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STABLE AND LABILE BASE CHANGE FOR $U(2)$

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Let E/F be a quadratic extension of local or global fields of characteristic 0, and $\mathbb{A}_E, \mathbb{A}$ the rings of adèles of E, F in the latter case. Denote by a bar or $\tilde{\sigma}$ the nontrivial element of the galois group $\mathrm{Gal}(E/F)$, and let G be the quasi-split form of $GL(2)$ defined by the twisted galois action of $\mathrm{Gal}(\bar{F}/F)$ ($\bar{F}$ is an algebraic closure of F) given by $\tau(g) = \tilde{\tau}(g)$ if the restriction of $\tilde{\tau}$ (in $\mathrm{Gal}(\bar{F}/F)$) to E is trivial, and $\tau(g) = w\,{}^t\tilde{\tau}(g)^{-1}w^{-1}$ if $\tilde{\tau}$ restricts to $\tilde{\sigma}$ on E. Here $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, ${}^tg$ denotes the transpose of g, and $\tilde{\tau}$ acts by mapping the matrix $g = (g_{ij})$ $(1 \leqslant i, j \leqslant 2)$ in $GL(2, \bar{F})$ (g_{ij} in $\bar{F}$) to the matrix $\tilde{\tau}(g) = (\tilde{\tau}(g_{ij}))$. Then $G(E) = GL(2, E)$ and $G(F)$ is the subgroup of σ-invariant g in $G(E)$, where $\sigma(g) = w\,{}^t\bar{g}^{-1}w^{-1}$.

If "upstairs" and "downstairs" refer to objects defined over E and F, the purpose of this work is to lift (L-packets $\{\pi\}$ of) admissible (locally) and automorphic (globally) representations π downstairs, to such representations π^E upstairs. The image consists of σ-invariant π^E, those with ${}^\sigma\pi^E \simeq \pi^E$ where ${}^\sigma\pi^E(g) = \pi^E(\sigma(g))$. Only one-half of the σ-invariant π^E are obtained by this lifting, in contrast with the base change theory of $GL(n)$ [1,4], where all $\tilde{\sigma}$-invariant π^E are obtained. More precisely, *there are two distinct liftings λ and λ_1 which inject the set of one-dimensional or discrete series L-packets $\{\pi\}$ downstairs into the set of π^E; the images of λ and λ_1 are disjoint and their union exhausts the set of σ-invariant representations π^E upstairs which are one-dimensional, discrete series or for which ${}^\sigma\pi^E$ is equivalent, but not equal, to π^E (globally and locally).* The central character of a σ-invariant irreducible one-dimensional or discrete series local (or global) representation π^E is trival on $F^\times$ (or the group $\mathbb{A}^\times$ of idèles of F), not only on $NE^\times$ (or $E^\times N\mathbb{A}_E^\times$).

To explain these results by means of the Langlands functoriality principle denote by G' the group $\mathrm{Res}_{E/F}G$ obtained from G by restricting scalars from E to F (thus $G'(F) \simeq G(E)$), and recall that the L-groups of G and G' are

$$ {}^LG = GL(2, \mathbb{C}) \rtimes W_{E/F}, \qquad {}^LG' = (GL(2, \mathbb{C}) \times GL(2, \mathbb{C})) \rtimes W_{E/F}; $$

the Weil group $W_{E/F}$ (of $(z, \tau), z$ in $E^\times$ or the idèle class group $E^\times \backslash \mathbb{A}_E^\times, \tau$ in $\mathrm{Gal}(E/F)$) acts through $\mathrm{Gal}(E/F)$ by

$$ \sigma(g) = w\,{}^tg^{-1}w^{-1}, \qquad \sigma((g, g')) = (\sigma(g'), \sigma(g)) \qquad (g, g' \text{ in } GL(2, \mathbb{C})). $$

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A key role will be played by two distinct embeddings λ and λ_1 of ${}^LG$ in ${}^LG'$ extending the diagonal embedding of $GL(2,\mathbf{C}) \times 1$ in ${}^LG'$, mapping each w in $W_{E/F}$ to $g(w) \times w$ in ${}^LG'$. They are defined by

$$\lambda : g \times z \times \tau \mapsto (g, g) \times z \times \tau$$

and

$$\lambda_1 : g \times z \times \tau \mapsto (g\kappa(z), g\kappa(z)\delta(\tau)) \times z \times \tau,$$

where $\delta(\tau)$ is 1 if $\tau = 1$ and -1 if $\tau = \sigma$, and κ is a unitary character of $E^\times/NE^\times$ or $\mathbf{A}_E^\times/E^\times N\mathbf{A}_E^\times$, whose restriction to $F^\times$ or $\mathbf{A}^\times$ is nontrivial. N is the norm from E to$F$.

The homomorphisms λ and λ_1 induce the two liftings mentioned above. The liftings are defined in §5, and studied in §7 by means of an identity of trace formulae §6. These formulae are similar to those of [3,4] and given in §§1,3. For the comparison the twisted formula is stabilized by means of the local calculations of §2. The comparison is based on a standard statement concerning matching orbital integrals (§5); the techniques of [4], §§5,6, have been applied to establish statements more difficult than the one of Lemma 5.2 and little will be gained by recording a proof here.

In addition to proving character relations and describing the image of the stable and labile liftings λ and λ_1, this work establishes the multiplicity-one theorem for the unitary group $G(F)$, as well as its "strong form",* results related to $G(F)$ alone. This study of base change for $U(2)$ is on the one hand introductory and on the other preparatory for the study of this problem for $U(3)$. Although some partial results concerning the latter problem can easily be obtained, a formulation of a full solution [1a] requires the results of §7.

The phenomenon of splitting the σ-invariant representations into two halves already occurs in the case of $U(1)$, and is easy to describe. Here $G(E)$ is $E^\times$, $G(\mathbf{A}_E)$ is the group $\mathbf{A}_E^\times$ of idèles of E, σ acts by mapping a to $\bar{a}^{-1}$, $G(F) = E^1$ and $G(\mathbf{A}) = \mathbf{A}_E^1$, where E^1 and $\mathbf{A}_E^1$ are the subgroups of $E^\times$ and $\mathbf{A}_E^\times$ consisting of the a whose norm $a\bar{a}$ is 1. An irreducible admissible, or automorphic, σ-invariant representation of $G(E)$ or $G(\mathbf{A}_E)$ is a character χ^E of $E^\times/NE^\times$, or $\mathbf{A}_E^\times/E^\times N\mathbf{A}_E^\times$. Its restriction to $F^\times$, or $\mathbf{A}^\times$, is of order two. If χ^E is trivial on $F^\times$, or $\mathbf{A}^\times$, then there exists a character χ of E^1, or $\mathbf{A}_E^1$, with $\chi(a/\bar{a}) = \chi^E(a)$; χ is said to lift to χ^E through λ. Otherwise there exists such χ with $\chi(a/\bar{a}) = \chi^E(a)$ $\kappa(a)$ for the fixed character κ, and χ lifts through λ_1 to χ^E.

This lifting is reflected by an identity of trace formulae. If $\phi = \otimes\phi_v$ is a function on $G(\mathbf{A}_E)$ whose components ϕ_v (indexed by all places v if F) are smooth and compactly supported, then in the standard notations the twisted

*Note how unfortunate the name "strong multiplicity one theorem" is: it neither implies nor is implied by "multiplicity one theorem". A name such as "(almost-all) rigidity theorem" could have been much better.

trace formula for $G(E)$ is given by

$$\begin{aligned} TF_{G(E)}(\phi\times\sigma) &= \sum_{\gamma\in G(E)}\int_{G(E)\backslash G(\mathbf{A}_E)}\phi(g\gamma\sigma(g^{-1}))\,dg \\ &= \sum_{x\in E^\times}\int_{E^\times\backslash\mathbf{A}_E^\times}\phi(xy\bar{y})\,dy = \sum_{x\in E}|E^1\backslash\mathbf{A}_E^1|\int_{E^\times\mathbf{A}_E^1\backslash\mathbf{A}_E^\times}\phi(xy\bar{y})\,dy \\ &= \sum_{x\in E^\times}|E^1\backslash\mathbf{A}_E^1|\int_{NE^\times\backslash N\mathbf{A}_E^\times}\phi(xy)\,dy \\ &= \sum_{x\in F^\times\backslash E^\times}|E^1\backslash\mathbf{A}_E^1|\int_{F^\times N\mathbf{A}_E^\times}\phi(xy)\,dy \\ &= \sum_{x\in F^\times\backslash E^\times}\frac{1}{2}|E^1\backslash\mathbf{A}_E^1|\sum_{\epsilon}\epsilon(x)\int_{\mathbf{A}^\times}\phi(xy)\epsilon(y)\,dy; \end{aligned}$$

the inner sum is taken over the set $\{1,\kappa\}$ of two characters. Put

$$f_v(x/\bar{x}) = \int_{F_v^\times}\phi_v(xy)\,dy, \qquad f_{1v}(x/\bar{x}) = \kappa_v(x)\int_{F_v^\times}\phi_v(xy)\kappa_v(y)\,dy$$

for x in $E_v^\times$, for all places v of F, and set $f=\otimes f_v$, $f_1=\otimes f_{1v}$. Choosing the global measure $dy = 2\otimes dy_v$ on $\mathbf{A}^\times$ we have

$$= \sum_{u\in G(F)}|E^1\backslash\mathbf{A}_E^1|f(u) + \sum_{u\in G(F)}|E^1\backslash\mathbf{A}_E^1|f_1(u) = TF_{G(F)}(f) + TF_{G(F)}(f_1).$$

This equality also follows from the global relations $\operatorname{tr}\chi^E(\phi\times\sigma)=\operatorname{tr}\chi(f)$ and $\operatorname{tr}\chi^E(\phi\times\sigma)=\operatorname{tr}\chi(f_1)$ if $\chi\to\chi^E$ through λ, or through λ_1, which is easy to prove if a product measure is taken on $\mathbf{A}_E^\times$.

1. The trace formula. The group $G(F)$ is essentially one of the groups of [3], and this section will summarize the results of [3] needed below. Let $S(F)$ be the group of g in $GL(2,F)$ whose determinant lies in $NE^\times$, where F is local or global and N denotes the norm from E to F. Denote by Z the centre of G; thus $Z(E)\simeq E^\times$ and $Z(F)\simeq E^1$. Then $G(F)Z(E)=S(F)Z(E)$ since each g in $G(F)$ can be written as

$$g=\begin{pmatrix} a & 0\\ 0 & \bar{a}^{-1}\end{pmatrix}s = \begin{pmatrix}\bar{a} & 0\\ 0 & \bar{a}\end{pmatrix}^{-1}\begin{pmatrix} a\bar{a} & 0\\ 0 & 1\end{pmatrix}s \qquad (s \text{ in } SL(2,F),\ a \text{ in } E^\times).$$

Writing $g=as$, $g'=as'$ for g, g' in $G(F)$, a in $E^\times$, s,s' in $S(F)$, it is clear that g and g' are (stably) conjugate if and only if s and s' are; moreover, if g and g' are (stably) conjugate then they have equal determinants (in E^1) and there is some a in $E^\times$ so that $g=as$, $g'=as'$ (s,s' in $S(F)$); recall that here stable conjugacy is conjugacy in $G(\bar{F})$.

For each torus $T(F)$ of $G(F)$ there exists a torus $T_S(F)$ of $S(F)$ with $T(F)Z(E) = T_S(F)Z(E)$, and for each $T_S(F)$ there is $T(F)$ with this property. It now follows from the results of [3], p. 729 (l. 2), concerning $S(F)$, that

LEMMA 1. *If $T(F)$ splits over E then either it is stably conjugate hence conjugate to the diagonal subgroup $A(F)$, or the conjugacy classes within its stable conjugacy class are parametrized by $F^\times/NE^\times$. The stable conjugacy classes of the $T(F)$ which do not split over E are parametrized by the set of quadratic extensions of F other than E. Each such stable class consists of a single conjugacy class.*

A set of representatives $H_u(F)$ for the conjugacy classes within the nontrivial stable conjugacy class of tori which split over E is given by $\left(\begin{smallmatrix}1&0\\0&u\end{smallmatrix}\right)^{-1}H(F)\left(\begin{smallmatrix}1&0\\0&u\end{smallmatrix}\right)$ (u in $F^\times/NE^\times$), where i is a fixed element of $E^\times$ with $i+\bar{i}=0$,

$$H(F) = \left\{\gamma = h^{-1}\begin{pmatrix}a&0\\0&b\end{pmatrix}h = \begin{pmatrix}\alpha & i\beta\\ \beta/i & \alpha\end{pmatrix};\ a,b \text{ in } E^1\right\},$$

and

$$\alpha = \tfrac{1}{2}(a+b),\quad \beta = \tfrac{1}{2}(a-b),\quad h = \begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\quad \left(\text{with } h\sigma(h^{-1}) = 2i\begin{pmatrix}0&1\\1&0\end{pmatrix}\right).$$

The $H_u(F)$ are nonconjugate for distinct u in $F^\times$ modulo $NE^\times$ since if

$$\begin{pmatrix}1&0\\0&u\end{pmatrix}^{-1}\gamma\begin{pmatrix}1&0\\0&u\end{pmatrix} = \delta^{-1}\gamma\delta \qquad (\delta \text{ in } G(F)),$$

then $\delta\left(\begin{smallmatrix}1&0\\0&u\end{smallmatrix}\right)^{-1}$ lies in the centralizer $H(E)$ of γ in $G(E)$, hence $\delta = h^{-1}\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right)h\left(\begin{smallmatrix}1&0\\0&u\end{smallmatrix}\right)$ for some a,b in $E^\times$, and from $\delta = \sigma(\delta)$ it follows that $u = (a\bar{a})^{-1} = (b\bar{b})^{-1}$ lies in $NE^\times$.

If the splitting field L of $T(F)$ has galois group $\langle\bar{\sigma}\rangle \oplus \langle\bar{\tau}\rangle$, where $\bar{\sigma}$ restricts to σ on E and $\bar{\sigma}^2 = \bar{\tau}^2 = 1$, then $T(F)$ is isomorphic to the subgroup of $L^\times \times L^\times$ consisting of $(\alpha, \bar{\sigma}\alpha^{-1})$ with $\bar{\sigma}\alpha = \bar{\tau}\alpha^{-1}$. $\mathrm{Gal}(L/F)$ acts on $T(L) \simeq L^\times \times L^\times$ by $\bar{\sigma}(a,b) = (\bar{\sigma}b^{-1}, \bar{\sigma}a^{-1})$ and $\bar{\tau}(a,b) = (\bar{\tau}b, \bar{\tau}a)$. For §2 we shall find the first cohomology group $H^1(F,T) = H^1(\mathrm{Gal}(L/F), T(L))$. Hence we have to find all $g(\bar{\sigma}) = (x,y)$, $g(\bar{\tau}) = (\alpha, \beta)$ in $T(L)$ with: (1) $1 = g(\bar{\sigma}^2) = \bar{\sigma}(g(\bar{\sigma}))g(\bar{\sigma})$ so that $g(\bar{\sigma}) = (x, \bar{\sigma}(x))$. Since $z\bar{\sigma}(z)^{-1} = (a\bar{\sigma}(b), b\bar{\sigma}(a))$ for $z = (a,b)$ we may assume that $g(\bar{\sigma}) = 1$ up to coboundaries; (2) $1 = g(\bar{\tau}^2) = \bar{\tau}(g(\bar{\tau}))g(\bar{\tau})$ so that $g(\bar{\tau}) = (\alpha, \bar{\tau}(\alpha)^{-1})$; (3) $g(\bar{\sigma}\bar{\tau}) = g(\bar{\tau}\bar{\sigma})$ implies that $\bar{\sigma}(g(\bar{\tau})) = g(\bar{\tau})$, hence $\alpha = \bar{\sigma}\bar{\tau}(\alpha)$, and α lies in L_0, the fixed field of $\bar{\sigma}\bar{\tau}$. But $g(\bar{\tau})$ may be changed by a coboundary $z\bar{\tau}(z^{-1}) = (\alpha\bar{\tau}(b^{-1}), b\bar{\tau}(a^{-1}))$ if $z = (a, \bar{\sigma}(a^{-1}))$ so that $z\bar{\sigma}(z^{-1}) = 1$ and the relation $g(\bar{\sigma}) = 1$ will not be changed. It follows that α lies in $L_0^\times/N_{L/L_0}L^\times$, and this last group is isomorphic to $H^1(F,T)$.

Several remarks have to be made before the trace formula is introduced. Let F be local. If γ and γ' are stably conjugate elements of $G(F)$ then there exists g in $G(\bar{F})$ ($\bar{F}$ is the algebraic closure of F) with $\gamma' = g^{-1}\gamma g$. The isomorphism

$h \mapsto g^{-1}hg$ from the centralizer G_γ of γ in G to $G_{\gamma'}$ will be used to transport invariant forms of highest degree from G_γ to $G_{\gamma'}$ and hence Haar measures from $G_\gamma(F)$ to $G_{\gamma'}(F)$.

Let f be a smooth function on $G(F)$ compactly supported modulo the centre $Z(F)$ which transforms under $Z(F)$ by a character ω^{-1} of E^1. For any γ in $G(F)$ put

$$\Phi_f(\gamma) = \int_{G_\gamma(F)/G(F)} f(g^{-1}\gamma g)\, dg;$$

implicit is a choice of invariant forms of highest degree on G_γ and G. Let γ be regular, write $T = G_\gamma$, and define

$$\Phi_f^{\mathrm{st}}(\gamma) = \Phi_f^{\mathrm{lab}}(\gamma) = \Phi_f(\gamma)$$

unless the torus $T(F)$ is stably conjugate to $H(F)$.

If γ lies in a compact torus which splits over E then its stable class contains γ_1 in $H_1(F)$ and γ_u in $H_u(F)$ for a fixed u in $F - NE$. Put

$$\Phi_f^{\mathrm{st}}(\gamma) = \Phi_f(\gamma_1) + \Phi_f(\gamma_u)$$

and

$$\Phi_f^{\mathrm{lab}}(\gamma) = \lambda\kappa(\beta)\Phi_f(\gamma_1) + \lambda\kappa(\beta u)\Phi_f(\gamma_u),$$

where λ is the constant $\lambda(E/F, \psi)$ of [3], p. 730, l. 3. As the notation indicates both expressions depend only on the stable class of γ, and not on the choice of γ_1, u or γ_u. Φ^{lab} depends on κ, and β was defined together with $H(F)$.

The trace formula can now be introduced. Then E/F is a quadratic extension of number fields, ω is a unitary character of $E^1 \backslash \mathbb{A}_E^1$, $L^2(\omega)$ is the space of functions ψ on $G(F)\backslash G(\mathbb{A})$ which transform under $Z(\mathbb{A})$ $(\simeq \mathbb{A}_E^1)$ by ω and are square-integrable on $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$, r denotes the restriction to the discrete spectrum L_0 of $L^2(\omega)$ of the right regular representation of $G(\mathbb{A})$ (by $r(g)\psi(h) = \psi(hg)$). Let $f = \otimes f_v$ be a function on $G(\mathbb{A})$ such that (1) f_v are smooth if v is archimedean, locally constant if v is p-adic, compactly supported on $G(F_v)$ modulo $Z(F_v)$, transform under the centre $Z(F_v)$ by the component ω_v^{-1} of ω^{-1} at v, (2) for almost all p-adic v each f_v is equal to the function f_v^0 which obtains the value 0 at g unless $g = zk$ (z in $Z(F_v)$, k in the standard maximal compact subgroup K_v of $G(F_v)$), when it is the quotient of $\omega_v(z)^{-1}$ by the volume $|K_v|$ of K_v.

The convolution operator $r(f)$ on L_0, defined by

$$r(f)\psi(h) = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \psi(hg) f(g)\, dg$$

is of trace class; an explicit expression for $\operatorname{tr} r(f)$ was essentially given in [3].

PROPOSITION 2. *The trace formula $TF(f)$ for* $\operatorname{tr} r(f)$ *is the sum of*

(1) $$|Z(\mathbf{A})G(F)\backslash G(\mathbf{A})| \prod_v f_v(1),$$

(2) $$\sum_{\{T\}_s} \epsilon(T)|Z(\mathbf{A})T(F)\backslash T(\mathbf{A})| \sum_\gamma{}' \prod_v \Phi^{\mathrm{st}}_{f_v}(\gamma),$$

(3) $$\frac{1}{2}\left(\lambda_0 - \sum_v \frac{L'(1,1_v)}{L(1,1_v)}\right) \prod_v \int_{F_v} f_v^K\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) da,$$

(4) $$\frac{1}{2}\lambda_{-1} \sum_\gamma \sum_v |a\bar{a}|^{1/2} \int_{F_v} f_v^K\left(\gamma\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) \log\|(1-(a\bar{a})^{-1}, n)\| \, dn \prod_{w \neq v} F(\gamma, f_w),$$

(5) $$-\frac{1}{4\pi} \sum_{\mu_0} \int_{-i\infty}^{i\infty} m(\eta)^{-1} m'(\eta) \operatorname{tr} I(\eta, f) \, ds,$$

(6) $$-\frac{1}{4\pi} \sum_{\mu_0} \sum_v \int_{-i\infty}^{i\infty} \operatorname{tr}\{R_v(\eta_v)^{-1} R_v(\eta_v)' I(\eta_v, f_v)\} \prod_{w \neq v} \operatorname{tr} I(\eta_w, f_w) \, ds,$$

(7) $$-\frac{1}{4} \sum_{\eta = w\eta} \operatorname{tr} M(\eta) I(\eta, f),$$

(8) $$\frac{1}{4}|Z(\mathbf{A})H(F)\backslash H(\mathbf{A})| \sum_\gamma{}' \prod_v f_v^H(\gamma),$$

(9) $$\frac{1}{2} L(1,\kappa) \prod_v \frac{L(1,1_v)}{L(1,\kappa_v)} \int_{F_v} f_v^K\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) \kappa_v(a) \, da.$$

Explanation of notations. Volumes of groups are denoted by $|\ldots|$. The first sum in (2) is taken over a set of representatives $T(F)$ for the stable conjugacy classes of compact tori in $G(F)$; $\epsilon(T)$ is $\frac{1}{2}$ unless T splits over E where $\epsilon(T) = \frac{1}{4}$; the sum over γ is taken over all regular elements in $T(F)$. $L(s,1_v)$ is the local factor of the Hecke–Tate L-function $L(s,1)$, λ_0 is the constant term in the Laurent expansion of $L(s,1)$ at $s=1$, and λ_{-1} is the residue. $f_v^K(g) = \int f_v(k^{-1}gk)\,dk$ (k in K_v) in (3), (9) and (4). The sum over γ in (4) is taken over *all* γ in the diagonal subgroup $A(F)$ of $G(F)$, modulo $Z(F)$.

The term (4) does not appear in the standard form; the term (5.3) of [3], p. 753, has been rewritten here in the style of [1], to afford an application of the summation formula. If γ is regular with eigenvalues γ_1, γ_2 put

$$\Delta_v(\gamma) = |(\gamma_1 - \gamma_2)^2/\gamma_1\gamma_2|_{F_v}^{1/2}, \qquad F(\gamma, f_v) = \Delta_v(\gamma)\Phi_{f_v}(\gamma).$$

$F(\gamma, f_v)$ extends to a smooth function on $A(F)$ whose value at $\gamma = 1$ is the local factor in the product of (3). Given g in G it can be written according to the decomposition $G = NAK$ (N is the unipotent radical of the upper triangular Borel subgroup $B = NA$ of G) as

$$g = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}k; \quad \text{put} \quad H(g) = \log|a\bar{a}|_{F_v} = \log|a|_{E_v}.$$

The term (4) normally appears in the form

$$-\frac{1}{2}\int_{N(\mathbf{A})}\sum_{\gamma}{}' f^K(n^{-1}\gamma n)H(wn)\,dn$$
$$= \frac{1}{2}\int_{\mathbf{A}}\sum_{\gamma}{}' f^K(n^{-1}\gamma n)\sum_{v}\log\|(1,x)\|_{F_v}\,dx \qquad \left(n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right),$$

where $\|(a,b)\|_v$ is the maximum of $|a|_v, |b|_v$ in the p-adic case, and the square root of $a\bar{a} + b\bar{b}$ in the archimedean case. The sum over γ is taken over the regular γ in $Z(F)\backslash A(F)$.

Let v_0 be a fixed place. If all components f_v of f for $v \neq v_0$ are fixed, then the γ are taken in a fixed finite set depending on the compact support of f_v for all $v \neq v_0$. The sum over γ taken in front of the integral, $\log\|(1,x)\|_v$ may be replaced (for all v) by

$$\log\|((1 - 1/a\bar{a}), (1 - 1/a\bar{a})x)\|_v \qquad \text{if} \quad \gamma = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix},$$

since $1 - 1/a\bar{a}$ lies in $F^{\times}$. Each summand in the sum over γ is equal to

$$\frac{1}{2}\lambda_{-1}\sum_{v}A(\gamma, f_v)\prod_{w \neq v}F(\gamma, f_w),$$

where

$$A(\gamma, f_v) = \Delta_v(\gamma)\int_{F_v} f_v^K(n^{-1}\gamma n)\log\|((1 - 1/a\bar{a}), (1 - 1/a\bar{a})x)\|_v\,dx,$$

because of the difference between the Tamagawa and product measures. As $\Delta_v(\gamma) = |a\bar{a}|_v^{1/2}|1 - 1/a\bar{a}|_v$, a standard change of variable shows that $A(\gamma, f_v)$ is the weighted orbital integral in (4).

Although nonsmooth the compactly supported function $A(\gamma, f_v)$ of γ extends continuously to $\gamma = 1$; its asymptotic behavior ([1], 2.7.1) is of the type discussed in [1], Lemma 2.8, in the p-adic case. The summation formula can therefore be applied to the global function of (4). Note that the sum over v in (4) is taken only over a finite set independent of v_0 since for almost all v we have $|1 - 1/a\bar{a}|_v = 1$ for any of the γ in the finite set; if in addition $f_v = f_v^0$ then $A(\gamma, f_v^0) = 0$. The limit of $A(\gamma, f_v)$ at $\gamma = 1$ is

$$A(1, f_v) = \int_{F_v} f_v^K(n)\log|x|\,dx.$$

A standard calculation shows that the term of [3], (5.11), indexed by $\kappa = 1$, is given by

$$\tfrac{1}{2}\lambda_0\theta(1) + \tfrac{1}{2}\lambda_{-1}\theta'(1).$$

where

$$\theta(s) = \prod_v \theta_v(s), \qquad \theta_v(s) = \frac{L(1, 1_v)}{L(s, 1_v)} \int_{F_v} f_v^K\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right)|a|^{s-1}\, da.$$

This explains the inclusion of the limit at $\gamma = 1$ in the sum of (4), and the appearance of the (finite) sum over v of $L'(1,1_v)/L(1,1_v)$ in (3).

For the terms (5), (6), (7), let $I(\eta)$ be the representation of $G(\mathbf{A})$ induced from the character η of $B(\mathbf{A})$ ($B = AN$) which obtains the value $\mu(a)$ at $\left(\begin{smallmatrix} a & x \\ 0 & \bar{a}^{-1} \end{smallmatrix}\right)$. Here μ is a character of $E^\times \backslash \mathbf{A}_E^\times$. The family of such μ is a one-dimensional analytic manifold with infinitely many connected components; if μ_0 is a unitary character then any μ in the connected component of μ_0 can be expressed in the form $\mu(a) = \mu_0(a)|a|^s$ (s in $\mathbf{C}$), and differentiation is defined with respect to s. Recall that $M(\eta) = m(\eta) \otimes R(\eta_v)$ [3]. For (5), (6) note that the μ_0 are taken over a set of representatives for the connected components of the manifold of μ, their restriction to the centre is ω, and the local components μ_{0v} are unramified whenever f_v is spherical. The sum over v in (6) is finite since when η_v is unramified the operator $R(\eta_v)$ is a scalar and its derivative $R'(\eta_v)$ is 0. A misprint of [3] (5.6) is corrected in (5).

The sum of (7) is taken over the η for which $w\eta$ (defined by $w\eta(a) = \eta(waw^{-1})$) is equal to η; equivalently: $\mu(a) = \mu(\bar{a}^{-1})$, and so μ is a character of $E^\times N\mathbf{A}_E^\times \backslash \mathbf{A}_E^\times$. If μ is trivial on $\mathbf{A}^\times$ there exists a character ν of $\mathbf{A}_E^\times$ such that $\mu(a) = \nu(a/\bar{a})$. The representation $I(\eta) = I(\nu, \nu)$ is irreducible, the operator $M(\eta)$ is a scalar, its value is -1 (evaluated as a limit). If the restriction of μ to $\mathbf{A}^\times$ is of order exactly 2 then $L(1_F, \mu^{-1}) = L(1_F, \mu)$ and $m(\eta) = 1$, $M(\eta) = \otimes R(\eta_v)$.

The sum of (8) is over the regular γ in $Z(F)\backslash H(F)$, and $f_v^H(\gamma)$ is the smooth compactly supported (modulo centre) function $\Delta_v(\gamma)\Phi_{f_v}^{\mathrm{lab}}(\gamma)$ ([3], Lemma 2.1). In (9) κ is the restriction to $\mathbf{A}^\times$ of our earlier character κ of $\mathbf{A}_E^\times/E^\times N\mathbf{A}_E^\times$, and the L-functions are defined with respect to F. Recall that (9) is the term of (8) indexed by $\gamma = 1$ ([3], p. 761), hence the term (9), and the prime in the sum of (8), will be erased.

A key role will be played by the *stable trace formula* $\mathrm{STF}(f)$, defined to be

$$\mathrm{STF}(f) = \mathrm{TF}(f) + \frac{1}{4}\sum_\eta \operatorname{tr} R(\eta) I(\eta, f) - \frac{1}{4}|H(F)Z(\mathbf{A})\backslash H(\mathbf{A})| \sum_\gamma \prod_v f_v^H(\gamma). \tag{10}$$

The sum over the η is taken over all $\mu: \mathbf{A}_E^\times/E^\times N\mathbf{A}_E^\times \to \mathbf{C}^\times$ whose restriction to

$\mathbb{A}^\times$ is nontrivial; γ ranges over $Z(F)\backslash H(F)$. This is the same as (1)–(6) and the subsum over μ trivial on $\mathbb{A}^\times$ in (7), of Proposition 2. Applying the summation formula to the smooth function $\gamma \to \prod f_v^H(\gamma)$ of $H(\mathbb{A})$, and the pair $Z(F)\backslash H(F)$, $Z(\mathbb{A})\backslash H(\mathbb{A})$, the last term in $\mathrm{STF}(f)$ can be expressed as

$$-\frac{1}{4}\sum_\theta \operatorname{tr}\theta(f^H),$$

where θ ranges over all characters of $H(F)\backslash H(\mathbb{A})$ which transform under $Z(\mathbb{A})$ by $\omega\kappa^{-1}$.

Extend ω to a character $\tilde{\omega}$ of $Z(\mathbb{A}_E)$, and recall that κ is a character of this group. Then each θ extends to a character of $H_u(\mathbb{A})Z(\mathbb{A}_E)$ which transforms under $Z(\mathbb{A}_E)$ by $\tilde{\omega}\kappa^{-1}$. Passing to local notations, the restriction of the local component θ to the intersection $H_u'(F)$ of $S(F)$ and $H_u(F)Z(E)$ defines [3] a set $\{\pi^+(\theta), \pi^-(\theta)\}$ of one or two irreducible representations of $S(F)$ such that for regular γ

$$\Delta(\gamma)(\chi_{\pi^+(\theta)}(\gamma) - \chi_{\pi^-(\theta)}(\gamma)) = \lambda\kappa(\mu\beta)\big(\theta(\gamma) + \theta(w^{-1}\gamma w)\big) \quad \text{or} \quad 0,$$

depending on whether the conjugacy class of γ intersects $H_u'(F)$ (u in $F^\times/NE^\times$) or not; w is a nontrivial element in the absolute Weyl group of H_u'. This statement is equivalent to the identity

$$\operatorname{tr}\theta(f^H) = \operatorname{tr}\pi^+(\theta, f) - \operatorname{tr}\pi^-(\theta, f),$$

by means of the Weyl integration formula. Extending π^+, π^- to $S(F)Z(E)$ and restricting to $G(F)$, both formulae remain valid, relating the character θ of $H(F)$ to the irreducible representations $\pi^+(\theta), \pi^-(\theta)$ of $G(F)$ so obtained.

If v splits in E then the set $\{\pi^+, \pi^-\}$ consists of a single representation π^+. When v does not split in E both π^+ and π^- exist. If θ does not split through the determinant then π^+, π^- are supercuspical. Otherwise there exists a character ν of E^1 so that $\theta(g) = \nu(\det g)$ (g in $H_u(F)$), and hence a character

$$\eta\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1}\end{pmatrix} = \nu(a/\bar{a})\kappa(a) \qquad (a \text{ in } E^\times)$$

of $A(F)$. The character η', where $\eta'\left(\begin{smallmatrix} a\bar{a} & 0 \\ 0 & 1\end{smallmatrix}\right) = \tilde{\omega}(\bar{a})\eta\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a}^{-1}\end{smallmatrix}\right)$, is of the type considered at [3], Lemma 3.6; hence

$$\operatorname{tr} R(\eta)I(\eta, f) = \operatorname{tr}\pi^+(\theta, f) - \operatorname{tr}\pi^-(\theta, f),$$

as representations of $S(F)$, and therefore also of $G(F)$. If E_v/F_v is unramified then π^- is not unramified; if f is spherical then $\operatorname{tr}\pi^-(f) = 0$.

Returning to (10), the sum over η is cancelled by the sum over the θ which split through the determinant. The coefficient $\frac{1}{4}$ can be replaced by $\frac{1}{2}$ if the sum over the remaining θ is taken over equivalence classes of characters θ of $H(\mathbb{A})$

which do not split through the determinant, instead of over distinct θ, as before. As in [3], let $P(\theta)$ be the set of representations $\otimes\pi_v$ of $G(\mathbf{A})$, where π_v is either $\pi_v^+(\theta_v)$ or $\pi_v^-(\theta_v)$ for all v, and it is the unramified $\pi_v^+(\theta_v)$ for almost all v. Put $\epsilon(\pi)=\prod\epsilon_v(\pi_v)$, where $\epsilon_v(\pi_v)$ is 1 if π_v is π_v^+ and -1 if π_v is π_v^-. Then

$$\mathrm{STF}(f)=\sum_{\pi} m(\pi)\operatorname{tr}\pi(f)-\frac{1}{2}\sum_{\theta}\sum_{\pi}\epsilon(\pi)\operatorname{tr}\pi(f),$$

where the first sum is taken over the set of equivalence classes of discrete series representations π of $G(\mathbf{A})$; the second over all equivalence classes of characters θ of $H(F)\backslash H(\mathbf{A})$ which transform under $Z(\mathbf{A})$ by $\omega\kappa^{-1}$ and do not split through the determinant; the last sum is over all π in $P(\theta)$.

Finally, an *L-packet* $\{\pi\}$ of irreducible admissible representations of $G(F_v)$ can be introduced as in [3] to be an equivalence class under the action of $GL(2,F_v)$ on the set of representations of $G(F_v)$; the action is given by $g:\pi\to{}^g\pi$, where ${}^g\pi(h)=\pi(g^{-1}hg)$. If v splits in E then $G(F_v)=GL(2,F_v)$ and any L-packet consists of a single equivalence class of representations of $G(F_v)$. Otherwise [3] an L-packet consists of one or two equivalence classes of representations of $G(F_v)$. An L-packet consists of two classes if and only if it is of the form $\{\pi^+(\theta),\pi^-(\theta)\}$ for some θ.

2. Twisted stable conjugacy. Analogous discussion has to be carried out over the quadratic field extension E. Consider first any cyclic extension $E\subset\bar{F}$ of degree l over F, and fix a generator σ of $\mathrm{Gal}(E/F)$. Let G be a reductive group defined over F and denote by G'' the product of G with itself l times, regarded as a group over F.

Let α be the automorphism

$$\alpha:(x_1,x_2,\ldots,x_l)\mapsto(x_l,x_1,\ldots,x_{l-1})$$

of $G''(\bar{F})$ over F, and let $\mathrm{Gal}(\bar{F}/F)$ act on $G''(\bar{F})$ through the action of $\mathrm{Gal}(E/F)$ by $\sigma^r\cdot x=\alpha^r(x)$ (x in $G''(\bar{F})$, $0\leqslant r\leqslant l$). This action of $\mathrm{Gal}(\bar{F}/F)$ defines an element of $H^1(F,\operatorname{Aut}G'')$, and hence a group $G'=\mathrm{Res}_{E/F}G$ over F. $G'(\bar{F})$ is realized as a product of l factors of $G(\bar{F})$ with the action

$$\tau((x_1,x_2,\ldots,x_l))=\alpha^{-r}((\tau x_1,\tau x_2,\ldots,\tau x_l))$$

(τ in $\mathrm{Gal}(\bar{F}/F)$, $\tau|_E=\sigma^r$). Then $G(E)$ is the product of l copies of $G(E)$ and

$$G'(F)=\{\delta=(x,\sigma^{-1}x,\sigma^{-2}x,\ldots,\sigma^{-(l-1)}x);\ x\text{ in }G(E)\}.$$

$G(F)$ embeds in $G'(F)$ via the diagonal map.

Definition 1. The elements δ_1,δ_2 of $G'(F)$ are *twisted conjugate* if $\delta_2=g^{-1}\delta_1\alpha(g)$ for some g in $G'(F)$, and they are *stably twisted conjugate* if $\delta_2=g^{-1}\delta_1\alpha(g)$ for some g in $G'(\bar{F})$.

For any δ in $G'(\bar{F})$ and x in $G(E)$ put

$$N'\delta = \delta\alpha(\delta)\alpha^2(\delta)\dots\alpha^{l-1}(\delta), \qquad N'x = x\sigma(x)\sigma^2(x)\dots\sigma^{l-1}(x).$$

In our case $l = 2$ and we shall verify directly the following:

LEMMA 2. *The map N' induces a bijection ("norm map N") from the set of stable twisted conjugacy classes of g in $G(E)$ ($\simeq G'(F)$) with regular $N'g$ to the set of stable conjugacy classes of regular elements in $G(F)$.*

Proof. Since g lies in $G(E)$ the determinant z of $g\sigma(g)$ lies in E^1, and $g\sigma(g)$ is equal to $h^{-1}\left(\begin{smallmatrix} c & 0 \\ 0 & z/c \end{smallmatrix}\right)h$ for some h in $G(L)$ where $L = E(c)$ is either E or a quadratic extension of E (since $g\sigma(g)$ lies in $G(E)$). We have to show that the stable conjugacy class of $g\sigma(g)$ intersects $G(F)$. The lemma is easy to establish for c in $E^\times$ since then c lies in E^1 or $z = c/\bar{c}$.

Suppose L is quadratic over E, and σ_1 is an embedding of L in $\bar{F}$ whose restriction to E is bar. Then

$$g^{-1}h^{-1}\begin{pmatrix} c & 0 \\ 0 & z/c \end{pmatrix}hg = g^{-1}g\sigma(g)g = \sigma(g\sigma(g)) = \sigma_1(h)^{-1}\begin{pmatrix} z\sigma_1 c & 0 \\ 0 & \sigma_1 c^{-1} \end{pmatrix}\sigma_1(h)$$

implies that (1) $c\sigma_1 c = 1$ or (2) $\sigma_1 c = \bar{z}c$. In both cases σ_1 stabilizes L, hence L/F can be seen to be galois. Let τ_1 denote an element of order 2 in $\mathrm{Gal}(L/F)$ whose restriction to E is trivial. Since $\sigma_1^2 c = c$ in both (1) and (2), L/F is biquadratic. The identity

$$h^{-1}\begin{pmatrix} c & 0 \\ 0 & z/c \end{pmatrix}h = g\sigma(g) = \tau_1(g\sigma(g)) = \tau_1(h^{-1})\begin{pmatrix} \tau_1 c & 0 \\ 0 & z/\tau_1 c \end{pmatrix}\tau_1(h)$$

implies that $\tau_1 c = z/c$; otherwise $c = \tau_1 c$ and c lies in $E^\times$, contradicting the assumption that $L \neq E$.

In case (2) it remains to find h in $GL(2, L)$ such that

$$h^{-1}\begin{pmatrix} c & 0 \\ 0 & \sigma_1 c^{-1} \end{pmatrix}h = \tau_1\left(h^{-1}\begin{pmatrix} c & 0 \\ 0 & \sigma_1 c^{-1} \end{pmatrix}h\right) = \tau_1(h^{-1})\begin{pmatrix} \tau_1 c & 0 \\ 0 & \tau_1\sigma_1 c^{-1} \end{pmatrix}\tau_1(h),$$

that is

$$\tau_1(h)h^{-1}\begin{pmatrix} c & 0 \\ 0 & \sigma_1 c^{-1} \end{pmatrix}\left(\tau_1(h)h^{-1}\right)^{-1} = \begin{pmatrix} \sigma_1 c^{-1} & 0 \\ 0 & c \end{pmatrix},$$

and such that

$$\sigma_1(h)h^{-1}\begin{pmatrix} c & 0 \\ 0 & \sigma_1 c^{-1} \end{pmatrix}\left(\sigma_1(h)h^{-1}\right)^{-1} = \begin{pmatrix} c & 0 \\ 0 & \sigma_1 c^{-1} \end{pmatrix}.$$

Take

$$h = \begin{pmatrix} a & \tau_1 a \\ \tau_1 a & a \end{pmatrix}, \qquad a = \sigma_1 a, \qquad a^2 \neq \tau_1 a^2.$$

In case (1) we have to find h in $GL(2, L)$ with

$$\tau_1(h)h^{-1}\begin{pmatrix} c & 0 \\ 0 & \tau_1 c \end{pmatrix}\left(\tau_1(h)h^{-1}\right)^{-1} = \begin{pmatrix} \tau_1 c & 0 \\ 0 & \tau_1 c \end{pmatrix}$$
$$= \sigma_1(h)h^{-1}\begin{pmatrix} c & 0 \\ 0 & \tau_1 c \end{pmatrix}\left(\sigma_1(h)h^{-1}\right)^{-1}.$$

It is given by $h = \left(\begin{smallmatrix} 1 & ij \\ 1 & -ij \end{smallmatrix}\right)$ with $j = -\tau_1 j$, $i = -\sigma_1 i$ in $L^\times$.

It follows that $g\sigma(g)$ is conjugate in $G(L)$ to an element of $G(F)$. The stable conjugacy class of $g\sigma(g)$ in $G(F)$ is uniquely determined by its eigenvalues, and it is easy to check (see below) that each stable conjugacy class of $G(F)$ is obtained (this comment is implicit in the statement of Lemma 5.2 (below)). The proof of [4], Lemma 4.2 (with α replacing σ, $G'(E)$ replacing $G(E)$, and h in $G'(E)$ on p. 33, l. -5) implies (for γ, δ in $G'(E)$) that $N'\delta$, $N'\gamma$ (in $G(F)$) are stably conjugate (if and) only if δ and γ are stably twisted conjugate, and the lemma follows.

To determine the twisted conjugacy classes within the stable twisted conjugacy class of some δ in $G'(F)$ with $N'\delta$ regular, let g be an element of $G'(\bar{F})$ so that $\gamma = g^{-1}N'\delta g$ lies in $G(F)$, and put $T' = \mathrm{Res}_{E/F}T$ where T is the Cartan subgroup of G containing γ. If $h^{-1}\delta\alpha(h)$ lies in $G'(F)$ for h in $G'(\bar{F})$ then $\tau(h^{-1}\delta\alpha(h)) = h^{-1}\delta\alpha(h)$ and

$$\left(\tau(h)h^{-1}\right)\delta\alpha\left(\tau(h)h^{-1}\right)^{-1} = \delta \qquad \text{for any } \tau \text{ in } \mathrm{Gal}(\bar{F}/F).$$

If $\delta' = g^{-1}\delta\alpha(g)$ then $N'\delta' = \gamma$ (in $T(F)$); hence δ' lies in $T'(\bar{F})$ and $T(\bar{F})$ is the group of all x in $G'(\bar{F})$ with $x\delta'\alpha(x^{-1}) = \delta'$. Thus $h_\tau = g^{-1}\tau(h)h^{-1}g$ lies in $T(\bar{F})$. But γ is in $T(F)$ and both δ and $N\delta = g\gamma g^{-1}$ are in $G'(F)$; hence $\tau(g^{-1})g$ is in $T'(\bar{F})$ and the map $\tau \to h_\tau$ is a one-cocycle of $\mathrm{Gal}(\bar{F}/F)$ in $T(\bar{F})$. The map $h \to \{\tau \to h_\tau\}$ is easily seen to be an injection of the set of twisted conjugacy classes in the stable twisted conjugacy class of δ in $G'(F)$ (with regular $N'\delta$) into $H^1(F, T)$.

In our case G is the quasi-split unitary group in two variables and $l = 2$. The groups $H^1(F, T)$ are described in §1. We shall need an explicit description of a set of representatives of the classes in a stable class.

If T splits over E it may be stably conjugate and since $H^1(F, A) = \{1\}$ also conjugate to the diagonal subgroup A of G over F. Otherwise $T(F)$ is stably conjugate to $H_u(F)$ (any u in $F^\times$, see §1), and isomorphic to $E^1 \times E^1$. Then $H^1(F, T) \simeq F^\times/NE^\times \times F^\times/NE^\times$. A set of representatives for the twisted classes within the stable class of

$$T'(F)=\left\{\left(h_1^{-1}\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}h_1, h_1^{-1}\begin{pmatrix}\bar{a}^{-1} & 0\\ 0 & \bar{b}^{-1}\end{pmatrix}h_1\right)\right\} \qquad (a,b \text{ in } E^\times)$$

is given by

$$\beta^{-1}T'(F)\alpha(\beta) \qquad \text{where } \beta=(\beta_1,1),\ \ \beta_1=h^{-1}\begin{pmatrix} u & 0\\ 0 & v\end{pmatrix}h,$$

and u,v are taken in $F^\times/NE^\times$ so that β lies in $T'(E)$. These tori are clearly stably twisted conjugate. They are not twisted conjugate. Indeed, if g_1 is an element in $G(E)$ such that for all a,b in $E^\times$ we have

$$(g_1,\sigma(g_1))^{-1}\left(h_1^{-1}\begin{pmatrix} a/u & 0\\ 0 & b/v\end{pmatrix}h_1, h_1^{-1}\begin{pmatrix} u/\bar{a} & 0\\ 0 & v/\bar{b}\end{pmatrix}h_1\right)\alpha(g_1,\sigma(g_1))$$

$$=\left(h_1^{-1}\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}h_1, h_1^{-1}\begin{pmatrix}\bar{a}^{-1} & 0\\ 0 & \bar{b}^{-1}\end{pmatrix}h_1\right)$$

(u,v in $F^\times$), then g_1 commutes with $h_1^{-1}\left(\begin{smallmatrix} a/\bar{a} & 0\\ 0 & b/\bar{b}\end{smallmatrix}\right)h_1$ and hence has the form h_1^{-1} $\left(\begin{smallmatrix} s & 0\\ 0 & t\end{smallmatrix}\right)^{-1}h_1$ (s,t in $E^\times$) and the displayed equality holds only if $u=s\bar{s}$, $v=t\bar{t}$ are in $NE^\times$. We shall use below the fact that modulo $Z'(F)\simeq Z(E)$ the classes in the stable class of $T(F)$ are parametrized by $F^\times/NE^\times$.

It remains to deal with a CSG (Cartan subgroup) T of G which does not split over E; it splits over a quadratic extension L of E. Adopting the notations of §1, $T'(F)$ is isomorphic to the group

$$\left\{\left(\begin{pmatrix} a & 0\\ 0 & \tau_1 a\end{pmatrix},\begin{pmatrix}\sigma_1\tau_1 a^{-1} & 0\\ 0 & \sigma_1 a^{-1}\end{pmatrix}\right);\ a \text{ in } L^\times\right\}.$$

Take $\beta=(\left(\begin{smallmatrix} s & 0\\ 0 & \tau_1 s\end{smallmatrix}\right),1)$ with s in $L_0^\times$. Then $\beta^{-1}\alpha(\beta)$ is σ_1-invariant and

$$T_1'(F)=\beta^{-1}T'(F)\alpha(\beta)=\left\{\left(\begin{pmatrix} a/s & 0\\ 0 & \tau_1 a/\tau_1 s\end{pmatrix},\begin{pmatrix} s/\sigma_1\tau_1 a & 0\\ 0 & \sigma_1 s/\sigma_1 a\end{pmatrix}\right);\ a \text{ in } L^\times\right\}$$

is stably twisted conjugate but not conjugate to $T'(F)$ if s lies in $L_0^\times$ but not in $N_{L/L_0}L^\times$. Indeed if β lies in $T'(F)$ then $\beta^{-1}\alpha(\beta)=(\left(\begin{smallmatrix} b^{-1} & 0\\ 0 & \sigma_1 b^{-1}\end{smallmatrix}\right),(\dots))$ with $b=a\sigma_1\tau_1 a$ in $N_{L/L_0}L^\times$.

Finally, the norm map can be extended to all stable twisted conjugacy classes in $G'(F)$. If x is in $G(E)$ and $x\sigma(x)$ is unipotent regular in $Z(E)\backslash G(E)$ (and hence the conjugacy class in $Z(\bar{F})\backslash G(\bar{F})$ of $x\sigma(x)$ intersects $Z(F)\backslash G(F)$), we say that Nx is in the stable conjugacy class of unipotent regular elements in $Z(F)\backslash G(F)$. If $n=x\sigma(x)$ is unipotent regular in $Z(F)\backslash G(F)$ then x commutes with $\sigma(x)$ and hence with n, thus it is a product of z in $Z(E)$ and a unipotent regular element; it follows that x lies in the unique stable twisted conjugacy class of unipotent regular elements in $Z(E)\backslash G(E)$. The conjugacy classes in the stable conjugacy class of unipotent regular elements in $G(F)$ are parametrized by $F^\times/N_{E/F}E^\times$. However there is only one twisted conjugacy class in $Z(E)\backslash G(E)$ of x such that Nx is unipotent regular in $Z(F)\backslash G(F)$.

If δ lies in $G(E)$ and $z = \delta\sigma(\delta)$ is in $Z(E)$ (hence in $Z(F)$) write $N\delta = z$. But $z = \bar{a}/a$ for some a in $E^\times$, hence $N(a\delta) = 1$ and the question of the description of the stable twisted conjugacy class of δ in $G(E)$ with scalar norm reduces to that of the δ with $N\delta = 1$. A δ is twisted stably conjugate to 1 if there exists $g = (g_1, g_2)$ in $G'(E)$ with $(\delta, \sigma(\delta)) = g\alpha(g^{-1}) = (g_2 g_1^{-1}, g_2 g_1^{-1})$, namely if $\sigma(\delta) = \delta^{-1}$. It is twisted conjugate to 1 if $\delta = g\sigma(g^{-1})$ with g in $G(E)$. Hence the classes within a stable class here are parametrized by $H^1(F, G)$. The determinant, from $G(F)$ to E^1, induces an exact sequence

$$\{0\} \to SL(2, F) \to G(F) \to E^1 \to \{0\}$$

and hence

$$H^1(F, SL(2)) \to H^1(F, G) \to H^1(F, U(1)),$$

where $U(1, E) = GL(1, E)$ and $U(1, F) = E^1$. The group on the left is trivial, and that on the right is $F^\times/NE^\times$.

A set of representatives for the twisted conjugacy classes within the stable conjugacy class of 1 is given by

$$\delta_f = \begin{pmatrix} 0 & -if \\ i^{-1} & 0 \end{pmatrix},$$

where f ranges over a set of representatives in $F^\times$ for $F^\times/NE^\times$; i is a fixed element of $E^\times$ with $\bar{i} + i = 0$. Indeed, for any g in $G(E)$, the determinant of $g\delta_f\sigma(g^{-1})$ lies in $fNE^\times$, and the subgroup $H^1(F, G)$ of $F^\times/NE^\times$ is the full group.

The same parametrization can be obtained without mentioning $H^1(F, G)$ on noting that if $\sigma(\delta) = \delta^{-1}$ then

$$\delta = \begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix} \qquad (b + \bar{b} = 0, c + \bar{c} = 0; a, b, c \text{ in } E)$$

and

$$\begin{pmatrix} c/a & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & \bar{c}/\bar{a} \end{pmatrix} = \begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix} \qquad (d = bc\bar{c}/a\bar{a} + c),$$

when $ac \neq 0$. If $c = 0$ then δ is twisted conjugate to 1. Note that δ_f is twisted conjugate to 1 if f lies in $NE^\times$.

The above description of conjugacy classes will now be used to introduce the twisted orbital integrals. Let F be local and E a quadratic extension of F. If δ and δ' are stably conjugate elements of $G'(F)$ then there is g in $G'(\bar{F})$ with $\delta' = \alpha(g^{-1})\delta g$. The isomorphism $h \mapsto g^{-1}hg$ from the α-centralizer $G_\delta'^\alpha$ of δ in $G' = \mathrm{Res}_{E/F}G$ to $G_{\delta'}'^\alpha$ will be used to transport invariant forms of highest degree from $G_\delta'^\alpha$ to $G_{\delta'}'^\alpha$ and hence Haar measures from $G_\delta'^\alpha(F)$ to $G_{\delta'}'^\alpha(F)$. If $\gamma = N\delta$ then $G_\delta'^\alpha$ is isomorphic to G_γ and Haar measures are again related by means of this isomorphism (this last comment will be used in the comparison of §§5, 6).

Let ϕ be a smooth compactly supported function on $G'(F) \simeq G(E)$. For any δ in $G'(F) \simeq G(E)$ put

$$\Phi_\phi(\delta) = \int_{G_\delta'^\alpha(F)\backslash G'(F)} \phi\big(\alpha(g^{-1})\delta g\big)\, dg$$

$$= \int_{G_\delta^\sigma(E)\backslash G(E)} \phi\big(\sigma(g^{-1})\delta g\big)\, dg.$$

In the first integral expression δ is viewed as an element of $G'(F)$, while in the second as an element of $G(E)$, and G_δ^σ is the σ-centralizer of δ in G. We shall adopt the second expression.

Let δ be an element of $G'(F) \simeq G(E)$ with $\gamma = N\delta$ in $G(F)$, and let $D(\delta)$ be a set of representatives in $G(E)$ for the twisted conjugacy classes in the stable twisted conjugacy class of δ (in $G'(F)$). Relating measures as above we set

$$\Phi_\phi^{\mathrm{st}}(\gamma) = \sum \Phi_\phi(\delta') \qquad (\delta' \text{ in } D(\delta)).$$

It depends only on the stable conjugacy class of $\gamma = N\delta$, as the notation indicates.

Let κ be a fixed character of $E^\times/NE^\times$ in $\mathbf{C}^\times$ whose restriction to $F^\times$ is nontrival (as in §1). Put

$$\Phi_\phi^{\mathrm{lab}}(\gamma) = \sum \kappa(\det \delta')\Phi_\phi(\delta') \qquad (\delta' \text{ in } D(\delta)).$$

Again this depends only on the stable conjugacy class of $\gamma = N\delta$, but not on δ itself.

The unstable orbital integrals Φ_ϕ^{lab} will be described in more detail for regular γ, using the explicit description of $D(\delta)$ above. If the centralizer of γ is conjugate to $A(F)$ then $D(\delta)$ consists of a single element and

$$\Phi_\phi^{\mathrm{st}}(\gamma) = \Phi_\phi(\delta), \qquad \Phi_\phi^{\mathrm{lab}}(\gamma) = \kappa(\det \delta)\Phi_\phi(\delta).$$

If the eigenvalues of γ are $a/\bar{a}$, $b/\bar{b}$ (a, b in $E^\times$) then

$$\Phi_\phi^{\mathrm{lab}}\left(h^{-1}\begin{pmatrix} a/\bar{a} & 0 \\ 0 & b/\bar{b} \end{pmatrix}h\right) = \kappa(ab)\left[\Phi_\phi\left(h^{-1}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}h\right) + \Phi_\phi\left(h^{-1}\begin{pmatrix} au & 0 \\ 0 & bu \end{pmatrix}h\right)\right.$$
$$\left. - \Phi_\phi\left(h^{-1}\begin{pmatrix} au & 0 \\ 0 & b \end{pmatrix}h\right) - \Phi_\phi\left(h^{-1}\begin{pmatrix} a & 0 \\ 0 & bu \end{pmatrix}h\right)\right];$$

here u lies in F but not in NE, and then $\kappa(u) = -1$. For all cases note that $\kappa(\det\sigma(g^{-1})g) = \kappa(x\bar{x}) = 1$ ($x = \det g$) since κ is trivial on $NE^\times$.

If the splitting field of the centralizer of γ is the quadratic extension L of E then

$$\Phi_\phi^{\mathrm{lab}}(\gamma) = \kappa(a\tau_1 a)\left(\Phi_\phi\left(\begin{pmatrix} a & 0 \\ 0 & \tau_1 a \end{pmatrix}\right) - \Phi_\phi\left(\begin{pmatrix} as & 0 \\ 0 & \tau_1(as) \end{pmatrix}\right)\right) \qquad (s \text{ in } L_0 - N_{L/L_0}L).$$

Now for t in $N_{L/L_0}L^\times$, $t\tau_1 t$ lies in $N_{L/F}L^\times \subset N_{E/F}E^\times$ and $\kappa(t\tau_1 t) = 1$. On the other hand $N_{L_0/F}L_0^\times$ is of index two in $F^\times$ and distinct from $N_{E/F}E^\times$, hence $\kappa(s\tau_1 s) = -1$, whence the above description.

Finally, if $\gamma = 1$ put

$$\Phi_\phi^{\mathrm{st}}(\gamma) = \Phi_\phi(1) + \Phi_\phi(\delta) \qquad \left(\delta = \begin{pmatrix} 0 & -if \\ i^{-1} & 0 \end{pmatrix}; f \text{ in } F^\times - NE^\times, \bar{i} = -i \text{ in } E^\times\right)$$

and

$$\Phi_\phi^{\mathrm{lab}}(\gamma) = \Phi_\phi(1) - \Phi_\phi(\delta) = \sum_f \kappa(\det\delta_f)\Phi_\phi(\delta_f) \qquad (f \in F^\times/NE^\times).$$

The same definitions also hold when Φ transforms under $Z(E)$ by the character ω_E^{-1} of $E^\times$, where $\omega_E(x) = \omega(x/\bar{x})$ (x in $E^\times$) and ω is a character of E^1.

3. Twisted trace formula. In the notations of §2 the centre $Z(\mathbb{A}_E)$ of $G(\mathbb{A}_E) = GL(2, \mathbb{A}_E)$ is isomorphic to $\mathbb{A}_E^\times$ and the norm map N takes z in $Z(\mathbb{A}_E)$ to $z/\bar{z}$ in $Z(\mathbb{A})$, the centre of $G(\mathbb{A})$. Let ω_E be the character of $Z(\mathbb{A}_E)$ defined by $\omega_E(z) = \omega(Nz)$; its restriction to $\mathbb{A}^\times$ is trivial. Denote by $L^2(\omega_E)$ the space of measurable functions ψ on $G(E)\backslash G(\mathbb{A}_E)$ which transform under $Z(\mathbb{A}_E)$ by ω_E and are square-integrable on $Z(\mathbb{A}_E)G(E)\backslash G(\mathbb{A}_E)$. Under the action of $G(\mathbb{A}_E)$ via r, $L^2(\omega_E)$ splits as a direct sum of two invariant mutually orthogonal subspaces L_0 and L_c which are discrete and continuous (respectively) sums of representations. Both are invariant under the action of σ.

Let $\phi = \otimes\phi_v$ be a function on $G(\mathbb{A}_E)$ whose properties are the same as those of $f = \otimes f_v$ (§1) with E_v and ω_{E_v} replacing F_v and ω_v. The restriction of the operator

$$r(\phi \times \sigma) = \int_{Z(\mathbb{A}_E)\backslash G(\mathbb{A}_E)} \phi(g) r(g \times \sigma)\, dg,$$

where

$$r(g \times \sigma)\psi(h) = \psi(\sigma(h)g),$$

to the discrete spectrum L_0 is of trace class. Since $G(\mathbb{A}_E)$ is of rank 1 the standard methods for $GL(2)$ are sufficient for the analysis required in the calculation of the twisted trace formula, which describes the trace of the above operator on L_0. This formula is a difference between two expressions.

As in the case of $U(1)$ in §0 we choose the global measure $dg = 2 \otimes dg_v$ on $G(\mathbf{A})\backslash G(\mathbf{A}_E)$, and the measure which assigns the volume 1 to each point in a finite set.

The first expression in the formula is the integral over $G(E)Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)$ of

$$\sum_{\gamma} \phi\big(\sigma(g^{-1})\gamma g\big) - \sum_{\delta} \sum_{\gamma \in Z(E)\backslash A(E)} \int_{N(\mathbf{A}_E)} \phi\big(\sigma(\delta g)^{-1}\gamma n \delta g\big)\tau(H(\delta g) - T)\,dn$$

(δ in $B(E)\backslash G(E)$). Here τ is the characteristic function of the positive real numbers, T is a large positive number, and $H(g) = \log|a/b|$ if $g = n\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)k$ according to the Iwasawa decomposition of $G(\mathbf{A}_E)$. The integral breaks as a sum of three terms. They are:

$$(1) \qquad \sum_{\{\delta\}} \int_{Z(\mathbf{A}_E)G_\delta^\sigma(E)\backslash G(\mathbf{A}_E)} \phi\big(\sigma(g^{-1})\delta g\big)\,dg;$$

the sum is taken over all σ-conjugacy classes of elements of $Z(E)\backslash G(E)$ whose norm is elliptic (including central) in $Z(F)\backslash G(F)$, and $G_\delta^\sigma(E)$ is the σ-centralizer of δ in $G(E)/Z(E)$.

$$(2) \qquad \int \Big[\sum_{\nu} \phi\big(\sigma(g^{-1})\nu g\big) - \int_{N(\mathbf{A}_E)} \phi\big(\sigma(g^{-1})ng\big)\,dn\,\tau(H(g) - T) \Big]\,dg$$

(g in $N(E)A(F)Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)$); the sum is over all ν in $N(E)$ such that $N\nu$ is unipotent regular in $G(F)$.

$$(3) \qquad \int \sum_{\eta} \Big[\frac{1}{2} \sum_{\nu \in N(E)} \phi\big(\sigma(\nu g)^{-1}\eta\nu g\big) - \int_{N(\mathbf{A}_E)} \phi\big(\sigma(g^{-1})\eta m g\big)\,dm\,\tau(H(g) - T) \Big]\,dg;$$

g in $B(E)Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)$, and η ranges over the elements of $A(E)/Z(E)$ whose norm is regular.

The terms of (1) indexed by the twisted conjugacy classes $\{\delta\}$ with $N\delta$ in $Z(F)$ will be taken separately. If $\delta\sigma(\delta) = z$ lies in $Z(E)$ then z lies in $Z(F)$ and $\delta = \sigma(g^{-1})\delta_f g$ (f in $F^\times/NE^\times$) for some g in $G(E)$; the δ_f were defined in §2. The natural map $F^\times/NE^\times \to F_v^\times/NE_v^\times$ permits us to regard δ_f as an element of $G(E_v)$ for all v and we can express the terms in the part of (1) under discussion as

$$\Phi_\phi(\delta_f) = 2\prod \Phi_{\phi_v}(\delta_f)$$

in the notations of §2, by the choice of the global measure. The product is absolutely convergent.

Using the embedding $\psi: F^\times/NE^\times \to \mathbf{A}^\times/N\mathbf{A}_E^\times$ we can write our sum

$$\sum_f \Phi_\phi(\delta_f) \quad \text{as} \quad \sum_{f \in \operatorname{Im}\psi} \Phi_\phi(\delta_f).$$

The last sum is finite. It can be written in the form

$$\frac{1}{2}\sum_f \Phi_\phi(\delta_f) + \frac{1}{2}\sum_f \kappa(\det \delta_f)\Phi_\phi(\delta_f) \qquad (f \text{ in } \mathbf{A}^\times/N\mathbf{A}_E^\times),$$

where κ, as in §1, is a character of $\mathbf{A}_E^\times/E^\times N\mathbf{A}_E^\times$ in $\mathbb{C}^\times$ whose restriction to $\mathbf{A}^\times$ is nontrivial. Note that the index in $\mathbf{A}^\times/N\mathbf{A}_E^\times$ of the image of $F^\times/NE^\times$ is 2. In the notations of §2 our sum can now be written as

$$\prod_v \Phi_\phi^{\mathrm{st}}(1) + \prod_v \Phi_\phi^{\mathrm{lab}}(1).$$

The remaining δ in (1) have regular norms, and we assume as we may that $N\delta$ lies in $G(F)$. The group G_δ^α of g in G' with $g^{-1}\delta\alpha(g) = \delta$ is isomorphic over F to the maximal torus T of G containing $N\delta$; T is not (stably) conjugate to A over F. Choose a representative T in each conjugacy class over F of such tori. The sum becomes

$$\sum_{\{T\}} \sum_\delta{}' \frac{|Z(\mathbf{A})T(F)\backslash T(\mathbf{A})|}{[W_0(T)]} \int_{G_\delta^\alpha(\mathbf{A})\backslash G'(\mathbf{A})} \phi\big(g^{-1}\delta\alpha(g)\big)\, dg.$$

The Weyl group $W_0(T)$ of T is the quotient by $T(F)$ of the normalizer $N(T(F))$ of $T(F)$ in $G(F)$. The Weyl group $W(T)$ will also be used; it is the quotient by $T(\overline{F})$ of the group of g in the normalizer $N(T(\overline{F}))$ of $T(F)$ in $G(\overline{F})$, such that ad g is defined over F. If $T(F)$ is any torus of $G(F)$ then $[W(T)] = 2$.

Let T be a torus of G whose splitting field L is a quadratic extension of E. The terms from the stable conjugacy class of T over F take the form

$$\frac{|Z(\mathbf{A})T(F)\backslash T(\mathbf{A})|}{[W(T)]} \sum_\delta{}' \sum_\epsilon \Phi_\phi\big(\epsilon^{-1}\delta\alpha(\epsilon)\big)$$

(δ with regular $N\delta$ in $Z(F)\backslash T(F)$). ϵ is a representative in $G'(\overline{F})$ of the elements in the group $H^1(F,T) \simeq L_0^\times/N_{L/L_0}L^\times$ which parametrizes the twisted conjugacy classes in the stable twisted class of δ (§2).

The natural map $L_0^\times/N_{L/L_0}L_0^\times \to L_{0v}^\times/N_{L_v/L_{0v}}L_v^\times$ permits us to regard ϵ as an element of the local group for all v, and we write

$$\Phi_\phi\big(\epsilon^{-1}\delta\alpha(\epsilon)\big) = \int \phi\big(g^{-1}\epsilon^{-1}\delta\alpha(\epsilon g)\big) = 2\prod_v \int \phi_v\big(g^{-1}\epsilon^{-1}\delta\alpha(\epsilon g)\big)$$

(g in $G^{\alpha}_{\epsilon^{-1}\delta\alpha(\epsilon)}\backslash G'$); the global groups are taken over $\mathbf{A}$ and the local over F_v. The absolute convergence of the product of local terms (denoted in §2 by $\Phi_{\phi_v}(\epsilon^{-1}\delta\alpha(\epsilon))$) can be proved as in the non-twisted case. The factor 2 appears by the choice of global measure on $G(\mathbf{A}_E)$.

Using the embedding $\psi: L_0^{\times}/N_{L/L_0}L^{\times}\to \mathbf{A}^{\times}_{L_0}/N_{L/L_0}\mathbf{A}^{\times}_L$ we may write

$$\sum_{\epsilon}\Phi_{\phi}(\epsilon^{-1}\delta\alpha(\epsilon))\qquad\text{as}\qquad\sum_{\epsilon\in\operatorname{Im}\psi}\Phi_{\phi}(\epsilon^{-1}\delta\alpha(\epsilon)).$$

Again, the last sum is finite. We choose the measure which assigns the volume 2 to the space $\mathbf{A}^{\times}_{L_0}/L_0^{\times}N_{L/L_0}\mathbf{A}^{\times}_L$ of two elements, and recall (§2) that $\kappa'=\kappa\circ N_{L_0/F}$ gives an isomorphism of this group with $\mathbf{Z}/2\mathbf{Z}$. In the obvious notations the last sum can be written as

$$\frac{1}{2}\sum_{\epsilon}\Phi_{\phi}(\epsilon^{-1}\delta\alpha(\epsilon))+\frac{1}{2}\sum_{\epsilon}\kappa'(\epsilon)\Phi_{\phi}(\epsilon^{-1}\delta\alpha(\epsilon));$$

ϵ ranges over a set of representatives in $G'(\overline{F})$ for the classifying group $\mathbf{A}^{\times}_{L_0}/N_{L/L_0}\mathbf{A}^{\times}_L$. The part of (1) under consideration is the (finite) sum over δ with regular $N\delta$ in $Z(F)\backslash T(F)$ of the product by $[W(T)]^{-1}|Z(\mathbf{A})T(F)\backslash T(\mathbf{A})|$ of

$$\prod_v\Phi^{\mathrm{st}}_{\phi_v}(\delta)+\prod_v\Phi^{\mathrm{lab}}_{\phi_v}(\gamma)$$

where $\gamma=N\delta$, the local distributions are those of §2 and the dependence on the component κ_v of κ at v is implicit in $\Phi^{\mathrm{lab}}_{\phi_v}$; here we used the fact that κ is trivial on $E^{\times}$.

A similar calculation can be repeated for $T=H$ in the unique stable conjugacy class of compact tori which split over E. For δ with regular $N\delta$ in $T(F)$ the classes in the stable twisted class of δ are parametrized by $H^1(F,T)\simeq(F^{\times}/NE^{\times})^2$, and we take, as usual, the measure which assigns $\mathbf{A}^{\times}/F^{\times}N\mathbf{A}^{\times}_E$ the volume 2. The contribution from the stable twisted class of H is the sum over δ with regular $N\delta$ in $H(F)$ of the product by $[W(H)]^{-1}|Z(\mathbf{A})H(F)\backslash H(\mathbf{A})|$ of

$$\frac{1}{4}\sum_{\kappa'}\sum_{\epsilon}\kappa'(\epsilon)\Phi_{\phi}(\epsilon^{-1}\delta\alpha(\epsilon));$$

κ' ranges over the group of (4) characters of $(\mathbf{A}^{\times}/F^{\times}N\mathbf{A}^{\times}_E)$, ϵ over a set of representatives in $G'(\overline{F})$ of $H^1(\mathbf{A},T)/H^1(F,T)$.

In §2 it was shown that

$$\beta^{-1}\delta\alpha(\beta)=h^{-1}\left(\begin{pmatrix}a/u&0\\0&b/v\end{pmatrix},\begin{pmatrix}u/\bar{a}&0\\0&v/\bar{b}\end{pmatrix}\right)h$$

($\beta=(\beta_1,1)$, $\beta_1=h^{-1}\left(\begin{smallmatrix}u&0\\0&v\end{smallmatrix}\right)h$), with u,v in $F^{\times}/NE^{\times}$, is a set of representatives for the stable twisted conjugates of

$$\delta = h^{-1}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} \bar{a}^{-1} & 0 \\ 0 & \bar{b}^{-1} \end{pmatrix}\right)h,$$

which are not twisted conjugate to each other. The function ϕ is assumed to be transforming under the centre by the character ω_E of $E^\times \backslash \mathbb{A}_E^\times$; ω_E is defined by $\omega_E(z) = \omega(z/\bar{z})$, and ω is a character on $E^1/\mathbb{A}_E^1$. Since ω_E is trivial on $\mathbb{A}^\times$, the sum over $\epsilon = \beta$ of $\kappa'(\epsilon)\Phi_\phi(\epsilon^{-1}\gamma\alpha(\epsilon))$ can be non-zero only if $\kappa'(\epsilon) = 1$ for all ϵ such that the product uv is in $NE^\times$. Hence the sum over κ' is taken over two elements only. Noting again that κ is trivial on $E^\times$ our sum is

$$\frac{1}{2}\prod_v \Phi_{\phi_v}^{\mathrm{st}}(\gamma) + \frac{1}{2}\prod_v \Phi_{\phi_v}^{\mathrm{lab}}(\gamma),$$

where $\gamma = N\delta$, the local integrals are as in §2 and $\Phi_{\phi_v}^{\mathrm{lab}}$ depends on the component κ_v of κ at v. The factor $\frac{1}{4}$ is replaced by $\frac{1}{2}$ by the choice of global measure.

The explicit expression for the twisted trace formula $\mathrm{TF}(\phi \times \sigma)$ will be recorded now, in the above notations, and those of the next section, where the remaining calculations will be done.

PROPOSITION 1. *The twisted trace formula* $\mathrm{TF}(\phi \times \sigma)$ *is given by*

(a) $$|Z(\mathbb{A})G(F)\backslash G(\mathbb{A})|\left[\prod_v \Phi_{\phi_v}^{\mathrm{st}}(1) + \prod_v \Phi_{\phi_v}^{\mathrm{lab}}(1)\right],$$

(b) $$\sum_{\{T\}_s} \epsilon(T)|Z(\mathbb{A})T(F)\backslash T(\mathbb{A})| \sum_{\gamma \in T(F)}{}' \left[\prod_v \Phi_{\phi_v}^{\mathrm{st}}(\gamma) + \prod_v \Phi_{\phi_v}^{\mathrm{lab}}(\gamma)\right],$$

(c) $$-\frac{1}{4}\sum_{\sigma\eta^E = w\eta^E} \mathrm{tr}\, M(\eta^E)I(\eta^E, \phi \times \sigma),$$

(d) $$\left(\lambda_0 - \sum_v \frac{L'(1, 1_v)}{L(1, 1_v)}\right)\prod_v \int\int\int \phi_v^K\left(\sigma(na)^{-1}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} na\right) dz\, dn\, da,$$

(e) $$\frac{1}{2}\lambda_{-1}\sum_{\gamma(=N\delta)}\sum_v \Delta_v(\gamma)\int\int \phi_v^K\left(\sigma(an)^{-1}\delta an\right)\log\|(u, un)\|_{E_v}\, dn\, da$$

$$\prod_{w \neq v} F(\gamma, \phi_w),$$

(f) $$-\frac{1}{4\pi}\sum_{\mu_0}\int_{-i\infty}^{i\infty} m(\eta^E)^{-1} m(\eta^E)' \mathrm{tr}\, I(\eta^E, \phi \times \sigma)\, ds,$$

(g) $$-\frac{1}{4\pi}\sum_{\mu_0}\sum_v \int_{-i\infty}^{i\infty} \mathrm{tr}\left\{R_v(\eta_v^E)^{-1} R_v(\eta_v^E)' I(\eta_v^E, \phi_v \times \sigma)\right\}$$

$$\times \prod_{w \neq v} \mathrm{tr}\, I(\eta_w^E, \phi_w \times \sigma)\, ds.$$

4. Other twisted terms. The Iwasawa decomposition $g = nzak$ of $Z(\mathbf{A}_E)\backslash G(\mathbf{A}_E)$ (with n in $N(\mathbf{A}_E)$, $z = \left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)$ and z in $\mathbf{A}^\times$, $a = \left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right)$ and a in $\mathbf{A}^\times\backslash\mathbf{A}_E^\times$, k in $K(\mathbf{A}_E)$) and the corresponding modular function $|az|_E$, can be used to put (2) of §3 in the form

$$\int\int\int\Bigg[\phi^K\left(\sigma(nza)^{-1}\begin{pmatrix}1&1\\0&1\end{pmatrix}nza\right)$$
$$-\int_{N(\mathbf{A}_E)}\phi^K\left(\sigma(nza)^{-1}mnza\right)dm\,\tau(H(z)-T)\Bigg]|az|_E\,dn\,da\,dz$$

(n in $N(\mathbf{A})\backslash N(\mathbf{A}_E)$, $\phi^K(g) = \int\phi(\sigma(k^{-1})gk)\,dk$ (k in K)). For any a and n put

$$F(a,n;z) = \phi^K\left(\sigma(na)^{-1}\begin{pmatrix}1&z\\0&1\end{pmatrix}na\right)|a|_E \qquad (z \text{ in } \mathbf{A}).$$

Our integral becomes the integral over a and n of

$$\int_{\mathbf{A}^\times}\left[F(a,n;z)|z|_F - \int_{\mathbf{A}}F(a,n;m)\,dm\,\tau(\log|z| < -T)\right]d^\times z.$$

A standard application of the summation formula shows that the constant term in T of the last integral (over z) is equal to the value at $s = 1$ of the analytic part of the integral (which converges for s with $\operatorname{Re} s > 1$)

$$\int_{\mathbf{A}^\times}F(a,n;z)|z|_F^s\,d^\times z.$$

The constant term in T of (2) in §3 is therefore equal to the value at $s = 1$ of the analytic part of the function $L(s, 1_F)\theta(s)$, where $\theta(s) = \prod_v\theta_v(s)$ and

$$\theta_v(s) = \frac{L(1,1_v)}{L(s,1_v)}\int\int\int_{F_v}\phi_v^K\left(\sigma(na)^{-1}\begin{pmatrix}1&z\\0&1\end{pmatrix}na\right)|z|_F^{s-1}\,dz|a|_E\,dz\,dn$$

(n in $N(F_v)\backslash N(E_v)$, $a = \left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right)$ and a in $F_v^\times\backslash E_v^\times$). The infinite product is absolutely convergent and can be differentiated term by term at a small neighborhood of $s = 1$.

The value at $s = 1$ of the analytic part of $L(s, 1_F)\theta(s)$ is given by

$$\lambda_0\theta(1) + \lambda_{-1}\theta'(1).$$

The derivative of the local factor $\theta_v(s)$ at $s = 1$ is equal to

$$\theta_v'(1) = \int\int\int\phi_v^K\left(\sigma(na)^{-1}\begin{pmatrix}1&z\\0&1\end{pmatrix}na\right)\log|z|_F\,dz|a|_E\,da\,dn$$
$$-\frac{L'(1,1_v)}{L(1,1_v)}\int\int\int\phi_v^K\left(\sigma(na)^{-1}\begin{pmatrix}1&z\\0&1\end{pmatrix}na\right)dz|a|_E\,da\,dn.$$

It is equal to 0 if $\phi_v = \phi_v^0$, namely for almost all v for any fixed ϕ.

To deal with (3) of §3 note that since $N\eta = (a/\bar{b}, b/\bar{a})$ (for $\eta = (a,b)$) is regular, the norm (from E to F) of the scalar a/b is not 1, and ηm in the last inner integral in (3) can be replaced by $\sigma(n^{-1})\eta n$ (n in $N(\mathbf{A}_E)$) so that we have

$$\int \sum \int_{N(\mathbf{A}_E)} \phi\big(\sigma(ng)^{-1}\eta ng\big)\big(\tfrac{1}{2} - \tau(H(g) - T)\big)\, dn\, dg$$

(g in $Z(\mathbf{A}_E)N(\mathbf{A}_E)A(E)\backslash G(\mathbf{A}_E)$). The sum is taken over the η in $Z(E)\backslash A(E)$ whose norm $N\eta$ is regular. The integral over $N(\mathbf{A}_E)$ can be combined with the integral over g, and the weight factor can be replaced by $\frac{1}{2}$ of $1 - \tau(H(g) - T) - \tau(H(wg) - T)$. The integral becomes

$$\frac{1}{2}\int_{N(\mathbf{A}_E)}\int_{Z(\mathbf{A}_E)A(\mathbf{A})A(E)\backslash A(\mathbf{A}_E)} \sum_{\eta} \phi^K\big(\sigma(an)^{-1}\eta an\big)v(n)\, da\, dn$$

where

$$v(n) = \int_{A_E^1 E^\times\backslash \mathbf{A}_E^\times}\big[1 - \tau(H(Z) - T) - \tau(H(wZ) + H(wn) - T)\big]\, d^\times z$$

($Z = \left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z}^{-1}\end{smallmatrix}\right)$). Explicitly $v(n)$ is equal to

$$\int_{\mathbf{A}_E^1 E^\times\backslash \mathbf{A}_E^\times}\big[1 - \tau(\log|z\bar{z}|_E - T) - \tau(-\log|z\bar{z}|_E + H(wn) - T)\big]\, d^\times z,$$

and since $|z\bar{z}|_E = |z|_E^2$ for any z in $\mathbf{A}_E^\times$ we obtain $T - \frac{1}{2}H(wn)$. Up to a scalar multiple of T the term (3) of §3 is equal to

$$-\frac{1}{4}|E^1\backslash \mathbf{A}_E^1| \sum_{\eta} \int_{N(\mathbf{A}_E)}\int \phi^K\big(\sigma(an)^{-1}\eta an\big)H(wn)\, da\, dn$$

(a in $Z(\mathbf{A}_E)A(\mathbf{A})A(E)\backslash A(\mathbf{A}_E)$), or

$$\frac{1}{2}|E^1\backslash \mathbf{A}_E^1| \sum_{\eta} \int\int \phi^K\big(\sigma(an)^{-1}\eta an\big)\log\|(1,n)\|_E\, dn\, da \quad \left(n = \begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix}\right).$$

As usual [1] we note that for any fixed place v_0 of F the sum over η is taken over a finite set independent of ϕ_{v_0}. Using the product formula on $F^\times$ we may correct (as in [1]) the last expression by replacing the weight factor with $\log\|(u, un)\|$, where $u = 1 - N(\beta/\alpha)$, for each $\eta = (\alpha, \beta)$ in our finite sum. This can be written as a sum over all places v of F of the local terms $\log\|(u,un)\|_v$; this sum can be taken over a fixed finite set of places v, independent of ϕ_{v_0}, simultaneously for all η in our finite set. Note that $da = 2(\lambda_{-1}^E)^{-1} \otimes d^\times a_v$ on $\mathbf{A}_E^1\backslash \mathbf{A}_E^\times \simeq Z(\mathbf{A})\backslash A(\mathbf{A})$, that $\frac{1}{2}|E^1\backslash \mathbf{A}_E^1| = L(1,\kappa)$ and $\lambda_{-1} = L(1,\kappa)\lambda_{-1}^E$, where λ_{-1}^E is the residue of $L(s, 1_E)$ at $s = 1$. We obtain $\frac{1}{2}\lambda_{-1}$ of the finite sum over v, and the sum over η, of the products over all $w \neq v$ of $F(\eta, \phi_w)$ and

$$A(\eta, \phi_w) = \Delta_v(N\eta)\int\int \phi_v^K\big(\sigma(na)^{-1}\eta na\big)\log\|(u, un)\|_{E_v}\, dn\, da;$$

as in §1 we put $\Delta_v(N\eta) = |\alpha/\bar\beta|_{E_v}^{1/2}|1 - \beta\bar\beta/\alpha\bar\alpha|_{F_v}$ and $F(\eta, \phi_w)$ is defined by the same expression with the weight factor omitted.

We have to find the limit of $A(\eta, \phi_v)$ as $N\eta$ approaches the singular set. If $\eta = (\alpha, \beta)$ then $N(\alpha/\beta) = \alpha\bar\alpha/\beta\bar\beta \to 1$ and we may assume that $\bar\beta/\alpha \to 1$, and in particular that it lies in $F_v^\times$. Hence $|u|_{F_v} = |2|\,|1 - \bar\beta/\alpha|_{F_v}$ if $|1 - \bar\beta/\alpha| < 1$. Replacing n by $n^F n$ with n^F in $N(F_v)$ and n in $N(F_v)\backslash N(E_v)$ and changing $(n^F)^{-1}\eta n^F$ to ηn^F, since η has entries in $F_v^\times$ we obtain the factor $|1 - \bar\beta/\alpha|_{F_v}^{-1}$, and $A(\eta, \phi_v)$ becomes

$$\int\int\int \phi_v^K\big(\sigma(an)^{-1}\eta n^F an\big)\log\|(u, un + 2n^F)\|_{E_v}\, dn\, da\, dn^F$$

($n^F = \left(\begin{smallmatrix}1 & n^F\\ 0 & 1\end{smallmatrix}\right)$). We have obtained the interesting part of $\theta_v'(1)$ of (2) of §3. Hence the sum over η can be extended to include $N\eta$ in $Z(F)$ on incorporating the term $\lambda_{-1}\theta'(1)$ of (2). The asymptotic behavior is of the type ([1], 3.7.1) which permits the application of the summation formula to the function whose values at η appear in our sum ([1], Lemma 2.8).

In the simple case at hand the calculations of the contribution to the twisted trace formula of $GL(2, E)$ from the continuous spectrum are almost identical to those in the case of the usual twisting for $GL(2, E)$ ([4], §10), which closely resemble the calculations for the usual trace formula for $GL(2)$ [2]. Let $\eta^E = (\mu_1, \mu_2)$ be the character of $A(\mathsf{A}_E)$ whose value at $\left(\begin{smallmatrix}a_1 & 0\\ 0 & a_2\end{smallmatrix}\right)$ is $\mu_1(a_1)\mu_2(a_2)$, $I(\eta^E)$ the representation of $G(\mathsf{A}_E)$ induced from the character η^E of $B(\mathsf{A}_E)$, $I(\eta^E, \phi \times \sigma)$ the operator twisted by σ so that

$$I(\eta^E, \phi \times \sigma)\psi(h) = \int \phi(g)\psi(\sigma(h)g)\, dg$$

for ψ in the space of $I(\eta^E)$, and $m(\eta^E)$ the normalizing factor $L(1, \mu_2/\mu_1)/L(1, \mu_1/\mu_2)$. The local normalized intertwining operators $R(\eta_v^E)$ are defined as usual ([4], §7); if η_v^E is unramified then $R(\eta_v^E)$ is a scalar.

One contribution is

$$\frac{1}{4\pi}\sum_{\mu_0}\int_{-i\infty}^{i\infty} m(\eta^E)^{-1} m(\eta^E)' \operatorname{tr} I(\eta^E, \phi \times \sigma)\, ds$$

$$+ \frac{1}{4\pi}\sum_{\mu_0}\sum_{v}\int_{-i\infty}^{i\infty} \operatorname{tr}\Big\{R_v(\eta_v^E)^{-1}R_v(\eta_v^E)' I(\eta_v^E, \phi_v \times \sigma)\Big\}\prod_{w \neq v}\operatorname{tr} I(\eta_w^E, \phi_w \times \sigma)\, ds.$$

The sum over v is clearly finite. The μ_0 are taken over a set of representatives for the connected components of the one-dimensional complex manifold of the η^E whose restriction to the centre is ω_E and whose local components μ_{0v} are unramified whenever ϕ_v is spherical.

The final term is

$$-\frac{1}{4}\sum \operatorname{tr} M(\eta^E) I(\eta^E, \phi \times \sigma).$$

The sum is over the η^E for which ${}^\sigma\eta^E$, defined by ${}^\sigma\eta^E((\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix})) = \eta^E((\begin{smallmatrix} \bar{b}^{-1} & 0 \\ 0 & \bar{a}^{-1} \end{smallmatrix}))$, is equal to $w\eta^E$; in other words the sum is over the η^E with $\mu_1(b\bar{b})\mu_2(a\bar{a}) = 1$ for all a, b in $\mathbf{A}_E^\times$. If $\mu_1 = \mu_2 = \mu$ then

$$\eta^E\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \mu(a/\bar{b}) = \eta\left(\begin{pmatrix} a/\bar{b} & 0 \\ 0 & b/\bar{a} \end{pmatrix}\right),$$

$I(\eta^E)$ is irreducible and both σ and $M(\eta^E)$ map $I(\eta^E)$ to itself. Hence $M(\eta^E)$ is a scalar (whose value is -1). The central character of $I(\eta^E)$ is ω_E.

5. Local theory. Let E/F be a quadratic extension of local fields, and ϕ, f, f_1 smooth compactly supported functions on $G(E), G(F), G(F)$ (respectively).

Definition 1. Write (1) $\phi \to f$ if $\Phi_\phi^{\mathrm{st}}(\gamma) = \Phi_f^{\mathrm{st}}(\gamma)$ for every regular γ in $G(F)$, and (2) $\phi \xrightarrow{1} f_1$ if $\Phi_\phi^{\mathrm{lab}}(\gamma) = \Phi_{f_1}^{\mathrm{st}}(\gamma)$ for every regular γ in $G(F)$.

The same definition applies also to ϕ, f, f_1 compactly supported modulo the centre, and then if ϕ transforms under $Z(E)$ by $\omega_E^{-1} = \omega^{-1} \circ N$ then f transforms under $Z(F)$ by ω^{-1} and f_1 by $\kappa^2\omega^{-1}$.

Suppose that E/F is unramified, and that κ and ω are unramified; then $\omega = 1$ and $\kappa^2 = 1$. The Satake isomorphism between the Hecke algebra $\mathbf{H}_G$ of spherical functions on $G(F)$ and the algebra of finite Laurent series on the conjugacy classes in ${}^LG$ of the coset of σ is given by

$$f \mapsto f^\vee\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \times \sigma\right) = \sum_n F_f(\gamma)(t_1/t_2)^n;$$

γ is a regular element of $A(F)$ with $\gamma = (\begin{smallmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{smallmatrix})$ and $|a| = |\tilde{\omega}|^{-n}$; $\tilde{\omega}$ is a local parameter for E. Put $F_f(\gamma) = \Delta(\gamma)\Phi_f(\gamma)$; $\Delta(\gamma)$ was defined in §1.

For any regular δ in $G(E)$ with eigenvalues a, b put

$$F_\phi(\delta) = \Delta(\delta)\int_{G_\delta(E)\backslash G(E)} \phi(g^{-1}\delta g)\,dg, \qquad \Delta(\delta) = \left|\frac{(a-b)^2}{ab}\right|_E^{1/2};$$

this is the standard nontwisted normalized orbital integral on $G(E)$.

The Satake isomorphism from the Hecke algebra $\mathbf{H}_{G'}$ of $G'(F) \simeq G(E)$ to the algebra of finite series on the conjugacy classes in ${}^LG'$ of the coset of σ is given by

$$\phi \mapsto \phi^\vee((t,1) \times \sigma) = \sum_{n_1, n_2} F_\phi(\delta)t_1^{n_1}t_2^{n_2}$$

($t = (\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix})$ in $GL(2, \mathbf{C})$). Here $\delta = (\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix})$ and $|a_i| = |\tilde{\omega}|^{-n_i}$ $(a_1 \neq a_2)$.

The homomorphisms λ and λ_1 (of §0) from ${}^LG$ to ${}^LG'$ induce dual maps λ^* and λ_1^* from $\mathbf{H}_{G'}$ to $\mathbf{H}_G$ defined by $f = \lambda^*(\phi)$ and $f_1 = \lambda_1^*(\phi)$ where

$$f^\vee(t \times \sigma) = \phi^\vee(\lambda(t \times \sigma)) = \phi^\vee((t,t) \times \sigma) = \phi^\vee((t\sigma(t), 1) \times \sigma)$$

and

$$f_1^\vee(t\times\sigma)=\phi^\vee(\lambda_1(t\times\sigma))=\phi^\vee((t,-t)\times\sigma)=\phi^\vee((-t\sigma(t),1)\times\sigma).$$

For any f the function $F_f\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a}^{-1}\end{smallmatrix}\right)$ of a in $E^\times - E^1$ extends smoothly to a in E^1 and its value at a in E^1 is denoted by $F_f(a)$. The same notation will be applied to the smooth $F_\phi^\sigma(\gamma)=\Delta(\gamma)\Phi_\phi(\delta)$, where $\delta=\left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right)$ in $A(E)$, at the singular set of $\gamma=\left(\begin{smallmatrix} a/\bar{b} & 0 \\ 0 & b/\bar{a}\end{smallmatrix}\right)$.

Since existing methods (based on [4], §§5–6) can establish more complicated results than the following, the proof will not be recorded here; see [5], Theorem 7.1 (and [4], Lemma 6.2) for the archimedean case.

LEMMA 2. *For each ϕ there exist f and f_1 with $\phi\to f$ and $\phi\xrightarrow{1} f_1$, and then*

$$f(1)=\Phi_\phi^{\mathrm{st}}(1),\qquad f_1(1)=\Phi_\phi^{\mathrm{lab}}(1),\qquad F_f(a)=\kappa(a)^{-1}F_{f_1}(a)=F_\phi^\sigma(a).$$

For every f, f_1 with $F_{f_1}\left(\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a}^{-1}\end{smallmatrix}\right)\right)=\kappa(a)F_f\left(\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a}^{-1}\end{smallmatrix}\right)\right)$ (a in $E^\times$) there is a ϕ with $\phi\to f$ and $\phi\xrightarrow{1} f_1$. For every ϕ in $\mathsf{H}_{G'}$ we have $\phi\to\lambda^*(\phi)$ and $\phi\xrightarrow{1}\lambda_1^*(\phi)$.

A special case of the last statement follows without effort from the definitions. A standard change of variables shows that for regular $\gamma=N\delta$ (δ in $A(E)$) we have

$$F_\phi^\sigma(\gamma)=\sum_{n_1-n_2=n}F_\phi(\delta).$$

By definition

$$\sum F_f(\gamma)(t_1/t_2)^n=f^\vee(t\times\sigma)=\phi^\vee((t\sigma(t),1)\times\sigma)=\sum F_\phi(\delta)(t_1/t_2)^{n_1-n_2}$$

if $f=\lambda^*(\phi)$. Hence $F_f(\gamma)=F_\phi^\sigma(\gamma)$ and $\Phi_f^{\mathrm{st}}(\gamma)=\Phi_f(\gamma)=\Phi_\phi(\delta)=\Phi_\phi^{\mathrm{st}}(\gamma)$. Similarly

$$f_1^\vee(t\times\sigma)=\phi^\vee((-t\sigma(t),1)\times\sigma)=\sum F_\phi(\delta)(-t_1/t_2)^{n_1-n_2}$$

if $f_1=\lambda_1^*(\phi)$. Thus

$$F_{f_1}(\gamma)=(-1)^n\sum_{n_1-n_2=n}F_\phi(\delta)=\kappa(\det\delta)F_\phi^\sigma(\gamma)$$

and

$$\Phi_{f_1}^{\mathrm{st}}(\gamma)=\Phi_{f_1}(\gamma)=\kappa(\det\delta)\Phi_\phi(\delta)=\Phi_\phi^{\mathrm{lab}}(\gamma).$$

In particular $f^0=\lambda^*(\phi^0)=\lambda_1^*(\phi^0)$. It is clear how to pass from the statement of the lemma for compactly supported ϕ to the statement concerning ϕ which transform under the centre by a character.

Suppose again that E/F is unramified. There is an isomorphism between the set of unramified representations of $G(F)$ and the representations $I(\eta)$ (§1) where

μ is unramified, in which each unramified representation is the unique unramified constituent in the composition series of the corresponding $I(\eta)$. The unramified representations are parametrized by conjugacy classes in ${}^LG$: that corresponding to $I(\eta)$ is the class of $t(\eta)\times\sigma$, where $t(\eta)=\left(\begin{smallmatrix}\mu(\tilde\omega) & 0\\ 0 & 1\end{smallmatrix}\right)$; $\tilde\omega$ is a uniformizing parameter of $F^\times$. A standard calculation shows that

$$\operatorname{tr} I(\eta, f) = f^\vee(t(\eta)\times\sigma)$$

(f in H_G).

Let $I(\eta^E)$ be the induced representation of §4 whose central character is $\mu_1\mu_2=\omega_E$. The operator $I(\eta^E,\phi\times\sigma)$ is nonzero only if η^E is σ-invariant, namely $\mu_1\bar\mu_2=1$. The manifold of unramified representations of $G(E)$ is the same as that of the $I(\eta^E)$ with unramified η^E, and the $I(\eta^E)$ are parametrized by conjugacy classes in ${}^LG'$: $I(\eta^E)$ corresponds to

$$\left(t(\eta^E),1\right)\times\sigma,\qquad\text{where}\quad t(\eta^E)=\begin{pmatrix}\mu_1(\tilde\omega) & 0\\ 0 & \mu_2(\tilde\omega)\end{pmatrix}.$$

We have

$$\operatorname{tr} I(\eta^E,\phi\times\sigma)=\phi^\vee\left(\left(t(\eta^E),1\right)\times\sigma\right)\qquad(\phi\text{ in }\mathsf{H}_{G'}),$$

as a result of a standard calculation.

From integrals we can now pass to local representations, always assumed to be admissible, and irreducible, unless otherwise specified. A σ-invariant representation π^E of $G(E)$ extends to a representation of $G(E)\times\mathrm{Gal}(E/F)$ by $\pi^E(\sigma)v=Av$ for v in the space of π^E, where A is an operator of order 2 with ${}^\sigma\pi^E=A\pi^EA^{-1}$. Any other extension is of the form $\pi^E\otimes\xi$ where ξ is a character of $\mathrm{Gal}(E/F)$. The extended representation is denoted again by π^E. Its character on $G(E)\times\mathrm{Gal}(E/F)$ can be introduced along well-known lines; it is denoted by χ_π^E. The character of π on $G(F)$ is denoted by χ_π; if $\{\pi\}$ is a finite set of representations then $\chi_{\{\pi\}}$ denotes the sum of $\chi_{\pi'}$ over all π' in $\{\pi\}$. Let $\{\pi\}$ denote the L-packet (§1) of π.

Definition 3. A representation π of $G(F)$ *lifts* to a representation π^E of $G(E)$ *through* λ ($\pi\to\pi^E$) if

$$\chi_{\pi^E}(\delta\times\sigma)=\chi_{\{\pi\}}(N\delta)$$

for all δ in $G(E)$ whose norm $N\delta$ is regular in $G(F)$. Further, π *lifts* to π^E *through* λ_1 ($\pi\xrightarrow{1}\pi^E$) if for all such δ we have

$$\chi_{\pi^E}(\delta\times\sigma)=\kappa(\det\delta)\chi_{\{\pi\}}(N\delta).$$

The Weyl integration formula and Lemma 2 imply that $\pi\to\pi^E$ if and only if

$$\operatorname{tr}\pi^E(\phi\times\sigma)=\operatorname{tr}\{\pi\}(f)\qquad(\text{for all }\phi\to f),$$

and $\pi \xrightarrow{1} \pi^E$ if and only if $\operatorname{tr}\pi^E(\phi \times \sigma) = \operatorname{tr}\{\pi\}(f_1)$ for all $\phi \xrightarrow{1} f_1$. We shall also say that $\{\pi\} \to \pi^E$ if $\pi \to \pi^E$ (same for λ_1). If $\pi \to \pi^E$ (resp. $\pi_1 \xrightarrow{1} \pi^E$) and the central character of π^E is ω_E then ω_E is trivial on $F^\times$ and $\omega(x/\bar{x}) = \omega_E(x)$ (resp. $\omega(x/\bar{x}) = \omega_E(x)\kappa(x)^{-2} = \omega_E(x)\kappa(\bar{x}/x)$), x in $E^\times$, is the central character of π (resp. π_1). The character $\chi_{\{\pi\}}$of $\{\pi\}$ depends only on the stable conjugacy class of $N\delta$. Note that $\{\pi\}$ lifts to $\pi^E \otimes \kappa$ through λ_1.

Several cases of the local lifting are easy to establish. Let μ be a character of $E^\times$. The characters of the induced representations $\pi = I(\eta)$ (§1) and $\pi^E = I(\mu, \bar{\mu}^{-1})$ (§4) can be found by standard techniques, and we have

LEMMA 4. *$I(\eta) \to I(\mu, \bar{\mu}^{-1})$ and $I(\eta) \xrightarrow{1} I(\mu, \bar{\mu}^{-1}) \otimes \kappa$ for all μ.*

LEMMA 5. *If π^E is the lift of π through λ and of π_1 through λ_1 then π^E is of the form $I(\mu, \bar{\mu}^{-1})$ for some μ.*

Proof. The identity

$$\chi_{\{\pi\}}(N\delta) = \kappa(\det \delta)\chi_{\{\pi_1\}}(N\delta)$$

implies that $\chi_{\{\pi\}}$ vanishes on all compact tori, since (§2) the stable twisted conjugacy class of any δ (such that the centralizer of $N\delta$ is a compact torus) contains elements whose determinants lie in distinct cosets of $NE^\times$ in $E^\times$. Hence $\{\pi\}$ is the set of components of an induced representation, and the lemma follows from Lemma 4.

A character χ of E^1 defines a one-dimensional representation $\pi(\eta)$ which is a constituent in the composition series (of length two) of $I(\eta)$, where $\mu = \chi_E\alpha^{1/2}$; here $\alpha(x) = |x|$ and $\chi_E(x) = \chi(x/\bar{x})$ (x in $E^\times$). The one-dimensional subquotient of $I(\mu, \bar{\mu}^{-1})$ is denoted by $\pi(\mu, \bar{\mu}^{-1})$; note that $\bar{\mu}^{-1} = \chi_E\alpha^{-1/2}$. The complement of $\pi(\eta)$ (resp. $\pi(\mu, \bar{\mu}^{-1})$) in $I(\eta)$ (resp. $I(\mu, \bar{\mu}^{-1})$) is the square-integrable special representation $\operatorname{sp}(\eta)$ (resp. $\operatorname{sp}(\mu, \bar{\mu}^{-1})$) of $G(F)$ (resp. $G(E)$).

LEMMA 6. *For all χ we have $\pi(\eta) \to \pi(\mu, \bar{\mu}^{-1})$, $\operatorname{sp}(\eta) \to \operatorname{sp}(\mu, \bar{\mu}^{-1})$, $\pi(\eta) \xrightarrow{1} s\pi(\mu, \bar{\mu}^{-1}) \otimes \kappa$, $\operatorname{sp}(\eta) \xrightarrow{1} \operatorname{sp}(\mu, \bar{\mu}^{-1}) \otimes \kappa$.*

Proof. By virtue of Lemma 4 it suffices to prove only the statements concerning the one-dimensional $\pi(\eta)$. But these follow at once from the definition.

It follows that all σ-invariant one-dimensional or special representations are obtained by the lifting either through λ (if $\pi = \pi(\mu\alpha^{1/2}, \mu\alpha^{-1/2})$, μ a character of $E^\times/F^\times$), or through λ_1 (if μ is a character of $E^\times/NE^\times$ whose restriction to $F^\times$ is nontrivial). Indeed, if $\pi = \pi(\mu\alpha^{1/2}, \mu\alpha^{-1/2})$ is a σ-invariant one-dimensional representation then μ is a character of $E^\times/NE^\times$ and the central character of π is trivial on $F^\times$.

A global definition is also needed. Let F be global; by a representation we mean an automorphic (irreducible) one.

Definition 7. A representation $\pi = \otimes\pi_v$ of $G(\mathbf{A})$ [quasi-] lifts to $\pi^E = \otimes\pi_v^E$ of $G(\mathbf{A}_E)$ through λ (resp. λ_1) if π_v lifts to π_v^E through λ (resp. λ_1) for [almost] all v. The liftings are denoted by $\pi \to \pi^E$ and $\pi \xrightarrow{1} \pi^E$.

Note that π (quasi-) lifts to π^E if and only if $\pi \otimes \chi$ (quasi-) lifts to $\pi^E \otimes \chi_E$, where $\chi_E(a) = \chi(a/\bar{a})$ (a in $\mathbf{A}_E^\times$). Further, $\pi \to \pi^E$ if and only if $\pi \xrightarrow{1} \pi^E \otimes \kappa$. Lemma 5 implies:

LEMMA 8. *The one-dimensional representation $\chi = \otimes\chi_v$ of $G(\mathbf{A})$ lifts to the one-dimensional representation $\chi_E(x) = \chi(x/\bar{x})$ of $G(\mathbf{A}_E)$ through λ, and to $\chi^E \otimes \kappa$ through λ_1. All σ-invariant one-dimensional representations of $G(\mathbf{A}_E)$ are so obtained.*

As noted after Lemma 6, all σ-invariant one-dimensional representations of $G(\mathbf{A}_E)$ have central character whose restriction to $\mathbf{A}^\times$ is trivial.

If v is a place of F which splits in E then $E_v = F_v \oplus F_v$, $G(F_v) = GL(2, F_v)$, and $\phi \to f$, $\phi \xrightarrow{1} f_1$ can be defined as in [1] §1.5 (with $l = 2$ and our σ). Note that now $F_v^\times = NE_v^\times$ and F_v embeds in E_v through the diagonal. The character κ of $E_v^\times$ is trivial on $F_v^\times$ and consists of two components $(\kappa_v, \kappa_v^{-1})$. Anticipating the current work, [1], §1.5, was written in a sufficiently general fashion so as to establish Lemma 2 here, that $\pi \to (\pi^\sigma, \pi)$ and $\pi \xrightarrow{1} (\pi \otimes \kappa, {}^\sigma\pi \otimes \kappa^{-1})$, for any (local) representation π of $G(F_v)$. In the sequel we shall therefore be able to discuss only the case of v where E_v is a field, without further comment about the split v.

6. Main identity. Let $\phi = \otimes\phi_v$, $f = \otimes f_v$, $f_1 = \otimes f_{1v}$ be functions on $G(\mathbf{A}_E)$, $G(\mathbf{A}), G(\mathbf{A})$ (respectively), such that ϕ_v, f_v, f_{1v} are smooth, compactly supported modulo $Z(E_v), Z(F_v)$, $Z(F_v)$, on $G(E_v), G(F_v), G(F_v)$ which transform under the centre by $\omega_{E_v}^{-1}, \omega_v^{-1}, \kappa_v^2\omega_v^{-1}$; for almost all v we have $\phi_v = \phi_v^0$, $f_v = f_v^0$, $f_{1v} = f_v^0$.

The key identity of trace formulae is given by the following:

PROPOSITION 1. *If $\phi_v \to f_v$ and $\phi_v \xrightarrow{1} f_{1v}$ for all v then*

$$\mathrm{TF}(\phi \times \sigma) + \frac{1}{2} \sum \mathrm{tr}\, R(\eta^E) I(\eta^E, \phi \times \sigma) = \mathrm{STF}(f) + \mathrm{STF}(f_1).$$

The sum is taken over all unordered pairs $\eta^E = (\mu_1, \mu_2)$ of distinct characters μ_i of $\mathbf{A}_E^\times / E^\times N\mathbf{A}_E^\times$ with $\mu_1\mu_2 = \omega_E$.

Proof. $\mathrm{STF}(f)$ is described by (1), ..., (6) and part of (7) in §1. $\mathrm{STF}(f_1)$ is similarly described by $(1)_1, \ldots, (7)_1$, obtained from (1), ..., (7) on replacing f by f_1. $\mathrm{TF}(\phi \times \sigma)$ is given by (a), ..., (g) in §3.

We have (a) = (1) + $(1)_1$, (b) = (2) + $(2)_1$ and (d) = (3) + $(3)_1$ by Lemma 5.2. The remaining terms of (c) are indexed by the η^E with $\mu_1 = \mu_2$. The $I(\eta^E)$ is irreducible and the scalar $M(\eta^E)$ is equal to -1 (evaluated as a limit). For the terms of (7) with μ trivial on $\mathsf{A}^\times$ the representation $I(\eta)$ of $G(\mathsf{A})$ is irreducible and $M(\eta)$ is again -1. It follows from Lemma 5.4 that the terms of (c) indexed by η^E with $\mu_1 = \mu_2$ trivial on $\mathsf{A}^\times$ cancel those of (7), while the terms with $\mu_1 = \mu_2$ non-trivial on $\mathsf{A}^\times$ cancel those of $(7)_1$. To show that (f) $= (5) + (5)_1$ note that $m(\eta^E) = m(\eta)m(\eta_1)$, where

$$\eta^E\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mu(a/\bar{b}), \qquad \eta\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} = \mu(a), \qquad \eta_1\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} = \kappa(a)\,\mu(a),$$

since

$$L(1, \mu' \circ N_{E/F}) = L(1, \mu')L(1, \kappa'\mu').$$

The prime indicates restriction to $\mathsf{A}^\times$. Taking logarithmic derivatives it follows that

$$m(\eta^E)^{-1}m(\eta^E)' = m(\eta)^{-1}m(\eta)' + m(\eta_1)^{-1}m(\eta_1)',$$

and the equality follows from Lemma 5.4.

Consider the difference between (e) and $(4) + (4)_1$. The sums over γ in the three terms are taken over the same group $Z(F)\backslash A(F)$, including the singular class. The sums over v are taken over a finite set of places of F. Let v_0 be a fixed place of F which splits in E, and assume that ϕ_{v_0} (as well as f_{v_0} and f_{1v_0}) are spherical, and that $\phi_{v_0}(g) = f_{v_0}(g_1)f^0_{v_0}(g_2)$ ($g = (g_1, g_2)$; see [1], §1.5). Then κ_{v_0} is trivial on $F^\times_{v_0}$; for $w \neq v_0$ we have

$$F(\gamma, f_w) = F^\sigma_{\phi_w}(\gamma) = \kappa_w(a)^{-1}F_{f_{1w}}(\gamma) \qquad \left(\gamma = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}, a \text{ in } E^\times_w\right)$$

and

$$\int\int \phi^K_{v_0}(\sigma(an)^{-1}\delta an)\log\|(u, un)\|_{E_{v_0}}\,dn\,da$$

$$= \int f^K_{v_0}(n^{-1}\gamma n)\log\|(u, un)\|\,dn + \kappa_{v_0}(a)\int f^K_{1v_0}(n^{-1}\gamma n)\log\|(u, un)\|\,dn,$$

where $u = 1 - (a\bar{a})^{-1}$, $n = \left(\begin{smallmatrix} 1 & \eta \\ 0 & 1 \end{smallmatrix}\right)$. It follows that the index $v = v_0$ can be deleted from the sums over v in the difference (e) $-$ (4) $-$ $(4)_1$, and the factor $F(\gamma, \phi_{v_0}), F(\gamma, f_{v_0}), F(\gamma, f_{1v_0})$ always appear in the product over w there.

By virtue of [1], Lemma 2.8, the summation formula can be applied to the pair $Z(F)\backslash A(F), Z(\mathsf{A})\backslash A(\mathsf{A})$ and the nonsmooth functions described by the finite sums over v (taken over sets independent of ϕ_{v_0} and excluding v_0), whose values at γ appear in (e), (4) and $(4)_1$. The Fourier transforms of $F(\gamma, \phi_{v_0})$, $F(\gamma, f_{v_0})$, $F(\gamma, f_{1v_0})$ are the traces $\operatorname{tr} I(\eta^E_{v_0}, \phi_{v_0} \times \sigma)$, $\operatorname{tr} I(\eta_{v_0}, f_{v_0})$, $\operatorname{tr} I(\eta_{1v_0}, f_{1v_0})$. The product over w in (g), (6) and $(6)_1$ always contains v_0 since ϕ_{v_0} is spherical. It follows that the difference between the two sides of the identity to be proved in the proposition, which is some discrete sum in the values of $f_{v_0}^{\vee}$, is equal to an integral obtained as above from (e) + (g) − (4) − (6) − $(4)_1$ − $(6)_1$, of values of $f_{v_0}^{\vee}$. A standard argument ([4], [1]) shows that this equality of discrete and continuous measures is impossible unless both are 0, and the proposition follows.

The trace formulae are sums of traces of representations. The identity of Proposition 1 takes the form

$$\sum_{\pi^E} \operatorname{tr} \pi^E(\phi \times \sigma) + \frac{1}{2} \sum_{\mu_1 \neq \mu_2} \operatorname{tr} R(\eta^E) I(\eta^E, \phi \times \sigma) + \frac{1}{2} \sum_{\mu_1 \neq \mu_2} \operatorname{tr} R(\eta_1^E) I(\eta_1^E, \phi \times \sigma)$$

$$= \sum_{\pi} m(\pi) \operatorname{tr} \pi(f_1) - \frac{1}{2} \sum_{\theta} \sum_{\pi} \epsilon(\pi) \operatorname{tr} \pi(f_1)$$

$$+ \sum_{\pi} m(\pi) \operatorname{tr} \pi(f) - \frac{1}{2} \sum_{\theta} \sum_{\pi} \epsilon(\pi) \operatorname{tr} \pi(f).$$

The sums over π^E and π are taken over all equivalence classes of representations which appear discretely in the spectrum of $G(\mathbb{A}_E)$ and $G(\mathbb{A})$. The multiplicity of π in the discrete spectrum of $G(\mathbb{A})$ is denoted by $m(\pi)$; the multiplicity of π^E is 1 by multiplicity one theorem for $GL(2)$. The second sum in the first row is over all unordered pairs η^E with $\mu_1 \neq \mu_2$, where μ_i are characters of $\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times$, and the third sum is over the η^E with $\mu_1 \neq \mu_2$ and μ_i are characters of $E^\times N\mathbb{A}_E^\times \backslash \mathbb{A}_E^\times$ whose restrictions to $\mathbb{A}^\times$ are non-trivial. The θ are taken over all representations of $H(F)\backslash H(\mathbb{A})$ which do not factor through the determinant, and the π in the inner sums are taken over $P(\theta)$ (§1).

A simplified version of the argument referred to at the end of proof of Proposition 1 establishes (as in [1]) the following.

Proposition 2. *Let ϕ, f, f_1 be as in Proposition 1, denote by V a set of places of F containing the infinite places and those which ramify in E, such that ϕ_v, f_v, f_{1v} are spherical for $v \notin V$. For each $v \notin V$ choose a class $t_v \times \sigma$ in ${}^LG$. Then we have* (1) = (2) *where*

$$(1) \qquad \sum_{\pi^E} \prod \operatorname{tr} \pi_v^E(\phi_v \times \sigma) + \frac{1}{2} \sum \prod \operatorname{tr} R(\eta_v^E) I(\eta_v^E, \phi_v \times \sigma)$$

$$+ \frac{1}{2} \sum \prod \operatorname{tr} R(\eta_{1v}^E) I(\eta_{1v}^E, \phi_v \times \sigma)$$

and

$$(2) \qquad \sum_{\pi} m(\pi) \prod \operatorname{tr} \pi_v(f_{1v}) - \frac{1}{2} \sum_{\theta} \sum_{\pi} \epsilon(\pi) \prod \operatorname{tr} \pi_v(f_{1v})$$

$$+ \sum_{\pi} m(\pi) \prod \operatorname{tr} \pi_v(f_v) - \frac{1}{2} \sum_{\theta} \sum_{\pi} \epsilon(\pi) \prod \operatorname{tr} \pi_v(f_v).$$

The products are taken over all v in V; the sums are over the π, π^E, θ for which $\operatorname{tr} \pi_v^E(\phi_v \times \sigma)$ *(resp.* $\operatorname{tr} R(\eta_v^E) I(\eta_v^E, \phi_v \times \sigma)$, $\operatorname{tr} R(\eta_{1v}^E) I(\eta_{1v}^E, \phi_v \times \sigma)$, $\operatorname{tr} \pi_v(f_{1v})$,

$\operatorname{tr}\pi_v(f_v))$ *is equal to* $f_v^\vee(t_v \times \sigma)$ *for all* v *outside* V *in the* 1*st* (*resp.* 2*nd*, 3*rd*, 4*th*, *and* 5*th*, 6*th*, *and* 7*th*) *sum*.

There is at most one nonzero entry in the sums of (1), by virtue of strong multiplicity one theorem for $GL(2)$.

7. The lifting. Let θ be a character of the torus $H(F)\backslash H(\mathbf{A})$ (§1) whose restriction to $Z(\mathbf{A})$ is $\omega\kappa^{-1}$. Denote by $\eta_1^E = (\mu_1, \mu_2)$ the character of $A(\mathbf{A}_E)$ given by

$$\eta_1^E\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \theta\left(h^{-1}\begin{pmatrix} a/\bar{a} & 0 \\ 0 & b/\bar{b} \end{pmatrix}h\right)\kappa(ab) \qquad (a, b \text{ in } \mathbf{A}_E^\times)$$

(h as in §1). The restriction of η_1^E to $Z(\mathbf{A}_E)$ is $\omega_E = \omega \circ N_{E/F}$. Analogous notations ($\theta_v$ and η_{1v}^E) will be employed in the local case. The representation $I(\eta_1^E)$ extends to a representation $I'(\eta_1^E)$ of the semidirect product $G(\mathbf{A}_E) \times \operatorname{Gal}(E/F)$ by setting [1,4]

$$I'(\sigma, \eta_1^E) = R({}^\sigma\eta_1^E)I(\sigma, \eta_1^E).$$

PROPOSITION 1. *Let E/F be a quadratic extension of local fields, and θ a character of $H(F)$. Then the L-packet $\{\pi^+(\theta), \pi^-(\theta)\}$ lifts to $I'(\eta_1^E)$ through λ.*

Proof. If θ splits through the determinant the claim follows from Lemma 5.4 and the fact that $\pi^+(\theta), \pi^-(\theta)$ are the irreducible components in the composition series of the $I(\eta)$ (§1) which lifts to $I(\eta_1^E)$; note that here $I(\eta_1^E)$ is irreducible and $R(\eta_1^E) = 1$.

It suffices to consider θ which do not factor through the determinant. The local field can be assumed to be the completion F_u of a global field F at u, where $F = \mathbf{Q}$ and E is a quadratic imaginary extension of F if $F_u = \mathbf{R}$, or F is totally imaginary if F_u is nonarchimedean. Extend θ_u to a character θ of $H(\mathbf{A})$ trivial on $H(F)$ and $H(F_v)$ for all $v \neq u$ which split in E, unramified for almost all v and invariant under a small compact subgroup for all v. Its restriction to $Z(\mathbf{A})$ defines a character ω by $\omega(a) = \theta\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\kappa(a)$; the global η_1^E is defined as above.

Let $\{t_v; v \notin V\}$ be a sequence of elements in $GL(2, \mathbf{C})$ so that the only entry in the sums of (1) (in §6) is the one corresponding to the above η_1^E in the third sum. Then θ makes a nonzero contribution to the last sum of (2). Indeed for $v \notin V$ where E_v is a field and ϕ_v, f_v are spherical, one has (in local notations)

$$(1) \quad \operatorname{tr}\pi^+(\theta, f) = \operatorname{tr} I(\eta, f) = \operatorname{tr} I(\eta_1^E, \phi \times \sigma) = \operatorname{tr} R(\eta_1^E) I(\eta_1^E, \phi \times \sigma)$$

(by Lemma 5.4), as $\eta\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) = \kappa(a)$, $\eta_1^E = (\kappa, \kappa)$ and $R(\eta_1^E) = 1$. The same identity holds for v split in E. It is the only contribution by [2], Lemma 12.3.

The second sum in (2) (of §6) vanishes since for $v \notin V$ where E_v is a field

$$\operatorname{tr}\pi^+(\theta, f_1) = \operatorname{tr} I(\eta^E, \phi \times \sigma), \qquad \eta^E = (1, 1),$$

and $I(\eta^E), I(\eta_1^E)$ are inequivalent. In the notations of Proposition 6.2 we now have

$$(2) \qquad \prod \operatorname{tr} R(\eta_{1v}^E) I(\eta_{1v}^E, \phi_v \times \sigma) = \sum 2m(\pi) \prod \operatorname{tr} \pi_v(f_v)$$
$$- \sum_{\pi \in P(\theta)} \epsilon(\pi) \prod \operatorname{tr} \pi_v(f_v)$$
$$+ \sum 2m(\pi) \prod \operatorname{tr} \pi_v(f_{1v}).$$

The finite products of (2) are taken over a set V which does not include any v which splits in E. We shall first show that (2) remains valid if the products are taken over a set V which consists of the single place u alone.

For any $v \neq u$ in the product of (2), as in (1) one has

$$\operatorname{tr} R(\eta_{1v}^E) I(\eta_{1v}^E, \phi_v \times \sigma) = \operatorname{tr} \pi_v^+(\theta_v, f_v) + \operatorname{tr} \pi_v^-(\theta_v, f_v).$$

In the obvious notations ($f = f_v$, etc.), (2) can be put in the form

$$(3) \qquad \alpha(\operatorname{tr} \pi^+(f) + \operatorname{tr} \pi^-(f))$$
$$- \sum 2\alpha(\pi) \operatorname{tr} \pi(f) + \beta(\operatorname{tr} \pi^+(f) - \operatorname{tr} \pi^-(f))$$
$$= \sum 2\beta(\pi) \operatorname{tr} \pi(f_1).$$

For any square integrable π we have $\alpha(\pi) = 0 = \beta(\pi)$. Indeed for such π we may take f to be compactly supported (modulo the centre) function whose stable orbital integrals are equal to those of a matrix coefficient of π (take the matrix coefficient itself if π is supercuspidal). We may still take $f_1 = 0$; evaluating (3) the identity $\alpha(\pi) = 0$ is obtained.

For any θ we have $\beta(\pi^+) = \beta(\pi^-)$. Indeed χ_{π^+} is the average of $\chi_{\pi^+} + \chi_{\pi^-}$ and $\chi_{\pi^+} - \chi_{\pi^-}$. If $\beta(\pi^+) \neq \beta(\pi^-)$ we could choose f_1 whose orbital integrals on $A(F)$ are 0, with $\operatorname{tr} \theta_1(f_1^H) = \delta(\theta_1, \theta)$. The last symbol is either 1 or 0, depending on whether θ_1 and θ are equivalent or not. We may still choose $f = 0$, and obtain a contradiction.

Hence we may assume that all π on the right of (3) are of the form $I(\eta_1')$ and write $\beta(\eta_1')$ for $\beta(\pi)$. If

$$\eta'\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} = \kappa(a)\eta_1'\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix},$$

as

$$F\left(\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}, f_1\right) = \kappa(a) F\left(\begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}, f\right) \qquad (a \text{ in } E^1)$$

we have

$$(4)\qquad \operatorname{tr} I(\eta_1', f_1)\left(=\int \eta_1'(a)F_{f_1}(a)\,da=\int \eta'(a)F_f(a)\,da\right)=\operatorname{tr} I(\eta', f).$$

Linear independence of characters [2] applied to (3) implies that

$$2\alpha(\pi^+)-\beta=2\alpha(\pi^-)+\beta$$

and

$$\alpha=2\alpha(\pi^+)-\beta+2\beta(\eta_1).$$

Repeating this argument to all $v \neq u$ in V we obtain (2) where the product is taken over the singleton $\{u\}$ only.

To complete the proof let $w \neq u$ be a place of F which does not split in E, and θ_w a character which does not factor through the determinant. In the same way as above we may obtain (2) where the product is taken over the set $\{u, w\}$ only. Choose f_w to be a matrix coefficient of the supercuspidal π_w^+. We may choose $f_{1w}=0$, and dropping u we obtain

$$(5)\qquad c \operatorname{tr} R(\eta_1^E) I(\eta_1^E, \phi \times \sigma)$$

$$=\sum 2m(\pi)\operatorname{tr}\pi(f)-\operatorname{tr}\pi^+(f)+\operatorname{tr}\pi^-(f).$$

The constant c is nonzero by linear independence as $m(\pi)$ are integers. The π here are not necessarily the same as the π of (2). The right side of (2) is now equal to the product of c^{-1} and the right side of (5). The arguments of the two paragraphs following (3) imply that the π_v of the last sum in (2) can be assumed to be of the form $I(\eta)$. It follows from (2) and (4) that (5) holds with $c=1$ (and new integers $m(\pi)$).

The value $\nu(a)$ of η_1^E at a diagonal matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ (a in $E^\times$) depends only on the image of a in $E^\times/F^\times$, since the value of η_1^E on $Z(E)$ is given by ω_E. Hence there is a character ν' of E^1 with $\nu(a)=\nu'(a/\bar{a})$ (a in $E^\times$). Extend ν' to a character $\tilde{\nu}$ of $E^\times$, and consider the representation $I(\eta^E)$ of $GL(2, E)$, where $\tilde{\eta}^E=(\tilde{\nu}, \bar{\tilde{\nu}})$ and $\bar{\tilde{\nu}}(a)=\tilde{\nu}(\bar{a})$. Then [4] there exists a representation $\pi(\tilde{\nu})$ of $GL(2, F)$ such that whenever $\tilde{\phi} \to \tilde{f}$ in the sense of [4] we have

$$(6)\qquad \operatorname{tr} R(\tilde{\eta}^E) I(\tilde{\eta}^E, \tilde{\phi} \times \tilde{\sigma})=\operatorname{tr}\pi(\tilde{\nu}, \tilde{f}).$$

The restriction of $\pi(\tilde{\nu})$ to $SL(2, F)$ consists of 1, 2 or 4 irreducible inequivalent representations [3]. We shall apply (5) and (6) with ϕ and $\tilde{\phi}$ supported on $SL(2, E)$. Then $\Phi_\phi^{\text{lab}}=\Phi_\phi^{\text{st}}$. Lemma 5.2 and [4], Lemma 6.2, imply that there exists $\tilde{\phi}$ on $G(E)$ such that $\tilde{\phi}$ is supported on $SL(2, E)$, and the orbital integrals $\Phi_{\tilde{\phi}}$ of $\tilde{\phi}$, twisted with respect to $\tilde{\sigma}$, at δ with $\gamma=N\delta$ regular in $SL(2, F)$, is equal to $\Phi_\phi^{\text{st}}(\gamma)$. In particular the stable orbital integrals of f on $G(F)$ are equal to those of $\tilde{f}$ on $GL(2, F)$; both are supported on $SL(2, F)$. Since the left sides of (5) and (6) depend only on Φ_ϕ^{st} and $\Phi_\phi^{\sim}$ (for our ϕ), both $I(\eta^E)$ and $I(\eta_1^E)$ restrict to the same L-packet (of one or two elements) of $SL(2,E)$, and $\tilde{\sigma}=\sigma$ on $SL(2,E)$, it follows that

$$\sum_i \operatorname{tr}\pi_i(f)=\sum 2m(\pi)\operatorname{tr}\pi(f)-\operatorname{tr}\pi^+(f)+\operatorname{tr}\pi^-(f).$$

The right side is as in (5); on the left ($1\leqslant i\leqslant j$; $j=1$, 2 or 4) the π_i are the irreducible components of the restriction of $\pi(\tilde{\nu})$ to $SL(2,F)$. On the left appears f, not $\tilde{f}$, since $\sum_i \operatorname{tr}\pi_i(\tilde{f})$ depends only on the stable orbital integrals [3] of $\tilde{f}$, regarded as a function on $SL(2,F)$, and these are equal to Φ_f^{st}.

By linear independence of characters on $SL(2)$ we obtain that $j=2$ and $m(\pi^+)=1$, while $m(\pi)=0$ for all π inequivalent to π^+. The proposition now follows from (5).

Let θ be a character of $H(F)$ (local F) whose restriction to $Z(F)$ is $\omega\kappa^{-2}$. Denote by η^E the character

$$\eta^E\left(\begin{pmatrix}a&0\\0&b\end{pmatrix}\right)=\theta\left(h^{-1}\begin{pmatrix}a/\bar{a}&0\\0&b/\bar{b}\end{pmatrix}h\right)\kappa(ab)^2$$

of $A(E)$. It follows from Definition 5.3 that $\pi\to\pi^E$ if and only if $\pi\xrightarrow{1}\pi^E\otimes\kappa$. Hence

COROLLARY 2. *The L-packet $\{\pi^+(\theta),\pi^-(\theta)\}$ lifts to $I'(\eta^E)$ through λ_1.*

It is now clear (also from the global theory of [3]) that the stable trace formula takes the form

$$\operatorname{STF}(f)=\sum_\pi m(\pi)\operatorname{tr}\pi(f)+\frac{1}{2}\sum_\theta\prod_v\left(\operatorname{tr}\pi_v^+(\theta_v,f_v)+\operatorname{tr}\pi_v^-(\theta_v,f_v)\right).$$

The second sum is over the characters θ of $H(F)\backslash H(\mathbb{A})$ which do not factor through the determinant. Almost all factors in the product consist of a single summand. The sum over π is taken over all discrete series representations $\pi=\otimes\pi_v$ of $G(\mathbb{A})$ such that there is no θ such that $\pi_v\simeq\pi_v^+(\theta_v)$ for almost all v. In particular we have part of "strong multiplicity one theorem for L-packets of $G(\mathbb{A})$": if $\pi=\otimes\pi_v$ is a discrete series representation of $G(\mathbb{A})$, $\pi'=\otimes\pi'_v$ lies in $P(\theta)$, and $\pi_v\simeq\pi'_v$ for almost all v, then π lies in $P(\theta)$.

For a suitable choice of $\{t_v\}$ there remains to explore the remaining identity

$$(7)\qquad \sum_{\pi^E}\prod\operatorname{tr}\pi_v^E(\phi_v\times\sigma)=\sum_\pi m(\pi)\prod\operatorname{tr}\pi_v(f_v)+\sum_\pi m(\pi)\prod\operatorname{tr}\pi_v(f_{1v})$$

from Proposition 6.2. The π,π^E are taken over the discrete series of $G(\mathbb{A})$, $G(\mathbb{A}_E)$, but not over any π in $P(\theta)$. By virtue of Lemma 5.8 none of the π,π^E is one-dimensional. Note that Lemmas 5.6 and 5.8 can also be deduced from (7) using only Lemma 5.4.

PROPOSITION 3. *For every supercuspidal representation π of $G(F)$, not in any L-packet $\{\pi(\theta)\}$, there exists a supercuspidal σ-invariant representation π^E of $G(E)$ so that π lifts to π^E through λ. For every σ-invariant supercuspidal representation π^E of $G(E)$ whose central character is trivial on $F^\times$ there exists a supercuspidal representation π of $G(F)$, not in any $\{\pi(\theta)\}$, so that either $\pi \to \pi^E$ or $\pi \xrightarrow{1} \pi^E$.*

Proof. The case of $\{\pi(\theta)\}$ was dealt with in Propositions 1 and 2. Suppose F is the completion at u of a global F, and E_u is a field. Construct, as in [1], Lemma 6.6, an automorphic representation π of $G(F)$ whose component (1) at u is the representation of the proposition, which we now denote by π_u', (2) at any $v \neq u$ in V is special, (3) at any other v is unramified. V is a (finite) set consisting of the infinite places, u, two v which split in E and all v which ramify in E. Since π does not lie in any $P(\theta)$, (7) is obtained. The representation π^E on the left of (7) does exist and it is σ-invariant; otherwise the left side is 0, and arguments as those following (3) would imply that the sum on the right is empty. Moreover, π_u^E is supercuspidal, since we know that induced or special representations of $G(E_u)$ are lifts but not from our π_u'.

Lemma 6.6 of [1] affords constructing also a σ-invariant representation π^E whose component at u is the above π_u^E, whose other components are either unramified or special representations which are lifts through λ of special representations of $G(F_v)$. (Note that [1], Lemma 6.6, cannot construct σ-invariant π^E with components π_{v_i} $(i = 1,2)$ where E_{v_i} are fields and π_{v_1} (resp. π_{v_2}) is a lift through λ (resp. λ_1), since the choice of $f_{1v_1} = 0$ and $f_{v_2} = 0$ is possible). Now at the v where E_v is a field and π_v^E is special and lifts from π_v through λ, we may choose f_v related as usual to a matrix coefficient of π_v, and $f_{1v} = 0$. Hence we have

$$\text{(8)} \qquad \operatorname{tr}\pi_u^E(\phi_u \times \sigma) = \sum n(\pi_u)\operatorname{tr}\pi_u(f_u) \qquad (n(\pi_u) \text{ in } \mathbf{Z}).$$

Using some $v \neq u$ in V such that E_v is a field, the identity (7) obtained here takes the form

$$c\operatorname{tr}\pi_u^E(\phi_u \times \sigma) = \sum m(\pi_u)\operatorname{tr}\pi_u(f_u)$$

where $m(\pi_u') \neq 0$. Hence $c \neq 0$, and $n(\pi_u') = m(\pi_u')/c$ is non-zero. Since the supercuspidal π_u' appears on the right of (8) a standard argument ([4], pp. 223–6) establishes the first claim.

To prove the second claim for a σ-invariant supercuspidal π_w^E whose central character is trivial on $F_u^\times$, let ϕ_u be a matrix coefficient of π_u^E. If all $\Phi_{\phi_u}^{\mathrm{st}}$ and $\Phi_{\phi_u}^{\mathrm{lab}}$ are 0 then all Φ_{ϕ_u} are 0 and we obtain a contradiction to the orthogonality relations (see e.g. [4], p. 111, *l.* -1; but note that l^{-1} there has to be replaced by 1 here since $NZ(E_u) = Z(F_u)$). It may be assumed that $\Phi_{\phi_u}^{\mathrm{st}}$ are nonzero, and we may obtain (8) again, arguing as above. The argument of [4], pp. 223–6, would apply once the existence of a square-integrable π_u on the right of (8) is shown; π_u will then be supercuspidal since π_u^E is not special.

If there are no square-integrable π_u on the right of (8) we may take $\phi_u = 0$ and f_u with $\Phi_{f_u}^{\mathrm{st}} = 0$ and $\Phi_{f_u}^{\mathrm{lab}}$ such that $\operatorname{tr}\theta(f_u^H) = \delta(\theta,\theta_1)$ for a fixed character θ_1 of $H(F_u)$, as in the second paragraph following (3). It follows that all π_u in (8) can be assumed to be of the form $I(\eta)$. Since π_u^E is supercuspidal we obtain a contradiction by means of Lemma 5.4, and the claim follows.

PROPOSITION 4. *Every σ-invariant cuspidal representation π^E of $G(\mathbf{A}_E)$ whose central character is trivial on $\mathbf{A}^\times$ is a lift of a cuspidal π of $G(\mathbf{A})$ either through λ or through λ_1 (but not both!). Every cuspidal representation π of $G(\mathbf{A})$ lifts to a representation π^E of $G(\mathbf{A}_E)$ which is cuspidal unless π lies in a global L-packet $P(\theta)$.*

Proof. The first claim was shown for other discrete series representations in Propositions 1 and 2, and Lemma 5.8. Given π^E we obtain (7) where π^E is the single entry on the left. The product is taken over a finite set V of places v. If π_v^E is induced but not that of Propositions 1 or 2, then v can be deleted from V (by the standard arguments, employed above). In particular the claim follows if all components of π^E at v where E_v is a field are induced but not those of Proposition 1, 2. Indeed, after deleting all v the left of (7) is equal to 1 while the right is a sum of integers $m(\pi)$. Otherwise π^E has a component π_u^E (E_u is a field) which is the lift of an L-packet $\{\pi_u\}$ of supercuspidal or special representations. We may assume that $\pi_u \to \pi_u^E$. Choosing f_u related as usual to a matrix coefficient of π_u and $f_{1u} = 0$, the claim follows from the standard arguments applied to the $v \neq u$ in V.

To prove the second claim take $\{t_v\}$ so that (7) holds and π defines a non-zero entry in the first sum on the right. We assume that π is not in any $P(\theta)$, a case discussed in Proposition 1, 2. The standard arguments establish a contradiction if no π^E exists on the left of (7), and that $\pi \to \pi^E$.

Clearly we have

COROLLARY 5. *Each cuspidal representation π of $G(\mathbf{A})$ occurs only once in the discrete spectrum of G.*

This is "multiplicity one theorem" for G; it asserts that all $m(\pi)$ in Proposition 6.2 and the expression for STF(f) following Corollary 2 are equal to 1.

Let P_v be an L-packet of $G(F_v)$ for all v, almost all of which contain an unramified representation π_v^0. A *global L-packet* P is defined to be the set of representations π of $G(\mathbf{A})$ such that π_v lies in P_v for all v and $\pi_v \simeq \pi_v^0$ for almost all v. Strong multiplicity one theorem for $G(F)$ also follows from the above arguments. Since it has nothing to do with multiplicity one theorem it will be less confusing to adopt the name "*rigidity theorem*". Of course in the case of $G(F)$ we have a rigidity theorem for L-packets; it asserts that *for any finite set V the family $\{P_v; v \notin V\}$ specifies P uniquely if P contains a discrete series (automorphic) representation.*

Propositions 3 and 4 assert that every σ-invariant (super) cuspidal representation π^E of $(G(E)$ or) $G(\mathbf{A}_E)$ whose central character is trivial on $(F^\times$ or) $\mathbf{A}^\times$ is a lift, through either λ or λ_1. In fact it is not necessary to put any condition on the central character of π^E, and every σ-invariant (super) cuspidal π^E is a lift, since we have

PROPOSITION 6. *If π^E is one-dimensional or discrete series σ-invariant representation then its central character is trivial on $F^\times$ (locally) or $\mathbf{A}^\times$ (globally).*

Proof. This was already observed for one-dimensional and special local representations after Lemma 5.6, and in the global case for one-dimensional representations in Lemma 5.8. All roads lead to Rome, that involving the trace formula is the nearest to the spirit of this paper. The proof for the (super) cuspidal representations will be based on an application of the twisted trace formula $\mathrm{TF}(\phi\times\sigma)$ with functions ϕ which transform under the centre by a character ω of $\mathbf{A}_E^\times/E^\times N\mathbf{A}_E^\times$, whose restriction to $\mathbf{A}^\times$ is non-trivial. It is convenient to take ω to be κ^{-1}.

Let E/F be a quadratic extension of local fields, and ϕ a smooth function on $G(E)$, compactly supported modulo $Z(E)$, which transforms under $Z(E)$ by κ^{-1}. The analysis of §2 shows that the twisted orbital integral $\Phi_\phi(\delta)$ is equal to 0 unless $N\delta$ is a regular element in a compact torus $H(F)$ which splits over E. As in §2 one defines two functions on $H(F)/Z(F)$ by

$$\Phi_\phi^1\big((a/\bar a, b/\bar b)\big) = \Phi_\phi(a,b) - \Phi_\phi(au,bu) + \Phi_\phi(au,b) - \Phi_\phi(a,ub)$$

$$= 2\kappa(b)\big[\Phi_\phi(a,b) + \Phi_\phi(au,b)\big],$$

$$\Phi_\phi^\kappa\big((a/\bar a, b/\bar b)\big) = \kappa(a)(\Phi_\phi(a,b) - \Phi_\phi(au,bu) - \Phi_\phi(au,b) + \Phi_\phi(a,ub))$$

$$= 2\kappa(a)\big[\Phi_\phi(a,b) - \Phi_\phi(au,b)\big].$$

Here a, b are in $E^\times$, u lies in $F^\times$ but not in $NE^\times$, and $\Phi_\phi(a,b)$ is $\Phi_\phi(h^{-1}(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix})h)$ of §2. If ϕ is supported on $Z(E)\mathrm{SL}(2,E)$ then $\Phi_\phi^\epsilon((a/\bar a, b/\bar b)) = 2\kappa(a)\Phi_\phi(a,b)$, ϵ in $\{1,\kappa\}$.

The work of §§3–4 shows that all terms in the twisted trace formula $\mathrm{TF}(\phi\times\sigma)$ vanish for our global function $\phi=\otimes\phi_v$ (with the usual properties, which transform under the centre by κ^{-1}), except two. They are the terms of the paragraph preceeding Proposition 3.1, and that of the end of §4. Hence

$$\mathrm{TF}(\phi\times\sigma) = \tfrac14 |Z(\mathbf{A})H(F)\backslash H(\mathbf{A})| \sum_{\gamma\in Z(F)\backslash H(F)}{}' \Big(\prod_v \Phi_{\phi_v}^1(\gamma) + \prod_v \Phi_{\phi_v}^\kappa(\gamma)\Big)$$

$$- \tfrac12 \sum \operatorname{tr} M(\eta^E) I(\eta^E, \phi\times\sigma).$$

The last sum is taken over all unordered pairs $\eta^E = (\mu_1, \mu_2)$ of characters μ_i of $\mathbf{A}_E^\times / E^\times N\mathbf{A}_E^\times$ with $\mu_1 \mu_2 = \kappa$.

Consideration involving the Shalika germs of $f_v^\epsilon(a) = \Delta(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right))\Phi^\epsilon_{\phi_v}(h^{-1}\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)h)$ and vanishing of $\Phi_\phi(\delta)$ unless $N\delta$ is regular (in $H(F)$) show that $f_v^\epsilon(a)$ is a regular function on $E^1 \simeq Z(F)\backslash H(F)$ which vanishes at $a = 1$ when E_v is a field. Hence the summation formula can be applied to the function $f^\epsilon = \otimes f_v^\epsilon$ and the pair $(E^1, \mathbf{A}_E^1)$, and one has

$$\mathrm{TF}(\phi \times \sigma) + \frac{1}{2} \sum \operatorname{tr} M(\eta^E) I(\eta^E, \phi \times \sigma) = \frac{1}{4} \sum_{\mu} \left(\operatorname{tr} \mu(f^1) + \operatorname{tr} \mu(f^\kappa)\right).$$

The last sum is taken over all unitary characters of $E^1\backslash\mathbf{A}_E^1$.

The argument referred to in the passage from Proposition 6.1 to Proposition 6.2, applied only to the places v which split in E, implies that

$$\text{(9)} \qquad \sum_{\pi} \prod_{v} \operatorname{tr} \pi_v(\phi_v \times \sigma) + \frac{1}{2} \sum_{\eta^E} \prod_{v} \operatorname{tr} M(\eta_v^E) I(\eta_v^E, \phi_v \times \sigma)$$
$$= \frac{1}{4} \sum_{\mu} \prod_{v} \operatorname{tr} \mu_v(f_v^1) + \frac{1}{4} \sum_{\mu} \prod_{v} \operatorname{tr} \mu_v(f_v^\kappa).$$

The sums are taken over discrete series representations π of $G(E)$, and unordered pairs $\eta^E = (\nu, \kappa/\nu)$, and μ, such that

$$\operatorname{tr} \pi_v(\phi_v \times \sigma) = \operatorname{tr} M(\eta_v^E) I(\eta^E, \phi_v \times \sigma) = \operatorname{tr} \mu_v(f_v^\epsilon)$$

for all v which split in E. The products are taken over the remaining v.

Let v be a place where E_v is an unramified quadratic extension, κ_v is unramified, and suppose that ϕ_v is spherical. Then π_v must be unramified principal series representation if $\operatorname{tr} \pi_v(\phi_v \times \sigma) \neq 0$. That is $\pi_v = I(\nu_v, \kappa/\nu_v)$ where ν_v is a character of $F_v^\times\backslash E_v^\times$. But since ν_v and E_v/F_v are unramified, $\nu_v = 1$, and

$$\operatorname{tr} \pi_v(\phi_v \times \sigma) = \operatorname{tr} M(\eta_v^E) I(\eta_v^E, \phi_v \times \sigma)$$

where $\eta_v^E = (1, \kappa_v)$. For the same set of v one also has $\mu_v = 1$ and $\operatorname{tr} \mu_v(f_v^1) = \operatorname{tr} \mu_v(f_v^\kappa)$. Hence (9) remains valid if all products are taken only over a finite set V (the complement of the set of v mentioned up to now) and the right side is multiplied by the nonzero number

$$\alpha = \prod_{v} \operatorname{tr} \mu_v(f_v^1) / \operatorname{tr} M(\eta_v^E) I(\eta_v^E, \phi_v^0 \times \sigma).$$

The last product is over the v of the first sentence in this paragraph.

Suppose π is a discrete series representation whose central character is κ (contrary to the assertion of the proposition); we may assume that it appears in the first sum of (9). The vanishing of the left side of (9) contradicts "linear independence of characters"; alternatively, "strong multiplicity one" theorem for

$\mathrm{GL}(2, E)$ implies that π is the sole entry on the left. Hence the right of (9) is nonzero, and there is a μ such that $\mathrm{tr}\, \pi_v(\phi_v \times \sigma) = \mathrm{tr}\ \mu_v(f_v^\epsilon)$ (fixed ϵ in $\{1, \kappa\}$) whenever v splits. Define a character ν on $\mathbf{A}_E^\times / E^\times \mathbf{A}^\times$ by $\nu(a) = \mu(a/\bar{a})$, and take $\eta^E = (\nu, \kappa/\nu)$. Then π_v is equivalent to $I(\eta_v^E)$ for almost all v, and by "strong multiplicity one" for $\mathrm{GL}(2, E)$ our π is not a discrete series representation, but $I(\eta^E)$. Hence $\mathrm{TF}(\phi \times \sigma) = 0$ for all ϕ, and the proposition follows for global π.

To prove the claim for the supercuspidal representations, suppose π_u is one whose central character is κ_u. The arguments of [1], Lemma 6.6, applied to $\mathrm{TF}(\phi \times \sigma)$, afford the construction of a global π with central character κ, whose component at u is the above π_u, and whose other components are unramified or special (at a finite number of places which split in E). But π does not exist (as above), hence π_u does not exist either, and the claim follows.

References

1. Y. Z. Flicker, *The trace formula and base change for* GL(3), Springer Lecture Notes **927** (1982).

1a. ———, *L-packets and liftings for* $U(3)$, preprint (Princeton, 1982).

2. H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Springer Lecture Notes **114** (1970).

3. J. P. Labesse and R. P. Langlands, L-indistinguishability for SL(2), Canad. J. Math. **31** (1979), 726–785.

4. R. P. Langlands, *Base change for* GL(2), Ann. of Math Studies **96** (1980).

5. D. Shelstad, *Endoscopic groups and base change for* $\mathbf{C}/\mathbf{R}$, preprint, Rutgers (1981).

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