denotes the measure of the set S and g runs through a set of representatives for the double coset space $\bar{T}\backslash\bar{G}/\bar{K}$. We take the representatives $g=\begin{pmatrix} 1 & 0 \\ 0 & \tilde{\omega} \end{pmatrix}^m$ $(m\geqq 0)$, and then

$$\delta_m=[TgK:ZK]$$

is given by

$$\delta_m=\begin{cases} 1, & (m=0) \\ (1+1/q)q^m, & (m>0) \end{cases} \quad T \text{ unramified},$$
$$\qquad 2q^m, \qquad T \text{ ramified}.$$

We write $a+b\sqrt{\theta}$ for an eigenvalue of γ (with a, b in F) and $A+B\sqrt{\theta}=(a+b\sqrt{\theta})^n$ for an eigenvalue of $\tilde{\gamma}$ (with A, B in F). To determine (1) we note that in $\bar{G}$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & \tilde{\omega} \end{pmatrix}^{-m}\left(\begin{pmatrix} A & \theta B \\ B & A \end{pmatrix}, s(\gamma^n)^{-1}\right)\begin{pmatrix} 1 & 0 \\ 0 & \tilde{\omega} \end{pmatrix}^m$$
$$=\left(\begin{pmatrix} A & \theta B\tilde{\omega}^m \\ B\tilde{\omega}^{-m} & A \end{pmatrix}, s\begin{pmatrix} A & \theta B\tilde{\omega}^m \\ B\tilde{\omega}^{-m} & A \end{pmatrix}^{-1}\right). \tag{2}$$

To prove (iv) we have to examine the case where γ is close to some z in Z. Then $b\to 0$, $a\to z$, $A\to z^n$, $B\to 0$, and the function s on the right side of (2) obtains the value 1. Since

$$\Delta(\tilde{\gamma})=\left|\frac{(2B\sqrt{\theta})^2}{A^2-\theta B^2}\right|^{\frac{1}{2}}=|2n\sqrt{\theta}\,b/a|,$$

(1) is the product of

$$2|2\sqrt{\theta}|\,|K/T\cap K|$$

and

$$\tfrac{1}{2}\sum_{m\geqq 0}|nb/a|\,\delta_m\tilde{f}\left(\begin{pmatrix} A & \theta B\tilde{\omega}^m \\ B\tilde{\omega}^{-m} & A \end{pmatrix}\right)=\tfrac{1}{2}\sum_{m\geqq 0}|nb/a|\,\delta_m\tilde{f}\left(\begin{pmatrix} z^n & 0 \\ B\tilde{\omega}^{-m} & z^n \end{pmatrix}\right).$$

This is

$$\int_{|x|\geqq|nb/a|}\tilde{f}\left(\begin{pmatrix} z^n & 0 \\ xz^n & z^n \end{pmatrix}\right)dx$$

when T is ramified, and when the measure on T is normalized, (1) is the difference between

$$F(z^n, \tilde{f}) = \int_{G_{\underline{z}} \backslash G} \tilde{f}(g^{-1}\underline{z}g)\,dg \qquad \left(\underline{z} = z^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

and

$$\int_{|x|<|nb/a|} \tilde{f}\left(\begin{pmatrix} z^n & 0 \\ xz^n & z^n \end{pmatrix}\right) dx = \tilde{f}(z^n)\,|\tilde{\omega} nb/a| \sum_{m\geqq 0} |\tilde{\omega}|^m$$

$$= \frac{|nb/a|}{q-1}\tilde{f}(z^n) = \frac{|\frac{1}{2}n/\sqrt{\theta}|}{q-1}\,\Delta(\gamma)\,\tilde{f}(z^n);$$

hence (iv) follows. The calculation for unramified T is similar and will be deleted.

The local constancy of $\bar{F}(\gamma, \tilde{f})$ on most of the set of regular elements in T can easily be obtained from (1). The only exception is at the element $\gamma_0 = \begin{pmatrix} 0 & \theta z \\ z & 0 \end{pmatrix}$ where the verification is more difficult, and therefore it will be given here. We write $\gamma = \begin{pmatrix} a & \theta b \\ b & a \end{pmatrix}$ for an element sufficiently close to γ_0; thus $a \to 0$ and $b \to z$. We have to distinguish between even and odd n.

Suppose n is even; then $\gamma^n = \begin{pmatrix} A & \theta B \\ B & A \end{pmatrix}$ is close to the singular element z^n.

Since $A \equiv b^n \theta^{\frac{1}{2}n} \pmod{a}$ we have $s\begin{pmatrix} A & \theta Bx \\ Bx^{-1} & A \end{pmatrix}^{-1} = s(Bx^{-1})$, where we put $s(x) = (x, \theta^{-\frac{1}{2}n})$ if n does not divide $ord(x)$, and $s(x) = 1$ otherwise. When T is ramified, we put $t = \theta^{\frac{1}{2}n} b^n$ and note that up to a constant, (1) is equal to

$$\sum_{m\geqq 0} |a/b\theta|\, q^m \tilde{f}\left(\begin{pmatrix} A & \theta B\tilde{\omega}^m \\ B\tilde{\omega}^{-m} & A \end{pmatrix}\right) s(B\tilde{\omega}^{-m}) = \int_{|x|\geqq|a/b|} \tilde{f}\left(\begin{pmatrix} t & 0 \\ nxt & t \end{pmatrix}\right) s(xnt)\,dx.$$

Clearly, (1) would be independent of a if we showed that $\tilde{f}(t) = 0$. To see this we note that since $ord(\theta)$ is odd, there is a unit ε with $(\theta^{n/2}, \varepsilon) = -1$, and that

$$\tilde{f}\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\right) = \tilde{f}\left(\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}^{-1} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}\right) = \tilde{f}\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\right)(\theta^{n/2}, \varepsilon);$$

hence $\tilde{f}(t) = 0$ and (1) is independent of a, as required.

When T is unramified, then up to a constant (1) is equal to

$$\tilde{f}\left(\begin{pmatrix} A & \theta B \\ B & A \end{pmatrix}\right) |a/b|\, s(B) + (1 + 1/q) \sum_{m\geqq 0} |a/b|\, q^m \tilde{f}\left(\begin{pmatrix} A & \theta B\tilde{\omega}^m \\ B\tilde{\omega}^{-m} & A \end{pmatrix}\right) s(B\tilde{\omega}^{-m})$$

$$= \tilde{f}(t)\,|a/b|\, s(B) + (1 + 1/q) \int_{|x|>|a/b|} \tilde{f}\left(\begin{pmatrix} t & 0 \\ nxt & t \end{pmatrix}\right) s(nxt)\,dx$$

$$= (1 + 1/q) \int_F \tilde{f}\left(\begin{pmatrix} t & 0 \\ nxt & t \end{pmatrix}\right) s(nxt)\,dx \qquad (t = \theta^{\frac{1}{2}n} b^n),$$

which is independent of a, since

$$\begin{aligned}(1+1/q)\int_{|x|\leqq|a/b|} s(nxt)\,dx&=(1+1/q)|a/b|\sum_{m\geqq 0} q^{-m}s(B\tilde{\omega}^m)\\ &=(1+1/q)|a/b|\sum_{m\geqq 0} q^{-m}s(B)(-1)^m=s(B)|a/b|\end{aligned}$$

when $|n|=1$, and $\tilde{f}(t)=0$ if $|n|<1$, as required.

Suppose n is odd. Then $A\to 0$, $B\to\theta^{\frac{1}{2}(n-1)}z^n$, as $a\to 0$, $b\to z$, and $\Delta(\tilde{\gamma})$ is independent of a, b. For any α near 0 in F and a unit ε of $F^\times$, we write

$$\gamma(x,y)=\begin{pmatrix} y & \theta z x B\\ Bz^{-1}x^{-1} & y\end{pmatrix}$$

with $x=\varepsilon$ or 1 and $y=\alpha$ or 0. Then

$$\begin{aligned}\tilde{f}(\gamma(\varepsilon,0))\,s(\gamma(1,\alpha))/s(\gamma(\varepsilon,\alpha))&=\tilde{f}(\gamma(\varepsilon,\alpha))\,s(\gamma(1,\alpha))/s(\gamma(\varepsilon,\alpha))\\ &=\tilde{f}\left(\begin{pmatrix}1&0\\0&\varepsilon\end{pmatrix}^{-1}\gamma(1,\alpha)\begin{pmatrix}1&0\\0&\varepsilon\end{pmatrix}\right)=\tilde{f}(\gamma(1,\alpha))=\tilde{f}(\gamma(1,0))\\ &=\tilde{f}(\gamma(\varepsilon,0))\,s(\gamma(1,0))/s(\gamma(\varepsilon,0)).\end{aligned}$$

In n does not divide $ord(Bz^{-1})$ and $\tilde{f}(\gamma(\varepsilon,0))\neq 0$, then

$$(Bz^{-1},\alpha)^{-1}(Bz^{-1}\varepsilon^{-1},\alpha)=(\alpha,\varepsilon)$$

must be 1. But there is a unit ε and some α as above with $(\alpha,\varepsilon)\neq 1$; hence $\tilde{f}(\gamma(\varepsilon,0))=\tilde{f}(\gamma(1,0))=0$, and (1) reduces to the sum over all m for which n divides $ord(B\tilde{\omega}^{-m})$. For each such m we have $s\left(\begin{pmatrix}A & \theta B\tilde{\omega}^m\\ B\tilde{\omega}^{-m} & A\end{pmatrix}\right)=1$, and we conclude that (1) is independent of a, as required.

To complete the proof, we still have to show that if $\{\bar{F}(\gamma)\}$ satisfies (i)–(v) of Lemma 2, then it is a $\bar{G}$-family. By (i) and (ii) it suffices to restrict outselves to one choice of measures and of representatives for the conjugacy classes of tori, and we make the obvious choice.

It is not difficult to find $\tilde{f}_1$ and $\tilde{f}_2$ such that $F(\gamma,\tilde{f}_1)=\bar{F}'(\gamma)$ and $\tilde{f}_1(z^n)=0$ while $F(z^n,\tilde{f}_2)=0$, $\tilde{f}_2(z^n)\neq 0$ and $\bar{F}''(\gamma,\tilde{f}_2)=\bar{F}''(\gamma)$. It suffices to show that the new family

$$\bar{F}(\gamma)-\bar{F}(\gamma,\tilde{f}_1)-\bar{F}(\gamma,\tilde{f}_2),$$

which we denote again by $\bar{F}(\gamma)$, is a $\bar{G}$-family. Its virtues are of course that $\bar{F}'=\bar{F}''=0$ at some neighborhood of each singular γ. The singular elements are the only intersections between the different tori and $\bar{F}(\gamma)$ vanishes there; hence we may restrict our attention to a single torus T (there are a finite number of conjugacy classes of these). We can also find some $\tilde{f}_3$ with $\bar{F}(\gamma,\tilde{f}_3)$ being $\bar{F}(\gamma)$ at $\gamma=\begin{pmatrix}0&\theta b\\ b&0\end{pmatrix}$ and 0 outside a small neighborhood of this element; hence we may assume in addition that $\bar{F}(\gamma)$ vanishes at $\begin{pmatrix}0&\theta b\\ b&0\end{pmatrix}$.

By (iii) it is possible to find a locally constant compactly supported function on $\bar{T}^n_{\text{reg}} \times \bar{T}\backslash\bar{G}$ whose integral over $\zeta_\gamma^{-1}\tilde{\gamma} \times \bar{T}\backslash\bar{G}$ has the value $\bar{F}(\gamma)$ for all γ with $\tilde{\gamma}$ in $\bar{T}^n_{\text{reg}}$. Here we have to use (v), which implies that $\bar{F}(\gamma)$ depends only on γ^n but not on γ. Finally, since the map

$$\bar{T}^n_{\text{reg}} \times \bar{T}\backslash\bar{G} \to \bar{G}_{\text{reg}'} \qquad (\gamma, g) \to g^{-1}\gamma g,$$

is a local homeomorphism, we obtain $\tilde{f}$ with $F(\tilde{\gamma}\zeta_\gamma^{-1}, \tilde{f}) = \bar{F}(\gamma)$ and $\bar{F}(\gamma, \tilde{f}) = \bar{F}(\gamma)$, as required.

1.4. Spherical Functions

The trace formula will be applied to operators which depend on global functions whose local components are almost all spherical; namely, they satisfy

$$\tilde{f}(kgk') = \tilde{f}(g) \qquad (k, k' \text{ in } K, g \text{ in } \bar{G}).$$

Of course, such $\tilde{f}$ may exist only when n is a unit in the (local) field F (see the definitions of 0.2), and we restrict our attention to such residual characteristics only and to nondiadic F (for convenience sake).

To apply the trace formula effectively, we need a more delicate form of Lemma 1.2.3, which will not only relate $\bar{F}(\gamma, \tilde{f})$ and $F(\gamma, f)$, but will also show that if $\tilde{f}$ is spherical, f can be chosen to be spherical, and vice versa.

The convolution algebra $\bar{\mathscr{H}}$ of all genuine spherical compactly supported functions on $\bar{G}$ is generated over $\mathbb{C}$ by the functions $\tilde{f}_\lambda$, where $\lambda = (m, m')$ is a pair of integers divisible by n, which obtain the value ζ at g if g is in

$$K(\lambda, \zeta)K = K\left(\begin{pmatrix} \tilde{\omega}^m & 0 \\ 0 & \tilde{\omega}^{m'} \end{pmatrix}, \zeta\right)K,$$

and the value 0 otherwise. The reason for considering only m and m' divisible by n is explained by the arguments of Lemma 1.1.1. Similarly, the convolution algebra $\mathscr{H}$ of spherical compactly supported functions on G is generated by the characteristic functions f_λ ($\lambda = (m, m')$ is a pair of integers) of $K\lambda K$. We put $\langle \alpha, \lambda \rangle = m - m'$, and define a map φ from $\bar{\mathscr{H}}$ to $\mathscr{H}$ by

$$\varphi(\tilde{f}_{n\lambda}) = q^{\frac{1}{2}\langle \alpha, (n-1)\lambda \rangle} f_\lambda.$$

This map is clearly one-to-one and onto; moreover, it takes the characteristic function of $\bar{K}$ to that of K. We prove:

Lemma. *For every γ in G and $\tilde{f}$ in $\bar{\mathscr{H}}$, we have*

$$\bar{F}(\gamma, \tilde{f}) = F(\gamma, \varphi(\tilde{f})).$$

Proof. It suffices to prove this only for $\tilde{f} = \tilde{f}_{n\lambda}$, $\langle \alpha, \lambda \rangle \geqq 0$. If $\gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ lies in

A^n we claim that

$$\int_{\bar{A}\backslash G} \tilde{f}_{n\lambda}(g^{-1}\gamma g)\,dg = \int_{A\backslash G} f_{n\lambda}(g^{-1}\gamma g)\,dg.$$

This follows from the Iwasawa descomposition $\bar{G}=\bar{A}NK$ and the following equality in $\bar{G}$ (not only in G!):

$$\begin{pmatrix}1 & 0\\ m & 1\end{pmatrix}=\begin{pmatrix}1 & m^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}m^{-1} & 0\\ 0 & m\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & m^{-1}\\ 0 & 1\end{pmatrix} \quad (m \text{ in } F^{\times}).$$

Indeed we may assume that $|\gamma_1|\geqq|\gamma_2|$, and moreover that $|\gamma_1|>|\gamma_2|$ (since the case $|\gamma_1|=|\gamma_2|$ is immediate). But $\tilde{f}_{n\lambda}$ is spherical, hence (for $|m|>1$) we have

$$\begin{aligned}
&\tilde{f}_{n\lambda}\left[\begin{pmatrix}1 & -m\\ 0 & 1\end{pmatrix}\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\right]\\
&\quad=\tilde{f}_{n\lambda}\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}\begin{pmatrix}1 & (1-\gamma_2/\gamma_1)m\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}\right]\\
&\quad=\tilde{f}_{n\lambda}\left[\begin{pmatrix}\gamma_2 & 0\\ 0 & \gamma_1\end{pmatrix}\begin{pmatrix}1 & 0\\ (1-\gamma_2/\gamma_1)m & 1\end{pmatrix}\right]\\
&\quad=\tilde{f}_{n\lambda}\left[\begin{pmatrix}\gamma_2 & 0\\ 0 & \gamma_1\end{pmatrix}\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix}\right]=\tilde{f}_{n\lambda}\left(\begin{pmatrix}\gamma_2/x & 0\\ 0 & \gamma_1 x\end{pmatrix}\right),
\end{aligned}$$

where $x=((\gamma_2/\gamma_1)-1)m$. By definition this is equal to

$$f_{n\lambda}\left(\begin{pmatrix}\gamma_2/x & \\ 0 & \gamma_1 x\end{pmatrix}\right)=f_{n\lambda}\left[\begin{pmatrix}1 & -m\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\right]$$

as required. The last equality follows on reversing in G the above calculations in $\bar{G}$ and noting that $f_{n\lambda}$ is spherical.

It follows that $\tilde{F}(\gamma,\tilde{f}_{n\lambda})=F(\gamma^n,f_{n\lambda})$ for all γ in A. Lemma 3.1 of Langlands [9] establishes that for $\lambda^{\vee}=(ord\,\gamma_1, ord\,\gamma_2)$ with $\langle\alpha,\lambda^{\vee}\rangle\geqq 0$,

$$F(\gamma,f_\lambda)=|K/T\cap K|\times\begin{cases} q^{\frac{1}{2}\langle\alpha,\lambda\rangle}, & \text{if } \lambda=\lambda^{\vee},\\ q^{\frac{1}{2}\langle\alpha,\lambda\rangle}\left(1-\dfrac{1}{q}\right), & \text{if } \lambda=\lambda^{\vee}+m\alpha, m>0,\\ 0, & \text{otherwise,}\end{cases}$$

where $\gamma=\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}$ and $\alpha=(1,-1)$. It is implicit that $F(\gamma,f)$ depends only on the pair $\lambda^{\vee}$ so that we can write $F(\lambda^{\vee},f)$ for $F(\gamma,f)$, and this function is invariant under permutation of the two coordinates of $\lambda^{\vee}$. The lemma follows at once for all γ in A.

We shall now deal with elliptic elements γ of the nonsplit tori $T=\left\{\begin{pmatrix}a & \theta b\\ b & a\end{pmatrix};\right.$ $\left. a \text{ or } b\neq 0\right\}$. We assume as we may that $|\theta|=1$ if T is unramified and $|\theta|=|\tilde{\omega}|$ if T

is ramified. To consider the functions $\tilde{f}_\lambda$, $\lambda=(m,m')$, with $m=m'$, it clearly suffices to study the case of $\lambda=(0,0)$ where $\tilde{f}=\tilde{f}_\lambda$ is the "genuine" characteristic function of $\bar{K}$. We compare (1) of 1.3 with its obvious analogue

$$F(\gamma,f)=\Delta(\gamma)|K/T\cap K|\sum_{m\geq 0}\delta_m f\left(\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{-m}\gamma\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{m}\right)$$

for G, and as usual we write $\gamma=\begin{pmatrix}a&\theta b\\b&a\end{pmatrix}$, $\gamma^n=\begin{pmatrix}A&\theta B\\B&A\end{pmatrix}$.

There are various cases to be considered. Both $\bar{F}(\gamma,\tilde{f})$ and $F(\gamma,\varphi(\tilde{f}))$ vanish unless $\det\gamma$ is a unit in $F^\times$. Suppose $a^2-\theta b^2$ is a unit. If $|a|>|b|$, then $A\equiv a^n$ and $B\equiv nba^{n-1}(\mathrm{mod}\,(1+\tilde{\omega}0))$; hence $|a|=|A|(=1)$ and $|b|=|B|$, and the expansions for $\bar{F}(\gamma,\tilde{f})$ and $F(\gamma,\varphi(\tilde{f}))$ are clearly equal term by term. There is only one term on both sides when $|a|=|b|=1$. Indeed, in this case $|B|=1$. This is obvious if $|\theta|<1$. If $|\theta|=1$, we repeat the argument of Lemma 0.3.2: If $B\equiv 0\,(\mathrm{mod}\,(1+\tilde{\omega}0))$ then $x=a/b$ satisfies the equation

$$\binom{n}{1}x^{n-1}+\binom{n}{3}x^{n-3}\theta+\binom{n}{5}x^{n-5}\theta^2+\ldots\equiv 0$$

whose zeros are obtained by multiplying by $\sqrt{\theta}$ the zeros of

$$\binom{n}{1}x^{n-1}+\binom{n}{3}x^{n-3}+\ldots=\tfrac{1}{2}((x+1)^n-(x-1)^n),$$

who all lie in $F^\times$; hence $\sqrt{\theta}$ lies in $F^\times$, but this is impossible since θ is a nonsquare.

Finally, when $|b|>|a|$, we must have $|\theta|=1$. If n is odd, then $B\equiv b^n\theta^{(n-1)/2}$ and $A\equiv nab^{n-1}\theta^{(n-1)/2}$; hence there is only one term in the expansions of both $\bar{F}(\gamma,\tilde{f})$ and $F(\gamma,\varphi(\tilde{f}))$, and these are clearly equal. The last case is of even n. Here $A\equiv b^n\theta^{n/2}$ and $B\equiv nab^{n-1}\theta^{(n/2)-1}$; hence $|B|<1$, and we have

$$\bar{F}(\gamma,\tilde{f})=t\,|a|\left(s(B)+\sum_{m=1}^{k}\delta_m\tilde{f}\begin{pmatrix}A&\theta B\tilde{\omega}^m\\B\tilde{\omega}^{-m}&A\end{pmatrix}s(B\tilde{\omega}^{-m})\right)\qquad (t=|2|\,|K/K\cap T|),$$

where $s(x)=(x,A)^{-1}=(x,\theta^{n/2})^{-1}$ if n does not divide $ord(x)$, and $s(x)=1$ otherwise, and $k=ord\,a(|a|=q^{-k})$. The inner sum is

$$\left(1+\frac{1}{q}\right)\sum_{m=1}^{k}(-q)^m s(B)=\left(\frac{(-1)^k}{|a|}-1\right)s(B);$$

hence

$$\bar{F}(\gamma,\tilde{f})=t(-1)^k s(B)=t,$$

and this is equal to $F(\gamma,\varphi(\tilde{f}))$, as required.

To deal with general $\tilde{f}_\lambda$, we distinguish between two cases.

(i) Suppose T is unramified, γ in T, r is the order of the eigenvalues of γ, and put $\lambda^\vee=(r,r)$. $F(\gamma,\varphi(\tilde{f}_{n\lambda}))$ is nonzero only if $\lambda=\lambda^\vee+(m,-m)$, $m\geqq 0$. The case $\lambda=\lambda^\vee$ has been dealt with above, so we assume that $m>0$, and by Langlands [9], Lemma 3.6, we have $F(\gamma,\varphi(\tilde{f}_{n\lambda}))=tq^{nm}(1+1/q)$, where $t=|K/T\cap K|$. The same lemma implies also that some conjugate of γ^n lies in $K(n\lambda)K$, and that $F(\gamma^n,f_{n\lambda})=tq^{nm}(1+1/q)$, since $n\lambda=n\lambda^\vee+(nm,-nm)$. We deduce from the usual formula

$$F(\gamma^n,f_{n\lambda})=t\Delta(\gamma^n)\sum_{k\geqq 0}\delta_k f_{n\lambda}\left(\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{-k}\gamma^n\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{k}\right) \tag{1}$$

that if $\Delta(\gamma^n)=q^{-\beta}$, then

$$\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{-k}\gamma^n\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{k}\text{ lies in }K(n\lambda)K \tag{2}$$

if and only if $k=\beta+nm$. But by definition $\beta=ord\,B-nr$, hence (for such k)

$$ord(B\tilde{\omega}^{-k})=ord\,B-k=ord\,B-\beta-nm=n(r-m)\equiv 0\,(\mathrm{mod}\,n).$$

It follows that

$$s\left(\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{-k}\gamma^n\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{k}\right) \tag{3}$$

is 1. Hence $\bar{F}(\gamma,\tilde{f}_{n\lambda})$ is equal to $F(\gamma,\varphi(\tilde{f}_{n\lambda}))$, as required.

(ii) Suppose that T is ramified and $|\theta|=|\tilde{\omega}|$. If $ord(\det\gamma)=2r$ (r integer), then $|a|\geqq|b|$ and A is an n-th power in $F^\times$; hence $s\left(\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{-k}\gamma^n\begin{pmatrix}1&0\\0&\tilde{\omega}\end{pmatrix}^{k}\right)=1$ for all $k\geqq 0$. $F(\gamma,\varphi(\tilde{f}_{n\lambda}))$ does not vanish only if $\lambda=(r+m,r-m)$, $m\geqq 0$, and in view of the above considerations, we may assume that $m>0$. Then $\Delta(\gamma)=\Delta(\tilde{\gamma})$, $\bar{F}(\gamma,\tilde{f}_{n\lambda})$ is equal to $F(\gamma^n,f_{n\lambda})$, both $\tilde{f}_{n\lambda}(a^n)$ and $\varphi(\tilde{f}_{n\lambda})(a)$ are 0, and the lemma follows on a double application of [9], Lemma 3.8, with $F(\gamma^n,f_{n\lambda})$ and with $F(\gamma,\varphi(\tilde{f}_{n\lambda}))$.

If $ord(\det\gamma)=2r+1$ is odd, we have $|a|<|b|$, and we deduce from [9], Lemma 3.8, that for even n $F(\gamma^n,f_{n\lambda})$ is 0 unless $n\lambda=(nr+\frac{1}{2}n+m,\ nr+\frac{1}{2}n-m)$ with $m>0$, where it is $tq^{m-\frac{1}{2}}$. If $\Delta(\gamma^n)=q^{-\beta}$, then β is a half of a positive odd integer, and from (1) we deduce that (2) holds if and only if $k=m+\beta-\frac{1}{2}$. Since $ord(B\tilde{\omega}^{-k})=ord(A\tilde{\omega}^{-m})$ is divisible by n, the expression (3) is 1. Hence $\bar{F}(\gamma,\tilde{f}_{n\lambda})=tq^{m-\frac{1}{2}}$ when $n\lambda=(nr+\frac{1}{2}n+m,\ nr+\frac{1}{2}n-m)$ if $m>0$, and $=0$ otherwise. Now $q^{\frac{1}{2}\langle\alpha,(n-1)\lambda\rangle}=q^{\frac{n-1}{n}m}$,

$$\lambda=\left(r+\tfrac{1}{2}+\frac{m}{n},\ r+\tfrac{1}{2}-\frac{m}{n}\right)=\left(r+1+\left(\frac{m}{n}-\tfrac{1}{2}\right),\ r-\left(\frac{m}{n}-\tfrac{1}{2}\right)\right),$$

and [9], Lemma 3.7, implies that $F(\gamma,f_\lambda)=tq^{\frac{m}{n}-\frac{1}{2}}$, hence that $F(\gamma,\varphi(\tilde{f}_{n\lambda}))=tq^{m-\frac{1}{2}}$, as required.

Finally, if $ord(\det\gamma)=2r+1$ is odd and n is odd, $F(\gamma^n,f_{n\lambda})$ is 0 unless $n\lambda=\left(nr+\frac{n+1}{2}+m,\ nr+\frac{n-1}{2}-m\right)$, $m\geqq 0$, when it is tq^m. The only term to contribute to

(1) is $k=m\equiv\frac{n-1}{2}\,(mod\,n)$, and since $ord(B\tilde{\omega}^{-k})$ is divisible by n, the expression (3) is 1; hence $\bar{F}(\gamma,f_{n\lambda})=tq^m$ for $n\lambda$ as above. Also $q^{\frac{1}{2}\langle\alpha,(n-1)\lambda\rangle}=q^{\frac{n-1}{n}(m+\frac{1}{2})}$ and [9], Lemma 3.7, implies that $F(\gamma,f_\lambda)=tq^{\frac{m}{n}+\frac{1}{2n}-\frac{1}{2}}$, hence that $F(\gamma,\varphi(\bar{f}_{n\lambda}))=tq^m$, as required.

We finish this section by modifying all results so as to apply to genuine locally constant functions $\bar{f}$ which have compact support modulo the centre $\bar{Z}^n$ of $\bar{G}$ and transform under the centre by a character μ^{-1} of $\bar{Z}^n$, namely,

$$\bar{f}(zg)=\mu^{-1}(z)\bar{f}(g)\qquad(z\text{ in }Z^n, g\text{ in }\bar{G}).$$

Similarly, we shall consider locally constant functions f which have compact support modulo the centre Z of G and transform under the centre by a character μ'^{-1} of Z; thus

$$f(zg)=\mu'^{-1}(z)f(g)\qquad(z\text{ in }Z, g\text{ in }G).$$

Lemma 1.2.3 is again valid if the characters μ and μ' of Z^n and Z are related by the equation

$$\mu(z^n)=\mu'(z)\qquad(z\text{ in }Z).$$

Lemma 1.4 will be applied with elements of the algebra $\bar{\mathscr{H}}'$ of bi-invariant with respect to K, genuine, compactly supported modulo Z^n functions $\bar{f}'$ on $\bar{G}$ which transform under the centre by μ^{-1}, where μ is an unramified character of Z^n. The convolution in $\bar{\mathscr{H}}'$ is defined by

$$\bar{f}_1'*\bar{f}_2'(g)=\int_{Z^n\backslash\bar{G}}\bar{f}_1'(gh^{-1})\bar{f}_2'(h)\,dh.$$

Similarly, we define an algebra $\mathscr{H}'$ of spherical functions f' on G with compact support modulo Z and which transform by μ'^{-1} under Z, where μ' is an unramified character of Z satisfying $\mu'(z)=\mu(z^n)$ (z in Z). The convolution in $\mathscr{H}'$ is defined by an integral over $Z\backslash G$. The map $\bar{f}\to\bar{f}'$ defined by

$$\bar{f}'(g)=\int_{Z^n}\bar{f}(zg)\mu(z)\,dz,$$

is a surjective homomorphism from $\bar{\mathscr{H}}$ to $\bar{\mathscr{H}}'$ and so is the map $f\to f'$ from $\mathscr{H}$ to $\mathscr{H}'$ which can be defined in a similar way. Now if f is the image of $\bar{f}$ under the map φ from $\bar{\mathscr{H}}$ to $\mathscr{H}$, we have

$$\bar{F}(\gamma,\bar{f}')=\int_Z\bar{F}(z\gamma,\bar{f})\mu(z^n)\,dz=\int_Z F(z\gamma,f)\mu'(z)\,dz=F(\gamma,f')$$

for every γ in G with γ^n regular.

Lemmas 1.2.3 and 1.4 will be applied with functions which transform under the centre by a character μ and μ' as above, and which have compact support modulo the centre and not with the functions which had previously been used.

2.0. Characters

Let μ be a character of Z^n, and let $\bar{\pi}$ be an anti-genuine admissible representation of $\bar{G}$ in a vector space V (necessarily infinite-dimensional) which transforms under the centre by μ; thus, by definition,

$$\bar{\pi}(\zeta z g)=\zeta^{-1}\mu(z)\bar{\pi}(g) \quad (z \text{ in } Z^n,\ g \text{ in } \bar{G}).$$

For any locally constant genuine function $\tilde{f}$ on $\bar{G}$ with compact support modulo Z^n and with

$$\tilde{f}(\zeta z g)=\zeta\mu^{-1}(z)\tilde{f}(g) \quad (z \text{ in } Z^n),$$

we consider the operator

$$\bar{\pi}(\tilde{f})=\int_{Z^n\backslash G} \tilde{f}(g)\bar{\pi}(g)\,dg.$$

Suppose that $\bar{\pi}(\tilde{f})$ has a finite trace for any such $\tilde{f}$. The antigenuine locally integrable function $\chi(g)$ on the conjugacy classes of $\bar{G}$ is called the character of $\bar{\pi}$ if for any $\tilde{f}$ we have

$$\operatorname{tr}\bar{\pi}(\tilde{f})=\int_{Z\backslash G} \tilde{f}(g)\chi(g)\,dg.$$

In this section the characters of admissible anti-genuine representations $\bar{\pi}$ of $\bar{G}$ will be studied.

2.1. Principal Series

An admissible irreducible representation $\bar{\pi}$ of $\bar{G}$ on a vector space V is said be supercuspidal if for every vector v in V there is an open compact subgroup U of N for which

$$\int_U \bar{\pi}(n)\,v\,dn=0.$$

Every anti-genuine admissible irreducible nonsupercuspidal representation $\bar{\pi}$ of $\bar{G}$ is intertwined with a subquotient of the representation $\operatorname{Ind}(\bar{B},\bar{G},\bar{\tau})$ induced from an anti-genuine irreducible representation $\bar{\tau}$ of $\bar{B}$ trivial on N ([7] (for G), [1] ($n=2$)). The (Heisenberg) group $\bar{A}\simeq\bar{B}/N$ is not abelian, but it contains an abelian subgroup $\bar{A}_0$ of minimal (finite) index. Note that $\bar{A}_0$ contains $\bar{A}^n$, let $\eta=(\nu_1,\nu_2)$ be a pair of characters (a one-dimensional anti-genuine representation) of the abelian group $\bar{A}^n$, and extend η in any way to $\bar{A}_0$. By Clifford's theory any $\bar{\tau}$ as above is of the form $\operatorname{Ind}(\bar{A}_0,\bar{A},\eta)$ ([1], p. 99 ($n=2$)); hence any $\bar{\pi}$ is a subquotient of some $\operatorname{Ind}(\bar{B}_0,\bar{G},\eta)$, where $\bar{B}_0=\bar{A}_0\times N$ and η is extended to N by the value 1. We write $\bar{\rho}(\eta)$ for $\operatorname{Ind}(\bar{B}_0,\bar{G},\eta)$. We shall show below that it depends only on the restriction of η to $\bar{A}^n$; hence it is not necessary to specify the values of η on $\bar{A}_0-\bar{A}^n$ or even to choose $\bar{A}_0$. Recall that $\bar{\rho}(\eta)$ acts by right translations on the space of locally constant anti-genuine functions $\tilde{\phi}$ on $\bar{G}$ satisfying

$$\tilde{\phi}\left[\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \zeta\right) g\right]=\zeta^{-1}|a/b|^{\frac{1}{2}} v_1(a)\, v_2(b)\, \tilde{\phi}(g) \qquad (g \text{ in } \bar{G}),$$

for any $\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \zeta\right)$ in $\bar{B}_0$, which are square-integrable on $\bar{K}$.

Lemma. *The character $\chi_{\bar{\rho}(\eta)}$ of $\bar{\rho}(\eta)$ at (g, ζ) is equal to*

$$\zeta^{-1}\frac{\eta(g')+\eta(w^{-1}g'w)}{\Delta(g)} \qquad \left(w=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$

if g is conjugate to a regular element g' of A^n, *and to* 0 *if g is any other regular element.*

Here we set $\eta\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}=v_1(a)\, v_2(b)$ for a, b in $F^{\times n}$.

Proof. This follows closely the proof of [7], Proposition 7,6; when $n=2$ a variant was given by [1], Theorem 5.14.

By the Iwasawa decomposition we can choose a measure on K so that

$$\begin{aligned}\int_{Z^n\backslash G} \tilde{f}(g)\,\bar{\pi}(g)\,dg&=t^{-1}\int_K\int_N\int_{Z^n\backslash \bar{A}_0}\int_{\bar{A}_0\backslash \bar{A}} \tilde{f}(a_0amk)\,\bar{\pi}(a_0amk)\\ &=t^{-1}\int\int\int\int |a_0a|^{-1}\tilde{f}\bar{\pi}(ma_0ak);\end{aligned}$$

here $t=|\bar{A}_0\backslash\bar{A}|$ is the measure of $\bar{A}_0\backslash\bar{A}$, and we put $|a|=|\alpha/\beta|$ for a matrix $a=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Observing that

$$\begin{aligned}(\bar{\rho}(\eta,\tilde{f})\,\tilde{\phi})(h)&=\int_{Z^n\backslash G}\tilde{f}(g)\,(\bar{\rho}(\eta)(g)\,\tilde{\phi})(h)\,dg\\ &=\int_{Z^n\backslash G}\tilde{f}(g)\,\tilde{\phi}(hg)\,dg=\int_{Z^n\backslash G}\tilde{f}(h^{-1}g)\,\tilde{\phi}(g)\,dg\\ &=t^{-1}\int_K\int_N\int_{Z^n\backslash \bar{A}_0}\int_{\bar{A}_0\backslash\bar{A}}|a_0a|^{-1}\tilde{f}(h^{-1}ma_0ak)\,\tilde{\phi}(ma_0ak)\\ &=t^{-1}\int\int\left(\int\int|a_0a|^{-1}|a_0|^{\frac{1}{2}}\eta(a_0)\tilde{f}(h^{-1}ma_0ak)\right)\tilde{\phi}(ak),\end{aligned}$$

we see that $\bar{\rho}(\eta,\tilde{f})$ is an integral operator with kernel

$$K_{\tilde{f}}(a'k',ak)=\int_{Z^n\backslash\bar{A}_0}\int_N|a_0|^{-\frac{1}{2}}|a|^{-1}\eta(a_0)\,\tilde{f}(k'^{-1}a'^{-1}ma_0ak)\,dm\,da_0,$$

whence

$$\operatorname{tr}\bar{\rho}(\eta,\tilde{f})=t^{-1}\int\int\int\int|a_0|^{-\frac{1}{2}}|a|^{-1}\eta(a_0)\,\tilde{f}(k^{-1}a^{-1}ma_0ak).$$

Since, in $\bar{G}$,

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}=\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \zeta\right),$$

where $z=\frac{x\alpha_2 a_2}{a_1(\alpha_1-\alpha_2)}$ and $\zeta=(\alpha_1, a_2)(\alpha_2, a_1)$, the trace is

$$=t^{-1}\int\int\int\int\left|\frac{(\alpha_1-\alpha_2)^2}{\alpha_1\alpha_2}\right|^{\frac{1}{2}}\eta(a_0)\tilde{f}(k^{-1}m^{-1}a_0 mk)(\alpha_1, a_2)(\alpha_2, a_1),$$

where $a_0=\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}$, $a=\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}$. The integral over $\bar{A}_0\backslash\bar{A}$ will vanish for every a_0 in $\bar{Z}^n\backslash\bar{A}_0$ which does not lie in $\bar{Z}^n\backslash\bar{A}^n$. Noting this and integrating the remaining terms over $\bar{A}_0\backslash\bar{A}$ we obtain

$$=\int_{\bar{Z}^n\backslash\bar{A}^n}\Delta(a_0)\eta(a_0)\Big(\int_K\int_N\tilde{f}(k^{-1}m^{-1}a_0 mk)\,dm\,dk\Big)\,da_0$$

$$=\tfrac{1}{2}\int_{\bar{Z}^n\backslash\bar{A}^n}\Delta(a_0)^2\,\frac{\eta(a_0)+\eta(w^{-1}a_0 w)}{\Delta(a_0)}\Big(\int_K\int_N\tilde{f}(k^{-1}m^{-1}a_0 mk)\,dm\,dk\Big)\,da_0,$$

and this is equal, as required, to

$$\int_{\bar{Z}^n\backslash\bar{G}}\chi_{\bar{\rho}(\eta)}(g)\tilde{f}(g)\,dg,$$

by virtue of the Weyl integration formula

$$\int_{\bar{Z}^n\backslash\bar{G}}\chi(g)\tilde{f}(g)\,dg=\tfrac{1}{2}\sum_T\int_{\bar{Z}^n\backslash\bar{T}}\chi(\gamma)\Delta(\gamma)^2\int_{\bar{G}_\gamma\backslash\bar{G}}\tilde{f}(g^{-1}\gamma g)\,dg\,d\gamma$$
$$=\tfrac{1}{2}\sum_T\int_{\bar{Z}^n\backslash\bar{T}^n}\chi(\gamma)\Delta(\gamma)^2\int_{\bar{T}\backslash\bar{G}}\tilde{f}(g^{-1}\gamma g)\,dg\,d\gamma$$

($\chi(g)$ is an anti-genuine function invariant under conjugation and transforms under Z^n by μ, the last equality follows from Lemmas 1.1.1 and 1.1.2, and the sum is taken over a set of representatives for the conjugacy classes of Cartan subgroups of G).

This lemma, together with Lemmas 1.2.3 and 1.4, and [7], Proposition 7.6, implies that

$$\operatorname{tr}\bar{\rho}(\eta,\tilde{f})=\int_{\bar{Z}^n\backslash\bar{G}}\chi(g)\tilde{f}(g)\,dg=\tfrac{1}{2}\int_{Z\backslash A}(\eta(a^n)+\eta(w^{-1}a^n w))\,\bar{F}(a,\tilde{f})\,da$$

for measures related so that $\int_{\bar{Z}^n\backslash\bar{A}^n}h(x)\,d^\times x=\int_{Z\backslash A}h(x^n)\,d^\times x$ for any compactly supported locally constant function h on $Z\backslash A$,

$$=\tfrac{1}{2}\int_{Z\backslash A}(\eta'(a)+\eta'(w^{-1}aw))\,F(a,f)\,da=\int_{Z\backslash G}\chi_{\rho(\eta')}(g)f(g)\,dg=\operatorname{tr}\rho(\eta',f),$$

where $\eta'=(v_1', v_2')$ is a character on A with $\eta'(z)=\eta(z^n)$ (z in Z), and $\rho(\eta')$ is the principal series representation $\operatorname{Ind}(B,G,\eta')$ of G. Thus:

Corollary. *We have the character relation*

$$\Delta(\tilde{g})\,\chi_{\bar{\rho}(\eta)}(\tilde{g}\zeta_g^{-1})=\Delta(g)\,\chi_{\rho(\eta')}(g)$$

with the ζ_g *defined in* 1.2, *and for every* $\tilde{f}$ *and* f *with matching orbital integrals as in Lemma* 1.2.3 *we have*

$$\operatorname{tr} \bar{\rho}(\eta, \tilde{f}) = \operatorname{tr} \rho(\eta', f).$$

A representation π (admissible, irreducible) is said to be of *class* 1 (or *unramified*) if it contains a K-invarient non-zero vector. If f is spherical, then it is easy to verify that $\pi(k)(\pi(f)v) = \pi(f)v$ for any k in K and v in the space of π. Hence $\pi(f) = 0$ and $\operatorname{tr} \pi(f) = 0$ if π is not of class 1 but f is spherical.

A representation π (admissible, irreducible) is said to be *square-integrable* if there is some $v \neq 0$ in its space and some $u \neq 0$ in the space of its contragredient representation, such that

$$\int_{Z\backslash G} |f(g)|^2 \, |\mu^{-1}(\det g)| \, dg < \infty,$$

where $f(g) = (\pi(g)v, u)$ and $(\, , \,)$ denotes the canonical pairing between π and its contragredient, Z denote the centre of G, and μ is the central character of π. The function f is called a matrix coefficient of π. The matrix coefficients of a supercuspidal representation π have compact support modulo the centre; hence such π is square-integrable.

The representations $\rho(\eta')$ of $G = GL(2)$ are class 1 if and only if $\eta' = (\nu_1', \nu_2')$ is a pair of unramified characters, and then it contains exactly one K-invariant vector ([7], Lemma 3.9). The $\rho(\eta')$ are reducible if and only if $\eta' = (\mu' \|^s, \mu' \|^{-s})$ with $s = \frac{1}{2}$ or $-\frac{1}{2}$, and then the decomposition series is of length two. It has a square-integrable subquotient $\sigma(\eta')$ which is called the special representation ([7], Lemma 15.2). The complement of $\sigma(\eta')$ in $\rho(\eta')$ is denoted by $\pi(\eta')$ (and it is one dimensional); $\pi(\eta')$ is of class 1 when η' is unramified.

The representations $\bar{\rho}(\eta)$ of $\bar{G}$ cannot be of class 1 if $|n|_v < 1$. Suppose that $|n|_v = 1$; then $\bar{\rho}(\eta)$ is of class 1 if and only if η is unramified, and it contains exactly one K-invariant vector in this case ([1], p. 103 ($n=2$)). The $\bar{\rho}(\eta)$ are reducible if and only if $\eta = (\mu \|^s, \mu \|^{-s})$ with $s = \dfrac{1}{2n}$ or $\dfrac{-1}{2n}$, and then the decomposition series is of length two (see Lemma 3.2 below). We observe that $\bar{\rho}(\eta)$ $(s = 1/2n)$ has a subrepresentation $\bar{\sigma}(\eta)$ (whose space contains the φ in $\bar{\rho}(\eta)$ with $\int_F \varphi\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] dx = 0$) which is square-integrable (for proof see the remark ending the proof of Lemma 2.3.2 below). The quotient of $\bar{\rho}(\eta)$ by $\bar{\sigma}(\eta)$ is denoted by $\bar{\pi}(\eta)$ and it is equivalent to a subrepresentation of $\bar{\rho}(\eta)$ $(s = -1/2n)$ which is again denoted by $\bar{\pi}(\eta)$; the corresponding quotient is denoted by $\bar{\sigma}(\eta)$, and it is equivalent to $\bar{\sigma}(\eta)$ $(s = 1/2n)$. In general $\bar{\rho}(\mu_1, \mu_2)$ is equivalent to $\bar{\rho}(\mu_2, \mu_1)$, and we shall deal below mostly with equivalence classes of representations. We call $\bar{\sigma}(\eta)$ "special". Its complement $\bar{\pi}(\eta)$ in $\bar{\rho}(\eta)$ is of class 1 if and only if $|n|_v = 1$ and η is unramified.

As in [7], where an irreducible $\rho(\eta')$ was denoted also by $\pi(\eta')$, we shall often write $\bar{\pi}(\eta)$ for $\bar{\rho}(\eta)$ if $\bar{\rho}(\eta)$ is irreducible. Note that $\rho(\eta')$ and $\bar{\rho}(\eta)$ are not square-integrable. Since $\bar{\sigma}(\eta)$ and $\sigma(\eta')$ are never of class 1, we deduce from the above corollary that for any spherical $\tilde{f}$ and f with matching orbital integrals as in Lemma 1.4 we have

$$\operatorname{tr} \bar{\pi}(\eta, \tilde{f}) = \operatorname{tr} \bar{\rho}(\eta, \tilde{f}) = \operatorname{tr} \rho(\eta', f) = \operatorname{tr} \pi(\eta', f), \tag{2}$$

where $\eta'(z) = \eta(z^n)$ (z in $F^\times$); both sides are 0 unless η and η' are unramified. If η is ramified, then there is some μ such that $\eta \otimes \mu^{-1}$ is unramified, and we have

$$\begin{aligned} \operatorname{tr} \bar{\pi}(\eta, \mu^{-1} \tilde{f}) &= \operatorname{tr}(\bar{\pi}(\eta) \otimes \mu^{-1})(\tilde{f}) = \operatorname{tr}(\bar{\rho}(\eta) \otimes \mu^{-1})(\tilde{f}) \\ &= \operatorname{tr}(\rho(\eta') \otimes \mu'^{-1})(f) = \operatorname{tr}(\pi(\eta') \otimes \mu'^{-1})(f) = \operatorname{tr} \pi(\eta', \mu'^{-1} f), \end{aligned}$$

where $\mu'(z) = \mu(z^n)$ (z in $F^\times$).

This formula relating $\bar{\pi}(\eta)$ to $\pi(\eta')$ holds not only with (twists of) spherical functions, but also with general functions as in Lemma 1.2.3. Moreover, it holds for all v, not only when v is such that $|n|_v = 1$. This will be deduced in 5.2 from the trace formula.

2.2. *Weil Representations* ($n = 2$)

These are certain genuine representations of the two-fold covering group $\bar{G}$ of G, which were introduced by Weil [16] in 1964 in his representation theoretic reformulation of theta-series in an odd number of variables. Although the Weil representations play no vital role in this work and could be ignored altogether, it seems suitable to discuss their characters here in view of the applications that this has to the classical theory of theta-series, and since the Weil representations have attracted much interest in the past. Thus in this subsection we shall calculate the character of the Weil representation associated to a quadratic form in one variable, and when our study of characters is complete, we shall be able to identify the various (even or odd) pieces of the Weil representation (cf. [4]).

Let $q(x) = qx^2$ be a quadratic form in one variable on F where q is a representative of the finite group $F^\times / F^{\times 2}$. We fix an additive order 0 character ψ of F and define the Fourier transform Φ' (with respect to q and ψ) of a square-integrable function Φ on F by

$$\Phi'(x) = \int_F \Phi(y)\, \psi(2qxy)\, |q|^{\frac{1}{2}}\, dy.$$

The subgroup $\bar{S}$ of all (g, ζ) in $\bar{G}$ with $\det g = 1$ is generated by the elements $\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta\right)$ and $\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \zeta\right)$ (b in F). The maps

$$r\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \zeta\right) \Phi(x) = \zeta \psi(b q(x))\, \Phi(x)$$

and

$$r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta\right) \Phi(x) = \zeta \gamma(q)\, \Phi'(x),$$

where $\gamma(q)$ is an 8-th root of unity ([16], Theorem 2), extend to a genuine admissible representation r of $\bar{S}$ on the space of square-integrable functions on F, and it satisfies

$$r\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \zeta\right) \Phi(x) = \zeta |a|^{\frac{1}{2}} \omega(a) \Phi(ax) \qquad (\omega(a) = \gamma(q)/\gamma(aq)),$$

for every a in $F^\times$. Note that by [16], p. 176, we have

$$\omega(a)\,\omega(b) = (a, b)\,\omega(ab).$$

The representation r of $\bar{S}$ decomposes as a direct sum of two irreducible subrepresentations on the subspaces of even ($\Phi(-x) = \Phi(x)$) and odd ($\Phi(-x) = -\Phi(x)$) functions ([1], p. 118). If μ is an even (resp. odd) character, then r extends to an irreducible genuine admissible representation r_μ of the subgroup $\bar{G}_+$ of all (g, ζ) in $\bar{G}$ with square $\det g$ on the space of even (resp. odd) square-integrable functions on F by

$$r_\mu\left(\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}, \zeta\right) \Phi(x) = \zeta \mu(a) |a|^{\frac{1}{2}} \Phi(ax) \qquad (a \text{ in } F^\times),$$

and it can easily be verified that r_μ satisfies

$$r_\mu\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \zeta\right) \Phi(x) = \zeta \mu(a)\,\omega(a)\,\Phi(x).$$

The map $\Phi \to \varphi$, given by

$$\varphi(x^2) = r_\mu\left(\begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}, 1\right) \Phi(1) = \mu(x) |x|^{\frac{1}{2}} \Phi(x),$$

intertwines r_μ with a (genuine admissible irreducible) representation ξ of $\bar{G}_+$ on a space of square-integrable functions φ on $F^{\times 2}$, satisfying

$$\xi\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \zeta\right) \varphi(x^2) = \zeta \psi(bqx^2)\,\varphi(ax^2) \qquad (a \text{ in } F^{\times 2}),$$

$$\xi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta\right) \varphi(x^2) = \zeta \gamma(q) |x|^{\frac{1}{2}} \mu(x) \int_F \Big(\int_{y^2 = z^2} \mu^{-1}(y)\,\psi(2qxy)\Big) |z|^{-\frac{1}{2}} \varphi(z^2) |q|^{\frac{1}{2}}\, dz,$$

where the inner integral denotes the average of the nonzero values at $y = z$ and $y = -z$ of the integrand, so that the right-hand side above depends only on x^2 but not on the choice of its square root x, and

$$\xi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \zeta\right) \varphi(x^2) = \zeta \mu(a)\,\omega(a)\,\varphi(x^2).$$

We write ξ_q to specify the dependence of ξ on the quadratic form $q(x)$. The induced representation $\operatorname{Ind}(\bar{G}_+, \bar{G}, \xi_q)$ of $\bar{G}$ is the "Kirillov model" of the representation $\mathrm{Ind}(\bar{G}_+, \bar{G}, \xi_q)$; it is independent of q, and its restriction to $\bar{G}_+$ is the direct sum of the ξ_q over q in $F^\times/F^{\times 2}$. Its character χ_μ, if exists, is a function on conjugacy classes in $\bar{G}$, and as such Lemma 1.1.1 shows that it vanishes at all regular elements of $\bar{G}$ which are not of the form $\zeta\tilde{g}$ (g in G).

Lemma. *The character* χ_μ *of the Weil representation is a locally integrable function and is given by*

$$\chi_\mu(\tilde{g}\zeta_g^{-1}) = \mu(-1)\,\mu(g)\,\Delta(g)/\Delta(\tilde{g})$$

if $\tilde{g}$ *is elliptic, and by*

$$\chi_\mu\left(\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}, 1\right) = \tfrac{1}{2}\left[\mu(-ab)\,\Delta\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \mu(ab)\,\Delta\begin{pmatrix} -a & 0 \\ 0 & b \end{pmatrix}\right]\Big/\Delta\begin{pmatrix} a^2 & 0 \\ & b^2 \end{pmatrix}$$

on the regular elements of $\bar{A}$.

Here we write $\mu(g)$ for $\mu(\det g)$. The right-hand sides above depend only on g^2 but not on g, since if g^2 is elliptic regular, the only solutions in G for $x^2 = g^2$ are $x = g$ and $x = -g$. Finally, we note that when μ is odd and $|a| \neq |b|$, we clearly have $\chi_\mu\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} = 0$; hence the odd Weil representation is supercuspidal.

Proof. Any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\bar{G}_+$ with $c \neq 0$ can be written in the form

$$g = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\alpha/c & 0 \\ 0 & -\alpha/c \end{pmatrix}\begin{pmatrix} c^2/\alpha & cd/\alpha \\ 0 & 1 \end{pmatrix}(-c, -1)\,s(g)^{-1},$$

where $\alpha = \det g$, and this allows us to calculate in stages the transformation $\varphi \to \xi_q(g)\,\varphi$ on functions φ in the space of ξ, using the values of ξ_q given above. Since γ is a character on the Witt group of F ([16], *p.* 173) and

$$\gamma(x^2 - ay^2 - bz^2 + abt^2) = (a, b) \qquad (a, b \text{ in } F^\times),$$

([16], *p.* 176), we have

$$\xi_q(g)\,\varphi(u^2) = s(g)^{-1}\gamma(-cq)\,\mu(-u\alpha/c)\,\psi(qau^2/c)$$
$$\cdot \int_F \left|\frac{u}{v}\right|^{\frac{1}{2}} \varphi(c^2v^2/\alpha)\,|q|^{\frac{1}{2}}\,dv \int_{y^2=v^2} \mu^{-1}(y)\,\psi(q(2uy + cdv^2/\alpha)).$$

On applying the transformation $v \to \dfrac{\alpha'}{c}v$, where α' in $F^\times$ satisfies $\alpha'^2 = \alpha$, we see that $\xi_q(g)$ is an integral operator with a kernel

$$s(g)^{-1}\gamma(-cq)\left|\frac{u}{v}\frac{\alpha'}{c}q\right|^{\frac{1}{2}} \int_{y^2=v^2} \mu\left(-\frac{u}{y}\alpha'\right)\psi\left(q\left(u^2\frac{a}{c} + v^2\frac{d}{c} + 2uy\frac{\alpha'}{c}\right)\right)$$
$$= s(g)^{-1}\gamma(-cq)\left|\frac{u}{v}\frac{\alpha'}{c}q\right|^{\frac{1}{2}} \int_{\alpha'^2=\alpha} \mu\left(-\frac{u}{v}\alpha'\right)\psi\left(q\left(u^2\frac{a}{c} + v^2\frac{d}{c} + 2uv\frac{\alpha'}{c}\right)\right).$$

It follows that for any genuine locally constant and compactly supported function $\tilde{f}$ on $\bar{G}$ which vanishes on the upper triangular subgroup, the equation

$$\xi_q(\tilde{f}) = \int_{\zeta_2 \backslash \bar{G}_+} \tilde{f}(g)\, \xi_q(g)\, dg$$

defines an integral operator whose kernel is easily obtained from the kernel of $\xi_q(g)$. The trace of $\xi_q(\tilde{f})$ is obtained by integrating the kernel on its diagonal $u = v$, namely,

$$\int_{\zeta_2 \backslash \bar{G}_+} \tilde{f}(g)\, s(g)^{-1}\, \gamma(-cq)\, |\alpha|^{\frac{1}{4}} \int_{\alpha'^2 = \alpha} \mu(-\alpha') \cdot \left(\int_F |q/c|^{\frac{1}{2}}\, \psi\left(qu^2 \frac{\beta + 2\alpha'}{c} \right) du \right) dg,$$

where $\beta = \operatorname{tr} g = a + d$. Using the transformation $t = qu^2/c$ $(du = |ct/q|^{\frac{1}{2}}\, d^\times t)$ we can write the inner integral in the form

$$\int_{cqF^{\times 2}} \psi(t(\beta + 2\alpha'))\, |t|^{\frac{1}{2}}\, d^\times t = \sum_v \int_{F^{\times 2}} (cqt, v)\, \psi(t(\beta + 2\alpha'))\, |t|^{\frac{1}{2}}\, d^\times t.$$

For brevity, we wrote $\sum_v$ for the sum over a set of representatives v of $F^\times/F^{\times 2}$ which is normalized in the sense that it is divided by the cardinality of $F^\times/F^{\times 2}$; note that

$$\sum_v (v, a) = \begin{cases} 1, & \text{if } a \text{ is in } F^{\times 2}, \\ 0, & \text{otherwise.} \end{cases}$$

The normalized sum of $\operatorname{tr} \xi_q(\tilde{f})$ over a set of representatives q for $F^\times/F^{\times 2}$ is therefore given by

$$\int_{\zeta_2 \backslash \bar{G}_+} \tilde{f}(g)\, s(g)^{-1}\, |\alpha|^{\frac{1}{4}} \int_{\alpha'^2 = \alpha} \mu(-\alpha')$$
$$\cdot \int_{F^\times} \sum_v (v, t) \left(\sum_q \gamma(-cq)(cq, v)\right) \psi(t(\beta + 2\alpha'))\, |t|^{\frac{1}{2}}\, d^\times t\, dg.$$

Since

$$\sum_q \gamma(-cq)\,(cq, v) = \gamma(v) = \varepsilon(\chi_v, \tfrac{1}{2}),$$

where $\chi_v(u) = (u, v)$ and $\varepsilon(\chi_v, \frac{1}{2})$ denotes the ε-factor in the Tate functional equation ([12], p. 503 and p. 537), this integral can be written in the form

$$\sum_v \int_{F^{\times 2}} \int_{\alpha'^2 = \alpha} \mu(-\alpha')\, |\alpha|^{\frac{1}{4}}\, \varepsilon(\chi_v, \tfrac{1}{2})$$
$$\cdot \int_{F^\times} (v, t)\, |t|^{\frac{1}{2}} \left(\int_{G_\alpha} \tilde{f}(g)\, s(g)^{-1}\, \psi(t(\beta + 2\alpha'))\, dg \right) d^\times t\, d^\times \alpha$$

the inner integral is taken over the set G_α of all g in $\zeta_2 \backslash \bar{G}_+$ with $\det g = \alpha$.

Theorem 11 of Harish-Chandra [5], p. 49, states that in our situation there is a map $\tilde{f} \to M = M_{\tilde{f},\alpha}$ such that for any locally constant function ρ with compact support on F we have

$$\int_{G_\alpha} \tilde{f}(g)\, s(g)^{-1}\, \rho(\operatorname{tr}(g+\alpha'))\, dg = \int_F M(t)\, \rho(t)\, dt.$$

Hence we can write the inner integral above in the form

$$\int_F M(s)\, \psi(ts)\, ds = \hat{M}(t).$$

On applying the Tate functional equation we see that

$$\varepsilon(\chi_v, \tfrac{1}{2}) \int_{F^\times} (v,t)\, |t|^{\frac{1}{2}}\, \hat{M}(t)\, d^\times t = \int_{F^\times} M(t)\, |t|^{\frac{1}{2}} (v,t)\, d^\times t,$$

and our integral is equal to

$$\sum_v \int_{F^{\times 2}} \int_{\alpha'^2 = \alpha} \mu(-\alpha')\, |\alpha|^{\frac{1}{4}} \int_{F^\times} (v,t)\, M(t)\, |t|^{\frac{1}{2}}\, dt\, d^\times \alpha$$

$$= \sum_v \int_{F^{\times 2}} \int_{\alpha'^2 = \alpha} \mu(-\alpha')\, |\alpha|^{\frac{1}{4}} \int_{G_\alpha} (v, \beta + 2\alpha')\, \tilde{f}(g)\, s(g)^{-1}\, |\beta + 2\alpha'|^{-\frac{1}{2}}\, dg\, d^\times \alpha.$$

If g is elliptic regular in $\bar{G}_+$, we denote its eigenvalues by x and $\bar{x}$ and note that $x\bar{x}$ lies in $F^{\times 2}$. It contributes a nonzero term to the above expression only if

$$\beta \pm 2\alpha' = x + \bar{x} \pm 2(x\bar{x})^{\frac{1}{2}} = x(1 \pm (\bar{x}/x)^{\frac{1}{2}})^2$$

lies in $F^{\times 2}$. If E denotes the quadratic extension of F generated by x, then $\bar{x}/x$ lies in $E^{\times 2}$; hence $(1 \pm (\bar{x}/x)^{\frac{1}{2}})^2$ lies in $E^{\times 2}$ and x lies in $E^{\times 2}$, and there is some z in $E^\times$ such that $z^2 = x$. We write $\bar{z}$ for the conjugate of z in E/F and note that $\bar{z}^2 = \bar{x}$. We have

$$\beta \pm 2\alpha' = (z \pm \bar{z})^2,$$

and we see that $(z+\bar{z})^2$ is a square in $F^\times$ while $(z-\bar{z})^2$ is not. Let h be the element of G whose eigenvalues are z and $\bar{z}$, and $g = h^2$. Let $\tilde{G}$ be the subset of $\bar{G}$ of all $g = \zeta \tilde{h}$ (h in G). If $\tilde{f}$ vanishes outside the set of elliptic elements, the trace is

$$\int_{\zeta_2 \backslash \tilde{G}} \tilde{f}(g)\, s(g)^{-1}\, \mu(-\alpha')\, \frac{|z\bar{z}|^{\frac{1}{2}}}{|z+\bar{z}|}\, dg$$

$$= \int_{\zeta_2 \backslash \tilde{G}} \tilde{f}(g)\, s(g)^{-1}\, \mu(-1)\, \mu(h)\, \frac{\Delta(h)}{\Delta(g)}\, dg,$$

as required.

Similarly, we see that if g in $\bar{G}_+$ has two distinct eigenvalues A and B in $F^\times$, then it contributes a nonzero term only if A and B lie in $F^{\times 2}$. We then write $a^2 = A$ and $b^2 = B$ with a, b in $F^\times$ and let h (resp. h') be an element with eigenvalues a (resp. $-a$) and b such that $g = h^2$ (resp. $g = h'^2$). Now $\beta \pm 2\alpha'$ is in $F^{\times 2}$ for both choices of sign, and assuming that $\tilde{f}$ vanishes on the set of elliptic elements, we obtain

$$\tfrac{1}{2}\int_{\zeta_2\backslash\tilde{G}} \tilde{f}(g)\, s(g)^{-1}\left(\frac{\mu(-ab)|ab|^{\frac{1}{2}}}{|a+b|}+\frac{\mu(ab)|ab|^{\frac{1}{2}}}{|a-b|}\right)dg$$
$$=\tfrac{1}{2}\int \tilde{f}(g)\, s(g)^{-1}\,[(\mu(h')\,\Delta(h)+\mu(h)\,\Delta(h'))/\Delta(g)]\,dg;$$

this completes the proof of the lemma.

For any character μ of $F^\times$ the equation $\mu(g)=\mu(\det g)$ defines a one-dimensional representation μ of G. The above lemma shows that for every g in G such that $\tilde{g}$ is regular, we have

$$\Delta(\tilde{g})\,\chi_\mu(\tilde{g}\zeta_g^{-1})=\begin{cases}\Delta(g)\,\mu(g), & \text{elliptic } g,\\ \tfrac{1}{2}(\Delta(g)\,\mu(g)+\Delta(g')\,\mu(g')), & \text{otherwise,}\end{cases} \tag{1}$$

when μ is even $\left(\text{as usual we put } g'=\begin{pmatrix}-a & 0\\ 0 & b\end{pmatrix} \text{ if } g=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\right)$.

When μ is odd, the lemma implies that

$$\Delta(\tilde{g})\,\chi_\mu(\tilde{g}\zeta_g^{-1})$$
$$=\begin{cases}-\Delta(g)\,\mu(g)=\Delta(g)\,\chi_\sigma(g), & \text{elliptic } g,\\ -\tfrac{1}{2}(\Delta(g)\,\mu(g)+\Delta(g')\,\mu(g'))=\tfrac{1}{2}(\Delta(g)\,\chi_\sigma(g)+\Delta(g')\,\chi_\sigma(g')), & \end{cases} \tag{2}$$

where $\sigma=\sigma(\mu\|^{-\frac{1}{2}},\mu\|^{\frac{1}{2}})$ is the special representation of [7], Sect. 3 (whose character is calculated at [7], end of Sect. 7).

According to the definitions of 5.0 below, (1) implies that the even Weil representation r_μ corresponds to the one-dimensional representation μ, and (2) implies that the odd Weil representation r_μ corresponds to the special representation $\sigma=\sigma(\mu)$ as above.

We shall extend (1) and (2) to all representations of $\bar{G}$ in 5.2. In particular, we shall see that the character $\chi_{\bar{\pi}(\eta)}$ of $\bar{\pi}(\eta)$ $(\eta=(\mu^{\frac{1}{2}}\|^{-\frac{1}{4}},\mu^{\frac{1}{2}}\|^{\frac{1}{4}})$, μ even) satisfies the equation obtained from (1) by replacing χ_μ by $\chi_{\bar{\pi}(\mu)}$; we shall then deduce that r_μ is equivalent to $\bar{\pi}(\eta)$. Also we shall see that there is a supercuspidal representation of $\bar{G}$ whose character χ satisfies (2) with χ in place of χ_μ. We shall deduce that r_μ is equivalent to this representation (μ odd), and that it is supercuspidal. As we have noted above, the fact that r_μ is supercuspidal if μ is odd can also be deduced from the fact that its character is compactly supported modulo the centre. Both results are well-known ([1], p. 118), and it is merely our proof which is new and depends on character theory only.

2.3. *Square-Integrable Representations*

The character of the principal series was determined in 2.1. The characters of the square-integrable representations will be studied in 5.2 below, using the trace formula. Here we prepare two auxiliary results which are of key importance in that study. First we shall record the intimate relationship between matrix coefficients and characters of square-integrable representations on the set of the elliptic regular elements. Then we shall give the orthogonality relations for such representations.

Lemma 1. *The character $\chi=\chi_{\bar\pi}$ of an anti-genuine square-integrable representation $\bar\pi$ of $\bar G$ is a locally integrable function, and it is given by*

$$\chi(g)=d(\bar\pi)\int_{Z^n\backslash G}(\bar\pi(h^{-1}gh)u,u)\,dh$$

on the set of elliptic regular elements. Here u is a vector of length one and $d(\bar\pi)$ is the formal degree of $\bar\pi$.

Lemma 1.1.1 now implies that χ vanishes at every elliptic regular g which is not of the form $g=\zeta\bar h$ for any h in G.

Proof. The proof of [7], Lemma 7.4.1, can be imitated to show that if $\tilde f$ is any genuine function on $\bar G$ as in 2.0, then the trace of the operator $\bar\pi(\tilde f)$ is finite and equal to

$$\operatorname{tr}\bar\pi(\tilde f)=d(\bar\pi)\int_{Z^n\backslash G}\Big(\int_{Z^n\backslash G}\tilde f(g)(\bar\pi(h^{-1}gh)u,u)\,dg\Big)\,dh.$$

To obtain the lemma we merely have to show that the two integrations can be interchanged. If $\bar\pi$ is supercuspidal its matrix coefficients are compactly supported modulo the centre, and the absolute convergence of $\int(\bar\pi(h^{-1}gh)u,u)\,dh$ (for any elliptic regular g) is proved exactly as in [7], Proposition 7.5, using the obvious analogue of Lemma 7.4.2 for $\bar G$.

For special $\bar\pi$ the absolute convergence of this integral (uniformly in $\tilde f$ with support in a fixed compact modulo the centre) can be proved using arguments generalizing those of [7], Lemma 15.2. Indeed, on multiplying $\bar\pi$ by a character we may assume that $\bar\pi=\bar\sigma(\|^{1/2n},\|^{-1/2n})$; its most general matrix coefficient is given by $f(g)=\langle\varphi,\bar\pi(g)\tilde\varphi\rangle$ with φ in the space of $\bar\sigma(\|^{1/2n},\|^{-1/2n})$ and $\tilde\varphi$ in the space of $\bar\rho(\|^{-1/2n},\|^{1/2n})$. Hence φ satisfies

$$\varphi\left(\begin{pmatrix}a_1 & x\\ 0 & a_2\end{pmatrix}g\right)=\left|\frac{a_1}{a_2}\right|^{1/2}\left|\frac{a_1}{a_2}\right|^{1/2n}\varphi(g)\qquad(a_1,a_2\ \text{in}\ F^{\times n})$$

and

$$\int_F\varphi\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)dx=0,$$

and $\tilde\varphi$ satisfies

$$\tilde\varphi\left(\begin{pmatrix}a_1 & x\\ 0 & a_2\end{pmatrix}g\right)=\left|\frac{a_1}{a_2}\right|^{1/2}\left|\frac{a_1}{a_2}\right|^{-1/2n}\tilde\varphi(g).$$

To estimate the integral $\int f(h^{-1}gh)\,dh$ we have (cf. (1) of 1.3) to estimate the sum

$$\sum_{m\geqq 0} q^m f\left[\begin{pmatrix}1 & 0\\ 0 & \tilde{\omega}^{-m}\end{pmatrix}\begin{pmatrix}a & b\theta\\ b & a\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \tilde{\omega}^{m}\end{pmatrix}\right]$$
$$=\sum_{m\geqq 0} q^m f\left[\begin{pmatrix}-a\tilde{\omega}^m/b & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\tilde{\omega}^{-m} & 0\\ 0 & \tilde{\omega}^{m}\end{pmatrix}\begin{pmatrix}-b & -a\tilde{\omega}^m\\ 0 & b\theta-a^2/b\end{pmatrix}\right],$$

and since f is K-finite it is enough to consider

$$\sum_{m\geqq 0} q^m f\left(\begin{pmatrix}\tilde{\omega}^{-m} & 0\\ 0 & \tilde{\omega}^{m}\end{pmatrix}\right). \tag{1}$$

It suffices to consider $f\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)$ for a in $F^{\times n}$. Since $\langle\varphi, \tilde{\varphi}\rangle = \int_F \varphi_1(x)\,\bar{\tilde{\varphi}}_1(x)\,dx$ this is given by

$$\int \varphi_1(x)\,\bar{\tilde{\varphi}}\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)dx$$
$$=\int \varphi_1(x)\,\bar{\tilde{\varphi}}\left(\begin{pmatrix}a^{-1} & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x/a^2\\ 0 & 1\end{pmatrix}\right)dx$$
$$=\int \varphi_1(x)\,\bar{\tilde{\varphi}}_1(xa^{-2})\,|a|^{-1+1/n}\,dx,$$

where $\bar{\tilde{\varphi}}_1(x)$ denotes the complex conjugate of $\tilde{\varphi}_1(x)$,

$$\varphi_1(x)=\varphi\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) \quad\text{and}\quad \tilde{\varphi}_1(x)=\tilde{\varphi}\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right).$$

Let $M>0$ have the property that $\tilde{\varphi}_1(x)=\tilde{\varphi}_1(0)$ for $|x|\leqq M$. If $|a|\geqq 1$ the last integral is equal to

$$|a|^{-1+1/n}\Big[\tilde{\varphi}_1(0)\int_{|x|\leqq|a|^2M}\varphi_1(x)\,dx+\int_{|x|>|a|^2M}\varphi_1(x)\,\tilde{\varphi}_1(a^{-2}x)\,dx\Big]$$

which is bounded by the sum of

$$|a|^{-1+1/n}\int_{|x|>|a|^2M}|\varphi_1(x)\,\bar{\tilde{\varphi}}_1(a^{-2}x)|\,dx$$
$$\leqq |a|^{-1+1/n}\Big(\int_{|x|>|a|^2M}|\varphi_1(x)|^s\,dx\Big)^{1/s}\Big(\int_{|x|>|a|^2M}|\tilde{\varphi}_1(a^{-2}x)|^t\,dx\Big)^{1/t}$$
$$\leqq c_1\,|a|^{-1+\frac{1}{n}+\frac{2}{t}}\Big(\int_{|x|>|a|^2M}|\varphi_1(x)|^s\,dx\Big)^{1/s} \tag{2}$$

$\left(\text{where } t>2,\ \frac{1}{s}+\frac{1}{t}=1 \text{ and } c_1^t=\int_{|x|>M}|\tilde{\varphi}_1(x)|^t\,dx \text{ is a finite constant}\right)$ and

$$|a|^{-1+1/n}\,|\tilde{\varphi}_1(0)|\int_{|x|>|a|^2M}|\varphi_1(x)|\,dx, \tag{3}$$

since $\int \varphi_1(x)\,dx=0$.

Without loss of generality we may assume that φ is the function of $\bar{\rho}(\|^{1/2n}, \|^{-1/2n})$ which is defined by

$$\varphi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} k\right) = |a_1/a_2|^{\frac{1}{2}+1/2n} \qquad (a_1, a_2 \text{ in } F^{\times n}),$$

with k in K (or in a suitable congruence subgroup if $|n|_v < 1$). Now since for $|x| > 1$ we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} k \qquad (k \text{ in } K),$$

we see that if $|a|^2 = |\tilde{\omega}|^{-nm} (m \to \infty)$ then

$$\int_{|x|>|a|^2} |\varphi_1(x)|^s \, dx = O\left(\sum_{k>m} q^{nk} |\tilde{\omega}^{2k}|^{s(n+1)/2}\right) = O\left(\sum_{k>m} |\tilde{\omega}|^{k(n(s-1)+s)}\right)$$
$$= O(|\tilde{\omega}|^{m(n(s-1)+s)}) = O(|a|^{-2(s-1)-2s/n}),$$

and hence that (2) is $O(|a|^{-1-1/n})$; similarly

$$\int_{|x|>|a|^2} |\varphi_1(x)| \, dx = O\left(\sum_{k>m} q^{nk} |\tilde{\omega}^{2k}|^{(n+1)/2}\right) = O\left(\sum_{k>m} |\tilde{\omega}|^k\right) = O(|\tilde{\omega}|^m) = O(|a|^{-2/n}),$$

and so (3) is also $O(|a|^{-1-1/n})$. It follows that $f\left(\begin{pmatrix} \tilde{\omega}^{-m} & 0 \\ 0 & \tilde{\omega}^m \end{pmatrix}\right)$ is $O(|\tilde{\omega}|^{m(1+1/n)})$, and therefore that the sum (1) is absolutely convergent, as required.

We could imitate the arguments of [7], pp. 268–271, to show that the character exists as a locally integrable function also on A, at least when $\bar{\pi}$ is supercuspidal, but it is easier to deduce this in general from the trace formula, as we shall do below.

Finally we note that the above estimations can also be used to prove that the special representation $\bar{\sigma}$ is square-integrable. Indeed, as in [7], p. 473, by virtue of the decomposition $G = KAK$ we have to consider the sum

$$\sum_{m \geq 0} q^m f\left(\begin{pmatrix} \tilde{\omega}^{-m} & 0 \\ 0 & 1 \end{pmatrix}\right)^2$$

instead of (1), and this is absolutely convergent since

$$f\left(\begin{pmatrix} \tilde{\omega}^{-2m} & 0 \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} \tilde{\omega}^{-m} & 0 \\ 0 & \tilde{\omega}^m \end{pmatrix}\right) = O(|\tilde{\omega}|^{m(1+1/n)}).$$

Let $\bar{\pi}_i$ be square-integrable anti-genuine representations of $\bar{G}$, and denote by $\bar{\pi}_i'$ the contragredient of $\bar{\pi}_i$ $(i = 1, 2)$. We have:

Lemma 2. *The function* $f(g) = \chi_{\bar{\pi}_1'}(g)\, \chi_{\bar{\pi}_2}(g)$ *satisfies*

$$\delta_{\bar{\pi}_1, \bar{\pi}_2} = \tfrac{1}{2} \sum |\bar{Z}^n \backslash \bar{T}^n|^{-1} \int_{\bar{Z}^n \backslash \bar{T}^n} f(g)\, \Delta(g)^2 \, dg,$$

where the sum is taken over a set of representatives for the nonsplit Cartan subgroups of G, *and* $\delta_{\bar{\pi}_1, \bar{\pi}_2}$ *is* 1 *if* $\bar{\pi}_1$ *and* $\bar{\pi}_2$ *are equivalent and* 0 *otherwise.*

Proof. We may assume that the central character μ (see 2.0) is unitary, in which case $\chi_{\bar{\pi}'_1}$ is the complex conjugate of $\chi_{\bar{\pi}_1}$.

The matrix coefficient

$$\varphi(g) = d(\bar{\pi}_1)(u, \bar{\pi}_1(g)u)$$

is square-integrable, and the Schur orthogonality relations imply that

$$\delta_{\bar{\pi}_1, \pi_2} = \operatorname{tr} \bar{\pi}_2(\varphi).$$

This is also given by

$$\operatorname{tr} \bar{\pi}_2(\varphi) = \int_{\bar{Z}^n \backslash G} \chi_{\pi_2}(g)\, \varphi(g)\, dg,$$

which, by Weyl's integration formula ((1) of 2.1) is

$$= \tfrac{1}{2} \sum_T \int_{\bar{Z}^n \backslash \bar{T}^n} \Delta(h)^2 \chi_{\bar{\pi}_2}(h) \Big(\int_{T \backslash G} \varphi(g^{-1} h g)\, dg \Big)\, dh.$$

If T is a nonsplit torus the inner integral is

$$d(\bar{\pi}_1)\, |\bar{Z}^n \backslash \bar{T}^n|^{-1} \int_{\bar{Z}^n \backslash G} (u, \bar{\pi}_1(g^{-1} h g) u)\, dg,$$

which, by virtue of Lemma 1, is equal to

$$|\bar{Z}^n \backslash \bar{T}^n|^{-1} \chi_{\bar{\pi}'_1}(g).$$

When $T = A$, we apply Iwasawa's decomposition and note that the inner integral is a constant times the integral over K of

$$\int_F \left(\bar{\pi}_1 \left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \bar{\pi}_1(k) u, \bar{\pi}_1(k) u \right) dx$$

$$= \left| 1 - \frac{b}{a} \right|^{-1} \int_F \left(\bar{\pi}_1(g)\, \bar{\pi}_1 \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \bar{\pi}_1(k) u, \bar{\pi}_1(k) u \right) dx \qquad \left(g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right).$$

If $\bar{\pi}_1$ is supercuspidal, then the last integral is 0 by a direct application of the definition of a supercuspidal representation. This conclusion holds also for any square-integrable representation $\bar{\pi}_1$ by virtue of [6], Theorem 29, and the lemma follows.

2.4. *Archimedean Places*

Here the work is simpler since there are no supercuspidal representations, and only principal series and their subrepresentations occur. The results of 2.1 hold also for genuine principal series of $\bar{G}(\mathbb{R})$ and $\bar{G}(\mathbb{C})$. Since the norm residue symbol is trivial on $\mathbb{C}$, all irreducible admissible genuine representations of $\bar{G}(\mathbb{C})$ are either principal series or a finite-dimensional quotient of two principal series, as in the case of $G(\mathbb{C})$ ([7], Theorem 6.2); no more needs to be said for the field $\mathbb{C}$. This is also all that we have to say for the case $n \geqq 3$, since our assumption that the global field F contains the n-th roots of unity implies that F is totally imaginary.

It remains to discuss the case $n=2$ and the group $\bar{G}=\bar{G}(\mathbb{R})$. Here the norm residue symbol is nontrivial and $\bar{G}$ has no finite-dimensional genuine admissible representations. The reducible principal series of $\bar{G}$ are of special interest. Their irreducible subrepresentations are the discrete series and the corresponding quotients. Here we shall recall the results that we need and refer to [1], Sect. 4, for a more complete discussion.

$\bar{G}$ is the direct product of the multiplicative group $\mathbb{R}_+^\times$ of the positive real numbers and the subgroup $\bar{G}'$ of all (g, ζ) in $\bar{G}$ with $\det g = \pm 1$. It suffices to deal with the discrete series of $\bar{G}'$; those of $\bar{G}$ are obtained by tensoring with a character of $\mathbb{R}_+^\times$. The discrete series of $\bar{G}'$ are denoted by $\bar{\pi}_{k/2}$ ($k>0$ odd) and are induced from the discrete series $\bar{\pi}_{k/2}^-$ of the subgroup $\bar{S}$ of (g, ζ) in $\bar{G}'$ with $\det g=1$.

The Iwasawa decomposition of $\bar{S}$ takes the form $\bar{S}=A_0 N\bar{K}$ where the inverse image $\bar{K}$ of $SO(2, \mathbb{R})$ in $\bar{S}$ can be parametrized by

$$r(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \qquad (0 \leqq \theta < 4\pi),$$

and A_0 consists of all $\underline{a}=\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}$ with a in $\mathbb{R}_+^\times$. The upper triangular subgroup of $\bar{S}$ is $\bar{B}=\bar{M}A_0 N$, where $\bar{M}$ is the cyclic subgroup of order 4 generated by $\gamma = \left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, 1\right)$. Every genuine representation of $\bar{B}$ trivial on N is of the form (λ, τ), with λ in $\mathbb{C}$ and $\tau=\pm\frac{1}{2}$, and

$$(\lambda, \tau)(\gamma^j \underline{a} n)=e^{\pi i j\tau} a^\lambda \qquad (1 \leqq j \leqq 4,\ n \text{ in } N,\ i=(-1)^{\frac{1}{2}}).$$

Every genuine principal series of $\bar{S}$ is of the form $\rho(\lambda, \tau)=\operatorname{Ind}(\bar{B}, \bar{G}, (\lambda, \tau))$ and can be realized as the right representation on the space of functions on $\bar{G}$ which are square-integrable on $\bar{K}$ and transform on the left under $\bar{b}=\gamma^j \underline{a} n$ (in $\bar{B}$) by multiplication by $a(\lambda, \tau)(\bar{b})=(\lambda+1, \tau)(\bar{b})$. An orthonormal basis for this space is given by the functions φ_m defined by

$$\varphi_m(\underline{a} n r(\theta))=a^\lambda e^{i(2m+\tau)\theta} \qquad (m \text{ integer}).$$

The representation $\rho(\lambda, \tau)$ is reducible only if $\lambda=\pm(\frac{1}{2}k-1)$ with odd positive integer k. Suppose that $\lambda=\frac{1}{2}k-1$ and $2\tau \equiv k \pmod 4$; then $\bar{\pi}_{\frac{1}{2}k}^-$ is the subrepresentation of $\rho(\lambda, \tau)$ whose space has the orthonormal basis of all φ_m with $2m+\tau \geqq \frac{1}{2}k$.

The character of $\bar{\pi}_{\frac{1}{2}k}^-$ on $\bar{K}$ can easily be calculated now. Since the eigenvalue of the operator $\bar{\pi}_{\frac{1}{2}k}^-(r(\theta))$ with respect to the eigenvector φ_m is $e^{i(2m+\tau)\theta}$, we merely have to sum these up for m with $2m \geqq k-\tau$. As usual, introducing a convergence factor $t(0<t<1)$, this sum is

$$\sum_{m \geqq \frac{1}{2}(\frac{1}{2}k-\tau)} t^m e^{i(2m+\tau)\theta} = t^{\frac{1}{2}(\frac{1}{2}k-\tau)} \frac{e^{\frac{1}{2}ik\theta}}{1-te^{2i\theta}},$$

which tends to

$$\frac{-e^{i(\frac{1}{2}k-1)\theta}}{e^{i\theta}-e^{-i\theta}}, \quad \text{as } t\to 1.$$

The character $\chi_{\frac{1}{2}k}$ of the representation $\bar{\pi}_{\frac{1}{2}k}=\operatorname{Ind}(\tilde{S}, \bar{G}', \bar{\pi}_{\frac{1}{2}k})$ is therefore

$$\frac{e^{-i(\frac{1}{2}k-1)\theta}-e^{i(\frac{1}{2}k-1)\theta}}{e^{i\theta}-e^{-i\theta}}.$$

The discrete series π_k ($k>0$ even) of G' $\left(\text{the subgroup of } GL(2) \text{ generated by } SL(2) \text{ and } \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)$ are well-known and the value of its character on K is given by

$$\chi_k(r(\theta))=\frac{e^{-i(k-1)\theta}-e^{i(k-1)\theta}}{e^{i\theta}-e^{-i\theta}} \qquad (0\leqq\theta<2\pi).$$

Since $\Delta(r(\theta))=e^{i\theta}-e^{-i\theta}$ we have

$$\Delta(r(2\theta))\,\chi_{\frac{1}{2}k}(r(2\theta))=\Delta(r(\theta))\,\chi_{k-1}(r(\theta)) \qquad (0\leqq\theta<2\pi), \tag{1}$$

for all odd integral $k\geqq 3$. This extends to $k=1$ on denoting by π_0 the trivial representation of G'. By tensoring $\bar{\pi}_{\frac{1}{2}k}$ with a character μ of $\mathbb{R}_+^\times$ and π_{k-1} by μ^2 this equality extends to $\bar{G}$ and G. We proved:

Lemma. *Let μ be a character of $\mathbb{R}_+^\times$ and k a positive odd integer. Suppose that $\bar{\pi}_{\frac{1}{2}k}$ and π_{k-1} are the discrete series which transform under $\mathbb{R}_+^\times$ by μ and μ^2, respectively. Then for any g in $\mathbb{R}_+^\times K$ with $\tilde{g}$ regular in $\mathbb{R}_+^\times \bar{K}$ we have*

$$\Delta(\tilde{g})\,\chi_{\frac{1}{2}k}(\tilde{g}\zeta_g^{-1})=\Delta(g)\,\chi_{k-1}(g). \tag{2}$$

It is now easy to deduce from Lemma 2.2 that the Weil representation r_μ, where μ is the character of $\mathbb{R}^\times$ with $\mu(x)=(\operatorname{sgn}(x))^j\mu(|x|)$ $(j=0,1)$, is equivalent to the discrete series $\bar{\pi}_{\frac{1}{2}+j}$, and that (2) extends to all g in G with $\tilde{g}$ regular for $k=1$ or 3. The identification of the even (resp. odd) Weil representation with $\bar{\pi}_{\frac{1}{2}}$ (resp. $\bar{\pi}_{\frac{3}{2}}$) is well known using other techniques ([1], p. 94). Finally, we note that (2) will be used only with the g specified in the lemma, although it holds for all g (this can be proved either directly or by using the trace formula below).

II. Global Theory

3.0. The Trace Formula

Our main purpose in this work is the study of the correspondence between representations of the adele groups $\bar{G}(\mathbb{A})$ and $G(\mathbb{A})$ and not only that of the local groups $\bar{G}(F_v)$ and $G(F_v)$ with which we have dealt in the previous sections. To do this we shall apply the trace principle. More precisely, we shall obtain a variant of the

Selberg trace formula for the covering group $\bar{G}(\mathbb{A})$ and using the local analysis of the previous sections we shall equate it to the trace formula for $G(\mathbb{A})$ for matching functions as in Lemmas 1.2.3 and 1.4. This will afford not only a good understanding of the global correspondence but we shall also be in a position to complete the study of the local correspondence, in particular for square-integrable representations.

We begin by discussing the trace formula for $\bar{G}(\mathbb{A})$. Although $\bar{G}$ is not an algebraic group, its trace formula is similar enough to the trace formula for G, and after some easy modifications the proof applies here as well. Since the proof has been fully exposed, at least in the context of $GL(2)$ ([7], [9] and recently [2]) and of other algebraic groups of rank one, there is no need to supply all details in our case. We shall content ourselves with writing the formula and indicating the new features here. The main phenomenon to be pointed out is the cancellation of all terms not indexed by n-th powers in the sense of Lemma 1.1.1.

3.1. *Preliminaries*

Let μ be a unitary character of $Z^n(\mathbb{A})\,Z(F)$ trivial on $F^\times = Z(F)$, and express it as a product $\mu = \bigotimes_v \mu_v$ of local characters μ_v on $F_v^{\times n}$, almost all of which are unramified. Signify by $L(\mu, \bar{G})$ the space of genuine measurable functions φ on $\bar{G}(\mathbb{A})$ with

$$\varphi(\zeta(\gamma, s(\gamma)^{-1})\,zg) = \zeta\mu(z)\,\varphi(g) \qquad (\gamma \text{ in } G(F),\ z \text{ in } Z^n(\mathbb{A})\,Z(F)),$$

and

$$\int_{Z^n(\mathbb{A})\,G(F)\backslash \bar{G}(\mathbb{A})} |\varphi(g)|^2\,dg < \infty;$$

here we recall (from 0.2) that $G(F)$ embeds as a subgroup of $\bar{G}(\mathbb{A})$ through the map $\gamma \mapsto (\gamma, s(\gamma)^{-1})$. The group $\bar{G}(\mathbb{A})$ acts on $L(\mu, \bar{G})$ by right translations: g maps the function $\{h \mapsto \varphi(h)\}$ to the function $\{h \mapsto \varphi(hg)\}$.

A function φ in $L(\mu, \bar{G})$ is called cuspidal if

$$\int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

for almost all g in $\bar{G}(\mathbb{A})$. The subspace $L_0(\mu, \bar{G})$ of cuspidal functions in $L(\mu, \bar{G})$ is invariant under the action of $\bar{G}(\mathbb{A})$. Its orthogonal complement is also invariant, and Langlands' theory of Eisenstein series decomposes it into two orthogonal invariant subspaces. One of these can be expressed as a direct integral of induced representations and the other, which we denote by $L_1(\mu, \bar{G})$, is the subspace of residues of Eisenstein series.

Let $\tilde{f} = \bigotimes_v \tilde{f}_v$ be an anti-genuine function on $\bar{G}(\mathbb{A})$ with

$$\tilde{f}(\zeta(\gamma, s(\gamma)^{-1})\,zg) = \zeta^{-1}\mu^{-1}(z)\,\tilde{f}(g) \qquad (\gamma \text{ in } G(F),\ z \text{ in } Z^n(\mathbb{A})\,Z(F)).$$

Suppose $\tilde{f}_v$ is a smooth anti-genuine compactly supported function on $\bar{G}(F_v)$ modulo $Z^n(F_v)$ satisfying

$$\tilde{f}_v(\zeta z g) = \zeta^{-1}\mu_v^{-1}(z)\tilde{f}_v(g) \qquad (z \text{ in } Z^n(F_v)),$$

for all v. For almost all (non-archimedean) v we let $\tilde{f}_v$ be the function which obtains the value $\zeta^{-1}\mu_v^{-1}(z)$ at $g = \zeta z k$ in $\bar{Z}^n(F_v)\bar{K}$, and the value 0 otherwise.

On the direct sum $L_0(\mu, \bar{G}) \oplus L_1(\mu, \bar{G})$ we define the operator

$$r(\tilde{f})\varphi(h) = \int_{G(F)\bar{Z}^n(\mathbb{A})\backslash \bar{G}(\mathbb{A})} \varphi(hg)\tilde{f}(g)\,dg.$$

It is of trace class and the Selberg trace formula provides an explicit expression for its trace. We shall now write a variant of the formula for this operator in a form similar to [7], Sect. 16, and [9], Sect. 8, modifying it as the change from $G(\mathbb{A})$ to $\bar{G}(\mathbb{A})$ and $Z(\mathbb{A})$ to $\bar{Z}^n(\mathbb{A})$ requires, and indicating the cancellations which occur. As in [9], Sect. 8, we shall express the trace as a sum of invariant distributions in order to simplify the comparison with the trace formula for $GL(2)$.

Following [9] and [2], Sect. 7, we shall use Tamagawa measures locally and globally to simplify some normalizing constants. In particular we fix a non-trivial additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}$ which is trivial on F. For each place v we denote by dx_v the self-dual Haar measure on F_v with respect to ψ_v. On $F_v^\times$ we take the Haar measure $d^\times x_v = L(1, 1_v)\,dx_v/|x_v|$ (note that $L(1, 1_v)$ is $1 - 1/q$ for finite v). On the subgroup $F_v^{\times n}$ we take the Haar measure (denoted again by $d^\times x_v$) so that the resulting quotient measure assigns the finite group $F^{\times n}\backslash F^\times$ the measure $|F^{\times n}\backslash F^\times| = 1$. The normalized Tamagawa measure on $\mathbb{A}^\times$ is given by

$$d^\times x = (\lambda_{-1})^{-1} \bigotimes_v d^\times x_v, \qquad \lambda_{-1} = \lim_{s\to 1}(s-1)L(s, 1_F).$$

The same formula defines also a measure on $\mathbb{A}^{\times n}$ with respect to the local measures $d^\times x_v$ on $F_v^{\times n}$. As in Tate's thesis, we can find a fundamental domain for the quotient $F^{\times n}\backslash \mathbb{A}^{\times n}$ and, by virtue of the relationship between the local measures, the global measure assigns to it the same volume which is assigned by $d^\times x$ to the fundamental domain of $F^\times\backslash \mathbb{A}^\times$. In other words, we have

$$|F^\times F^0(\mathbb{A})^n \backslash F^0(\mathbb{A})| = |(F^{\times n}\backslash F^0(\mathbb{A})^n)\backslash (F^\times \backslash F^0(\mathbb{A}))| = 1,$$

where $F^0(\mathbb{A})$ denote the group of ideles of volume 1.

Finally, for any subgroups $\bar{H} \subset \bar{H}'$ of $\bar{G}$ we define the measure on the quotient space $\bar{H}\backslash \bar{H}'$ to be the pull-back of the measure on the quotient $H\backslash H'$ of the subgroups $H \subset H'$ of G; (H and H' are the groups defined by the projection $\bar{g} = (g, \zeta) \mapsto g$ from $\bar{G}$ to G).

3.2. *Explicit Expression (I)*

The first term in the trace formula for $\bar{G}(\mathbb{A})$, corresponding to the term (*i*) of the formula for $GL(2)$ on pp. 516–7 of [7], is

$$|\bar{Z}^n(\mathbb{A})\,G(F)\backslash \bar{G}(\mathbb{A})|\,\tilde{f}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right); \tag{1}$$

the second, corresponding to (ii) of [7], is

$$\tfrac{1}{2}\sum_{T}\sum_{\gamma}|\bar{Z}^n(\mathbb{A})\,G_\gamma(F)\backslash\bar{G}_\gamma(\mathbb{A})|\int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})}\tilde{f}(g^{-1}(\gamma,s(\gamma)^{-1})g)\,dg. \tag{2}$$

We use the fact that $|\bar{Z}^n(\mathbb{A})\,Z(F)\backslash\bar{Z}(\mathbb{A})|=1$ to replace $\bar{Z}^n(\mathbb{A})$ by $\bar{Z}(\mathbb{A})$ in the constants of (1) and (2). The first sum in (2) is taken over all non-split tori T of G, and the second is taken over all regular γ in $T(F)$ modulo $Z^n(F)$. Since

$$\int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})}\tilde{f}(g^{-1}(\gamma,s(\gamma)^{-1})g)\,dg=\prod_v\int_{G_\gamma(F_v)\backslash G(F_v)}\tilde{f}_v(g^{-1}(\gamma,s(\gamma)^{-1})g)\,dg,$$

we can apply Lemma 1.1.1 to deduce that the left hand side contributes a non-zero term to (2) only if γ is an n-th power in $G(F_v)$ for all v, and hence it lies in $T^n(F)$. We can now apply Lemma 1.1.2 to deduce that $\bar{G}_\gamma(\mathbb{A})=\bar{T}(\mathbb{A})$, hence (2) is

$$\tfrac{1}{2}\sum_{T}\sum_{1\neq\gamma\in T^n(F)}|\bar{Z}(\mathbb{A})\,T(F)\backslash\bar{T}(\mathbb{A})|\int_{T(\mathbb{A})\backslash G(\mathbb{A})}\tilde{f}(g^{-1}(\gamma,s(\gamma)^{-1})g)\,dg. \tag{2'}$$

Let $\bar{D}$ be the set of quasi-characters $\eta=(\nu_1,\nu_2)$ on $A^n(F)\,Z(F)\backslash A^n(\mathbb{A})\,Z(F)$ with $\nu_1\nu_2=\mu$ on $Z^n(\mathbb{A})\,Z(F)$. The only η of order n is clearly the trivial one. The set $\bar{D}$ has a structure of a complex manifold of dimension 1 with infinitely many connected components. The component $(\nu_1\|\,\|^{s/2n},\nu_2\|\,\|^{-s/2n})$ (s in $\mathbb{C}$) is parametrized by the complex variable s. Differentiation with respect to s is well-defined (and denoted by a prime) and we introduce the Haar measure $|ds|$ on the subset $\bar{D}_0$ of (unitary) characters on $\bar{D}$.

For each v we fix a subgroup $\bar{A}_0(F_v)$ of $\bar{A}(F_v)$ containing $\bar{A}^n(F_v)$, as in 2.1, and let $\bar{A}_0(\mathbb{A})$ be the restricted direct product of the $\bar{A}_0(F_v)$. We extend η from $A^n(\mathbb{A})\,Z(F)$ to the abelian group $A_0(\mathbb{A})\,Z(F)$ in any way.

The representation η of $A_0(\mathbb{A})\,Z(F)$ extends to $\bar{A}_0(\mathbb{A})\,Z(F)\,N(\mathbb{A})$ by setting $\eta=1$ on $N(\mathbb{A})$ and $\eta(\zeta)=\zeta$ on $\underline{\zeta}_n$. Let $\rho(\eta,g)$ be the genuine representation of $\bar{G}(\mathbb{A})$ induced from η on $\bar{A}_0(\mathbb{A})\,\bar{Z}(F)\,N(\mathbb{A})$, and put

$$\rho(\eta,\tilde{f})=\int_{\bar{Z}^n(\mathbb{A})\,Z(F)\backslash G(\mathbb{A})}\tilde{f}(g)\,\bar{\rho}(\eta,g)\,dg.$$

For each place v we denote by η_v the component of η at v and introduce the (normalized) intertwining operator $\bar{R}(\eta_v)$ from the space of $\bar{\rho}(\eta_v,g)$ to the space of $\bar{\rho}(\tilde{\eta}_v,g)$, where $\tilde{\eta}_v=(\nu_{2v},\nu_{1v})$. It is defined by

$$\bar{R}(\eta_v)\,\varphi(g)=\bar{m}(\eta_v)^{-1}\int_{F_v}\varphi\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)dx,$$

where

$$\bar{m}(\eta_v)=\varepsilon(0,\nu_{2v}^n\,\nu_{2v}^{-n},\psi_v)\,L(1,\nu_{1v}^n\,\nu_{2v}^{-n})/L(0,\nu_{2v}^n\,\nu_{1v}^{-n});$$

(see [7] for the definition of the ε and L factors). The products

$$\bar{M}(\eta)=\bar{m}(\eta)\bigotimes_v\bar{R}(\eta_v)$$

and

$$\bar{m}(\eta)=\prod_{v}\bar{m}(\eta_v)=\frac{L(1,\nu_1^n\nu_2^{-n})}{L(1,\nu_2^n\nu_1^{-n})},$$

are well-defined.

With these definitions we can introduce the terms

$$\frac{1}{4\pi}\int_{\bar{D}_0}\bar{m}(\eta)^{-1}\,\bar{m}'(\eta)\,\mathrm{tr}\,\bar{\rho}(\eta,\bar{f})\,|ds| \tag{3}$$

and

$$-\tfrac{1}{4}\sum_{\eta=(\nu,\nu)}\bar{M}(\eta)\,\mathrm{tr}\,\bar{\rho}(\eta,\bar{f}) \tag{4}$$

of the formula, corresponding to (vii) and (vi) of [7], respectively. It is implicit in (3) that $\bar{m}(\eta)$, $\bar{m}'(\eta)$ and $\mathrm{tr}\,\bar{\rho}(\eta,\bar{f})$ depend only on the restriction of η to $Z(F)\backslash A^n(\mathbb{A})\,Z(F)$. We write $\bar{M}(\eta)$ in (4) outside the trace since $\bar{M}(\eta)$ intertwines the irreducible representation space of $\bar{\rho}(\eta,g)$ with itself, hence it must be a scalar. Moreover we have:

Lemma. *If* $\eta=(\nu,\nu)$ *then* $\bar{M}(\eta)=-1$.

Proof. Let $\lambda_{-1},\lambda_0,\ldots$ be the coefficients of the terms $(s-1)^j$ $(j=-1,0,\ldots)$ in the Laurent expansion of the global L-function $L(s,1_F)$ at 1. The scalar $\bar{m}(\eta)$ can be calculated as the limit over $t\to 0$ of

$$\frac{L(1-t,1_F)}{L(1+t,1_F)}=\frac{\dfrac{\lambda_{-1}}{-t}+\lambda_0+\ldots}{\dfrac{\lambda_{-1}}{t}+\lambda_0+\ldots},$$

and this clearly tends to -1.

It suffices to show that each $\bar{R}(\eta_v)$, which is clearly a scalar, is equal to 1. For non-archimedean places v with $|m|_v=1$, this can be shown by modifying the arguments of [9], pp. 5.9–11; the only change is that we have to assume that the function φ_0 which is defined on p. 5.9 is genuine and vanishes on the set of $\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}k$ with $\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}$ outside A_0, in the notations of [9]. At other places k has to be restricted to a maximal subgroup of K which splits in $\bar{G}$. (An alternative and detailed discussion of intertwining operators for $\bar{G}(F_v)$ and their normalization will be given by C. Moen in a work still in progress.)

The next term of the trace formula, corresponding to the first term of (v) in [7], is

$$\lambda_0\prod_{v}L(1,1_v)^{-1}\,\bar{F}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\bar{f}_v\right), \tag{5}$$

in the notations of Lemma 1.1.3.

3.3. Explicit Expression (II)

Let $\lambda(g)$ be the function on $\bar{A}(F_v)\backslash\bar{G}(F_v)$ obtained by writing $g=amk$ (a in $\bar{A}$, m in N, k in K) and setting $\lambda(g)=|x|^{-2}$ if $m=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ with $|x|\geqq 1$ and $\lambda(g)=1$ if $|x|\leqq 1$. Consider the distribution

$$A_1(\gamma,\tilde{f}_v)=\Delta_v(\gamma)\int_{\bar{A}(F_v)\backslash\bar{G}(F_v)}\tilde{f}_v(g^{-1}\gamma g)\log\lambda(g)\,dg.$$

Noting that $\prod_v |a|_v=1$ for any a in $F^\times$, the term corresponding to (iv) of [7] is

$$-\tfrac{1}{2}\lambda_{-1}\sum_v\sum_\gamma A_1(\gamma,\tilde{f}_v)\prod_{w\neq v}F(\gamma,\tilde{f}_w). \tag{6}$$

The inner sum here is taken over all $\gamma\neq 1$ in $Z^n(F)\backslash A^n(F)$ and not in $Z(F)\backslash A(F)$ as for $GL(2)$. This follows for example by integrating (6.19) of [2] over $A(F)\bar{Z}^n(\mathbb{A})\backslash\bar{G}(\mathbb{A})$, and noting as in Lemma 1.1.1 that

$$\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}=\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix},(\alpha,z)\right)\qquad (z\text{ in }\mathbb{A}^\times).$$

To deal with (6) we need the following:

Lemma. *For every place v of F there exist two distributions $\tilde{f}_v\mapsto A_2(\gamma,\tilde{f}_v)$ and $\tilde{f}_v\mapsto A_3(\gamma,\tilde{f}_3)$ (γ regular in $A^n(F_v)$) with the following properties:*

(i) $\frac{1}{2}A_1(\gamma,\tilde{f}_v)=A_2(\gamma,\tilde{f}_v)+A_3(\gamma,\tilde{f}_v)$,

(ii) *$A_2(\gamma,\tilde{f}_v)$ is an invariant distribution,*

(iii) *For each genuine $\tilde{f}_v$, $A_3(\gamma,\tilde{f}_v)$ extends to a genuine continuous function on $\bar{A}^n(F_v)$ with compact support modulo $\bar{Z}^n(F_v)$,*

(iv) *If μ_v is unramified and n is a unit in F_v let $\tilde{f}_v^0$ be the genuine spherical function which vanishes outside $\bar{Z}^n(F_v)K$. Then $A_3(\gamma,\tilde{f}_v^0)$ is 0 for all γ,*

(v) *If $\tilde{f}_v$ and f_v have matching orbital integrals as in Lemma 1.2.3 and $A_2(\gamma,f_v)$ is the distribution defined in [9], p. 7.3, then*

$$A_2\left(\begin{pmatrix}a^n & 0\\ 0 & b^n\end{pmatrix},\tilde{f}_v\right)=\sum_{\zeta\text{ in }\underline{\zeta}_n}A_2\left(\begin{pmatrix}\zeta a & 0\\ 0 & b\end{pmatrix},f_v\right)\qquad (a,b\text{ in }F_v^\times),$$

(vi) *If v is archimedean and $\tilde{f}_v$ satisfies $F\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\tilde{f}_v\right)=0$, then*

$$\frac{d}{ds}\theta(s,\tilde{f}_v)|_{s=0}=-A_3\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\tilde{f}_v\right)$$

where

$$\theta(s,\tilde{f}_v)=\frac{L(1,1_v)}{L(1+s,1_v)}\int_K\int_{F_v}\tilde{f}_v\left(k^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k\right)|x|^s\,dx\,dk.$$

Proof. For brevity we shall omit the index v from the notations below. We shall deal with non-archimedean local fields first. For any regular $\gamma = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ in $\bar{A}^n$ we define

$$A_2(\gamma, \tilde{f}) = \log \left|1 - \frac{b^n}{a^n}\right| F(\gamma, \tilde{f}) + \sum_{\zeta \text{ in } \zeta_n} b_\zeta(\gamma, \tilde{f}) + n\, c(\gamma, \tilde{f}).$$

Here $b_\zeta(\gamma, \tilde{f}) = c(\gamma, \tilde{f}) = 0$ if $\left|1 - \frac{b^n}{a^n}\right| > 1$; otherwise we put

$$b_\zeta(\gamma, \tilde{f}) = \left|1 - \zeta \frac{b}{a}\right| \tilde{f}(a^n) \int_K \int_{|x| \leq 1} \log|x|\, dx\, dk$$

and

$$c(\gamma, \tilde{f}) = -|\tilde{\omega}| \log|\tilde{\omega}|\, F\left(\begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}, \tilde{f}\right).$$

Assertion (ii) follows at once and so does (v), which motivates the present definition.

Since

$$\tfrac{1}{2} A_1(\gamma, \tilde{f}) = -\Delta(\gamma) \int_K \int_{|x|>1} \tilde{f}\left[k^{-1}\gamma \begin{pmatrix} 1 & \left(1 - \frac{b^n}{a^n}\right) x \\ 0 & 1 \end{pmatrix} k\right] \log|x|\, dx\, dk$$

$$= -\left|\frac{a}{b}\right|^{n/2} \int_K \int_{|x| > \left|1 - \frac{b^n}{a^n}\right|} \tilde{f}\left(k^{-1}\gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) \left(\log|x| - \log\left|1 - \frac{a^n}{b^n}\right|\right) dx\, dk,$$

if we want (i) to hold we must have

$$A_3(\gamma, \tilde{f}) = \left|\frac{a}{b}\right|^{n/2} \int_K \int_F \tilde{f}\left(k^{-1}\gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) \omega(x, y)\, dx\, dk - \sum_{\zeta \text{ in } \zeta_n} b_\zeta(\gamma, \tilde{f}) - n\, c(\gamma, \tilde{f}),$$

where

$$\omega(x, \gamma) = \begin{cases} -\log|x|, & |x| \geqq \left|1 - \frac{b^n}{a^n}\right|, \\ -\log\left|1 - \frac{b^n}{a^n}\right|, & |x| \leqq \left|1 - \frac{b^n}{a^n}\right|. \end{cases}$$

To verify (iii) we need only show that $A_3(\gamma, \tilde{f})$ extends to a continuous function at $\gamma = 1$, namely as $a \to 1$ and $b \to 1$. We may ignore the terms $c(\gamma, \tilde{f})$ and $b_\zeta(\gamma, \tilde{f})$ ($\zeta \neq 1$), since they are clearly locally constant at $a = b = 1$. The difference between the remaining terms is equal to

$$-\int_K \int_F \tilde{f}\left(k^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) \log|x|\, dx\, dk$$

near $a = b = 1$, and the assertion follows.

To see (iv) we note that all terms in $A_3(\gamma, \tilde{f}^0)$ are 0 unless $\left|1 - \frac{b^n}{a^n}\right| \leqq 1$. We may assume that $\left|1-\frac{b}{a}\right| \leq \left|1-\zeta\frac{b}{a}\right|$ for any $\zeta \neq 1$. The difference between the first term and $b_1(\gamma, \tilde{f})$ is

$$-\tilde{f}^0(a^n) \int_K \int_{|x|\leq 1} \log|x|\, dx\, dk = \frac{-|\tilde{\omega}|\log|\tilde{\omega}|}{1-|\tilde{\omega}|} \tilde{f}(a^n) \int_K \int_{|x|\leq 1} dx\, dk,$$

which is equal to $c(\gamma, \tilde{f}^0)$. Also $b_\zeta(\gamma, \tilde{f}^0)$ $(\zeta \neq 1)$ and $-c(\gamma, \tilde{f}^0)$ are equal (when $|n|_v = 1$), hence $A_3(\gamma, \tilde{f}^0)$ is always 0.

Finally if F is archimedean we write

$$A_2(\gamma, \tilde{f}) = \log\left|1-\frac{b^n}{a^n}\right| F(\gamma, \tilde{f}) + n\, c(\gamma, \tilde{f}),$$

where

$$c(\gamma, \tilde{f}) = \frac{-L'(1,1)}{L(1,1)^2} F\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tilde{f}\right).$$

Assertion (v) follows at once on comparison with [9], pp. 7.5–6. Also (ii) is obvious and to satisfy (i) we write

$$A_3(\gamma, \tilde{f}) = -\tfrac{1}{2}\left|\frac{a}{b}\right|^{n/2} \int_K \int_F \tilde{f}\left[k^{-1}\gamma \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} k\right] \log\left(\left|1-\frac{b^n}{a^n}\right|^2 + |x|^2\right) dx\, dk - n\, c(\gamma, \tilde{f}).$$

It is easy to verify (iii), and (vi) follows since under our assumption $c(\gamma, \tilde{f})$ is now 0. The proof is complete.

We can use the Lemma to write (6) as the sum of

$$-\lambda_{-1} \sum_v \sum_\gamma A_2(\gamma, \tilde{f}_v) \prod_{w \neq v} F(\gamma, \tilde{f}_w) \tag{7}$$

and

$$-\lambda_{-1} \sum_v \sum_\gamma A_3(\gamma, \tilde{f}_v) \prod_{w \neq v} F(\gamma, \tilde{f}_w). \tag{8}$$

The inner sums are taken over all $\gamma \neq 1$ in $Z^n \backslash A^n$.

Following [9], p. 8.7, we would like to extend the sum in (8) to include $\gamma = 1$, and we would like the new term to be equal to the second half of (v) in [7]. By virtue of (vi) of the above Lemma the last requirement will be satisfied if we apply the trace formula to a function $\tilde{f} = \bigotimes_v \tilde{f}_v$ whose component $\tilde{f}_{v_1}$ at some archimedean place v_1 satisfies $F\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tilde{f}_{v_1}\right) = 0$. Henceforth we shall assume that $\tilde{f}$ is of this form and extend the sum in (8) to include $\gamma = 1$.

We can now apply the Poisson summation formula to the sum (8), the function

$$\sum_v A_3(\gamma, \tilde{f}_v) \prod_{w \neq v} F(\gamma, \tilde{f}_w),$$

and the group $Z(F)\bar{Z}^n(\mathbb{A})\backslash\bar{A}^n(\mathbb{A})Z(F)$. This function transforms under $Z(F)\bar{Z}^n(\mathbb{A})$ by the character μ^{-1}, and the Fourier transform will be concentrated on $\bar{D}_0$. The global Fourier transform can be calculated locally after dividing by λ_{-1}, since the Tamagawa measure on $\mathbb{A}^\times$ is given by $d^\times x=(\lambda_{-1})^{-1}\bigotimes_v d^\times x_v$. The Fourier transform of $F(\gamma,\tilde{f}_v)$ is $\operatorname{tr}\bar{\rho}(\eta_v,\tilde{f}_v)$ and we denote the transform of $A_3(\gamma,\tilde{f}_v)$ by $B_1(\eta_v,\tilde{f}_v)$. Finally writing

$$B(\eta_v,\tilde{f}_v)=\tfrac{1}{2}\operatorname{tr}(\bar{R}^{-1}(\eta_v)\bar{R}'(\eta_v)\bar{\rho}(\eta_v,\tilde{f}_v))-B_1(\eta_v,\tilde{f}_v)$$

we can put (8) together with the term corresponding to (viii) of [7] to obtain

$$\frac{1}{2\pi}\int_{\bar{D}_0}|ds|\sum_v B(\eta_v,\tilde{f}_v)\prod_{w\neq v}\operatorname{tr}\bar{\rho}(\eta_w,\tilde{f}_w). \tag{9}$$

The full expression for the trace formuly for the operator $\bar{r}(\tilde{f})$ is given by the sum of (1), (2′), (3), (4), (5), (7) and (9). It is noteworthy that the formula simplifies considerably if at least two components $\tilde{f}_{w_i}$ $(i=1,2)$ of $\tilde{f}$ satisfy $F(\gamma,\tilde{f}_{w_i})=0$ for all γ in $\bar{A}^n(F_{w_i})$. In this case $\operatorname{tr}\bar{r}(\tilde{f})$ is given by the sum of (1) and (2), and all other terms vanish.

4.0. Equality of Traces

Let $\tilde{f}=\bigotimes_v \tilde{f}_v$ be a function satisfying the conditions of the previous section. In particular we require that $F\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\tilde{f}_{v_1}\right)$ vanishes for some archimedean place v_1. It follows that $F\left(\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\tilde{f}_{v_1}\right)=0$ if v_1 is complex; if v_1 is real the last equality follows from

$$\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}-t&0\\0&-1\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix}=\left(\begin{pmatrix}-t&0\\0&-1\end{pmatrix},(-1,-1)\right)\qquad(t>0).$$

If $\tilde{f}_v$ is spherical we let f_v be the spherical function on $G(F_v)$ associated to $\tilde{f}_v$ by matching orbital integrals as in Lemma 1.4. At other places v we choose f_v on $G(F_v)$ as in Lemma 1.2.3, so that $\tilde{f}_v$ and f_v have matching orbital integrals. This, and the condition that we put on $\tilde{f}_{v_1}$, implies that

$$F\left(\begin{pmatrix}0&1\\\pm1&0\end{pmatrix},f_{v_1}\right)=F\left(\begin{pmatrix}\pm1&0\\0&1\end{pmatrix},f_{v_1}\right)=0. \tag{1}$$

We shall continue to denote by $\bar{r}(\tilde{f})$ the trace class operator on $L_0(\mu,\bar{G})$ defined in 3.1. Also with $f=\bigotimes_v f_v$ we shall denote by $r(f)$ the trace class operator on the subspace $L_0(\mu',G)\oplus L_1(\mu',G)$ of $L(\mu',G)$. Here $\mu'(z)=\mu(z^n)$ (z in $Z(\mathbb{A})$) and $L(\mu',G)$ denotes the space of square-integrable functions on $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$ which transform under $Z(\mathbb{A})$ by the character μ' of $\mathbb{A}^\times$. $L_0(\mu',G)$ denotes the subspace of cuspidal functions and $L_1(\mu', G)$ denotes the span of the one-dimensional invariant subspaces.

Our aim in this section is to prove:

Theorem. *For all functions $\tilde{f}$ on $\bar{G}(\mathbb{A})$ and f on $G(\mathbb{A})$ related as above we have*

$$\operatorname{tr} \bar{r}(\tilde{f}) = \operatorname{tr} r(f).$$

This simple equality is of key importance on our work. Using the local analysis of Chap. I we shall show in the next section that it affords both a thorough description of the global correspondence and a completion of the description, begun in Chap. I, of the local correspondence.

To prove the Theorem we shall use the Selberg trace formula. In fact we shall compare the expression for the trace of $\bar{r}(\tilde{f})$ given in the previous section with the expression for the trace of $r(f)$ given in [7], Sect. 16 (with Φ replaced by f, and modified as in [9], Sect. 8, with $E=F$), and show that they are equal.

Finally we note that the Theorem holds even without the restriction on $\tilde{f}_{v_1}$; this will follow from the results of 5.2 and 5.3 below.

4.1. Direct Comparison

We shall now start the proof of Theorem 4.0. All terms, except (9), of the trace formula for $\bar{r}(\tilde{f})$ can be directly compared with the corresponding terms of the trace formula for $r(f)$. In this subsection we shall carry out this comparison.

We begin with (1) (of 3.2). Our choice of measures (in 3.1) guarantees that

$$|\bar{Z}^n(\mathbb{A})\, G(F)\backslash \bar{G}(\mathbb{A})| = |Z(\mathbb{A})\, G(F)\backslash G(\mathbb{A})|.$$

The relation between $\tilde{f}_v$ and f_v, together with their asymptotic expansions on the non-split tori, implies that

$$\tilde{f}_v\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = |n|_v f_v\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

for all non-archimedean v, and for real v (note that $(e^{2i\theta}-e^{-2i\theta})/(e^{i\theta}-e^{-i\theta}) = e^{i\theta} + e^{-i\theta}$ is equal to 2 at $\theta=0$). For complex v this follows from the Plancherel formula

$$f(\alpha) = c(\alpha)\frac{\partial^2}{\partial z_2\, \partial \bar{z}_2} F(\alpha, T) \qquad (\alpha \text{ in } Z(\mathbb{C})),$$

where $T(\mathbb{C})$ is the group of $\begin{pmatrix}\exp(z_1+z_2) & 0\\ 0 & \exp(z_1-z_2)\end{pmatrix}$ $(z_1, z_2$ in $\mathbb{C})$, and $c(\alpha)$ is a function on $Z(\mathbb{C})$ (cf. [9], p. 4.6); we only have to note that $|n|_{\mathbb{C}} = n^2$ for the normalized valuation on $F_v = \mathbb{C}$. Hence (1) is equal to (i) of [7], p. 5.16, by virtue of the product formula on the number field F.

To deal with (2′) (of 3.2) we note that our choice of measures (in 3.1) implies that

$$|\bar{Z}^n(\mathbb{A})\, T(F)\backslash \bar{T}(\mathbb{A})| = |Z(\mathbb{A})\, T(F)\backslash T(\mathbb{A})|$$

for any non-split torus T of G and its lift $\bar{T}$ in $\bar{G}$. If H is a normal subgroup of a group H' we shall denote by $(H\backslash H')^\times$ the set $H\backslash H'$ with the coset of H excluded. The map $\gamma\mapsto\gamma^n$ from $(Z(F)\backslash T(F))^\times$ to $(\bar{Z}^n(F)\backslash \bar{T}^n(F))^\times$ is clearly surjective and also injective, since ζ_n is contained in $F^\times$. Lemma 0.3.2 implies that all elements of $(\bar{Z}^n(F)\backslash \bar{T}^n(F))^\times$ are regular when n is odd. If n is even $(\bar{Z}^n(F)\backslash \bar{Z}(F)\cap \bar{T}^n(F))^\times$ is non-empty; its pull-back consists of $G(F)$-conjugates in $T(F)$ of the elements $\begin{pmatrix}0 & 1\\ x & 0\end{pmatrix}$ with x in $(F^{\times n}\backslash F^\times)^\times$. But (1) of 4.0 implies that

$$F\left(\begin{pmatrix}0 & 1\\ x & 0\end{pmatrix}, f_{v_1}\right) = F\left(\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}^{n/2}, \tilde{f}_{v_1}\right) = 0$$

for all x in $F^\times$. Hence all terms in (ii) of [7], p. 5.16, must vanish unless they are indexed by a regular γ in $Z(F)\backslash T(F)$ for which γ^n is regular in $Z^n(F)\backslash T^n(F)$. We conclude that the sums over the non-zero terms in (2) and (ii) of [7] are taken over isomorphic sets. The corresponding terms are equal since $\tilde{f}_v$ and f_v have matching orbital integrals for all v and by virtue of the product formula

$$\int_{T(\mathbb{A})\backslash G(\mathbb{A})} \tilde{f}(g^{-1}(\gamma, s(\gamma)^{-1})\, g)\, dg = \prod_v \Delta_v(\gamma) \int_{T(F_v)\backslash G(F_v)} \tilde{f}_v(g^{-1}(\gamma, s_v(\gamma)^{-1})\, g)\, dg.$$

We shall now compare (3) and the term (3) of [9], p. 8.4 (with $E=F$; this is (vii) of [7]). The map

$$\eta = (\nu_1, \nu_2) \mapsto \eta' = (\nu'_1, \nu'_2),$$

where $\nu'_i(z) = \nu_i(z^n)$ (z in $\mathbb{A}^\times$; $i=1,2$), from the subset of η in $\bar{D}_0$ with $\operatorname{tr}\bar{\rho}(\eta_v, \tilde{f}_v) \neq 0$ for all v to the subset of η' in D_0 with $\operatorname{tr}\rho(\eta'_v, f_v) \neq 0$ for all v, is both injective and surjective. Here D_0 denotes the set of characters $\eta_1 = (\nu'_1, \nu'_2)$ on $A(F)\backslash A(\mathbb{A})$ for which $\nu'_1\nu'_2 = \mu$ on $Z(\mathbb{A})$, where $\mu'(z) = \mu(z^n)$ (z in $\mathbb{A}^\times$). The injectivity follows from the fact that all elements of $\bar{D}_0$ which are mapped to η' must differ by a character of order n, and the only such character in $\bar{D}_0$ is the trivial one. To see the surjectivity we note that each $\eta_1 = (\nu'_1, \nu'_2)$ in D_0 is of the form η' with η in $\bar{D}_0$ if $\nu'_{1v}(\zeta) = \nu'_{2v}(\zeta) = 1$ for all v and all ζ in ζ_n. If η_1 is such that there exists some v for which this does not hold, say $\nu'_{1v}(\zeta) \neq 1$ ($\zeta \neq 1$), then $\operatorname{tr}\rho(\eta_{1v}, f_v) = 0$. The last claim is a consequence of the equalities

$$\begin{aligned}\operatorname{tr}\rho(\eta_{1v}, f_v) &= \int_{Z(F_v)\backslash A(F_v)} F\left(\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}, f_v\right) \nu_{1v}(a)\, \nu_{2v}(b)\, d^\times a\, d^\times b\\ &= \int_{Z(F_v)\backslash A(F_v)} F\left(\begin{pmatrix}\zeta a & 0\\ 0 & b\end{pmatrix}, f_v\right) \nu_{1v}(\zeta a)\, \nu_{2v}(b)\, d^\times a\, d^\times b,\end{aligned}$$

and (by Lemma 1.2.3)

$$F\left(\begin{pmatrix}\zeta a & 0\\ 0 & b\end{pmatrix}, f_v\right)=F\left(\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}, f_v\right).$$

Since the measure on $\bar{Z}^n\backslash\bar{A}^n$ was chosen to be the pull-back of the measure on $Z\backslash A$ under the map $\gamma\mapsto\gamma^n$ we deduce that

$$\operatorname{tr}\bar{\rho}(\eta_v, \tilde{f}_v)=\operatorname{tr}\rho(\eta'_v, f_v) \qquad \text{(for all } v). \tag{1}$$

By definition we have $\bar{m}(\eta)=m(\eta')$ and $\bar{m}'(\eta)=m'(\eta')$; hence (3) (of 3.2) and (3) of [9] are equal, as required.

The term (4) (of 3.2) can be shown to be equal to (vi) of [7] ((2) of [9]) in the same way. The map $\eta\mapsto\eta'$ gives an isomorphism between the sets indexing the non-zero terms in the sums occuring in (4) and (vi). The corresponding terms are equal by virtue of (1) (above) and the fact (Lemma 3.2) that $\bar{M}(\eta)=M(\eta')=-1$.

The fact that (5) (of 3.2) and the first term of (v) in [7] ((4) of [9]) are equal follows from the equality of $F\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tilde{f}_v\right)$ and $F\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, f_v\right)$ for all v. In fact both are 0 since we assumed that for some archimedean v_1 we have $F\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tilde{f}_{v_1}\right)=0$.

It is also easy to see that (7) is equal to (5) of [9] since we defined $A_2(\gamma, \tilde{f}_v)$ with Lemma 3.3(v) in mind, and $A_2\left(\begin{pmatrix}\zeta & 0\\ 0 & 1\end{pmatrix}, f_{v_1}\right)$ is 0 for $\zeta\neq 1$ by our assumption on f_{v_1}.

The proof of Theorem 4.0 will be complete when we have shown that (9) (of 3.3) is equal to (7) of [9]. Rather indirectly we shall show this by establishing in the next subsection the following lemma, the assumption in which has just been proved. No simpler proof of this is available as yet.

Lemma. *If* $\operatorname{tr}\bar{r}(\tilde{f})-\operatorname{tr}r(f)$ *is equal to*

$$\frac{1}{2\pi}\int_{D_0}|ds|\sum_v (B(\eta_v, \tilde{f}_v)-B(\eta'_v, f_v))\prod_{w\neq v}\operatorname{tr}\rho(\eta'_w, f_w), \tag{2}$$

then both are 0.

Here we write

$$B(\eta'_v, f_v)=\tfrac{1}{2}\operatorname{tr}(R(\eta'_v)^{-1}R'(\eta'_v)\,\rho(\eta'_v, f_v))-B_1(\eta'_v, f_v),$$

where $R(\eta'_v)$ is the (normalized) intertwining operator of [9], Sect. 5, and $B_1(\eta'_v, f_v)$ is the Fourier transform of

$$\sum_{\zeta \text{ in } \underline{\zeta}_n} A_3\left(\begin{pmatrix}\zeta a & 0\\ 0 & b\end{pmatrix}, f_v\right);$$

$A_3(\gamma, f_v)$ is defined in [9], Sect. 7. Finally η is the element of $\bar{D}_0$ which is determined uniquely by η' of D_0 and the map $\eta\mapsto\eta'$ $(\eta'(z)=\eta(z^n)$, z in $\mathbb{A}^\times)$, provided that η' satisfies $\operatorname{tr}\rho(\eta'_w, f_w)\neq 0$ for all $w\neq v$.

4.2. *Indirect Comparison*

We shall now prove Lemma 4.1. First we shall rewrite $\operatorname{tr}\tilde{r}(\tilde{f}) - \operatorname{tr} r(f)$ and then we shall rewrite (2) of 4.1 so that they are put in forms which are sufficiently easy to compare. We shall then choose a function $\tilde{f}$ for which these expressions are different unless both are identically 0, thus completing the proof.

Let V be a finite set of places containing all the v for which $|n|_v \neq 1$ (in particular it contains all of the archimedean places). Suppose that $\tilde{f} = \bigotimes_v \tilde{f}_v$ is such that $\tilde{f}_v$ is spherical outside V. Let $\tilde{\pi} = \bigotimes_v \tilde{\pi}_v$ be an automorphic tensor product (of genuine irreducible admissible representations $\tilde{\pi}_v$ of $\bar{G}(F_v)$) in $L_0(\mu, \bar{G}) \oplus L_1(\mu, \bar{G})$. Suppose that $\tilde{\pi}_v$ belongs to the unramified principal series for v outside V. Then

$$\operatorname{tr}\tilde{\pi}_v(\tilde{f}_v) = \operatorname{tr}\pi_v(f_v) = f_v^\vee(t(\tilde{\pi}_v)) \qquad (v \text{ outside } V),$$

where π_v is the unramified principal series representation of $G(F_v)$ which is obtained from $\tilde{\pi}_v$ by (2) of 2.1. Here $f_v^\vee$ is the Satake transform of f_v (which, in the notations of [9], Sect. 3, is given by

$$f_v^\vee(\gamma) = |A(0_v)| \sum_{\lambda^\vee} F(\lambda^\vee, f_v)\, \lambda^\vee(\gamma) \qquad (\gamma \text{ in } A(\mathbb{C}));$$

$\lambda^\vee$ runs through the lattice of rational characters on $A(\mathbb{C})$ which is the same as $A(0_v)\backslash A(F_v)$ and $\mathbb{Z}^2$), and

$$t(\tilde{\pi}_v) = \begin{pmatrix} a(\tilde{\pi}_v) & 0 \\ 0 & b(\tilde{\pi}_v) \end{pmatrix}, \qquad \text{with } a(\tilde{\pi}_v)\, b(\tilde{\pi}_v) = \mu_v'(\tilde{\omega}_v) = \mu_v(\tilde{\omega}_v^n),$$

is an element of $A(\mathbb{C})$.

For brevity we put

$$\alpha(\tilde{\pi}) = \prod_{v \text{ in } V} \operatorname{tr}\tilde{\pi}_v(\tilde{f}_v),$$

and then

$$\operatorname{tr}\tilde{r}(\tilde{f}) = \sum \alpha(\tilde{\pi}) \prod_{v \text{ outside } V} f_v^\vee(t(\tilde{\pi}_v)),$$

where the sum is taken over all $\tilde{\pi} = \bigotimes_v \tilde{\pi}_v$ with $\tilde{\pi}_v$ in the unramified genuine principal series for v outside V.

The trace of the operator $r(f)$ can also be treated in this way, and we write

$$\operatorname{tr} r(f) = \sum_\pi \alpha(\pi) \prod_{v \text{ outside } V} f_v^\vee(t(\pi_v)),$$

with

$$t(\pi_v) = \begin{pmatrix} a(\pi_v) & 0 \\ 0 & b(\pi_v) \end{pmatrix} \quad \text{in } A(\mathbb{C}), \qquad a(\pi_v)\, b(\pi_v) = \mu_v'(\tilde{\omega}_v).$$

The difference between $\operatorname{tr}\bar{r}(\tilde{f})$ and $\operatorname{tr} r(f)$ can now be expressed in the form

$$\sum_k \beta_k \prod_{v \text{ outside } V} f_v^\vee(t_{k,v}).$$

Here for each $k(=0,1,\ldots)$, $\{t_{k,v};\ v \text{ outside } V\}$ is a sequence of elements from $A(\mathbb{C})$. The sequences are distinct in the sense that for each $k \neq k'$ there exists some v such that $t_{k,v}$ and $t_{k',v}$ are not conjugate. The β_k are non-zero complex numbers; the products and the sum are absolutely convergent.

Finally we fix some v_0 outside V and let $r_j = r_{j,v_0}$ $(j=1,2,\ldots)$ denote the distinct elements in the set $\{t_{k,v_0}\}$. We proved:

Lemma 1. *Put*

$$c_j = \sum_{t_{k,v_0}=r_j} \beta_k \prod_{\substack{w \text{ outside } V \\ w \neq v_0}} f_w^\vee(t_{k,w}).$$

Then $\operatorname{tr}\bar{r}(\tilde{f}) - \operatorname{tr} r(f)$ *is equal to*

$$\sum_j c_j f_{v_0}^\vee(r_j). \tag{1}$$

We shall now rewrite (2) of 4.1 in a similar way. In the notation of Lemma 4.1, we set

$$\beta(\eta') = \sum_{v \text{ in } V} (B(\eta_v, \tilde{f}_v) - B(\eta'_v, f_v)) \prod_{\substack{w \text{ in } V \\ w \neq v}} \operatorname{tr}\rho(\eta'_w, f_w),$$

and prove:

Lemma 2. *The integral* (2) *of* 4.1 *is equal to*

$$\frac{1}{2\pi} \int_{-i\infty}^{i\infty} d(s) f_{v_0}^\vee \left(\begin{pmatrix} |\tilde{\omega}_{v_0}|^s & 0 \\ 0 & \delta|\tilde{\omega}_{v_0}|^{-s} \end{pmatrix} \right) |ds|, \tag{2}$$

where

$$d(s) = \sum \beta(\eta'_0 \alpha_s) \prod_{\substack{w \text{ outside } V \\ w \neq v_0}} f'_w(t(\eta'_{0w} \alpha_s))$$

and $\alpha_s\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = |a/b|^s$, *for any imaginary* s. *Here the sum is taken over all connected components of* D_0 *at which the* η'_v *are unramified for* v *outside* V *and satisfy* $\eta'_v(\zeta)=1$ *for all* v *and* ζ. *In each of these components there is some* η'_0 *with*

$$t(\eta'_{0v}) = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \quad \textit{where} \quad \delta = \mu'(\tilde{\omega}_v) \ (=\mu(\tilde{\omega}_v^n)),$$

and any other η' *is of the form* $\eta'_0 \alpha_s$.

Proof. The integral (2) above is clearly equal to

$$\frac{1}{2\pi} \int \beta(\eta') \prod_{v \text{ outside } V} f_v^\vee(t(\eta'_v)) |ds|, \tag{3}$$

where the integral is taken over all η' in D_0 for which η'_v is unramified for v outside V and η'_v satisfies $\eta'_v(\zeta)=1$ for all v and ζ, and where

$$t(\eta'_v)=\begin{pmatrix} v'_{1v}(\tilde{\omega}_v) & 0 \\ 0 & v'_{2v}(\tilde{\omega}_v)\end{pmatrix}.$$

To prove that (3) is equal to (2) of 4.1, we merely have to show that

$$B(\eta_v, \tilde{f}_v)=B(\eta'_v, f_v)$$

for all v outside V. For these v the function f_v is spherical and η'_v is unramified; thus we have

$$B(\eta'_v, f_v)=-B_1(\eta'_v, f_v) \quad \text{and} \quad B(\eta_v, \tilde{f}_v)=-B_1(\eta_v, \tilde{f}_v).$$

It suffices to show that

$$B_1(\eta'_v, f_v)=\int_{F_v^\times} A_3\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}, f_v\right) \eta'_v\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right) d^\times a$$

is equal to

$$B_1(\eta_v, \tilde{f}_v)=\int_{F_v^{\times n}} A_3\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}, \tilde{f}_v\right) \eta_v\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right) d^\times a,$$

and this will follow if we knew that

$$A_3\left(\begin{pmatrix} a^n & 0 \\ 0 & 1\end{pmatrix}, \tilde{f}_v\right)=\sum_{\zeta \text{ in } \zeta_n} A_3\left(\begin{pmatrix} \zeta a & 0 \\ 0 & 1\end{pmatrix}, f_v\right),$$

for any a in $F_v^\times$. For brevity we shall drop the index v from now on.

Lemma 3.3(iv) and [9], pp. 7.4–5 show that both sides vanish identically if $\tilde{f}$ is the genuine spherical function which vanishes outside $\bar{Z}^n\bar{K}$. By virtue of Lemma 3.3(v) it suffices to show that for any spherical $\tilde{f}$ we have

$$A_1\left(\begin{pmatrix} a^n & 0 \\ 0 & 1\end{pmatrix}, \tilde{f}\right)=\sum_{\zeta \text{ in } \zeta_n} A_1\left(\begin{pmatrix} \zeta a & 0 \\ 0 & 1\end{pmatrix}, f\right)$$

for any a in $F^\times$. Since the left side is equal to

$$|a|^{n/2}\int_K \int_{|x|\geqq 1} \tilde{f}\left(k^{-1}\begin{pmatrix} a^n & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} k\right)(\log|x|^{-2}-\log|1-a^n|^{-2})\, dx\, dk,$$

and a similar treatment can be given to the right side, it suffices to prove that

$$|a|^{n/2}\int_K \int_{|x|>1} \tilde{f}\left(k^{-1}\begin{pmatrix} a^n & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} k\right)\log|x|^{-2}\, dx\, dk$$
$$=n|a|^{1/2}\int_K \int_{|x|>1} f\left(k^{-1}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} k\right)\log|x|^{-2}\, dx\, dk.$$

As in 1.4 we may restrict ourselves to the genuine characteristic function $\tilde{f}_{n\lambda}$ of a double coset $\bar{K}(n\lambda)\bar{K}$ where $\lambda=(k'\ k)$ and $k'\geqq k$. The case of $k'=k$ has already been dealt with, and we now assume that $k'>k$.

As in the proof of Lemma 1.4 we note that

$$A_1\left(\begin{pmatrix} a^n & 0\\ 0 & 1\end{pmatrix}, \tilde{f}_{n\lambda}\right)=A_1\left(\begin{pmatrix} a^n & 0\\ 0 & 1\end{pmatrix}, f_{n\lambda}\right).$$

The value of this was calculated in [9], pp. 3.18–9, to be

$$(2\log|\tilde{\omega}|\,nr(1-1/q)\,q^{nr+\frac{1}{2}mn},$$

if

$$\lambda^{\vee}\left(\begin{pmatrix} a^n & 0\\ 0 & 1\end{pmatrix}\right)=(nm,0)=(nk'-nr,\ nk+nr)$$

since $\Delta\left(\begin{pmatrix} a^n & 0\\ 0 & 1\end{pmatrix}\right)=q^{mn/2}$. Also we have

$$A_1\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}, f_\lambda\right)=(2\log|\tilde{\omega}|)\,r(1-1/q)\,q^{r+\frac{1}{2}m}.$$

But $\tilde{f}_{n\lambda}$ corresponds to $q^{\frac{1}{2}\langle\alpha,(n-1)\lambda\rangle}f_\lambda$ under the map φ of 1.4 and

$$q^{\frac{1}{2}\langle\alpha,(n-1)\lambda\rangle}=q^{\frac{1}{2}(n-1)(k'-k)}=q^{(r+\frac{1}{2}m)(n-1)},$$

hence the proof is complete.

Lemma 4.1 can now be made to follow from the equality of (1) and (2). Indeed, on noting that the trace formula gives absolutely convergent sums and integrals and that all contributions to (2) are unitary, this can be done by choosing a suitable function f_v for which (1) and (2) are different unless all c_j are 0. It is easy to see that such a suitable function is given by the function f_v which is defined in [9], p. 9.16, and used for the same purpose. We deduce that the c_j (and also the β_k) are all 0, and Lemma 4.1 follows. This completes the proof of Theorem 4.0.

4.3. Reformulation

The following is a more practical form of Theorem 4.0.

Theorem. *Let V be a finite set of places containing all v with $|n|_v \neq 1$. Suppose that for every v outside V we are given*

$$r_v=\begin{pmatrix} a_v & 0\\ 0 & b_v\end{pmatrix}\quad \textit{in } A(\mathbb{C})\quad \textit{with } a_v b_v=\mu'(\tilde{\omega}_v)=\mu(\tilde{\omega}_v^n).$$

Then for any matching $\tilde{f}_v$ and f_v (v in V) as in Lemma 1.2.3 such that

$F\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tilde{f}_{v_1}\right)=0$ *for some archimedean* v_1, *we have*

$$\sum_{\tilde{\pi}} \prod_{v \text{ in } V} \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v)=\sum_{\pi} \prod_{v \text{ in } V} \operatorname{tr} \pi_v(f_v). \tag{1}$$

The sum on the left (resp. right) is taken over all irreducible constituents $\tilde{\pi}=\bigotimes_v \tilde{\pi}_v$ *of* $L_0(\mu, \bar{G}) \oplus L_1(\mu, \bar{G})$ *(resp.* $\pi=\bigotimes_v \pi_v$ *of* $L_0(\mu', G) \oplus L_1(\mu', \bar{G})$*), such that for all* v *outside* V *the* $\tilde{\pi}_v$ *and* π_v *are unramified and satisfy*

$$\operatorname{tr} \tilde{\pi}_v(\tilde{f}_v)=f_v^{\vee}(r_v)=\operatorname{tr} \pi_v(f_v)$$

for all spherical functions $\tilde{f}_v$ *and* f_v *matching as in Lemma* 1.4.

Proof. We fix some v outside V and claim that

$$\alpha_{r_v}=\sum_{\tilde{\pi}} \prod_{w \neq v} \operatorname{tr} \tilde{\pi}_w(\tilde{f}_w)-\sum_{\pi} \prod_{w \neq v} \operatorname{tr} \pi_w(f_w)$$

is 0. Theorem 4.0 implies that

$$\sum \alpha_{r_v} f_v^{\vee}(r_v)=0$$

for all spherical f_v, where the sum is absolutely convergent and taken over all matrices r_v in $A(\mathbb{C})$ with determinant $\mu'(\tilde{\omega}_v)=\mu(\tilde{\omega}_v^n)$. Considerations as at the end of 4.2 (cf. [9], pp. 9.11–15) afford a choice of f_v (as in [9], p. 9.16) for which $\sum \alpha_{r_v} f_v^{\vee}(r_v)$ does not vanish unless all α_{r_v} are 0. The claim follows. The theorem now follows on ordering the countable set of v outside V and applying induction. An alternative proof for the induction step, or the above claim, is given by noting that characters of inequivalent representations are linearly independent ([10], Lemma 6.1).

We note that fixing r_v in the theorem is equivalent to fixing unramified $\tilde{\pi}_v'$ and π_v' with

$$\operatorname{tr} \tilde{\pi}_v'(\tilde{f}_v)=\operatorname{tr} \pi_v'(f_v)$$

for all matching spherical $\tilde{f}_v$ and f_v, and summing over all $\tilde{\pi}=\bigotimes_v \tilde{\pi}_v$ with $\tilde{\pi}_v \approx \tilde{\pi}_v'$ and $\pi=\bigotimes_v \pi_v$ with $\pi_v \approx \pi_v'$ for all v outside V.

Finally we deduce from the fact that both strong multiplicity one and multiplicity one theorems hold for $L_0(\mu', G) \oplus L_1(\mu', G)$, that the sum on the right (over π) contains at most one term.

5.0. *The Correspondence*

The preparations are over and we can begin the discussion of the local and global correspondence.

Definition 1. A genuine irreducible admissible representation $\tilde{\pi}_v$ of $\bar{G}(F_v)$ with a central character μ corresponds to an admissible representation π_v of $G(F_v)$ with a central character μ' if $\mu'(z)=\mu(z^n)$ for all z in $F_v^\times$ and if

$$\Delta(\tilde{g})\,\chi_{\tilde{\pi}_v}(\tilde{g}\zeta_g^{-1})=\begin{cases}\Delta(g)\,\chi_{\pi_v}(g), & \text{elliptic } g,\\ \dfrac{1}{n}\sum\limits_{\zeta \text{ in } \zeta_n}\Delta(g_\zeta)\,\chi_{\pi_v}(g_\zeta), & \text{otherwise,}\end{cases}$$

whenever g^n is regular.

Here $\chi_{\tilde{\pi}_v}$ and χ_{π_v} are the characters of $\tilde{\pi}_v$ and π_v (see 2.0), $\tilde{g}$ was defined in 0.3 and ζ_g in 1.2, and for any $g=h^{-1}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}h$ we write $g_\zeta=h^{-1}\begin{pmatrix}\zeta a & 0\\ 0 & b\end{pmatrix}h$. In all cases the right side of the character identity above is equal to the average of the values of $\Delta(h)\,\chi_{\pi_v}(h)$ over all (conjugacy classes in G of) the h with $\tilde{h}=\tilde{g}$ (equality of conjugacy classes). An equivalent identity is given by

$$\operatorname{tr}\tilde{\pi}_v(\tilde{f}_v)=\operatorname{tr}\pi_v(f_v),$$

for every $\tilde{f}_v$ on $\bar{G}(F_v)$ and f_v on $G(F_v)$ with matching orbital integrals as in Lemma 1.2.3.

For example, we saw in 2.1 that the principal series $\tilde{\rho}(\nu_1,\nu_2)$ corresponds to $\rho(\nu_1',\nu_2')$. One of our aims in this section is to extend this for every such representation of $\bar{G}(F_v)$.

The definition of the local correspondence affords the following definition of the global correspondence:

Definition 2. An irreducible constituent $\tilde{\pi}=\bigotimes_v\tilde{\pi}_v$ of the space $L(\mu,\bar{G})$ of automorphic forms on $\bar{G}(\mathbb{A})$ corresponds to the constituent $\pi=\bigotimes_v\pi_v$ of the space $L(\mu',G)$ of automorphic forms of $G(\mathbb{A})$ if $\tilde{\pi}_v$ corresponds to π_v for all v.

Our main aim in this section is to give a full description of the global correspondence; this will be obtained as an application of Theorem 4.3.

5.1. Lemmas

The following is fundamental in obtaining a description of the correspondence from the trace formula.

Let V be a finite set of places $v_1,\dots,v_r$ such that v_1 is archimedean. Denote by $(\tilde{f}_1,\dots,\tilde{f}_r)$ an r-tuple of anti-genuine locally constant compactly supported (modulo the centre) functions on $\bar{G}(F_{v_i})$ $(1\leqq i\leqq r)$ which transform under $Z^n(F_{v_i})$ by $\mu_{v_i}^{-1}$, and assume that $F\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\tilde{f}_{v_1}\right)=0$.

Lemma 1. *Let $\{(\tilde{\pi}_{1k},\dots,\tilde{\pi}_{rk});\,k\geqq 0\}$ be a sequence of r-tuples of genuine, irreducible, admissible and unitary representations of $\bar{G}(F_{v_i})$ $(1\leqq i\leqq r)$ which transform under $Z^n(F_{v_i})$ by μ_{v_i}. If for every $(\tilde{f}_1,\dots,\tilde{f}_r)$ the series*

$$\sum_{k\geqq 0}\prod_{i=1}^{r}\operatorname{tr}\tilde{\pi}_{ik}(\tilde{f}_i)$$

is absolutely convergent and its sum is 0 then the sequence is empty.

Proof. The lemma would have followed at once from [7], Lemma 16.1.1, applied to the direct product G of $\bar{G}(F_{v_i})$ $(1\leqq i\leqq r)$ and with $\pi_1=\sum_i\bigotimes\tilde{\pi}_{ik}$ and $\pi_2=0$, had we not restricted $\tilde{f}_{v_1}$ by the condition $F\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\tilde{f}_{v_1}\right)=0$. It suffices therefore to show that v_1 can be omitted from V. Our argument is a variant of that given in [9], pp. 9.24–30. We assume that v_1 is real (and hence that $n=2$); the proof for a complex place is similar but simpler.

Let $\{\tilde{\pi}_j; j\geqq 0\}$ denote the set of inequivalent discrete series or their complements in the suitable principal series which occur in the sequence $\tilde{\pi}_{ik}$ $(k\geqq 0)$. We write

$$\alpha_j=\sum\prod_{i=2}^{r}\operatorname{tr}\tilde{\pi}_{ik}(\tilde{f}_i),$$

where the sum is taken over all k with $\tilde{\pi}_{1k}\approx\tilde{\pi}_j$. Our assumption implies that

$$\sum_{j\geqq 0}\alpha_j\operatorname{tr}\tilde{\pi}_j(\tilde{f}_1)=0;$$

we assert that if the discrete series $\tilde{\pi}_{j_1}$ occurs then its complement $\tilde{\pi}_{j_2}$ occurs (and vice versa), and $\alpha_{j_1}=\alpha_{j_2}$. To see this we choose $\tilde{f}_1$ with $F(\gamma,\tilde{f}_1)=0$ if γ lies in $A^2(F_{v_1})$ and $F(\gamma,\tilde{f}_1)=\Delta(\gamma)\bar{\chi}(\gamma)$, where χ denotes the character of $\tilde{\pi}_{j_1}$, if γ is a regular elliptic element. Clearly $\operatorname{tr}\tilde{\pi}_j(\tilde{f}_1)$ is 0 unless $j=j_1$ when it is equal to some positive constant c or $j=j_2$ when it is $-c$. We deduce that $\alpha_{j_1}-\alpha_{j_2}=0$, as asserted.

The above discussion shows that we may assume that all $\tilde{\pi}_{1k}$ belong to the principal series. Hence $\operatorname{tr}\tilde{\pi}_{1k}(\tilde{f}_1)$ depends only on the values of $\varphi(t)=F(t,\tilde{f}_1)$ on $A^2(F_{v_1})$. The α_j's can now be seen to be 0 as in [9], pp. 9.26–30; we merely have to note that the condition $\varphi\begin{pmatrix}1&0\\0&1\end{pmatrix}=0$ that $\tilde{f}_1$ satisfies does not introduce any restriction on the Fourier transforms $\varphi^{\vee}(s)$ of [9]. Hence the proof is complete.

Let $\pi=\bigotimes_v\pi_v$ be as in Theorem 4.3 and choose a set V so that (1) of 4.3 takes the form

$$\sum_k\prod_{v\text{ in }V}\operatorname{tr}\tilde{\pi}_{vk}(\tilde{f}_v)=\prod_{v\text{ in }V}\operatorname{tr}\pi_v(f_v).$$

We may assume that for every v in V there is some f_v, obtained from an $\tilde{f}_v$, for which $\operatorname{tr}\pi_v(f_v)\neq 0$; otherwise the sum on the left is empty, by Lemma 1. This implies that the central character of π_v obtains the value 1 at any ζ in $\underline{\zeta}_n$ for all v in V, and hence for all v. We have:

Lemma 2. *If for some v in V there exists a $\tilde{\pi}_v$ which corresponds to π_v then $\tilde{\pi}_{vk}$ is equivalent to $\tilde{\pi}_v$ for all k.*

Proof. Let $\bar{\pi}_j$ be the inequivalent representations among the $\bar{\pi}_{vk}$ $(k \geqq 0)$ and put

$$\alpha_j = \sum \prod_{\substack{w \text{ in } V \\ w \neq v}} \operatorname{tr} \bar{\pi}_{wk}(\tilde{f}_w), \qquad \alpha = \prod_{\substack{w \text{ in } V \\ w \neq v}} \operatorname{tr} \pi_w(f_w),$$

where the sum is taken over all the $\bar{\pi}_k$'s with $\bar{\pi}_{vk} \approx \bar{\pi}_j$. Hence

$$\sum_j \alpha_j \operatorname{tr} \bar{\pi}_j(\tilde{f}_v) = \alpha \operatorname{tr} \bar{\pi}_v(\tilde{f}_v).$$

Arguments similar to those used in the proof of Lemma 1 show that for some $j = j_0$ we have $\bar{\pi}_{j_0} \approx \bar{\pi}_v$ and $\alpha_{j_0} = \alpha$, while for $j \neq j_0$ the α_j is 0 and is defined by an empty sum, as required.

Lemma 2 allows us to deduce from (1) of 4.3 that

$$\sum_{k \geqq 0} \prod_{v \text{ in } V'} \operatorname{tr} \bar{\pi}_{vk}(\tilde{f}_v) = \prod_{v \text{ in } V'} \operatorname{tr} \pi_v(f_v), \tag{1}$$

where V' is the set of v in V for which no $\bar{\pi}_v$ corresponds to π_v; the sum is taken over all $\bar{\pi} = \bigotimes_v \bar{\pi}_v$ for which $\bar{\pi}_v$ corresponds to π_v for all v in V but not in V', and such that $\operatorname{tr} \bar{\pi}_v(\tilde{f}_v) = \operatorname{tr} \pi_v(f_v)$ for all v outside V and for all matching spherical $\tilde{f}_v$ and f_v. We shall eventually show that V' is empty. At present we know only that any v in V' is non-archimedean and that π_v is either one-dimensional or square-integrable.

5.2. *Local Results*

We shall discuss here the local correspondence. This has already been completed for the principal series in 2.1 and for the archimedean places in 2.4. Hence it remains to deal with the supercuspidal representations and the subquotients of the reducible principal series of $G(F_v)$ and $\bar{G}(F_v)$ where v is a non-archimedean place.

We have to determine which representations π_v of $G(F_v)$ are obtained by the correspondence. Clearly such π_v must satisfy $\operatorname{tr} \pi_v(f_v) \neq 0$ for some f_v obtained from $\tilde{f}_v$ as in Lemma 1.2.3, and this implies that the central character of π_v is even. Here and below we say that a character is even if it obtains the value 1 at any ζ in $\underline{\zeta}_n$. Moreover we saw that any principal series π_v so obtained is of the form $\rho(\nu_1, \nu_2)$, where both ν_1 and ν_2 are even.

Let μ'_v be a character of $F_v^\times$ and set $\eta'_v = (\mu'_v \|{}^{-\frac{1}{2}}, \mu'_v \|{}^{\frac{1}{2}})$. The central character of $\sigma(\eta'_v)$ is μ'^2_v. If n is odd then μ'^2_v is even if and only if μ'_v is even. However when n is even there exist μ'_v such that μ'^2_v is even but μ'_v is not; such μ'_v will here be called odd. We say that η'_v and $\sigma(\eta'_v)$ are even (or odd) if μ'_v is even (or odd). Finally when η'_v is even we define η_v by $\eta_v(z^n) = \eta'_v(z)$ (z in $F_v^\times$).

Theorem. *Every genuine irreducible admissible representation of $\bar{G}(F_v)$ corresponds to an irreducible admissible representation of $G(F_v)$. All supercuspidal representations of $G(F_v)$ whose central character is even are obtained by the correspondence from supercuspidal representations of $\bar{G}(F_v)$. Any even special repre-*

sentation $\sigma(\eta'_v)$ is obtained from the square-integrable subquotient $\bar{\sigma}(\eta_v)$ of $\bar{\rho}(\eta_v)$, hence any even one-dimensional representation $\pi(\eta'_v)$ is obtained from the quotient $\bar{\pi}(\eta_v)$ of $\bar{\rho}(\eta_v)$ by $\bar{\sigma}(\eta_v)$. Any odd special representation is obtained from a supercuspidal representation of $\bar{G}(F_v)$.

In view of the above comments this Theorem determines the image of the local correspondence. Also we have:

Corollary. *The correspondence is one-to-one, and maps class 1 representations to class 1 representations and square-integrable to square-integrable representations. If n is odd then supercuspidal representations correspond to supercuspidal representations, while if n is even there are supercuspidal representations which correspond to (odd) special representations (which are not supercuspidal). If $n=2$ then $\bar{\pi}(\eta_v)$ is equivalent to the even Weil representation and the pull-back of an odd $\sigma(\eta'_v)$ is equivalent to the odd Weil representation. If a compactly supported (modulo the centre) genuine function on $G(F_v)$ whose orbital integrals are 0 on $\bar{A}(F_v)$ and which transforms under $\bar{Z}^n$ by a character is orthogonal to the characters of all square-integrable representations on $\bar{G}(F_v)$ then its orbital integrals are all 0.*

Proof of Corollary. The first assertion follows from the orthogonality relations for square-integrable representations on $\bar{G}(F_v)$, the second from Lemma 1.4 and the third is obvious; note that class 1 representations exist on $\bar{G}$ only for v with $|n|_v=1$, when the cocycle β splits on K_v. The second sentence is obvious, and the third follows from the results of 2.2. The last assertion, called the completeness of square-integrable representations on $\bar{G}(F_v)$, follows from the completeness of square-integrable representations on $G(F_v)$, through the correspondence.

Proof of Theorem. All representations of $G(F_v)$ to be mentioned below are assumed to have an even central character. The proof is based on a repeated application of (1) of 5.1.

Let π_v be a square-integrable representation of $G(F_v)$, and $\pi' = \otimes \pi'_w$ a cusp form with $\pi'_v \approx \pi_v$ and π'_w in the unramified principal series for all non-archimedean $w \neq v$. The existence of π' was proved in [9], Proposition 9.6 (p. 9.40) if F is totally real. In general it suffices to consider $f = \otimes f_w$ with f_v being a locally constant compactly supported (modulo the centre) function whose orbital integrals are equal to those of a matrix coefficient of the given π_v (see subsection 2.3 and below), f_w is the spherical function which vanishes outside $Z(F_w)K_w$ for any non-archimedean $w \neq v$, and f_w with sufficiently small compact support (modulo the centre) at all archimedean places. Then f has a compact support containing the identity 1 of $G(\mathbb{A})$, which can be chosen to be so small that its intersection with the discrete subgroup $G(F)$ of $G(\mathbb{A})$ is only 1 (modulo $Z(F)$). The automorphic function

$$\varphi(g) = \sum_{Z(F)\backslash G(F)} f(\gamma g),$$

which lies in the space of square-integrable cusp forms on $G(\mathbb{A})$, is therefore non-zero, since $\varphi(1) = f(1) \neq 0$. Hence there exists an irreducible cuspidal representation π' such that the projection φ_1 of φ to the space of π' is non-zero. Since projections commute with the right representation, φ_1 is right invariant under $\prod_w K_w$ (product over non-archimedean $w \neq v$) and transforms on the right by π_v.

Hence $\pi' = \bigotimes_w \pi'_w$ is with $\pi'_v \approx \pi_v$ and π'_w spherical for all non-archimedean $w \neq v$, as required.

Now for this π' we have

$$\operatorname{tr} \pi_v(f_v) = \sum n_{\tilde{\pi}_v} \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v) \tag{1}$$

by (1) of 5.1, where $\tilde{\pi}_v$ are inequivalent and irreducible, and $n_{\tilde{\pi}_v}$ are positive integers.

There occur some non-principal series $\tilde{\pi}_v$ on the right of (1) since otherwise (1) would contradict the linear independence of characters on $G(F_v)$. By Lemma 2.3.1 there exists a function (matrix coefficient of $\tilde{\pi}_v$) $\tilde{f}'_v$ with $F(\gamma, \tilde{f}'_v) = 0$ for γ in $\bar{A}^n$ and $F(\gamma, \tilde{f}'_v) = t^{-1} \Delta(\gamma) \overline{\chi}_{\tilde{\pi}_v}(\gamma)$ for a regular γ in $\bar{T}^n$ where T is a non-split torus, $t = |Z \backslash T|$ and $\overline{\chi}_{\tilde{\pi}_v}$ denotes the complex conjugate of the character $\chi_{\tilde{\pi}_v}$ of $\tilde{\pi}_v$.

We have $\operatorname{tr} \tilde{\pi}'_v(\tilde{f}') = 0$ for all $\tilde{\pi}'_v$ unless $\tilde{\pi}'_v \approx \tilde{\pi}_v$ where we have $\operatorname{tr} \tilde{\pi}_v(\tilde{f}') = 1$, or $\tilde{\pi}_v$ is a subquotient of a reducible principal series $\bar{\rho}_v$, and $\tilde{\pi}'_v$ is equivalent to the quotient of $\bar{\rho}_v$ by $\tilde{\pi}_v$, where $\operatorname{tr} \tilde{\pi}'_v(\tilde{f}'_v) = -1$. This is clear for a principal series $\tilde{\pi}'_v$ and follows in the other cases from the orthogonality relations for square-integrable representations.

The trace formula is made of absolutely convergent sums and products, hence for $\tilde{f}_v = \tilde{f}'_v$ the right hand side of (1) is equal to an integer. We may assume that this integer is non-zero since otherwise we would be able to re-arrange the $\tilde{\pi}_v$ in (1) so that they include only (reducible or not) principal series, and deduce a contradiction from the linear independence of characters on $G(F_v)$. If $\tilde{\pi}_v$ is supercuspidal then this integer is clearly positive.

On the other hand we can consider the left side of (1) at $f_v = f'_v$ (that is, at $\tilde{f}_v = \tilde{f}'_v$), and deduce from Schwarz's inequality that the square of

$$\alpha_v = \operatorname{tr} \pi_v(f'_v) = \tfrac{1}{2} \sum{}' |Z(F_v) \backslash T(F_v)|^{-1} \int_{Z(F_v) \backslash T(F_v)} \Delta(g)\, \chi_{\pi_v}(g)\, F(g, f_v)\, dg$$

is bounded from above by the product of

$$\tfrac{1}{2} \sum{}' |Z(F_v) \backslash T(F_v)|^{-1} \int_{Z(F_v) \backslash T(F_v)} |\chi_{\pi_v}(g)|^2\, \Delta(g)^2\, dg = 1$$

and

$$\tfrac{1}{2} \sum{}' |Z(F_v) \backslash T(F_v)|^{-1} \int_{\bar{Z}^n(F_v) \backslash \bar{T}^n(F_v)} |\chi_{\tilde{\pi}_v}(g)|^2\, \Delta(g)^2\, dg,$$

which is also equal to 1, since we chose the measures so as to satisfy

$$|Z(F_v) \backslash T(F_v)| = |\bar{Z}^n(F_v) \backslash \bar{T}^n(F_v)|.$$

Since we have already seen that $\operatorname{tr} \pi_v(f'_v)$ is a non-zero integer, we deduce that it is equal either to 1 or to -1, and that $\alpha_v \operatorname{tr} \pi_v(f_v) = \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v)$ for any $\tilde{f}_v$ with $F(\gamma, \tilde{f}_v) = 0$ for all γ in $\bar{A}^n$. It is now possible to rewrite (1) in the form

$$\operatorname{tr} \pi_v(f_v) = \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v) + \sum_{j \geq 1} n_{1j} \operatorname{tr} \bar{\rho}_j(\tilde{f}_v), \tag{1'}$$

where $\bar{\pi}_v$ is not a principal series, $\bar{\rho}_j$ are inequivalent (reducible or not) principal series and $n_{1j}>0$.

Consider (1′) with $\pi_v=\sigma(\eta'_v)$ and even η'_v, and assert that $\bar{\pi}_v$ is a subquotient of a principal series. Indeed, if $\bar{\pi}_v$ was supercuspidal its character would have compact support modulo the centre and the arguments of [9], pp. 9.39–40, could be applied to show that all n_{1j} are 0 and $\bar{\pi}_v$ corresponds (recall Definition 5.0.1) to $\sigma(\eta'_v)$. This is impossible since at $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ we have (as $|a|\to\infty$)

$$\Delta\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\chi_{\sigma(\eta'_v)}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$=\eta'_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)+\eta'_v\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\right)-\Delta\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\mu'_v(a)=\mu'_v(a)\,|a|^{-\frac{1}{2}}$$

if $\eta'_v=(\mu'_v\|\ \|^{-\frac{1}{2}}, \mu'_v\|\ \|^{\frac{1}{2}})$, which is not compactly supported.

The central character χ_v of $\bar{\pi}_v$ must satisfy $\mu'^2_v(z)=\chi_v(z^n)$ since μ'^2_v is the central character of $\sigma(\eta'_v)$. Hence $\bar{\pi}_v$ is of the form $\bar{\sigma}(\eta_v\otimes\varepsilon_v)$ or $\bar{\pi}(\eta_v\otimes\varepsilon_v)$ where $\eta_v(z^n)=\eta'_v(z)$ and ε_v is a character of $Z^n(F_v)$ of order 2. In particular (1′) implies that

$$\operatorname{tr}\sigma(\eta'_v\otimes\varepsilon'_v)(f_v)=\pm\operatorname{tr}\bar{\sigma}(\eta_v)(\tilde{f}_v)+\sum_{j\geqq 1} n_{2j}\operatorname{tr}\bar{\rho}_j(\tilde{f}_v),$$

with integral positive n_{2j} and inequivalent principal series $\bar{\rho}_j$, and some determination of the sign. The advantage of the last equality over (1′) is that here $\bar{\pi}_v$ is identified to be either $\bar{\sigma}(\eta_v)$ or $\bar{\pi}(\eta_v)$. Now on applying (1′) once again with either a supercuspidal or an odd special π_v we deduce that $\bar{\pi}_v$ cannot be a subquotient of $\bar{\rho}(\eta_v)$ (by the orthogonality relations for square-integrable representations), hence $\bar{\pi}_v$ is supercuspidal. In particular $\operatorname{tr}\sigma(\eta'_v)(f'_v)=1$ for odd η'_v as noted in the discussion of (1) above.

Now suppose that n is even and consider the one-dimensional $\pi(\eta')$ in $L_1(\mu'^2, G)$, where $\mu'=\bigotimes_w \mu'_w$ is a (unitary) character of $F^\times\backslash\mathbb{A}^\times$ which is even at v, odd at a non-empty even set S of places containing all $w(\neq v)$ with $|n|_w<1$, and unramified at any other w. For any w in S we have $\operatorname{tr}\rho_w(\eta'_w)(f_w)=0$ for any f_w obtained from a $\tilde{f}_w$ by Lemma 1.2.3, hence

$$\operatorname{tr}\pi(\eta'_w)(f_w)=-\operatorname{tr}\sigma(\eta'_w)(f_w)\qquad (w \text{ in } S).$$

Noting that S has a non-zero even cardinality we apply (1) of 5.1 with $V'=S\cup\{v\}$ and obtain

$$\operatorname{tr}\pi(\eta'_v)(f_v)\prod_{w \text{ in } S}\operatorname{tr}\sigma(\eta'_w)(f_w)=\sum\prod_{\substack{w \text{ in } S\\ \text{or } w=v}}\operatorname{tr}\bar{\pi}_w(\tilde{f}_w).$$

The above discussion shows that at $\tilde{f}_w=\tilde{f}'_w$ for all w in S this equality yields

$$\operatorname{tr}\pi(\eta'_v)(f_v)=\sum_{\pi_v} n'_{\pi_v}\operatorname{tr}\bar{\pi}_v(\tilde{f}_v), \tag{2}$$

where $\bar{\pi}_v$ are irreducible and inequivalent, and $n'_{\bar{\pi}_v}$ are positive integers. Note that

$$\operatorname{tr}\bar{\pi}(\eta_v)(\tilde{f}_v)+\operatorname{tr}\bar{\sigma}(\eta_v)(\tilde{f}_v)=\operatorname{tr}\bar{\rho}(\eta_v)(\tilde{f}_v)$$
$$=\operatorname{tr}\rho(\eta'_v)(f_v)=\operatorname{tr}\pi(\eta'_v)(f_v)+\operatorname{tr}\sigma(\eta'_v)(f_v).$$

Applying (1') with $\pi_v=\sigma(\eta'_v)$ and adding it (side by side) to (2) we deduce that all n_{1j} are 0 and that all $n'_{\bar{\pi}_v}$ are 0 except one of $n'_{\bar{\pi}(\eta_v)}$ or $n'_{\bar{\sigma}(\eta_v)}$, which is equal to 1. Moreover we claim that

$$\operatorname{tr}\pi(\eta'_v)(f_v)=\operatorname{tr}\bar{\pi}(\eta_v)(\tilde{f}_v)\qquad(\text{all }\tilde{f}_v),\tag{3}$$

and hence that

$$\operatorname{tr}\sigma(\eta'_v)(f_v)=\operatorname{tr}\bar{\sigma}(\eta_v)(\tilde{f}_v)\qquad(\text{all }\tilde{f}_v).\tag{4}$$

This follows at once for v with $|n|_v=1$ from the remarks (involving spherical functions) at the end of subsection 2.1. At the other places we use a result of Harish-Chandra (private communication) asserting that the product of $\Delta(g)$ and the character of a square-integrable representation ($\bar{\sigma}$) is square-integrable on $\bar{Z}^n\backslash\bar{A}^n$; thus $\bar{\sigma}(\eta_v)$ cannot correspond to $\pi(\eta'_v)$.

We have already noted that if (1') is applied with a supercuspidal or an odd special π_v, then $\bar{\pi}_v$ is supercuspidal. In particular, the character of $\bar{\pi}_v$ is bounded on $\bar{A}^n$, and its support is compact modulo $\bar{Z}^n$. Hence we can repeat the argument of [9], pp. 9.39–40, using the function f_v as chosen in [9], pp. 9.37–38 (with the obvious modifications) and deduce that in this case all n_{1j} ($j\geq 1$) are 0. It follows that each supercuspidal and odd special representation of $G(F_v)$ is obtained from a supercuspidal of $\bar{G}(F_v)$ by the correspondence.

With the exception of the first sentence we have completed the proof of the theorem in the case that n is even. Suppose then that n is odd. The number field F contains a primitive n-th root ζ of 1, hence also the primitive $2n$-th root $-\zeta$ of 1. Let $\bar{\bar{G}}$ denote the $2n$-fold covering group over F and by $\bar{G}$ the n-fold group. Since $\bar{\bar{G}}$ is the 2-fold covering group of $\bar{G}$ we can apply our arguments to this situation and deduce in a parallel fashion that any special $\bar{\sigma}$ of $\bar{G}(F_v)$ (namely any square-integrable subquotient of a reducible principal series) is obtained by the correspondence from $\bar{\bar{G}}$ to $\bar{G}$; thus we have

$$\Delta(\tilde{\gamma}')\,\chi_{\bar{\bar{\pi}}}(\tilde{\gamma}'\zeta_\gamma'^{-1})=\Delta(\tilde{\gamma})\,\chi_{\bar{\sigma}}(\tilde{\gamma}\zeta_\gamma^{-1})\qquad(\tilde{\gamma}'=(\gamma,1)^{2n},\ \tilde{\gamma}=(\gamma,1)^n)$$

for any elliptic γ in $G(F_v)$ for which γ^{2n} is regular and any γ in A sufficiently far from the centre. But $\bar{\bar{G}}$ is the $2n$-fold covering of G, hence $\bar{\bar{\pi}}$ corresponds to some π on $G(F_v)$ and

$$\Delta(\gamma)\,\chi_\pi(\gamma)=\Delta(\tilde{\gamma})\,\chi_{\bar{\bar{\pi}}}(\tilde{\gamma}'\zeta_\gamma'^{-1})$$

for the above γ's. We deduce that

$$\Delta(\tilde{\gamma})\,\chi_{\bar{\sigma}}(\tilde{\gamma}\zeta_\gamma^{-1})=\Delta(\gamma)\,\chi_\pi(\gamma),$$

and hence that (1′) holds with $\pi_v \approx \pi$, $\tilde{\pi}_v \approx \tilde{\sigma}$ and $n_{1j}=0$ ($j \geqq 1$). Clearly π is special since it must be square-integrable but not supercuspidal, in fact it is of the form $\sigma(\eta'_v)$ if $\tilde{\sigma}=\tilde{\sigma}(\eta_v)$, where $\eta'_v(z)=\eta_v(z^n)$. Thus (3) and (4) are valid, and this $\pi=\sigma(\eta'_v)$ is even since its central character obtains the value 1 at any ζ in $\underline{\zeta}_n$ (but not in $\underline{\zeta}_{2n}$).

Since n is odd there are no odd special representations of $G(F_v)$ and it remains to deal with supercuspidals on $G(F_v)$. We apply (1′) again with such π_v, deduce as usual that $\tilde{\pi}_v$ is also supercuspidal, and apply [9], pp. 9.37–40, to deduce that all n_{1j} are 0.

Finally to establish that every $\tilde{\pi}_v$ corresponds to some π_v we may restrict our attention to a supercuspidal $\tilde{\pi}_v$ and consider the function $\tilde{f}'_v$ defined after (1). From Lemma 1.2.3 we obtain a matching f'_v and by completeness of square-integrable characters on $G(F_v)$ we deduce that $\operatorname{tr}\pi_v(f'_v) \neq 0$ for some square-integrable π_v. Hence π_v is obtained from some square-integrable $\tilde{\pi}'_v$ by the correspondence, and $\operatorname{tr}\tilde{\pi}'_v(\tilde{f}'_v) \neq 0$. The orthogonality relations on $\tilde{G}(F_v)$ imply that $\tilde{\pi}_v \approx \tilde{\pi}'_v$ and hence that $\tilde{\pi}_v$ corresponds to π_v.

The proof of the Theorem is now complete. Note that when n is of the form p^m with a prime p and integral $m>0$ there is no need to consider the n-fold and $2n$-fold covering groups, as we did. Indeed, in this case it is easy to show that without loss of generality we may assume that there is only one v with $|n|_v<1$ and hence (1) of 5.1 can be applied with V' which consists of a single element. We deduce that (2) holds, and by virtue of (1′) the proof is easily completed.

5.3. *Global Results*

The global correspondence was defined in 5.0 in terms of the local correspondence, and this in turn was fully described in 5.2. The description of the global correspondence readily follows.

Theorem. *Every irreducible admissible genuine automorphic representation $\tilde{\pi}$ of $\tilde{G}(\mathbb{A})$ corresponds to an irreducible admissible automorphic representation π of $G(\mathbb{A})$. The correspondence is one-to-one and its image consists of all $\pi=\bigoplus_v \pi_v$ such that π_v has even central character for all v and such that if $\pi_v=\pi(v'_1, v'_2)$ then both v'_1 and v'_2 are even.*

It is clear that: (i) Cuspidal representations π will be obtained from cuspidal representations $\tilde{\pi}$.

(ii) The continuous series representation

$$\pi=\Big(\bigotimes_{v \text{ in } V} \sigma(\mu'_v \|^{-\frac{1}{2}}, \mu'_v \|^{\frac{1}{2}})\Big) \otimes \Big(\bigotimes_{v \text{ outside } V} \pi(\mu'_v \|^{-\frac{1}{2}}, \mu'_v \|^{\frac{1}{2}})\Big),$$

(where V is empty if n is odd and it is a finite set with even cardinality if n is even, and μ'_v is even for v outside V and odd for v in V), will be obtained from

$$\tilde{\pi}=\Big(\bigotimes_{v \text{ in } V} r_{\mu'_v}\Big) \otimes \Big(\bigotimes_{v \text{ outside } V} \pi(\mu_v \|^{\frac{-1}{2n}}, \mu_v \|^{\frac{1}{2n}})\Big),$$

where for any v outside V we have $\mu_v(z^n)=\mu'_v(z)$ (z in $F_v^\times$). $\bar{\pi}$ is cuspidal if V is non-empty and it lies in $L_1(\mu^2, \bar{G})$, where $\mu=\bigotimes_v \mu_v$, if V is empty.

Here $r_{\mu'_v}$ denotes the representation which corresponds to an odd $\sigma(\mu'_v\|^{-\frac{1}{2}}, \mu'_v\|^{\frac{1}{2}})$. If $n=2$ then $r_{\mu'_v}$ is an odd Weil representation, $\bar{\pi}(\mu_v\|^{\frac{-1}{2n}}, \mu_v\|^{\frac{1}{2n}})$ is an even Weil representation, and $\bar{\pi}$ is a global Weil representation.

(iii) The continuous series representation $\pi(v'_1, v'_2)=\bigotimes_v \pi(v'_{1v}, v'_{2v})$ (where $v'_i=\bigotimes_v v'_{iv}$ and v'_{iv} are even for all $v (i=1,2)$) is obtained from the continuous series $\bar{\pi}(v_1, v_2)=\otimes\bar{\pi}(v_{1v}, v_{2v})$, where $v_i=\bigotimes_v v_{iv}$ and $v_{iv}(z^n)=v'_{iv}(z)$ (z in $F_v^\times$; $i=1,2$).

This list exhausts all representations in the image of the global correspondence and hence all representations of $\bar{G}(\mathbb{A})$. In particular any cuspidal representation of $\bar{G}(\mathbb{A})$ corresponds to a cuspidal representation of $G(\mathbb{A})$ if n is odd, while if n is even there are cuspidal representations of $\bar{G}(\mathbb{A})$ which correspond to continuous series of $G(\mathbb{A})$, namely the $\bar{\pi}$ of (ii) with non-empty V.

The theorem has the following

Corollary. (a) *Both multiplicity one and strong multiplicity one theorems hold in the space* $L_0(\mu, \bar{G})\otimes L_1(\mu, \bar{G})$.

(b) *An automorphic representation* $\bar{\pi}=\bigotimes_v \bar{\pi}_v$ *of* $\bar{G}(\mathbb{A})$ *must be of the form described in* (ii) *above if there is some* v *for which* $\bar{\pi}_v\approx\bar{\pi}(\mu_v\|^{\frac{-1}{2n}}, \mu_v\|^{\frac{1}{2n}})$. *Thus, when* $n=2$, $\bar{\pi}=\bigotimes_v \bar{\pi}_v$ *must be a Weil representation if for any* v $\bar{\pi}_v$ *is an even Weil representation.*

Clearly (a) follows from the fact that the correspondence is one-to-one and both theorems hold for $L_0(\mu', G)\oplus L_1(\mu', G)$; (multiplicity one theorem asserts that each constituent of $L_0\oplus L_1$ occurs only once, and strong multiplicity one asserts that if $\bar{\pi}'=\otimes\bar{\pi}'_v$ and $\bar{\pi}=\otimes\bar{\pi}_v$ lie in $L_0\oplus L_1$ and $\bar{\pi}'_v\approx\bar{\pi}_v$ for almost all v then this holds for all v and $\bar{\pi}'\approx\bar{\pi}$; although it is suggested by their names, neither theorem implies the other; in the special case of 2-fold $\bar{G}$ the strong multiplicity one theorem has also been proved in [3], using the (ε, L)-techniques). Assertion (b) is implicit in (i), (ii), (iii) above since any automorphic representation $\pi=\otimes\pi_v$ of $G(\mathbb{A})$ which has a one-dimensional component must be of the form described in (ii).

Proof of Theorem. Suppose $\bar{\pi}$ occurs in $L_0(\mu, \bar{G})\oplus L_1(\mu, \bar{G})$ and it is irreducible. For some set V and elements r_v (v outside V) we obtain (1) of 4.3 such that $\bar{\pi}$ occurs on the left. We have noted in 4.3 that the right side of (1) contains at most one π, and it is not empty by virtue of Lemma 5.1.1. Each component π_v of $\pi=\bigotimes_v \pi_v$ is clearly obtained by the local correspondence; suppose $\bar{\pi}'_v$ corresponds to π_v. Hence (1) of 4.3 can be written in the form

$$\prod_{v \text{ in } V} \operatorname{tr}\bar{\pi}_v(\tilde{f}_v)+\sum_{j\geqq 1}\prod_{v \text{ in } V}\operatorname{tr}\bar{\pi}_{jv}(\tilde{f}_v)=\prod_{v \text{ in } V}\operatorname{tr}\bar{\pi}'_v(\tilde{f}_v).$$

By linear independence of characters ([7], Lemma 16.1.1) we deduce that the above sum is empty, that $\bar{\pi}'_v\approx\bar{\pi}_v$ for all v in V, and hence that $\bar{\pi}$ corresponds to π.

The first claim is proved for $\bar{\pi}$ in $L_0 \oplus L_1$, as well as the assertion that the correspondence is one-to-one.

Let $\pi = \bigotimes_v \pi_v$ be a representation of $G(\mathbb{A})$ as described in the theorem, and suppose that it is either cuspidal or of the kind described in (ii). We rewrite (1) of 4.3 in the form

$$\sum_{j \geqq 1} \prod_{v \text{ in } V} \operatorname{tr} \bar{\pi}_{jv}(\tilde{f}_v) = \prod_{v \text{ in } V} \operatorname{tr} \bar{\pi}_v(\tilde{f}_v) \tag{*}$$

where $\bar{\pi}_v$ corresponds to π_v (v in V); (we note that

$$\operatorname{tr} \pi(\eta_v')(f_v) = -\operatorname{tr} \sigma(\eta_v')(f_v)$$

for any f_v obtained from $\tilde{f}_v$, where $\eta_v' = (\mu_v' \|^{-\frac{1}{2}}, \mu_v' \|^{\frac{1}{2}})$ is odd, and the set V of (ii) has even cardinality; then $\bar{\pi}_v$ corresponds to $\sigma(\eta_v')$ and not to $\pi(\eta_v')$ whenever $\pi(\eta_v')$ occurs). The usual arguments imply that the left of (*) reduces to exactly one term, say $\bar{\pi} = \bigotimes_v \bar{\pi}_v$, and $\bar{\pi}$ corresponds to π.

It remains to deal with continuous series representations π of $G(\mathbb{A})$ and $\bar{\pi}$ of $\bar{G}(\mathbb{A})$. Indeed π is automorphic but not cuspidal if and only if it is a constituent of $\rho(\eta')$ for some $\eta' = (v_1', v_2')$, where v_i' are quasi-characters of $F^\times \backslash \mathbb{A}^\times$ $(i = 1, 2)$; see [9], Lemma 10.1. The $\rho(\eta')$ are reducible only if $v_1'(z)/v_2'(z) = |z|^s$ with $s = 1$ or -1, and their subquotients are all of the form (ii). Similarly it can be shown that $\bar{\pi}$ is automorphic but not cuspidal if and only if it is a constituent of $\bar{\rho}(\eta)$ for some $\eta = (v_1, v_2)$, where v_i are quasi-characters of $F^\times \backslash \mathbb{A}^{\times n} F^\times$ $(i = 1, 2)$. The $\bar{\rho}(\eta)$ are reducible only if $v_1(z^n)/v_2(z^n) = |z|^s$ with $s = 1$ or -1, and their subquotients are all of the form (ii). Clearly $\bar{\rho}(\eta)$ corresponds to $\rho(\eta')$, where $\eta'(z) = \eta(z^n)$ (z in $\mathbb{A}^\times$), and all $\rho(\eta')$ (such that $v_i = \bigotimes_v v_{iv}$ and v_{iv} are even for all $v (i = 1, 2)$) are obtained, as required.

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