CHARACTERS, GENERICITY, AND MULTIPLICITY ONE FOR U(3)

By

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Abstract. Let $\psi : U \to \mathbb{C}^\times$ be a generic character of the unipotent radical U of a Borel subgroup of a quasisplit p-adic group G. The number (0 or 1) of ψ-Whittaker models on an admissible irreducible representation π of G was expressed by Rodier in terms of the limit of values of the trace of π at certain measures concentrated near the origin. An analogous statement holds in the twisted case. This twisted analogue is used in [F, p. 47] to provide a local proof of the multiplicity one theorem for U(3). This asserts that each discrete spectrum automorphic representation of the quasisplit unitary group U(3) associated with a quadratic extension E/F of number fields occurs in the discrete spectrum with multiplicity one. It is pointed out in [F, p. 47] that a proof of the twisted analogue of Rodier's theorem does not appear in print. It is then given below. Detailing this proof is necessitated in particular by the fact that the attempt in [F, p. 48] at a global proof of the multiplicity one theorem for U(3), although widely quoted, is incomplete, as we point out here.

Introduction

Let E/F be a quadratic extension of p-adic fields, $p \neq 2$, and consider the basechange lifting from the quasisplit unitary group $H = \mathrm{U}(3, E/F)$ in 3 variables to $G = \mathrm{GL}(3, E)$. Our main result is that there exists a family of suitably related functions $\psi'_{H,n}$ on $H = \mathrm{U}(3, E/F)$ and ψ'_n on $G = \mathrm{GL}(3, E)$ which are supported near the origin, such that the traces $\mathrm{tr}\, \pi_H(\psi'_{H,n} dh)$ and twisted traces $\mathrm{tr}\, \pi(\psi'_n dg \times \sigma)$ stabilize for sufficiently large n and become equal to the Whittaker multiplicity (multiplicity in the space of Whittaker vectors) of the irreducible admissible representations π_H of H and π of G. This is a twisted analogue of a theorem of Rodier for the involution that defines H in G.

Our motivation for considering such a twisted analogue of Rodier's theorem is that we found a gap in an attempted global proof ([F, p. 48]) of the multiplicity one

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theorem for automorphic representations of $\mathrm{U}(3, E/F)(\mathbb{A}_F)$. We use our result to provide a local proof, as suggested in [F, p. 47].

This gap is different from the observation of Harder [H, p. 173]. The latter deals with the question of which representation in a packet is automorphic, assuming multiplicity one holds.

In Section 1, we state a theorem of Rodier and define the Whittaker models, characters, and measures concentrated near the origin which enter the statement. We then state its twisted analogue of interest to us. In Section 2, we use these theorems to complete a key step in the proof of [F, p. 47] of the multiplicity one theorem for the global quasisplit unitary group U(3). In Section 3, we explain why the global proof [F, p. 48] of this multiplicity one theorem is not complete. In Section 4, we recall the main lines of Rodier's proof. In Section 5, the twisted analogue is reduced to Rodier's theorem. This completes the proof of the theorems of Sections 1 and 2. In the Appendix, Section 6, we derive another description of the Whittaker multiplicity, in terms of the coefficient of the regular orbit in the germ expansion of the character.

1 Whittaker models and characters

Rodier's theorem [R, p. 161] (for a split group H) computes the number of $\boldsymbol{\psi}_H$-Whittaker models of the admissible irreducible representation π_H of H in terms of values of the character $\operatorname{tr}\pi_H$ or χ_{π_H} of π_H at the measures $\psi_{H,n}dh$ which are supported near the origin.

We proceed to explain the notations to be used in Rodier's theorem. For simplicity and clarity, instead of working with a general connected reductive (quasi) split p-adic group H as in [R], we let H be a specific unitary group. To define it, we take $G = \mathrm{GL}(r, E)$, where E/F is a quadratic extension of p-adic fields of characteristic zero, $p \neq 2$. Let $x \mapsto \overline{x}$ denote the generator of $\mathrm{Gal}(E/F)$. For $g = (g_{ij})$ in G we put $\overline{g} = (\overline{g}_{ij})$ and ${}^t g = (g_{ji})$. Then $\sigma(g) = J^{-1}{}^t\overline{g}^{-1}J$, $J = ((-1)^{i-1}\delta_{i,r+1-j})$, defines an involution σ on G. The group $H = G^\sigma$ of $g \in G$ fixed by σ is a quasisplit unitary group $\mathrm{U}(r, E/F)$.

Denote by $\boldsymbol{\psi}_H : U_H \to \mathbb{C}^\times$ a character on the unipotent upper triangular subgroup U_H of H. It is necessarily unitary, i.e., its values lie in the unit circle $\mathbb{C}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

We assume that ψ_H is generic, i.e., nontrivial on each simple root subgroup. There is only one orbit of generic $\boldsymbol{\psi}_H$ under the action of the diagonal subgroup of H on U_H by conjugation. Hence we can and do work with the specific character $\boldsymbol{\psi}_H : U_H \to \mathbb{C}^1$, defined by $\boldsymbol{\psi}_H((u_{ij})) = \psi(\sum_{1\leq j<r} u_{j,j+1})$. Here $\psi : F \to \mathbb{C}^1$ is an

additive character which is 1 on R and not identically 1 on $\boldsymbol{\pi}^{-1}R$. Further, R is the ring of integers of F, and $\boldsymbol{\pi}$ is a generator of the maximal ideal of R. Note that $u_{r-j,r-j+1} = \overline{u}_{j,j+1}$.

By $\boldsymbol{\psi}_H$-*Whittaker vectors* we mean vectors in the space of the induced representation $\operatorname{ind}_{U_H}^H(\boldsymbol{\psi}_H)$. They are the functions $\varphi : H \to \mathbb{C}$ with $\varphi(uhk) = \boldsymbol{\psi}_H(u)\varphi(h)$, $u \in U_H$, $h \in H$, $k \in K_\varphi$, where K_φ is a compact open subgroup of H depending on φ, which are compactly supported on $U_H\backslash H$. The group H acts by right translation.

The multiplicity $\dim_{\mathbb{C}} \operatorname{Hom}_H(\operatorname{ind}_{U_H}^H \boldsymbol{\psi}_H, \pi_H)$ of any irreducible admissible representation π_H of H in the space of $\boldsymbol{\psi}_H$-Whittaker vectors is known to be 0 or 1. In the latter case, we say that π_H has a $\boldsymbol{\psi}_H$-Whittaker model or that it is $\boldsymbol{\psi}_H$-*generic*.

Let $\mathcal{G}_0$ be the ring of $r \times r$ matrices with entries in the ring of integers R_E of E. It is a subring of the ring $\mathcal{G}$ of $r \times r$ matrices with entries in E. Let $d\sigma$ be the involution $d\sigma(X) = -J^{-1}{}^t\overline{X}J$. Its set of fixed points in $\mathcal{G}$ is denoted by $\mathcal{H}$. Let $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0$ be the set of X in $\mathcal{G}_0$ fixed by the involution $d\sigma$. Write $H_n = \exp(\mathcal{H}_n)$, $\mathcal{H}_n = \boldsymbol{\pi}^n\mathcal{H}_0$. For $n \geq 1$, we have $H_n = {}^tU_{H,n}A_{H,n}U_{H,n}$, where $U_{H,n} = U_H \cap H_n$ and $A_{H,n}$ is the group of diagonal matrices in H_n. Define a character $\psi_{H,n} : H \to \mathbb{C}^1$ supported on H_n by $\psi_{H,n}({}^tbu) = \psi(\sum_{1\leq j<r} u_{j,j+1}\boldsymbol{\pi}^{-2n})$ at ${}^tb \in {}^tU_{H,n}A_{H,n}$, $u = (u_{ij}) \in U_{H,n}$. Alternatively, by

$$\psi_{H,n}(\exp X) = \operatorname{ch}_{\mathcal{H}_n}(X)\psi(\operatorname{tr}[X\boldsymbol{\pi}^{-2n}\beta_H]),$$

where $\operatorname{ch}_{\mathcal{H}_n}$ indicates the characteristic function of $\mathcal{H}_n = \boldsymbol{\pi}^n\mathcal{H}_0$ in $\mathcal{H}$ and β_H is the $r \times r$ matrix whose nonzero entries are 1 at the places $(j, j-1)$, $1 < j \leq r$. Denote by e_{H_n} the constant measure of volume one supported on the compact subgroup H_n in the Hecke algebra of H, i.e., $e_{H_n} = |H_n|^{-1} \operatorname{ch}_{H_n} dh$, where $|H_n|$ denotes the volume of H_n in dh.

Since π_H is admissible, for each test measure $f_H\,dh$ (dh is a Haar measure on H and f_H is a locally constant compactly supported complex-valued function on H), the image of $\pi_H(f_H\,dh)$ is finite dimensional and its trace $\operatorname{tr}\pi_H(f_H\,dh)$ is finite.

Rodier's theorem is

Theorem 1. *The multiplicity* $\dim_{\mathbb{C}} \operatorname{Hom}_H(\operatorname{ind}_{U_H}^H \boldsymbol{\psi}_H, \pi_H)$ *is equal to*

$$\lim_n \operatorname{tr}\pi_H(\psi_{H,n}e_{H_n}).$$

In fact, the limit stabilizes for sufficiently large n. Throughout this paper, "$= \lim_n a_n$" means "equals a_n for all sufficiently large n".

We need a twisted analogue of Rodier's theorem. It can be described as follows.

Let π be an admissible irreducible representation of G which is σ-invariant: $\pi \simeq {}^\sigma\pi$, where ${}^\sigma\pi(\sigma(g)) = \pi(g)$. Then there exists an intertwining operator

$A : \pi \to {}^{\sigma}\pi$ with $A\pi(g) = \pi(\sigma(g))A$ for all $g \in G$. Since π is irreducible, by Schur's lemma, A^2 is a scalar, which we may normalize by $A^2 = 1$. Thus A is unique up to a sign. Denote by G' the semidirect product $G \rtimes \langle\sigma\rangle$. Then π extends to G' by $\pi(\sigma) = A$.

Define $\psi_E : E \to \mathbb{C}^1$ by $\psi_E(x) = \psi(x + \overline{x})$. Define a character $\boldsymbol{\psi} : U \to \mathbb{C}^1$ on the unipotent upper triangular subgroup U of G by $\boldsymbol{\psi}((u_{ij})) = \psi_E(\sum_{1\leq j<r} u_{j,j+1})$. This one dimensional representation has the property that $\boldsymbol{\psi}(\sigma(u)) = \boldsymbol{\psi}(u)$ for all u in U. Note that $\boldsymbol{\psi}(u) = \boldsymbol{\psi}_H(u^2)$ at $u \in U_H = U \cap H$. There is only one orbit of generic σ-invariant characters on U under the adjoint action of the group of σ-invariant diagonal elements in G.

Suppose that π is $\boldsymbol{\psi}$-generic, namely $\mathrm{Hom}_{\mathbb{C}}(\mathrm{ind}_U^G \boldsymbol{\psi}, \pi) \neq \{0\}$. Here $\mathrm{ind}_U^G \boldsymbol{\psi}$ consists of the functions $\varphi : G \to \mathbb{C}$ with $\varphi(ug) = \boldsymbol{\psi}(u)\varphi(g)$, $u \in U$, $g \in G$, which are compactly supported on $U\backslash G$. Then we normalize A by $A\varphi' = {}^{\sigma}\varphi'$, where ${}^{\sigma}\varphi(g) = \varphi(\sigma(g))$, on the image φ' in π of the φ.

Write $G_n = \exp(\mathcal{G}_n)$, where $\mathcal{G}_n = \boldsymbol{\pi}^n\mathcal{G}_0$. For $n \geq 1$, we have $G_n = {}^tU_nA_nU_n$, where $U_n = U \cap G_n$ and A_n is the group of diagonal matrices in G_n. Define a character $\psi_n : G \to \mathbb{C}^1$ supported on G_n by $\psi_n({}^tbu) = \psi_E(\sum_{1\leq j<r} u_{j,j+1}\boldsymbol{\pi}^{-2n})$, where ${}^tb \in {}^tU_nA_n$, $u = (u_{ij}) \in U_n$. Alternatively, $\psi_n : G \to \mathbb{C}^1$ is defined by

$$\psi_n(\exp X) = \mathrm{ch}_{\mathcal{G}_n}(X)\psi_E(\mathrm{tr}[X\boldsymbol{\pi}^{-2n}\beta]),$$

where β is the $r \times r$ matrix with entries 1 at the places $(j, j-1)$, $1 < j \leq r$ and 0 elsewhere.

A first version of a σ-twisted analogue of Rodier's theorem asserts that the Whittaker multiplicity is equal to the twisted trace $\mathrm{tr}\,\pi(\psi_n e_{G_n} \times \sigma)$ for all sufficiently large n, where e_{G_n} is defined analogously to e_{H_n}.

A more useful version for us is stated in terms of the twisted character χ_π^σ. Let us first restate Rodier's theorem this way.

Denote by χ_{π_H} the *character* of π_H. It is a complex-valued conjugacy invariant function on H which is locally constant on the regular set and locally integrable on H (Harish-Chandra [HC], Theorem 1) defined by $\mathrm{tr}\,\pi_H(f_H\,dh) = \int_H \chi_{\pi_H}(h)f_H(h)dh$ for all $f_H\,dh$.

Rodier's theorem can be stated as asserting that the Whittaker multiplicity of Theorem 1 is equal to

$$\lim_n \int_H \chi_{\pi_H}(h)\psi_{H,n}(h)e_{H_n}(h).$$

Analogously, the twisted character χ_π^σ of π is a complex valued σ-conjugacy invariant function on G (that is, its value on $\{hg\sigma(h)^{-1}\}$ is independent of $h \in$

G) which is locally constant on the σ-regular set (g with regular $g\sigma(g)$), locally integrable (Clozel [C], Thm. 1, p. 153) and defined by $\operatorname{tr}\pi(f\,dg)A = \int_G \chi_\pi^\sigma(g) f(g) dg$ for all test measures $f\,dg$.

The σ-twisted analogue of Rodier's theorem of interest to us is as follows. Let $e_{G_n^\sigma}$ denote the constant measure of volume 1 supported on the compact subgroup $G_n^\sigma = \{g = \sigma g; g \in G_n\}$ of G. As G_n^σ is H_n, $e_{G_n^\sigma}$ lies in the Hecke algebra of H.

Theorem 2. *The multiplicity* $\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\operatorname{ind}_U^G \psi, \pi) = \dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{ind}_U^G \psi, \pi)$ *is equal to* $\int_{G_n^\sigma} \chi_\pi^\sigma(g)\psi_n(g) e_{G_n^\sigma}(g)$ *for all sufficiently large* n.

Remark. Recall that $\operatorname{Hom}_G(\pi_1, \pi_2^\vee) = \operatorname{Hom}_G(\pi_2, \pi_1^\vee)$, as both spaces can be identified with the space of (π_1, π_2)-invariant bilinear forms. The contragredient $(\operatorname{ind}_H^G \rho)^\vee$ is $\operatorname{Ind}_H^G(\frac{\Delta_G}{\Delta_H}\rho^\vee)$ [BZ1, 2.25(c)]; Ind indicates noncompact induction, H is a closed subgroup of G, ρ is a representation of H, π of G). Frobenius reciprocity asserts

$$\operatorname{Hom}_G(\operatorname{ind}_H^G \rho, \pi) = \operatorname{Hom}_H\left(\frac{\Delta_H}{\Delta_G}\rho, \pi|H\right)$$

and, equivalently, $\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \rho) = \operatorname{Hom}_H(\pi|H, \rho)$ ([BZ1], (2.28) and (2.29)).

2 Multiplicity one for U(3)

Let us recall how the Theorems are used in the proof of Proposition 3.5 of [F, p. 47]. Thus, in this section, we work only with $H = \mathrm{U}(r, E/F)$ and $G = \mathrm{GL}(r, E)$, $r = 2$ or 3, of Section 1. We are given a square integrable irreducible admissible representation ρ of the quasisplit group $\mathrm{U}(2, E/F)$. Its stable basechange to $\mathrm{GL}(2, E)$ is denoted by τ. The unstable basechange is $\tau \otimes \kappa$. Let $\pi = I(\tau \otimes \kappa)$ be the normalizedly induced representation of $\mathrm{GL}(3, E)$. This π is invariant under the involution σ (ours and of [F]). It is generic. For all matching measures fdg and $f_H dh$ on $G = \mathrm{GL}(3, E)$ and $H = \mathrm{U}(3, E/F)$, using an identity of trace formulae and orthogonality relations for characters [F], we obtain an identity

$$\operatorname{tr}\pi(fdg \times \sigma) = (2m+1)\sum_{\pi_H} \operatorname{tr}\pi_H(f_H dh).$$

The sum ranges over finitely many (in fact, two times the cardinality of the packet of ρ) inequivalent square integrable irreducible admissible representations π_H of $\mathrm{U}(3, E/F)$. The number m is a nonnegative integer, independent of π_H.

Proposition 3.5 of [F]. *The nonnegative integer* m *is zero, and there is a unique generic* π_H *in the sum. The other* $2[\{\rho\}] - 1$ *representations* π_H *are not generic.*

Note that our π, π_H, fdg, $f_H dh$, G, H, are denoted in [F] by Π, π, $\phi dg'$, fdg, G', G.

Proof. The identity for all matching test measures fdg and $f_H dh$ implies an identity of characters:

$$\chi_\pi^\sigma(\delta) = (2m+1)\sum_{\pi_H} \chi_{\pi_H}(\gamma)$$

for all $\delta \in G = \mathrm{GL}(3, E)$ with regular norm $\gamma \in H = \mathrm{U}(3, E/F)$. Note that $\delta \mapsto \chi_\pi^\sigma(\delta)$ is a stable σ-conjugacy class function on G, while $\gamma \mapsto \sum_{\pi_H} \chi_{\pi_H}(\gamma)$ is a stable conjugacy class function on H. We use Theorem 2 with $G = \mathrm{GL}(3, E)$ and $H = G^\sigma$. Then $G_n^\sigma = H_n$. On $\delta \in G_n^\sigma$, the norm $N\delta$ of the stable σ-conjugacy class δ is just the stable conjugacy class of δ^2. Hence $\chi_\pi^\sigma(\delta) = (2m+1)\sum_{\pi_H} \chi_{\pi_H}(\delta^2)$ at $\delta \in G_n^\sigma = H_n$.

We claim that for $\delta = \exp X$, $X \in \mathcal{G}_n^\sigma = \mathcal{H}_n$, we have

$$\psi_E(\mathrm{tr}[X\boldsymbol{\pi}^{-2n}\beta]) = \psi(\mathrm{tr}[2X\boldsymbol{\pi}^{-2n}\beta_H]).$$

For this, we note that $\beta = \beta_H$ and $\psi_E(x) = \psi(x + \overline{x})$.

Moreover, we claim that $\boldsymbol{\psi}_n(\delta) = \boldsymbol{\psi}_{H,n}(\delta^2)$ for $\delta \in G_n^\sigma = H_n$. For this, note that $\boldsymbol{\psi}_n({}^t bu) = \psi_E((x+y)\boldsymbol{\pi}^{-2n})$ if

$$u = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix};$$

and this is $= \psi(2(x+\overline{x})\boldsymbol{\pi}^{-2n})$ if $y = \overline{x}$. But $\psi_{H,n}({}^t bu) = \psi((x+\overline{x})\boldsymbol{\pi}^{-2n})$ at such $u \in U_H$ (thus with $y = \overline{x}$).

Now $d(g^2) = dg$ when $p \neq 2$. It follows that

$$\begin{aligned} 1 = \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_U^G \boldsymbol{\psi}, \pi) &= \int_{G_n^\sigma} \chi_\pi^\sigma(\delta)\psi_n(\delta) e_{G_n^\sigma}(\delta) \\ &= \int_{H_n} (2m+1) \sum_{\pi_H} \chi_{\pi_H}(\delta^2)\psi_{H,n}(\delta^2) e_{H_n}(\delta^2) \\ &= (2m+1)\sum_{\pi_H} \dim_{\mathbb{C}} \mathrm{Hom}_H(\mathrm{ind}_{U_H}^H \boldsymbol{\psi}_H, \pi_H). \end{aligned}$$

Hence $m = 0$, and there is just one generic π_H in the sum ($\dim_{\mathbb{C}} \neq 0$, necessarily $= 1$). $\square$

The excluded case of $p = 2$ might follow on counting the factors of 2 in our argument.

We repeat the conclusion of [F]. Note that the unrestricted trace identity is proved in [F1] and the fundamental lemma in [F2]. Both proofs employ simple methods: the usage of regular-Iwahori functions in [F1] removes the need to compute and compare weighted orbital integrals, and explicit double coset decomposition reduces the fundamental lemma in [F2] to elementary volume computations.

Corollary. *Each discrete spectrum automorphic representation of* $\mathrm{U}(3, E/F)(\mathbb{A})$ *occurs in the discrete spectrum with multiplicity one. Each packet (defined in* [F]*) of such an infinite dimensional representation contains precisely one irreducible representation which is generic. Each packet (defined in* [F]*) of tempered admissible representations of* $\mathrm{U}(3, E/F)$ *contains precisely one irreducible representation which is generic. There are no generic representations in any (nontempered) quasipacket, locally and globally.*

This type of argument, relating the number of Whittaker models to the values of the characters at suitable measures supported near the origin, was first employed in [FK], which appeared in 1987, in an analogous situation of the metaplectic correspondence between $\mathrm{GL}(r)$ and its n-fold covering group.

Remark. On the space $L^2(\mathrm{GL}(r, F)\backslash \mathrm{GL}(r, \mathbb{A}))$ of automorphic forms ϕ, we define the involution σ by $(r(\sigma)\phi)(g) = \phi(\sigma(g))$. On the space of Whittaker functions W ($W(ng) = \psi(n)W(g)$, $g \in \mathrm{GL}(r, \mathbb{A})$, $n \in N(\mathbb{A})$, where N denotes the unipotent upper triangular subgroup of $\mathrm{GL}(r)$), we choose the natural action of σ, by ${}^{\sigma}W(g) = W(\sigma g)$. The map $\phi \mapsto W_\phi$, where $W_\phi(g) = \int_{N(\mathbb{A})\backslash N(\mathbb{A})} \phi(ng)\overline{\psi}(n)dn$, respects the action of σ. Thus the global normalization of the action of σ is the product of the local normalizations ${}^{\sigma}W_v(g) = W_v(\sigma g)$.

Note also that underlying the character identity is an embedding of the local representation π, which is σ-elliptic, as a component of a global σ-invariant cuspidal representation Π of $\mathrm{GL}(3, \mathbb{A}_E)$ where E is a totally imaginary field. The character identity follows from using the trace formulae identity, applying "generalized linear independence of characters" and thus isolating Π and, further, π. The normalization of σ on the generic (Π and) π implies that no sign occurs in the character identity.

Had we not fixed the choice of the sign of $r(\sigma)$ at all places, the twisted character in our trace identity (at our place) might in principle be replaced by its negative. However, the identity (in two avatars)

$$-\operatorname{tr}\pi(fdg \times \sigma) = (2m+1)\sum_{\pi_H} \operatorname{tr}\pi_H(f_H dh), \qquad -\chi_\pi^\sigma(\delta) = (2m+1)\sum_{\pi_H} \chi_{\pi_H}(\gamma),$$

cannot hold, as evaluating it with our $fdg = \psi_n e_{G_n^\sigma}$ and $f_H dh = \psi_{H,n} e_{H_n}$, n large, would give -1 on the left, and a nonnegative integer on the right. This provides an independent verification that our normalization of the sign of $r(\sigma)$ is correct.

3 Incomplete global proof

The second proof of Proposition 3.5 of [F], on p. 48, is global, but incomplete. The false assertion is on lines 21–22: "Proposition 8.5(iii) (p. 172) and 2.4(i) of [GP] imply that for some π with $m(\pi) \neq 0$ above, we have $m(\pi) = 1$". Indeed, [GP], Prop. 2.4, defines $L^2_{0,1}$ to be the orthocomplement in the space L^2_0 (of cusp forms) of "all hypercusp forms", and claims: "(i) $L^2_{0,1}$ has multiplicity 1". ([GP], 8.5 (iii), asserts that π is in $L^2_{0,1}$.) Now the sentence of [F], p. 48, l. 21–22 assumes that [GP], 2.4(i), means that any irreducible π in $L^2_{0,1}$ occurs in L^2_0 with multiplicity one. But the standard techniques of [GP], 2.4, show only that any irreducible π in $L^2_{0,1}$ occurs in $L^2_{0,1}$ with multiplicity one. *A-priori* there can exist π' in L^2_0, isomorphic and orthogonal to $\pi \subset L^2_{0,1}$. In such a case, we would have $m(\pi) > 1$. Such a π' is locally generic (all of its local components are generic), isomorphic to a generic cuspidal π; and the question boils down to whether this implies that π' is generic (the linear form $L(\phi) = \int_{U_H(F)\backslash U_H(\mathbb{A})} \phi(u)\boldsymbol{\psi}_H(u)du$ is nonzero on $\pi \subset L^2_0$). This last claim might follow on using the theory of the Theta correspondence, but this has not been done as yet. In summary, a clear form of [GP], 2.4(i) is: "Any irreducible π in $L^2_{0,1}$ occurs in $L^2_{0,1}$ with multiplicity one." In the analogous situation of GSp(2), such a statement is made in [So]. It is not sufficiently strong to be useful for us.

We noticed that the global argument of [F, p. 48], which was first proposed in a preprint version of [F] in 1983, is incomplete while generalizing it in [F3] to the context of the symplectic group, where work of Kudla, Rallis, Langlands, Shahidi on the Siegel–Weil formula and on L-functions is available to show that a locally generic cuspidal representation which is equivalent at almost all places to a generic cuspidal representation is generic. A local proof, based on a twisted analogue of Rodier's result, is also used in [F4], in the context of the symmetric square lifting.

4 Review of Rodier's proof

We shall reduce Theorem 2 to Theorem 1 for G (not H), so we begin by recalling the main lines in Rodier's proof in the context of G. Choose $d = \mathrm{diag}(\boldsymbol{\pi}^{-r+1}, \boldsymbol{\pi}^{-r+3}, \dots, \boldsymbol{\pi}^{r-1})$. It lies in the unitary group, namely $\sigma(d) = d$, since $\boldsymbol{\pi}$ is in F. Put $V_n = d^n G_n d^{-n}$ and $\boldsymbol{\psi}_n(v) = \psi_n(d^{-n}vd^n)$ $(v \in V_n)$. Recall that ψ_n is defined to be supported on G_n. Note that $\sigma(G_n) = G_n$, $\sigma(U_n) = U_n$, $\sigma\psi_n = \psi_n$, and that the entries in the jth line ($j \neq 0$) above or below the diagonal of $v = (v_{ij})$ in V_n lie in $\boldsymbol{\pi}^{(1-2j)n}R_E$ (thus $v_{i,i+j} \in \boldsymbol{\pi}^{(1-2j)n}R_E$ if $j > 0$, and also when $j < 0$). Thus $V_n \cap U$ is a σ-invariant strictly increasing sequence of compact and open subgroups of U whose union is U, while $V_n \cap ({}^tUH)$ — where ${}^tUH$ is the lower triangular subgroup of G — is a strictly decreasing sequence of compact open subgroups of G whose intersection is the element I of G. Note that $\boldsymbol{\psi}_n = \boldsymbol{\psi}$ on $V_n \cap U$.

Consider the induced representations $\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n$ and the intertwining operators

$$A_n^m : \operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n \to \operatorname{ind}_{V_m}^G \boldsymbol{\psi}_m,$$

$$(A_n^m \varphi)(g) = ((e_{V_m}\boldsymbol{\psi}_m) * \varphi)(g) = \int_G \boldsymbol{\psi}_m(u)\varphi(u^{-1}g)e_{V_m}(u)$$

(g in G, φ in $\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n$, $e_{V_m} = |V_m|^{-1}1_{V_m}dg$, $|V_m|$ denotes the volume of V_m and 1_{V_m} denotes the characteristic function of V_m). For $m \geq n \geq 1$, we have

$$(A_n^m \varphi)(g) = ((e_{V_m \cap U}\boldsymbol{\psi}) * \varphi)(g) = \int_G \boldsymbol{\psi}(u)\varphi(u^{-1}g)e_{V_m \cap U}(u).$$

Hence $A_m^\ell \circ A_n^m = A_n^\ell$ for $\ell \geq m \geq n \geq 1$. So $(\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n, A_n^m\ (m \geq n \geq 1))$ is an inductive system of representations of G. Denote by $(I, A_n : \operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n \to I)$ $(n \geq 1)$ its limit.

The intertwining operators $\phi_n : \operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n \to \operatorname{ind}_U^G \boldsymbol{\psi}$,

$$(\phi_n(\varphi))(g) = (\boldsymbol{\psi}1_U * \varphi)(g) = \int_U \boldsymbol{\psi}(u)\varphi(u^{-1}g)du,$$

satisfy $\phi_m \circ A_n^m = \phi_n$ if $m \geq n \geq 1$. Hence there exists a unique intertwining operator $\phi : I \to \operatorname{ind}_U^G \boldsymbol{\psi}$ with $\phi \circ A_n = \phi_n$ for all $n \geq 1$. Proposition 3 of [R] asserts that

Lemma 1. *The map ϕ is an isomorphism of G-modules.* □

Lemma 2. *There exists $n_0 \geq 1$ such that $\boldsymbol{\psi}_n * \boldsymbol{\psi}_m * \boldsymbol{\psi}_n = |V_n||V_m \cap V_n|\boldsymbol{\psi}_n$ for all $m \geq n \geq n_0$.*

Proof. This is Lemma 5 of [R]. We review its proof (the first displayed formula in the proof of this Lemma 5, [R], p. 159, line -8, should be erased).

There are finitely many representatives u_i in $U \cap V_m$ for the cosets of V_m modulo $V_n \cap V_m$. Denote by $\varepsilon(g)$ the Dirac measure in a point g of G. Consider

$$\begin{aligned}(\varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m \cap V_n})(g) &= \int_G \varepsilon(u_i)(gh^{-1})(\boldsymbol{\psi}_n 1_{V_m \cap V_n})(h)dh \\ &= \boldsymbol{\psi}_n(u_i^{-1}g) = \boldsymbol{\psi}_m(u_i)^{-1}\boldsymbol{\psi}_m(g).\end{aligned}$$

Note here that if the left side is nonzero, then $g \in u_i(V_m \cap V_n) \subset V_m$. Conversely, if $g \in V_m$, then $g \in u_i(V_m \cap V_n)$ for some i. Hence $\boldsymbol{\psi}_m = \sum_i \boldsymbol{\psi}_m(u_i)\varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m \cap V_n}$; thus

$$\boldsymbol{\psi}_n * \boldsymbol{\psi}_m * \boldsymbol{\psi}_n = \sum_i \boldsymbol{\psi}_m(u_i)\boldsymbol{\psi}_n * \varepsilon(u_i) * \boldsymbol{\psi}_n 1_{V_m \cap V_n} * \boldsymbol{\psi}_n.$$

Since $\boldsymbol{\psi}_n 1_{V_m \cap V_n} * \boldsymbol{\psi}_n = |V_m \cap V_n| \boldsymbol{\psi}_n$, this is

$$= \sum_i \boldsymbol{\psi}_m(u_i) |V_m \cap V_n| \boldsymbol{\psi}_n * \varepsilon(u_i) * \boldsymbol{\psi}_n.$$

But the key Lemma 4 of [R] asserts that $\boldsymbol{\psi}_n * \varepsilon(u) * \boldsymbol{\psi}_n \neq 0$ implies that $u \in V_n$. Hence the last sum reduces to a single term, with $u_i = 1$, and we obtain

$$= |V_m \cap V_n| \boldsymbol{\psi}_n * \boldsymbol{\psi}_n = |V_m \cap V_n||V_n| \boldsymbol{\psi}_n.$$

This completes the proof of the lemma. □

Lemma 3. *For an inductive system* $\{I_n\}$, $\operatorname{Hom}_G(\varinjlim I_n, \pi) = \varprojlim \operatorname{Hom}_G(I_n, \pi)$.

Proof. See, e.g., Rotman [Ro], Theorem 2.27. It is also verified in [R]. □

Corollary. *We have* $\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n |G_n|^{-1} \operatorname{tr} \pi(\psi_n dg)$.

Proof. As the numbers $\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$ increase with n, if they are bounded they are independent of n for sufficiently large n. Hence the left side of the corollary equals $\lim_n \dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$, which is equal to $\lim_n \dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{ind}_{G_n}^G \psi_n, \pi)$ since $\boldsymbol{\psi}_n(v) = \psi_n(d^{-n} v d^n)$. This equals $\lim_n \dim_{\mathbb{C}} \operatorname{Hom}_{G_n}(\psi_n, \pi | G_n)$ by Frobenius reciprocity, which is equal to the right side of the corollary since $|G_n|^{-1} \pi(\psi_n dg)$ is a projection from π to the space of ξ in π with $\pi(g)\xi = \psi_n(g)\xi$ $(g \in G_n)$, a space whose dimension is then $|G_n|^{-1} \operatorname{tr} \pi(\psi_n dg)$. □

5 The twisted case

We now reduce Theorem 2 to Theorem 1 for G. Note that since $\sigma \boldsymbol{\psi}_n = \boldsymbol{\psi}_n$, the representations $\operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n$ are σ-invariant, where σ acts on $\varphi \in \operatorname{ind}_{V_n}^G \boldsymbol{\psi}_n$ by $\varphi \mapsto \sigma\varphi$, $(\sigma\varphi)(g) = \varphi(\sigma g)$. Similarly, $\sigma \boldsymbol{\psi} = \boldsymbol{\psi}$ and $\operatorname{ind}_U^G \boldsymbol{\psi}$ is σ-invariant. We then extend these representations ind of G to the semidirect product $G' = G \rtimes \langle \sigma \rangle$ by putting $(i(\sigma)\varphi)(g) = \varphi(\sigma(g))$.

Let π be an irreducible admissible representation of G which is σ-invariant. Thus there exists an intertwining operator $A : \pi \to {}^\sigma\pi$, where ${}^\sigma\pi(g) = \pi(\sigma(g))$, with $A\pi(g) = \pi(\sigma(g))A$. Then A^2 commutes with every $\pi(g)$ $(g \in G)$, hence A^2 is a scalar by Schur's lemma and can be normalized to be 1. This determines A up to a sign. We extend π from G to $G' = G \rtimes \langle \sigma \rangle$ by putting $\pi(\sigma) = A$ once A is chosen.

If $\operatorname{Hom}_G(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi) \neq 0$, its dimension is 1. Choose a generator $\ell : \operatorname{ind}_U^G \boldsymbol{\psi} \to \pi$. Define $A : \pi \to \pi$ by $A\ell(\varphi) = \ell(i(\sigma)\varphi)$. Then

$$\operatorname{Hom}_G(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi) = \operatorname{Hom}_{G'}(\operatorname{ind}_U^G \boldsymbol{\psi}, \pi).$$

Similarly, we have $\mathrm{Hom}_G(\mathrm{ind}_{V_n}^G \boldsymbol{\psi}_n, \pi) = \mathrm{Hom}_{G'}(\mathrm{ind}_{V_n}^G \boldsymbol{\psi}_n, \pi)$.
The right side in the last equality can be expressed as

$$\mathrm{Hom}_{G'}(\mathrm{ind}_{G_n}^G \psi_n, \pi) = \mathrm{Hom}_{G'_n}(\psi'_n, \pi|G'_n) \qquad (G'_n = G_n \rtimes \langle\sigma\rangle).$$

The last equality follows from Frobenius reciprocity, where we extended ψ_n to a character ψ'_n on G'_n by $\psi'_n(\sigma) = 1$. Thus $\psi'_n = \psi^1_n + \psi^\sigma_n$, with $\psi^\alpha_n(g \times \beta) = \delta_{\alpha\beta}\psi_n(g)$, $\alpha, \beta \in \{1, \sigma\}$.

In this case, $\mathrm{Hom}_{G'_n}(\psi'_n, \pi|G'_n)$ is isomorphic to the space π_1 of vectors ξ in π with $\pi(g)\xi = \psi_n(g)\xi$ for all g in G'_n. In particular, $\pi(g)\xi = \psi_n(g)\xi$ for all g in G_n and $\pi(\sigma)\xi = \xi$. Clearly, $|G'_n|^{-1}\pi(\psi'_n dg')$ is a projection from the space of π to π_1 (it is independent of the choice of the measure dg'). Its trace is then the dimension of the space Hom. We conclude a twisted analogue of the theorem of [R]:

Proposition 1. *The integer* $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{ind}_U^G \boldsymbol{\psi}, \pi)$ *is equal to*

$$|G'_n|^{-1} \operatorname{tr} \pi(\psi'_n dg')$$

for all sufficiently large n. □

Note that G'_n is the semidirect product of G_n and the two-element group $\langle\sigma\rangle$. With the natural measure assigning 1 to each element of the discrete group $\langle\sigma\rangle$, we have $|G'_n| = 2|G_n|$. The result is then, for all sufficiently large n,

$$\frac{1}{2} \operatorname{tr} \pi(\psi_n e_{G_n}) + \frac{1}{2} \operatorname{tr} \pi(\psi_n e_{G_n} \times \sigma)$$

(as $\psi'_n = \psi^1_n + \psi^\sigma_n$, $\psi^1_n = \psi_n$ and $\operatorname{tr} \pi(\psi^\sigma_n dg) = \operatorname{tr} \pi(\psi_n dg \times \sigma)$). By (the nontwisted version of) Rodier's Theorem 1,

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n \operatorname{tr} \pi(\psi_n e_{G_n}),$$

we conclude

Proposition 2. *We have* $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{ind}_U^G \boldsymbol{\psi}, \pi) = \lim_n \operatorname{tr} \pi(\psi_n e_{G_n} \times \sigma)$ *for all* σ*-invariant irreducible representations* π *of* G. □

The terms in the limit on the right can be written in terms of Harish-Chandra's twisted character as

$$\int_G \chi^\sigma_\pi(g)\psi_n(g)e_{G_n}(g).$$

Again, put $e_{G^\sigma_n} = |G^\sigma_n|^{-1} \operatorname{ch}_{G^\sigma_n} dg$, where $\operatorname{ch}_{G^\sigma_n}$ is the characteristic function of G^σ_n in G.

Proposition 3. *The last displayed integral is equal to*

$$\int_{G_n^\sigma} \chi_\pi^\sigma(g)\psi_n(g)e_{G_n^\sigma}(g).$$

Proof. Consider the map $G_n^\sigma \times G_n^\sigma\backslash G_n \to G_n$, $(u,k) \mapsto k^{-1}u\sigma(k)$. It is a closed immersion. More generally, given a semisimple element s in a group G, we can consider the map $Z_{G^0}(s) \times Z_{G^0}(s)\backslash G^0 \to G^0$ by $(u,k) \mapsto k^{-1}usks^{-1}$. Our example is $(s,G) = (\sigma, G_n \times \langle\sigma\rangle)$.

Our map is, in fact, an analytic isomorphism, since G_n is a small neighborhood of the origin where the exponential $e : \mathcal{G}_n \to G_n$ is an isomorphism. Indeed, we can transport the situation to the Lie algebra $\mathcal{G}_n$. Thus we write $k = e^Y$, $u = e^X$, $\sigma(k) = e^{(d\sigma)(Y)}$, $k^{-1}u\sigma(k) = e^{X-Y+(d\sigma)(Y)}$, up to smaller terms. Here $(d\sigma)(Y) = -J^{-1t}\overline{Y}J$. So we just need to show that $(X,Y) \mapsto X - Y + (d\sigma)(Y)$, $Z_{\mathcal{G}_n}(\sigma) + \mathcal{G}_n(\operatorname{mod} Z_{\mathcal{G}_n}(\sigma)) \to \mathcal{G}_n$ is bijective. But this is obvious, since the kernel of $(1 - d\sigma)$ on $\mathcal{G}_n$ is precisely $Z_{\mathcal{G}_n}(\sigma) = \{Y \in \mathcal{G}_n : (d\sigma)(Y) = Y\}$.

Changing variables on the terms on the right of Proposition 2, we get

$$\int_{G_n} \chi_\pi^\sigma(g)\psi_n(g)e_{G_n}(g) = |G_n|^{-1} \int_{G_n^\sigma} \int_{G_n^\sigma\backslash G_n} \chi_\pi^\sigma(k^{-1}u\sigma(k))\psi_n(k^{-1}u\sigma(k))dkdu.$$

But $\sigma\psi_n = \psi_n$, ψ_n is a homomorphism (on G_n), G_n is compact, and χ_π^σ is a σ-conjugacy class function, so we end up with the expression of the proposition. Note that χ_π^σ is locally integrable on G_n^σ and locally constant on its regular set by the character relation stated in the proof of Prop. 3.5 of [F] above. The proposition, and Theorem 2, follow. □

6 Appendix. Germs of twisted characters

Harish-Chandra [HC] showed that χ_π is locally integrable (Thm. 1, p. 1) and has a germ expansion near each semisimple element γ (Thm. 5, p. 3), of the form

$$\chi_\pi(\gamma \exp X) = \sum_{\mathcal{O}} c_\gamma(\mathcal{O},\pi)\widehat{\mu}_{\mathcal{O}}(X).$$

Here $\mathcal{O}$ ranges over the nilpotent orbits in the Lie algebra $\mathcal{M}$ of the centralizer M of γ in G, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$, $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform with respect to a symmetric nondegenerate G-invariant bilinear form B on $\mathcal{M}$ and a selfdual measure, $c_\gamma(\mathcal{O},\pi)$ are complex numbers, and X ranges over a small neighborhood of the origin in $\mathcal{M}$. Both $\mu_{\mathcal{O}}$ and $c_\gamma(\mathcal{O},\pi)$ depend on the choice of a Haar measure $d_{\mathcal{O}}$ on the centralizer $Z_G(X_0)$ of $X_0 \in \mathcal{O}$, but their product does not. We are interested only in the case of $\gamma = 1$ and therefore omit

γ from the notation. The size of the domain where the germ expansion holds is studied in Waldspurger [W].

Suppose that G is quasisplit over F and U is the unipotent radical of a Borel subgroup B. Let $\psi : U \to \mathbb{C}^1$ be the nondegenerate character of U (its restriction to each simple root subgroup is nontrivial) specified in [R], p. 153. The number $\dim_{\mathbb{C}} \mathrm{Hom}(\mathrm{ind}_U^G \psi, \pi)$ of ψ-Whittaker functionals on π is known to be zero or one. Let $\mathcal{G}_0$ be a selfdual lattice in the Lie algebra $\mathcal{G}$ of G. Denote by ch_0 the characteristic function of $\mathcal{G}_0$ in $\mathcal{G}$. Rodier ([R], p. 163) showed that there is a regular nilpotent orbit $\mathcal{O} = \mathcal{O}_\psi$ such that $c(\mathcal{O}, \pi)$ is not zero iff $\dim_{\mathbb{C}} \mathrm{Hom}(\mathrm{ind}_U^G \psi, \pi)$ is one; in fact, $\widehat{\mu}_{\mathcal{O}}(\mathrm{ch}_0)c(\mathcal{O}, \pi)$ is one in this case. Alternatively put, normalizing $\mu_{\mathcal{O}}$ by $\widehat{\mu}_{\mathcal{O}}(\mathrm{ch}_0) = 1$, we have $c(\mathcal{O}, \pi) = \dim_{\mathbb{C}} \mathrm{Hom}(\mathrm{ind}_U^G \psi, \pi)$. This is shown in [R] for all p if $G = \mathrm{GL}(r, F)$, and for general quasisplit G for all $p \geq 1+2\sum_{\alpha\in S} n_\alpha$, if the longest root is $\sum_{\alpha\in S} n_\alpha \alpha$ in a basis S of the root system. A generalization of Rodier's theorem to degenerate Whittaker models and nonregular nilpotent orbits is given by Moeglin–Waldspurger [MW]. See [MW], I.8, for the normalization of measures. In particular, they show that $c(\mathcal{O}, \pi) > 0$ for the nilpotent orbits $\mathcal{O}$ of maximal dimension with $c(\mathcal{O}, \pi) \neq 0$. For applications to minimal representations, see Savin [S].

Harish-Chandra's results extend to the twisted case. The twisted character is locally integrable (Clozel [C], Thm. 1, p. 153), and there exist unique complex numbers $c^\theta(\mathcal{O}, \pi)$ ([C], Thm. 3, p. 154) with $\chi_\pi^\theta(\exp X) = \sum_{\mathcal{O}} c^\theta(\mathcal{O}, \pi)\widehat{\mu}_{\mathcal{O}}(X)$. Here $\mathcal{O}$ ranges over the nilpotent orbits in the Lie algebra $\mathcal{G}^\theta$ of the group G^θ of $g \in G$ with $g = \theta(g)$. Further, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$ (it is unique up to a constant, not unique as stated in [HC], Thm. 5, and [C], Thm. 3); $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform, and X ranges over a small neighborhood of the origin in $\mathcal{G}^\theta$.

In this section, we compute the expression displayed in Proposition 3 using the germ expansion $\chi_\pi^\sigma(\exp X) = \sum_{\mathcal{O}} c^\sigma(\mathcal{O}, \pi)\widehat{\mu}_{\mathcal{O}}(X)$. This expansion means that for any test measure $f dg$ supported on a small enough neighborhood of the identity in G, we have

$$\int_{\mathcal{G}^\sigma} f(\exp X)\chi_\pi^\sigma(\exp X)dX = \sum_{\mathcal{O}} c^\sigma(\mathcal{O}, \pi) \int_{\mathcal{O}} [\int_{\mathcal{G}^\sigma} f(\exp X)\psi(\mathrm{tr}(XZ))dX]d\mu_{\mathcal{O}}(Z).$$

Here $\mathcal{O}$ ranges over the nilpotent orbits in $\mathcal{G}^\sigma$, $\mu_{\mathcal{O}}$ is an invariant distribution supported on the orbit $\mathcal{O}$, $\widehat{\mu}_{\mathcal{O}}$ is its Fourier transform, and X ranges over a small neighborhood of the origin in $\mathcal{G}^\sigma$. Since we are interested only in the case of the unitary group, and to simplify the exposition, we take $G = \mathrm{GL}(r, E)$ and the involution σ whose group of fixed points is the unitary group $H = \mathrm{U}(r, E/F)$. In this case, there is a unique regular nilpotent orbit $\mathcal{O}_0$.

We normalize the measure $\mu_{\mathcal{O}_0}$ on the orbit $\mathcal{O}_0$ of β in $\mathcal{G}^\sigma$ by requiring that $\widehat{\mu}_{\mathcal{O}_0}(\mathrm{ch}_0^\sigma)$ is 1, so that $\int_{\beta+\boldsymbol{\pi}^n\mathcal{G}_0^\sigma} d\mu_{\mathcal{O}_0}(X) = q^{n\dim(\mathcal{O}_0)}$ for large n. Equivalently, a measure on an orbit $\mathcal{O} \simeq G/Z_G(Y)$ ($Y \in \mathcal{O}$) is defined by a measure on its tangent space $m = \mathcal{G}/Z_{\mathcal{G}}(Y)$ ([MW], p. 430) at Y, taken to be the selfdual measure with respect to the symmetric bilinear nondegenerate F-valued form $B_Y(X, Z) = \mathrm{tr}(Y[X, Z])$ on m.

Proposition 4. *If π is a σ-invariant admissible irreducible representation of G and $\mathcal{O}_0$ is the regular nilpotent orbit in $\mathcal{G}^\sigma$, then the coefficient $c^\sigma(\mathcal{O}_0, \pi)$ in the germ expansion of the σ-twisted character χ_π^σ of π is equal to*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{ind}_U^G \boldsymbol{\psi}, \pi) = \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{ind}_U^G \boldsymbol{\psi}, \pi).$$

This number is one if π is generic and zero otherwise.

Proof. We compute the expression displayed in Proposition 3 as in [MW], I.12. It is a sum over the nilpotent orbits $\mathcal{O}$ in $\mathcal{G}^\sigma$ of $c^\sigma(\mathcal{O}, \pi)$ times

$$|G_n^\sigma|^{-1}\widehat{\mu}_{\mathcal{O}}(\psi_n \circ e) = |G_n^\sigma|^{-1}\mu_{\mathcal{O}}(\widehat{\psi_n \circ e}) = |G_n^\sigma|^{-1}\int_{\mathcal{O}} \widehat{\psi_n \circ e}(X) d\mu_{\mathcal{O}}(X).$$

The Fourier transform (with respect to the character ψ_E) of $\psi_n \circ e$,

$$\widehat{\psi_n \circ e}(Y) = \int_{\mathcal{G}^\sigma} \psi_n(\exp Z)\overline{\psi}_E(\mathrm{tr}\, ZY)dZ = \int_{\mathcal{G}_n^\sigma} \psi_E(\mathrm{tr}\, Z(\boldsymbol{\pi}^{-2n}\beta - Y))dZ,$$

is the characteristic function of $\boldsymbol{\pi}^{-2n}\beta + \boldsymbol{\pi}^{-n}\mathcal{G}_0^\sigma = \boldsymbol{\pi}^{-2n}(\beta + \boldsymbol{\pi}^n\mathcal{G}_0^\sigma)$ multiplied by the volume $|\mathcal{G}_n^\sigma| = |G_n^\sigma|$ of $\mathcal{G}_n^\sigma$. Hence we get

$$= \int_{\mathcal{O}\cap(\boldsymbol{\pi}^{-2n}(\beta+\boldsymbol{\pi}^n\mathcal{G}_0^\sigma))} d\mu_{\mathcal{O}}(X) = q^{n\dim(\mathcal{O})}\int_{\mathcal{O}\cap(\beta+\boldsymbol{\pi}^n\mathcal{G}_0^\sigma)} d\mu_{\mathcal{O}}(X).$$

The last equality follows from the homogeneity result of [HC], Lemma 3.2, p. 18. For sufficiently large n, we have that $\beta + \boldsymbol{\pi}^n\mathcal{G}_0^\sigma$ is contained only in the orbit $\mathcal{O}_0$ of β. Then only the term indexed by $\mathcal{O}_0$ remains in the sum over $\mathcal{O}$, and

$$\int_{\mathcal{O}_0\cap(\beta+\boldsymbol{\pi}^n\mathcal{G}_0^\sigma)} d\mu_{\mathcal{O}_0}(X) = \int_{\beta+\boldsymbol{\pi}^n\mathcal{G}_0^\sigma} d\mu_{\mathcal{O}_0}(X)$$

equals $q^{-n\dim(\mathcal{O}_0)}$ (cf. [MW], end of proof of Lemme I.12). The proposition follows. □

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REFERENCES

[BZ1] J. Bernstein and A. Zelevinskii, *Representations of the group* $\mathrm{GL}(n, F)$ *where* F *is a nonarchimedean local field*, Russian Math. Surveys **31** (1976), 1–68.

[BZ2] J. Bernstein and A. Zelevinsky, *Induced representations of reductive p-adic groups*, Ann. Sci. École Norm. Sup. **10** (1977), 441–472.

[C] L. Clozel, *Characters of non-connected, reductive p-adic groups*, Canad. J. Math. **39** (1987), 149–167.

[F] Y. Flicker, *Packets and liftings for* $\mathrm{U}(3)$, J. Analyse Math. **50** (1988), 19–63.

[F1] Y. Flicker, *Base change trace identity for* $\mathrm{U}(3)$, J. Analyse Math. **52** (1989), 39–52.

[F2] Y. Flicker, *Elementary proof of a fundamental lemma for a unitary group*, Canad. J. Math. **50** (1998), 74–98.

[F3] Y. Flicker, *Lifting Automorphic Forms on* $\mathrm{PGSp}(2)$ *and* $\mathrm{SO}(4)$ *to* $\mathrm{PGL}(4)$, Research monograph, 2001; see also: *Automorphic forms on* PGSp(2), Electron. Res. Ann. Amer. Math. Soc. **10** (2004), 39–50; *Automorphic forms on* $\mathrm{SO}(4)$, Proc. Japan Acad. **80** (2004), 100–104.

[F4] Y. Flicker, *Automorphic Representations of Low Rank Groups*, Research monograph, 2003.

[FK] Y. Flicker and D. Kazhdan, *Metaplectic correspondence*, Publ. Math. IHES **64** (1987), 53–110.

[GP] S. Gelbart and I. Piatetski-Shapiro, *Automorphic forms and L-functions for the unitary group*, in *Lie Groups Representations II*, Lecture Notes in Math. **1041**, Springer, Berlin, 1984, pp. 141–184.

[H] G. Harder, *Eisensteinkohomologie und die Konstruktion gemischter Motive*, Lecture Notes in Math. **1562**, Springer, Berlin, 1993.

[HC] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, Notes by S. DeBacker and P. Sally, AMS Univ. Lecture Series **16** (1999); see also: Queen's Papers in Pure and Appl. Math. **48** (1978), 281–346.

[MW] C. Moeglin and J. L. Waldspurger, *Modèles de Whittaker dégénérés pour des groupes p-adiques*, Math. Z. **196** (1987), 427–452.

[R] F. Rodier, *Modèle de Whittaker et caractères de représentations*, in *Non-commutative Harmonic Analysis*, Lecture Notes in Math. **466**, Springer, Berlin, 1975, pp. 151–171.

[Ro] J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.

[S] G. Savin, *Dual pair* $G_J \times \mathrm{PGL}_2$ *where* G_J *is the automorphism group of the Jordan algebra* J, Invent. Math. **118** (1994), 141–160.

[So] D. Soudry, *A uniqueness theorem for representations of* $\mathrm{GSO}(6)$ *and the strong multiplicity one theorem for generic representations of* $\mathrm{GSp}(4)$, Israel J. Math. **58** (1987), 257–287.

[W] J.-L. Waldspurger, *Homogénéité de certaines distributions sur les groupes p-adiques*, Publ. Math. IHES **81** (1995), 25–72.

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