Contemporary Mathematics
Volume 53, 1986

Dedicated to A. Selberg

UNITARY QUASI-LIFTING: PREPARATIONS

Yuval Z. Flicker

0. INTRODUCTION. Our aim in the unpublished manuscript [U(3)] was to discuss in a special case a problem which seems to us to be central in representation theory. It is to classify the automorphic, and admissible, representations of classical groups, in terms of those of the general linear group. The first step, quasi-lifting, is to obtain a global result in terms of almost all places. This can be done using several techniques. The trace formula makes it possible to carry out the second step, which from another point of view is an aim in itself. It is to obtain a classification in the local case, by means of local lifting. Finally the local study at all places is put together to obtain by means of the global lifting a complete description, useful for applications.

The aim of [U(3)] was to carry out a preliminary stage of the entire project in the special case of the unitary group in three variables, thus giving a precise statement of the expected results, and reducing them by means of the trace formula to "standard" assumptions in local harmonic analysis. This helps to see which of the assumptions are really required. As it turns out, some results in local harmonic analysis, which are hard to prove by local techniques, can be deduced on using the global trace formula.

We intend to publish the work of [U(3)] as a series of papers, the present one is the first, and we refer to it here as [I]. The first unit is composed of [I] and [II], and deals with the first step, of quasi-lifting from [U(3)], where it is perhaps overshadowed by other results. The key result ([II], Theorem 4.4) is that the "unitary" lifting from U(2) to U(3) can be established by means of the study of base change only, and using only available results from local harmonic analysis for base change (namely transfer of orbital integrals of spherical functions on the regular set [I], Lemma 3.3).

1980 Mathematics Subject Classification: 12A85.

LEMMA. The map $T \to \{x'_\tau\}$ injects the set of stable conjugacy classes of tori in G into $H^1(F, W)$. If G is quasi-split, and its derived group is simply connected, then this map is a bijection.

Proof. If $\underline{T} = g\underline{A}g^{-1}$ and $\underline{T}'$ are stably conjugate, then there is x in $\underline{G}$ with $\underline{T}' = x\underline{T}x^{-1} = xg\underline{A}(xg)^{-1}$, and $(xg)_\tau = g^{-1}x_\tau g \cdot g_\tau$ has the image g'_τ in $H^1(F, W)$, since $g^{-1}x_\tau g$ lies in $\underline{A}$ (x_τ in $\underline{T}$). Conversely, if $\underline{T} = g\underline{A}g^{-1}$, $\underline{T}' = g'\underline{A}g'^{-1}$, and $g_\tau = a(\tau)g'_\tau$ with $a(\tau)$ in $\underline{A}$, then $a(\tau) = g'^{-1}x(\tau)\,g'$ with $x(\tau)$ in $\underline{T}'$, and the map $t \mapsto gg'^{-1}t(gg'^{-1})^{-1}$ [t in $\underline{T}'$] is defined over F.

For the second claim, if $\{g_\tau\}$ lies in $H^1(F, W)$, then it defines a new $\mathrm{Gal}(\underline{F}/F)$-action by $\hat{\tau}(h) = g_\tau^{-1}\,\tau(h)\,g_\tau$ (h in $\underline{A}$). If h is a fixed $\hat{\tau}$-invariant regular element, then $\tau(h) = g_\tau h g_\tau^{-1}$, and the conjugacy class of h in $\underline{G}$ is defined over F. A theorem of Steinberg and Kottwitz [Ko] implies the existence of h' in G which is conjugate to h in $\underline{G}$, since the field F is perfect. The centralizer of h' in G is a torus whose stable conjugacy class corresponds to $\{g_\tau\}$.

1.2 The conjugacy classes within the stable conjugacy class of T are parametrized by the set

$$B(T/F) = G \backslash A(T/F)/\underline{T}.$$

$A(T/F)$ is the set of x in $\underline{G}$ so that $T' = T^x = xTx^{-1}$ is defined over F. The map $x \mapsto \{\tau \mapsto x_\tau = x^{-1}\tau(x);\ \tau \text{ in } \mathrm{Gal}(\underline{F}/F)\}$ defines a bijection

$$B(T/F) \simeq \ker\,[H^1(F, T) \to H^1(F, G)].$$

Let G_{sc} be the simply connected covering group of the derived group of G, T_{sc} the inverse image of T, $C(T/F)$ the image of $H^1(F, T_{sc})$ in $H^1(F, T)$. $B(T/F)$ is a subset of $C(T/F)$. If $H^1(F, G_{sc}) = \{0\}$, for example when F is a non-archimedean local field, then $B(T/F) = C(T/F)$. If F is a global field with a ring $\mathbb{A}$ of adeles, then we put $C(T/\mathbb{A}) = \oplus\, C(T/F_v)$, $B(T/\mathbb{A}) = \oplus\, B(T/F_v)$; the sums range over all places.

Let K be a finite galois extension of F over which T splits. Write $H^{-1}(X)$ for $H^{-1}(\mathrm{Gal}(K/F), X)$, and $X_*(T)$ for $\mathrm{Hom}(G_m, T)$. In the local case the Tate-Nakayama duality (see [T]) identifies $C(T/F)$ with the image of $H^{-1}(X_*(T_{sc}))$ in $H^{-1}(X_*(T))$. In the global case it yields an exact sequence

$$C(T/F) \to C(T/\mathbb{A}) \to \mathrm{Im}\,[H^{-1}(X_*(T_{sc})) \to H^{-1}(X_*(T))].$$

The last term here is the quotient of the $\mathbb{Z}$-module of μ in $X_*(T_{sc})$ with $\Sigma\tau\mu = 0$ (sum over τ in Gal(K/F)), by the submodule spanned by $\mu - \tau\mu$, where μ ranges over $X_*(T)$ and τ over Gal(K/F).

1.3 We shall now discuss <u>the above definitions in our case</u> where $G = U(3)$. The centralizer E' of T in the algebra $M(3,E)$ of 3×3 matrices over E, is a maximal commutative semi-simple subalgebra. Hence it is isomorphic to a direct sum of field extensions of E. There are three possibilities. (1) $E' = E \oplus E \oplus E$. (2) $E' = E'' \oplus E$, $[E'':E] = 2$. (3) E' is a cubic extension of E. The Weyl group is the symmetric group on three letters, generated by the reflections (12), (23), (13). Note that $\sigma(12) = (23)$, $\sigma(13) = (13)$. In view of Lemma 1.1, the stable conjugacy classes are determined by $H^1(F, W)$. We also note that if the eigenvalues of x in G are α, β, γ in K, then τ in Gal(K/F) whose restriction to E is non-trivial, maps α, β, γ to $\tau\alpha^{-1}$, $\tau\beta^{-1}$, $\tau\gamma^{-1}$. $X_*(T)$ is the group of $\mu = (x, y, z)$ in $\mathbb{Z}^3$, and $X_*(T_{sc})$ is the subgroup of μ with $x+y+z = 0$. Indeed $G_{sc} = SU(3)$. If $\tau|E \neq 1$ it maps the set $\{x, y, z\}$ to the set $\{-x, -y, -z\}$. We deal with each case separately.

(1) T splits over E, and we take $K = E$. $H^1(F, W)$ is $H^1(\mathrm{Gal}(E/F), W)$; it consists of $w = w(\sigma)$ in W with $w\sigma(w) = 1$, up to coboundaries. Hence w is 1, (123), (132) or (13). As $(123) = (12)\sigma(12)$ and $(132) = (23)\sigma(23)$, there are only two non-cohomologous cycles $w = 1$, $w = (13)$. If $w = 1$, then $\sigma(x, y, z) = (-z, -y, -x)$, T is stably conjugate to the diagonal torus A of G, and $C(T/F) = \{0\}$ in the local case. If $w = (13)$, then $\sigma(x, y, z) = (-x, -y, -z)$, and $C(T/F) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ in the local case.

(2) T splits over $K = E''$. E'' is galois over F, since the involution $x \mapsto x^{-1} = J\,{}^t(\sigma x_j)J$ stabilizes T, hence E'', and induces an automorphism of E'' whose restriction to E is σ. It is clear that $\mathrm{Gal}(K/F) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Indeed, if it is $\mathbb{Z}/4$, and τ is a generator, then τ^2 acts trivially on E. The cocycle w in $H^1(\mathrm{Gal}(K/F), W)$ defined by τ satisfies $1 = w(\tau^4) = (w(\tau^2))^2 = (\tau(w(\tau)w(\tau))^2$, hence $w(\tau) = 1$ or (13) up to coboundaries. But then $w(\tau)^2 = 1$, and T splits already over E.

Hence Gal(K/F) is generated by σ with $\sigma|E \neq 1$, and $\tau \neq 1$, $\tau|E = 1$. Any w in $H^1(\mathrm{Gal}(K/F), W)$ satisfies $w(\sigma)w(\tau) = w(\tau\sigma) = w(\sigma\tau) = \sigma(w(\tau))w(\sigma)$, or $\sigma(w(\tau)) = w(\sigma)w(\tau)w(\sigma)^{-1}$. Up to coboundaries the possibilities are (i) $w(\sigma) = (13)$, $w(\tau) = (13)$, $w(\sigma\tau) = 1$; (ii) $w(\sigma) = 1$, $w(\tau) = (13)$, $w(\sigma\tau) = (13)$;

(iii) $w(\sigma) = (13)$, $w(\tau) = 1$, $w(\sigma\tau) = (13)$. In case (iii) T splits over E. As both σ and $\sigma\tau$ have non-trivial restrictions to E, we may assume (i) holds. Then $C(T/F) \simeq \mathbb{Z}/2$ in the local case.

(3) If E' is a cubic extension of E, then there is τ in $\mathrm{Gal}(K/F)$ with $\tau(x, y, z) = (z, x, y)$ (up to order). But $\mu - \tau\mu = (x, y, -x-y)$ if $\mu = (x, x+y, 0)$. Hence $C(T/F)$ is $\{0\}$, in the local case.

1.4 <u>In the case of</u> $H = U(2)$, each torus T splits over a biquadratic extension of F, and $C(T/F)$ is trivial, unless T splits over E and σ acts by $\sigma(x, y) = (-x, -y)$, where $C(T/F)$ is $\mathbb{Z}/2$ in the local case.

1.5. We also need <u>a twisted analogue of the above discussion.</u> Let $\underline{G}' = \mathrm{Res}_{E/F}\underline{G}$ be the group obtained from $\underline{G}$ upon restricting scalars from E to F. It is defined over F. In fact, $\underline{G}' \simeq \underline{G} \times \underline{G}$, and $\mathrm{Gal}(\underline{F}/F)$ acts on $\underline{G}'$ by $\tau(x, y) = (\tau x, \tau y)$ if $\tau|E = 1$, or by $\tau(x, y) = \alpha(\tau x, \tau y)$ if $\tau|E \neq 1$. Here $\alpha(x, y) = (y, x)$. Further we have $G'(E) = G(E) \times G(E)$, and $G' = G'(F)$ consists of all $(x, \sigma x)$, x in $G(E)$. G embeds in G' as the diagonal, x and x' in G' are called <u>(stably)</u>σ-<u>conjugate</u> if there is y in G' (resp. $\underline{G}'$) so that $yx = x'\alpha(y)$. In this case $\tau x = x$ for all τ in $\mathrm{Gal}(\underline{F}/F)$, and $\tau(y)x = x'\alpha(\tau y)$. Hence the σ-conjugacy classes within the stable σ-conjugacy class of x are parametrized by the elements $\{\tau \mapsto y_\tau = y^{-1}\tau(y)\}$ of the kernel $B''(T/F)$ of the natural map from $H^1(F, T)$ to $H^1(F, G')$. Here $\underline{T}$ denotes the α-centralizer of $x = (x', x'')$ in $\underline{G}'$, namely the $y = (y', y'')$ with $(y', y'')(x', x'') = (x', x'')\alpha(y', y'')$. These y satisfy $y'x'x'' = x'x''y'$, $y'' = x'^{-1}y'x'$. It is defined over F, since α is. The group T of F-rational points consists of such y with $y'' = \sigma y'$. T is isomorphic to the σ-centralizer of x' in G. The conjugacy class in $\underline{G}$ of $x'x'' = x'\sigma(x')$ is defined over F. Hence it contains a member Nx of G by [Ko]. Nx is determined only up to stable conjugacy. T is isomorphic to the centralizer of Nx in G, over F, by the map $(y', y'') \mapsto y'$. It is clear that $H^1(F, G')$ is trivial; hence $B''(T/F) = H^1(F, T)$.

The map $z \mapsto \{z_\tau = (z, 1)\tau(z, 1)^{-1}\}$ embeds $F^\times$ in $B''(T/F)$. z_τ acts on x in G' by $(z, 1)x\ \alpha(z, 1)^{-1} = zx (= (zx', \sigma(zx'))$ if $x = (x', \sigma x'))$. z maps the member $\{y_\tau = y^{-1}\tau(y)\}$ of $B''(T/F)$ to $\{(zy)_\tau\}$, which sends x to $[(z, 1)y]x\ \alpha[(z, 1)y]^{-1} = (z, z^{-1})yx\,\alpha(y^{-1})$. The quotient of $B''(T/F)$ under this action of $F^\times$ is denoted by $B'(T/F)$. Put $B'(T/\mathbb{A}) = \oplus B'(T/F_v)$ if F is global. Then the Tate-Nakayama theory implies that $B'(T/F)$ (in the local case) or $B'(T/\mathbb{A})/\mathrm{Image}\ B'(T/F)$ (in the global case), is the quotient of the $\mathbb{Z}$-module of the μ in $X_*(T)$ modulo $\mathbb{Z}$ with $\Sigma\tau\mu = 0$ (τ in $\mathrm{Gal}(K/F)$), by the span of $\mu - \tau\mu$ for all μ in $X_*(T)$ and τ in $\mathrm{Gal}(K/F)$.

The map $x \mapsto Nx$ gives a bijection of the set of stable σ-conjugacy classes in G', and the set of stable conjugacy classes in G. In fact, for the present discussion it suffices to consider regular x in G (x with distinct eigenvalues), and σ-regular x in G' (Nx is regular). Hence there are three types of stable σ-conjugacy classes of σ-regular elements in G', denoted by (1), (2), (3) as in the non-twisted case. It is clear that (in the local case) $B'(T/F)$ is trivial if T is A, and in case (3); it is $\mathbb{Z}/2$ in case (2); it is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ if T splits over E but T is not (stably) conjugate to A.

2. STABLE TRACE FORMULA

2.1 Let F be a global field with a ring $\mathbb{A}$ of adeles. Then E denotes a quadratic extension, and $\mathbb{A}^1$ the group of ideles of E whose norm from E to F is 1. The center $Z(\mathbb{A})$ of $G(\mathbb{A})$ is isomorphic to $\mathbb{A}^1$. Fix a character ω of $Z(\mathbb{A})/Z$ (Z is $Z(F)$). Denote the action of $\mathrm{Gal}(E/F)$ on the idele x in $\mathbb{A}_E^\times$ by $\overline{x}$. Then $\omega'(x) = \omega(\overline{x}/x)$ defines a character of the center $Z'(\mathbb{A}) = \mathbb{A}_E^\times$ of $G'(\mathbb{A}) = G(\mathbb{A}_E)$, which is trivial on $E^\times\mathbb{A}^\times$. For each place v of F, let f_v be a smooth (this means locally constant in the non-archimedean case) complex valued function on $G_v = G(F_v)$, which satisfies $f_v(zx) = \omega_v(z)^{-1} f_v(x)$ for all z in Z_v, x in G_v, where ω_v is the component of ω at v; further, the support of f_v is compact modulo Z_v. Let R_v be the ring of integers in F_v if v is non-archimedean. Let K_v be the hyperspecial maximal compact subgroup $G(R_v)$ of G_v. At almost all v the character ω_v is unramidifed, and we take f_v to be the function f_v^0, which obtains the value $\omega_v(z)^{-1}/|K_v|$ at zk in Z_vK_v, and 0 elsewhere. $|K_v|$ denotes the volume of K_v with respect to a Haar measure fixed below. Put $f = \otimes f_v$.

2.2 The conjugacy-class analysis of (1) is motivated by the appearance in the trace formula of the absolutely convergent sum

$$\sum_x \delta(x)^{-1}\,|T(\mathbb{A})/Z(\mathbb{A})T|\,\Phi(x, f) \tag{2.2.1}$$

over all conjugacy classes x of regular elliptic elements in G modulo Z. $\delta(x)$ is the index $[(G/Z)_x : T/Z]$ of T/Z in the centralizer $(G/Z)_x$ of x in

G/Z. T is the centralizer G_x of x in G. Recall that x is called elliptic regular if the quotient of its centralizer $T(\mathbb{A})$ in $G(\mathbb{A})$, by $Z(\mathbb{A})$ and T, is compact. T is called an elliptic torus of G, in this case. The volume $|T(\mathbb{A})/Z(\mathbb{A})T|$ of the quotient (with respect to a Haar measure fixed below) is then finite. We fix differential forms of highest degree defined over F on $\underline{G}/\underline{Z}$ and $\underline{T}/\underline{Z}$, and define Haar measures dg and dt on G_v/Z_v and T_v/Z_v at all v. $\Phi(x, f_v)$ is the orbital integral $\int f_v(gxg^{-1})dg/dt$ (over G_v/T_v), if x is regular with centralizer T_v. We put $\Phi(x, f) = \prod \Phi(x, f_v)$ for regular x in G (with centralizer T).

2.3 <u>The sum</u> (2.2.1) <u>can be written</u> as a sum over the conjugacy classes in G of elliptic tori T, and a sum over the regular x in T/Z. But we have to note that $\delta(x)$ equals the number of w in the Weyl group $W(T)$ of T in G with $wxw^{-1} = zx$ for some z in Z, and the conjugacy class of x in G/Z intersects T/Z precisely $[W(T)]/\delta(x)$ times. So we have

$$\sum_T |T(\mathbb{A})/Z(\mathbb{A})T| \,/[W(T)] \sum_{x \text{ in } T/Z}' \Phi(x, f) \;.$$

Σ_x' indicates sum over regular elements. This is equal to

$$\sum_T{}' |T(\mathbb{A})/Z(\mathbb{A})T| \,/\, [W'(T)] \sum_x{}' \sum_{b \text{ in } B(T/F)} \Phi(x^b, f) \;. \tag{2.3.1}$$

Σ_T' indicates sum over stable conjugacy classes of elliptic T. $W'(T)$ is the Weyl group of T in $A(T/F)$. x^b is $\underline{b}x\underline{b}^{-1}$, where $\underline{b}$ is a representative of b in $\underline{G}$. Note that $\Phi(x^b, f)$, as a function of $\underline{b}$, depends only on the projection of $\underline{b}$ in $B(T/F)$.

2.4 <u>For a fixed regular</u> x <u>the sum over</u> b <u>is finite</u>. $B(T/F)$ is a subset of $C(T/F)$, and we extend the sum to $C(T/F)$, setting $\Phi(x^b, f) = 0$ if b lies in $C(T/F) - B(T/F)$. Note that if the image in $C(T/\mathbb{A})$, of b in $C(T/F)$, lies in $B(T/\mathbb{A})$, then b lies in $B(T/F)$. Since $\Phi(x^b, f) = \prod_v \Phi(x^b, f_v)$, it depends only on the image of b in $C(T/\mathbb{A})$. It remains to note that in our case the map $C(T/F) \to C(T/\mathbb{A})$ is injective (in general the kernel is finite), and we obtain a sum

$$[k(T)]^{-1} \sum_\kappa \Phi(x, f, \kappa) \;.$$

κ ranges over the finite group $k(T)$ of characters of $C(T/\mathbb{A})/C(T/F)$, which is described above. If κ_v is the restriction of κ to $C(T/F_v)$ we put

$$\Phi(x, f_v, \kappa_v) = \sum \kappa_v(b)\, \Phi(x^b, f_v) \qquad (b \text{ in } C(T/F_v)) ,$$

where we set $\Phi(x^b, f_v) = 0$ if b lies in $C(T/F_v) - B(T/F_v)$. $\Phi(x, f, \kappa)$ is the product over all places v of the local sums (which are almost all trivial).

2.5 When κ is trivial we put $\Phi'(x, f)$ for $\Phi(x, f, 1)$, and $\Phi'(x, f_v)$ for $\Phi(x, f_v, 1_v)$. $k(T)$ is trivial unless T is quadratic, when $[k(T)] = 2$, or T splits over E, when $[k(T)] = 4$. We obtain the sum of

$$(2^*) \qquad \sum_T{}' \left(|T(\mathbb{A})/Z(\mathbb{A})T| \,/\, [W'(T)]\,[k(T)]\right) \sum_x{}' \Phi'(x, f)$$

and

$$(2^{**})'' \qquad \frac{1}{2} \sum_T{}'' \left(|T(\mathbb{A})/Z(\mathbb{A})T| \,/\, [W'(T)][k'(T)]\right) \sum_{\kappa \neq 1} \sum_x{}' \Phi(x, f, \kappa) .$$

Σ''_T ranges over the T with even $[k(T)]$, where we put $[k'(T)] = [k(T)]/2$.

Consider the stable conjugacy class of the elliptic T which splits over E. Fix $\kappa \neq 1$. We claim that $\Sigma'_x \Phi(x, f, \kappa') = \Sigma'_x \Phi(x, f, \kappa)$ for any $\kappa' \neq 1$. Indeed, the group $W'(T)$ acts (transitively) on the group $\mathrm{Im}[H^{-1}(X_*(T_{sc})) \to H^{-1}(X_*(T))]$, hence on its dual group $k(T)$. For b in $B(T/\mathbb{A})$ and w in $W'(T)$, we have

$$(bw)_\tau = (bw)^{-1}\, \tau(bw) = w^{-1} b_\tau w \cdot w_\tau \qquad (w_\tau = w^{-1}\tau(w)) .$$

If $\kappa^w(\{b_\tau\}) = \kappa(\{w^{-1} b_\tau w\})$, then

$$\Phi(x^w, f, \kappa^w) = \sum_b \kappa(\{w^{-1} b_\tau w\})\Phi(x^{bw}, f) = \kappa(\{w_\tau\})^{-1} \sum_b \kappa(\{b_\tau\})\Phi(x^b, f) = \Phi(x, f, \kappa) .$$

The last equality follows from the triviality of κ on $C(T/F)$. Hence the claim.

2.6 Note that there is a bijection between the stable conjugacy classes of T in $(2^{**})''$, and the stable conjugacy classes of elliptic tori in $H = U(2) \simeq U(2) \times U(1)/Z$ (where $U(1) \simeq Z \simeq E^1$). If T is quadratic (its splitting field is a biquadratic extension of F), then $[k'(T)] = 1$, and $[W'(T)] = 2$ is the cardinality of the Weyl group of T in A(T/F) with respect to H. If T splits over E, there are three $\kappa \neq 1$ in $(2^{**})''$, $[k'(T)] = 2$ and $[W'(T)] = 6$. With respect to H, $[k(T)] = 2$ and $[W'(T)] = 2$. Hence we can write $(2^{**})''$ in the form

$$(2^{**})' \qquad \frac{1}{2}\sum_T{}''(|T(\mathbb{A})/T|/[W'(T)][k(T)])\sum_x{}'\Phi(x,f,\kappa).$$

Σ_T'' now indicates the sum over the stable conjugacy classes of elliptic T in H. $W'(T)$ and $k(T)$ are defined with respect to H, and Σ_x' is the sum over all regular x in T with eigenvalues not equal to 1. κ is non-trivial.

2.7 <u>To write $(2^{**})'$ as a part of the trace formula</u> for H we need to express $\Phi(x,f,\kappa)$ as the stable orbital integral $\Phi'(x,{}'f)$ of a smooth compactly supported function ${}'f = \otimes {}'f_v$ on $H(\mathbb{A})$, where ${}'f_v$ is the characteristic function ${}'f_v^0$ of the maximal compact subgroup ${}'K_v = H(R_v)$, divided by the volume $|{}'K_v|$, for almost all v. To formulate the desired relation, suppose that E/F is a quadratic extension of non-archimedean local fields. Consider the stably-conjugate but non-conjugate elliptic tori T, T′ of H which split over E as tori of G, and complete this to a set T, T′, ′T, ′T′ of representatives for the conjugacy classes in the stable conjugacy class of T in G. Put

$$\Phi(x,f,\kappa) = \Phi(x,f) + \Phi(x',f) - \Phi({}'x,f) - \Phi({}'x',f) \quad ,$$

and

$$\Phi'(x,{}'f) = \Phi(x,{}'f) + \Phi(x',{}'f) \ ,$$

where x in T, x′ in T′, and so on. x is viewed as an element of the subgroup $M = \{(a_{ij});\ a_{ij} = 0 \text{ if } i+j \text{ is odd}\}$ of G. Modulo Z, we may assume that the entry a_{22} in x is 1. Its eigenvalues are then $1, \epsilon, \epsilon'$. Let κ denote a character of $E^\times$ which is trivial on $NE^\times$, but non-trivial on $F^\times$. Put $\kappa(x) = \kappa(-(\epsilon-1)(\epsilon'-1))$. In his thesis ([24] of [L]), Rogawski uses the Bruhat-Tits building for U(3) to prove

<u>LEMMA</u>. Suppose that E/F and κ are unramified. Then

$$\kappa(x)\Delta(x)\Phi(x,f^0,\kappa) = \Delta'(x)\Phi'(x,{}'f^0) \ .$$

Here $\Delta(x) = |(\epsilon-1)(\epsilon'-1)(\epsilon-\epsilon')|$ and $\Delta'(x) = |\epsilon'-\epsilon|$; $|\epsilon|^2$ is $|N\epsilon|$. It will be interesting to find a simpler proof of this Lemma. In references [24] (p. 74) and [25] (Prop. 6.4) of [L] it is stated that this

Lemma holds for all quadratic tori T of G, but the proof is not given there. When T is quadratic the terms of x' and $'x'$ do not appear. We do not need this Lemma for our applications in [II], but we need it for the study of the local lifting of [U(3)], which will appear elsewhere.

2.8 For the study of the local lifting, which will not be executed here, we will need an approximation argument based on a generalization of Rogawski's Lemma to the context of an arbitrary spherical function. We give this generalization here as it explains the appearance of the unitary lifting. So we fix an unramified quadratic extension E/F, and an unramified character κ of $E^\times/NE^\times$ which is non-trivial on $F^\times$. The Hecke convolution algebra $\mathbb{H}$ consists of K-biinvariant compactly supported functions, named spherical. The Satake isomorphism identifies $\mathbb{H}$ with the algebra $\mathbb{C}[\hat{G}^0\times\sigma]^W$ of W-invariant finite Laurent series on the conjugacy classes in the dual group $\hat{G}$ (denoted by ^LG in [B]; we keep to the original notations) of G of the form $t\times\sigma$, where t lies in the connected component $\hat{G}^0 = GL(3,\mathbb{C})$. We do not repeat the definition of the Satake transform $f \mapsto f^\vee$ here (see [U(2)], p.714). But note that a spherical function is completely determined by the coefficients of $f^\vee$, which are the orbital integrals $F(x,f) = \Delta(x)\Phi(x,f)$ at the diagonal regular elements $x = (u\tilde{\omega}^n, 1, *)$, where u is a unit, and $\tilde{\omega}$ a uniformizer. $F(x,f)$ is independent of u, and we denote it by $F(n,f)$. Note that the dual group $\hat{G}$ used here is the semi-direct product $\hat{G}^0\times W_{E/F}$. $W_{E/F}$ is the Weil group of E/F, namely an extension of Gal(E/F) by $E^\times$. The non-trivial element σ of Gal(E/F) has σ^2 in F-NE; it acts on $\hat{G}^0$ by $\sigma x = J^t x^{-1} J$.

Similarly, we have the Hecke algebra $'\mathbb{H}$ on H and dual group $\hat{H} = \hat{H}^0\times W_{E/F}$, where σ acts on $\hat{H}^0 = GL(2,\mathbb{C})$ by $\sigma x = w\,{}^t x^{-1} w^{-1}$. Here $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We write $F(n,'f)$ for the value of $F(x,'f) = \Delta'(x)\Phi(x,'f)$ at $x = (u\tilde{\omega}^n, *)$.

To relate f and 'f it suffices to relate $F(n,f)$ and $F(n,'f)$. We need to observe that when $x = (\varepsilon, 1, \varepsilon')$ with $\varepsilon' = \bar{\varepsilon}^{-1}$, we have $\kappa(x) = \kappa(\varepsilon)$. So we want $(-1)^n F(n,f) = F(n,'f)$, and in fact use this as a definition of a map $\mathbb{H} \to {}'\mathbb{H}$, $f \mapsto {}'f$. Note that this map is dual to the homomorphism $\hat{H} \to \hat{G}$, which we call endo-lift, defined by

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & 0 & b \\ 0 & x & 0 \\ c & 0 & d \end{pmatrix} = h_1 \quad (1 = x(ad-bc));$$

$$\sigma \mapsto (1,1,-1)\times\sigma; \quad E^\times \ni z \mapsto (\kappa(z), 1, \kappa(z)) \times z .$$

2.9 In these notations, a method of Clozel [Cl], which we used [Sym^2], section 2, in a situation similar to the current one, and hence will not repeat here, implies

LEMMA. For spherical f, f' related by the map $\mathbb{H} \to \mathbb{H}'$, we have

$$F(x, f, \kappa) = F'(x, 'f) ,$$

where $F(x,f,\kappa)$ is $\kappa(x)\Delta(x)\Phi(x,f,\kappa)$, and $F'(x,'f)$ is $\Delta'(x)\Phi'(x,'f)$.

Homogeneity properties of germs [H], Theorem 14, imply

LEMMA. Suppose E/F and κ are unramified. For each smooth compactly supported f on G there exists a smoothly compactly supported $'f$ on H, and for each $'f$ these exists an f, so that $F(x,f,\kappa) = F'(x,'f)$.

This statement is trivial if $\Phi(x,f)$ is supported on the regular set. In fact we need this Lemma for $\Phi(x,f)$ supported on the elliptic set in a future work, but it is not needed for [II]. We say that f, $'f$ are matching if $F(x,f,\kappa) = F'(x,'f)$ for all regular x.

The above Lemmas imply that we can put $(2^{**})'$ in the form

$$(2^{**}) \qquad \frac{1}{2} \sum_T{}'' (|T(\mathbb{A})/T| / [W'(T)][k(T)]) \sum_x{}'' \Phi'(x, 'f) .$$

Indeed, we choose a global character κ of $\mathbb{A}_E^\times/E^\times N\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}^\times$ is non-trivial. At v which splits in E we take $'f(x) = f_M(x)/\kappa(-(\epsilon-1)(\epsilon'-1))$; f_M is defined as usual. The sum of (2^{**}) is the stabilized elliptic regular part of the trace formula of H.

3. TWISTED TRACE FORMULA

3.1 Analogous discussion has to be given in the twisted case. Again F is global, and we fix $\omega'(x) = \omega(\overline{x}/x)$ on $Z'(\mathbb{A})/Z'$, namely on $\mathbb{A}_E^\times/E^\times$, which is trivial on $\mathbb{A}^\times$. We use $\phi = \otimes\phi_v$, where ϕ_v is smooth, transforms under Z'_v by $\omega_v'^{-1}$, and is compactly supported modulo the center. For almost all v the component ϕ_v is ϕ_v^0, the function supported on $Z'_v K'_v$, whose value on $K'_v = G'(R_v)$ is the volume $|K'_v|^{-1}$. When v splits we take $\phi_v = (f_v, f_v^0)$ if f_v is spherical; otherwise f_v^0 is a measure of volume one with $f_v = f_v * f_v^0$. So for almost all split v, we have $\phi_v^0 = (f_v^0, f_v^0)$. In the trace formula, twisted by σ, there appears a sum

$$\sum_x \delta'(x)^{-1} |T(\mathbb{A})/TZ(\mathbb{A})| \Phi(x, \phi) ,$$

over the σ-conjugacy classes x of σ-regular σ-elliptic elements in G'/Z'. T is the α-centralizer of x in G'; $\delta'(x)$ is the index of TZ' in the α-centralizer of x in G'/Z'. Here $\Phi(x,\phi)$ is the integral $\int \phi(yx\alpha(y^{-1}))\,dy$ over $G'(\mathbb{A})/T(\mathbb{A})Z'(\mathbb{A})$. As ϕ transforms by ω'^{-1}, we have $\phi(z\bar{z}x) = \phi(x)$ for z in $Z'(\mathbb{A}) \simeq \mathbb{A}_E^\times$. $\Phi(x,\phi)$ is a product of local integrals $\Phi(x, \phi_v)$.

<u>LEMMA</u>. If Σ'_T indicates the sum over the stable conjugacy classes of elliptic T in G, and $\Sigma'_{x'}$ the sum over regular x' in T/Z, our sum is

$$\sum_T{}' |T(\mathbb{A})/TZ(\mathbb{A})| / [W'(T)] \sum_{x'}{}' \sum_{b \text{ in } B'(T/F)} \Phi(x^b, \phi) .$$

<u>Proof</u>. The sum over b is defined to be 0 unless there is x in G' with $Nx = x'$. If $Nx = x'$, we let $W'(x')$ be the set of g in $\underline{G}/\underline{G}_{x'}$ with $gx'g^{-1} = zx'$ for some z in Z; and $W'(x)$ the set of g in $\underline{G}'/F^\times \underline{G}'_x$ with $gx\alpha(g^{-1}) = zx'$ for some z in Z'. $\underline{G}'_x$ is the α-centralizer of x in $\underline{G}'$; $F^\times$ is the group of (z, z^{-1}), z in $F^\times$. It is clear that the map $W'(x) \to W'(x')$, by $g = (g', g'') \mapsto g'$, is an isomorphism. Also we put $W(x)$ for the g in $G'/Z'G'_x$. It is clear that $\delta'(x) = [W(x)]$, and that $W(x) \to W'(x)$ is injective. Further we note that the stable conjugacy class of x' intersects T/Z in $[W'(T)]/[W'(x)]$ points. If $\delta''(x^b)$ is the number of b' in $B'(T/F)$ with $x^{b'}$ σ-conjugate to zx^b for some z in Z', it remains to show that $[W'(x)]$ is $\delta'(x^b)\delta''(x^b)$, or $\delta''(x) = [W'(x) : W(x)]$, as we can take $b = 1$. But this is clear. Note that it suffices to deal only with x so that $W'(x')$, $W'(x)$ are trivial, by virtue of our assumptions below about the support of ϕ.

3.2 <u>The sum over</u> b <u>can be replaced</u> by the quotient $[k''(T)]$ of the sum over κ in $k''(T)$ of $\Phi(x, \phi, \kappa)$. $k''(T)$ is the dual group of the quotient of $B'(T/\mathbb{A})$ by (the image of) $B'(T/F)$. This is computed above. Note that $[k''(T)] = [k(T)]$. Hence we obtain the twisted analogue of (2^*) and $(2^{**})'$, namely

$$(3^*) \qquad \sum_T{}' (|T(\mathbb{A})/Z(\mathbb{A})T|/[W'(T)][k(T)]) \sum_x{}' \Phi'(x,\phi) \ ,$$

and

$$(3^{**})' \qquad \frac{1}{2}\sum_T{}'' (|T(\mathbb{A})/T|/[W'(T)][k(T)]) \sum_x{}' \Phi(x,\phi,\kappa) \ .$$

The notations in $(3^{**})'$ are taken with respect to H.

3.3 The twisted and non-twisted stable terms (2^*), (3^*) are related by the base change map. We briefly indicate the notations and results. The dual group $\hat{G}'$ of G' is the semi-direct product of the connected component $\hat{G}'^0 = GL(3,\mathbb{C}) \times GL(3,\mathbb{C})$ with $W_{E/F}$. $W_{E/F}$ acts through its quotient Gal(E/F), by $\sigma(x,y) = (\sigma y, \sigma x)$. The diagonal map $\hat{G} \to \hat{G}'$, $x \mapsto (x,x)$, $w \mapsto (1,1)\times w$, induces a dual map $\mathbb{H}' \to \mathbb{H}$ of Hecke algebras.

LEMMA. (1) Suppose E/F is unramified, and ϕ maps to f under the map $\mathbb{H}' \to \mathbb{H}$. Then $\Phi'(x,\phi) = \Phi'(Nx,f)$ for all σ-regular x in G'. (2) If E/F is quadratic, and $\Phi(x,\phi)$ is a σ-stable function supported on the σ-elliptic set, then there exists a matching f.

We say that ϕ, f are matching if $\Phi'(x,\phi) = \Phi'(Nx,f)$ for all σ-regular x in G'. The proof is omitted. The case of $(\phi, f) = (\phi^0, f^0)$ is due to Kottwitz (see [GL(n)], section 6.1). The general case of (1) again follows as in [Sym^2], section 2 and [C1]. (2) is not hard; it is not needed in [II]. For matching global functions ϕ and f, the twisted sum (3^*) is equal to the non-twisted sum (2^*). At a split, v, if $\phi_v = (f'_v, f''_v)$ then $f_v = f'_v * f''_v$.

3.4 The twisted unstable sum $(3^{**})'$ can be related to the stable sum of the elliptic terms in the trace formula of H, as in the case of the non-twisted unstable sum $(2^{**})'$. For that we need a suitable lemma, which we formulate, but not prove. It is likely to follow in the case of ϕ^0, $'\phi^0$ on combining the corresponding proofs in the previous cases of base-change and endo-lift. The extension to any spherical functions will then follow by Clozel's technique as in [Sym^2], and homogeneity yields the matching result for unramified E/F and κ. Note that we need the following lemma for the local study of unstable σ-invariant local G'-modules, and for complete study of the automorphic G-modules, but it is not needed for the local study of G-modules, or the study of automorphic G-modules with elliptic components. So we need a Hecke algebras σ-endo-lift map $\mathbb{H}' \to \mathbb{H}$, $\phi \mapsto {}'\phi$, dual to the dual groups σ-endo-lift homomorphism $\hat{H} \to \hat{G}'$, by $h \mapsto (h_1, h_1)$, $\sigma \mapsto [(1,1,-1),(-1,1,1)] \times \sigma$. We denote smooth compactly supported functions on H by ${}'\phi$ when they are matched with ϕ on G'. Recall that a global character κ on $\mathbb{A}_E^\times / E^\times N\mathbb{A}_E^\times$ was fixed, and we refer to its local components in the local case. Then the lemma asserts the existence of a transfer factor $\kappa'(x)$, which is defined in [U(3)] but better not be specified here until the lemma is proven, so that the following holds. Put $F(x, \phi, \kappa)$ for $\kappa'(x)\Delta(x)\Phi'(x, \phi, \kappa)$. We say that ϕ and ${}'\phi$ are <u>matching</u> if $F(x, \phi^0, \kappa) = F'(Nx, {}'\phi^0)$ for all σ-regular x in G'.

<u>LEMMA</u>. Suppose E/F and κ are unramified. Then $F(x, \phi^0, \kappa) = F'(Nx, {}'\phi^0)$. If ϕ maps to ${}'\phi$ under $\mathbb{H}' \to {}'\mathbb{H}$, then ϕ and ${}'\phi$ are matching. For every ϕ (resp. ϕ'), there is a matching ϕ' (resp. ϕ).

Assuming that ϕ and ${}'\phi$ are global matching functions, $(3^{**})'$ can be put in the form

$$(3^{**}) \qquad \frac{1}{2}\sum_{T}'' (|T(\mathbb{A})/T| / [W'(T)][k(T)]) \sum_{x}' \Phi'(x, {}'\phi) .$$

This is the stabilized elliptic part of the trace formula for H and ${}'\phi$.

4. COMPARISON

4.1 <u>The trace formula for</u> G, and the trace formula for G' twisted with respect to σ, are computed in [U(3)] for any functions f and ϕ. We do not reproduce the computations, and even not record the formulae in full. This will be done by Arthur in the context of any reductive group. Fix two places u, u' of F. We shall work here with global functions f, ϕ, ${}'f$, ${}'\phi$ whose components at u, u' have (twisted in the case of ϕ) orbital integrals which vanish on the (σ-)regular split set. An element is called split if its conjugacy class intersects the diagonal torus non-trivially. Further, we fix a non-archimedean place u'', and require that the (σ-)orbital integral of the component at u'' be zero on the (σ-)semi-simple singular set.

Let L(G) be the space of measurable functions on $G\backslash G(\mathbb{A})$, which transform by the fixed character ω under the center $Z(\mathbb{A})$, and are (absolutely) square-integrable on $G\backslash G(\mathbb{A})$. $G(\mathbb{A})$ acts on L(G) by right translations. Any irreducible constituent π is called an automorphic G-module.

The theory of Eisenstein series decomposes $L(G)$ into a direct sum of two subspaces. One decomposes discretely as a direct sum of $G(\mathbb{A})$-modules π, which appear with finite multiplicities; such π are called discrete-series. The other decomposes as a direct integral. We write $\operatorname{tr}\pi(f)$ for the trace of the trace class convolution operator $\pi(f) = \int f(x)\pi(x)dx$ (x in $G(\mathbb{A})/Z(\mathbb{A})$; dx is a Haar measure), for an irreducible π. Under the above restrictions at u, u', u'' on f, the trace formula (see [U(3)], but the computation is similar to the case of $GL(2)$) asserts

<u>LEMMA</u>. The sum $\Sigma \operatorname{tr}\pi(f)$ over all discrete series π is equal to the sum of (2^*), (2^{**}), and $-\frac{1}{4}\Sigma_\mu \operatorname{tr} M(\mu)I(\mu, f)$. All sums here are absolutely convergent.

The new sum extends over all characters μ of $\mathbb{A}_E^\times/E^\times N\mathbb{A}_E^\times$. The diagonal subgroup $A(\mathbb{A})$ of $G(\mathbb{A})$ consists of $(a, b, \bar{a}^{-1})$, a in $\mathbb{A}_E^\times$, b in $\mathbb{A}^1$. Any character of $A(\mathbb{A})/A$ whose restriction to $Z(\mathbb{A})$ is ω, is of the form $(a, b, \bar{a}^{-1}) \mapsto \mu(a)(\omega/\mu)(b)$, where μ is a character of $\mathbb{A}_E^\times/E^\times$. We denote the $G(\mathbb{A})$-module unitarily induced from the character of $A(\mathbb{A})$, by $I(\mu)$. We shall also use the analogous notations in the local case. $M(\mu)$ is an intertwining operator which is defined in the theory of Eisenstein series.

4.2 <u>We now record the twisted trace formula of</u> G'. The center $Z'(\mathbb{A})$ of $G'(\mathbb{A}) = GL(3, \mathbb{A}_E)$ is isomorphic to $\mathbb{A}_E^\times$. The norm map N takes z in $Z'(\mathbb{A})$ to $z/\bar{z}$ in $Z(\mathbb{A})$. The restriction to $\mathbb{A}^\times$ of the character $\omega' = \omega \circ N$ of $Z'(\mathbb{A})$ is trivial. $L(G')$ denotes the space of measurable functions ψ on $G'\backslash G'(\mathbb{A})$, which transform under $Z'(\mathbb{A})$ by ω', and are absolutely square-integrable on $G'Z'(\mathbb{A})\backslash G'(\mathbb{A})$. $G'(\mathbb{A})$ acts on $L(G')$ by right translation. The irreducible constituents Π are called automorphic. The discrete and continuous spectra are invariant under the action of σ, which maps ψ to $\sigma\psi$, where $(\sigma\psi)(x) = \psi(\sigma x)$. We say that the $G'(\mathbb{A})$-module Π is σ-invariant if Π is equivalent to $\sigma\Pi$, where $(\sigma\Pi)(x) = \Pi(\sigma x)$. In this case there is an intertwining operator $\Pi(\sigma)$ of Π with $\sigma\Pi$, whose square is the identity. We write $\operatorname{tr}\Pi(\phi\times\sigma)$ for the trace of the operator $\Pi(\phi\times\sigma) = \int\phi(x)\Pi(x)\Pi(\sigma)\,d'x$ (x in $G'(\mathbb{A})/Z'(\mathbb{A})$; $d'x$ is a Haar measure).

As usual unitarily induced $G'(\mathbb{A})$-modules are denoted by $I(\eta)$, where $\eta = (\mu, \mu', \mu'')$ is a character of the diagonal subgroup $A'(\mathbb{A})$ of $G'(\mathbb{A})$. μ, μ', μ'' are characters of $\mathbb{A}_E^\times/E^\times$. For each element w of the Weyl group

W of A in G, there is an intertwining operator $M(w,\eta)$ from $I(\eta)$ to $I(w\eta)$, where $(w\eta)(a) = \eta(waw^{-1})$. The $I(\eta)$ which appear in the trace formula are those whose central character $\mu\mu'\mu''$ is equal to ω'.

Suppose τ is an irreducible H'-module, where $H' = GL(2,E)$. Denote by $I(\tau)$ the G'-module unitarily induced from the $H' \times \mathbb{G}_m$ module $\tau \otimes \omega_\tau$, where ω'/ω_τ is the central character of τ. The central character of $I(\tau)$ is then ω'. $I(\tau)$ is σ-invariant if and only if $\tau \simeq w\tau$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(w\tau)(x) = \tau(w^t\bar{x}^{-1}w^{-1})$, and $\omega_\tau(a\bar{a}) = 1$ for all a in $\mathbb{A}_E^\times$.

Recall that the twisted orbital integrals of the components of ϕ at u, u′ are assumed to be zero on the σ-regular split set. The integral of $\phi_{u''}$ at the σ-semi-simple-singular elements, is assumed to be 0. Then the twisted trace formula for G' and ϕ asserts the following. (For a similar case see [Sym].)

<u>LEMMA</u>. The sum $\Sigma \operatorname{tr} \Pi(\phi\times\sigma)$ over the σ-invariant discrete series automorphic G'-modules Π, is equal to the sum of (3^*), (3^{**}), and

$$-\frac{1}{4}\sum \operatorname{tr} I(\eta,\phi\times\sigma) + \frac{1}{8}\sum \operatorname{tr} I(\eta,\phi\times\sigma) + \frac{3}{8}\sum \operatorname{tr} I(\eta,\phi\times\sigma)$$

$$-\frac{1}{2}\sum \operatorname{tr} I(\tau;\phi\times\sigma).$$

All G'-modules $I(\eta)$, $I(\tau)$ here are σ-invariant. The characters μ, μ', μ'' in η are trivial on $E^\times N\mathbb{A}_E^\times$. The first sum is over all unordered triples of pairwise distinct μ, μ', μ''. The second is over all (μ,μ',μ), $\mu' \neq \mu$. In the third $\mu = \mu' = \mu''$. The $I(\eta)$, $I(\tau)$ here are all irreducible.

In fact the way in which the $I(\eta)$ appear in the trace formula is as

$$\frac{1}{24}\sum \operatorname{tr} M((13),\eta)\, I(\eta,\phi\times\sigma) + \frac{1}{6}\sum_{w=(12),(23)}\sum_{\eta} \operatorname{tr} M(w,\eta)\, I(\eta,\phi\times\sigma).$$

The non-zero contributions are given by the η for which η, acted upon by σ and then the reflection w, is equal to η. Thus the first sum is over the η with μ, μ', μ'' trivial on $N\mathbb{A}_E^\times$; the others are over the η with $\mu = \mu' = \mu''$, μ trivial on $N\mathbb{A}_E^\times$.

The intertwining operators $M(w,\eta)$ can be written as local products $m(w,\eta) \otimes_v R(w,\eta_v)$ (see [Sh]). $R(w,\eta_v)$ are the local normalized intertwining operators. They are the trivial operators in our case. The normalizing factors $m(w,\eta)$ are given by

$$m((12),\eta) = L(1,\mu'/\mu)/L(1,\mu/\mu'), \quad m((23),\eta) = L(1,\mu''/\mu')/L(1,\mu'/\mu'') ,$$

and

$$m((13),\eta) = [L(1,\mu''/\mu')/L(1,\mu'/\mu'')][L(1,\mu''/\mu)/L(1,\mu/\mu'')][L(1,\mu'/\mu)/L(1,\mu/\mu')].$$

If at least two of the μ's are equal, $m(w,\eta)$ has to be evaluated as a limit; the value is -1. If the μ are all distinct, then $m((13),\eta)$ is 1. Indeed, $L(1,\mu) = L(1,\bar{\mu})$, and here $\bar{\mu} = \mu^{-1}$. Up to equivalence each $I(\eta)$ appears in the first sum 6 times if the μ are distinct, 3 times if exactly two of the μ are equal, and once if $\mu = \mu' = \mu''$. Whence the expression of the lemma.

4.3 The character μ of $\mathbb{A}_E^\times/E^\times$ defines a character of the diagonal subgroup $'A(\mathbb{A})$ of $H(\mathbb{A})$, by $(a,\bar{a}^{-1}) \mapsto \mu(a)$, and an induced representation $I(\mu)$. Under the usual restriction on $'f$ at u, u' and u'', the trace formula for H and $'f$ asserts the following (see [U(2)]).

LEMMA. The sum $\Sigma\, n(\rho)\operatorname{tr}\{\rho\}('f)$ over all automorphic packets $\{\rho\}$ of H, is equal to the sum of (2^{**}) (multiplied by 2), and $\frac{1}{4}\Sigma_\mu \operatorname{tr} I(\mu, 'f)$. The sum over μ is taken over all characters of $\mathbb{A}_E^\times/E^\times\mathbb{A}^\times$.

The automorphic, and local, packets of H-modules, and the global multiplicities $n(\rho)$ (=1 or 1/2), are defined in [U(2)].

4.4 We now obtain an identity of trace formulae. Let E/F be a glocal quadratic extension, and $'\phi$, ϕ, f, $'f$ matching functions on $H(\mathbb{A})$, $G'(\mathbb{A})$, $G(\mathbb{A})$, $H(\mathbb{A})$. We assume that the (twisted) orbital integrals of the components at u, u' are 0 on the (σ-)regular split set, and that the corresponding integral of the component at the non-archimedean place u'' vanishes on the (σ-) semi-simple singular set. Combining the Lemmas 4.1, 4.2 and 4.3, we deduce

PROPOSITION. In the above notations, we have

$$\sum \prod \operatorname{tr} \Pi_v(\phi_v \times \sigma) + \frac{1}{2}\sum \prod \operatorname{tr} I(\tau_v;\ \phi_v \times \sigma) + \frac{1}{4}\sum \prod \operatorname{tr} I(\eta_v;\ \phi_v \times \sigma)$$

$$-\frac{1}{8}\sum \prod \operatorname{tr} I((\mu,\mu',\mu);\ \phi_v \times \sigma) - \frac{3}{8}\sum \prod \operatorname{tr} I(\mu,\mu,\mu);\ \phi_v \times \sigma)$$

$$= \sum \prod \operatorname{tr} \pi_v(f_v) - \frac{1}{2}\sum n(\rho) \prod \operatorname{tr}\{\rho_v\}('f_v) + \frac{1}{2}\sum n(\rho) \prod \operatorname{tr}\{\rho_v\}('\phi_v)$$

$$+\frac{1}{4}\sum m(\eta)\prod \operatorname{tr} R(\mu)\, I(\mu, f_v) + \frac{1}{8}\sum \prod \operatorname{tr} I(\mu, {}'f_v) - \frac{1}{8}\sum \prod \operatorname{tr} I(u, {}'\phi_v) .$$

The products $\prod$ are taken over all places v of F. It is useful to fix a finite set V of places, which includes u, u', u'', the archimedean and those which ramify in E/F, such that ${}'\phi_v$, ϕ_v, f_v, ${}'f_v$ are spherical outside V. Then the components Π_v, π_v and ρ_v are unramified, and correspond to the conjugacy classes $t'_v\times\sigma$, $t_v\times\sigma$, ${}'t_v\times\sigma$ in the dual groups $\hat{G}'$, $\hat{G}$, $\hat{H}$, by the definition of the Satake transform. For each v outside V we fix $t_v\times\sigma$, and let $t'_v\times\sigma$ be its image under the base change map $\hat{G}\to\hat{G}'$, ${}'t_v\times\sigma$ the pullback via the endo-map $\hat{H}\to\hat{G}$, and ${}'t'_v\times\sigma$ the pullback of $t'_v\times\sigma$ via the σ-endo-map $\hat{H}\to\hat{G}'$. A standard approximation argument, based on (1) the fact that the sums in the Proposition are absolutely convergent, and (2) the matching result of Lemmas 2.9, 3.3(1) and 3.4, for corresponding spherical functions, implies the following

<u>COROLLARY</u>. Fix $\{t_v\times\sigma;\ v \text{ outside } V\}$. Then all products in the Proposition extend over V. The sums range over Π, π, ρ whose component at v outside V is parametrized by $t_v\times\sigma$.

The rigidity theorem for $G' = GL(3)$ of [JS] implies that at most one of the first five sums involving G'-modules is non-empty, and this sum consists of a single G'-module by multiplicity one theorem.

4.5 <u>To use this we briefly recall the lifting in the case where the place</u> v splits in E (see [GL(3)], section 1.5, for a fuller discussion in the case of base change). In this case $E_v = F_v \oplus F_v$ and H, G, G' are $GL(2)$, $GL(3)$ and $GL(3)\times GL(3)$. We now omit v for brevity. σ acts on $G'(F) = G(E)$ by mapping (x,x') to $(\sigma x', \sigma x)$, where $\sigma x = J\,{}^t x^{-1} J$ for x in G. The component at v of the global character κ is a character of $E^\times = F^\times \times F^\times$ invariant under σ. It is a pair (κ, κ^{-1}) of characters of $F^\times$.

The notion of local lifting which we choose is that π' lifts to π if $\operatorname{tr}\pi(f) = \operatorname{tr}\pi'(f')$ for all matching f, f', in the given situation. Recall that matching functions is a relation defined by Lemmas 2.9, 3.3, 3.4. It is then easy to check ([GL(3)], section 1.5, in the case of base change; computation of the character of an induced G-module in the endo-cases), and (1) π lifts to $\Pi = \pi \oplus \sigma\pi$ by base-change, (2) τ lifts to $I(\tau\otimes\kappa)$, where κ is the character of $F^{\times}$ mentioned above, in the case of endo-lifting, (3) τ lifts to $I(\tau) \oplus I(\sigma\tau) = I(\tau \oplus \sigma\tau)$ in the case of σ-endo-lifting. Here $(\sigma\pi)(x) = \pi(\sigma x)$, and $(\sigma\tau)(x) = \tau(\sigma x)$, as usual.

As noted in the introduction, first applications of Proposition 4.4 are given in [II].

REFERENCES

[B] A. Borel, Automorphic L-functions, Proc. Symp. Pure Math. 33 (1979), II, 27-63.

[C] W. Casselman, Characters and Jacquet modules, Math. Ann. 230 (1977), 101-105.

[Cl] L. Clozel, Local base change for GL(n), lectures at IAS, 1984.

[U(2)] Y. Z. Flicker, Stable and labile base change for U(2), Duke Math. J. 49 (1982), 691-729.

[U(3)] ———, L-packets and liftings for U(3), unpublished, Princeton, (1982).

[GL(3)] ———, The trace formula and base change for GL(3), SLN 927 (1982).

[Sym] ———, Twisted trace formula and symmetric square comparison, preprint, Princeton (1984).

[Sym^2] ———, Symmetric square: applications of a trace formula, preprint, Princeton (1984). See also; Outer automorphisms and instability, Theorie de Nombres, Paris, 1980-1981, Progress in Mathematics 22, Birkhauser (1982), 57-65.

[GL(n)] ———, On twisted lifting, Trans. AMS (1985).

[II] ———, Unitary quasi-lifting: applications, Trans. AMS (1985).

[GP] S. Gelbart, I. Piatetski-Shapiro, Automorphic forms on unitary groups, SLN 1041 (1984), 141-184.

[H] Harish-Chandra, Admissible invariant distributions on reductive p-adic groups. Queen's papers in Pure and Applied Math. 48 (1978), 281-346.

[JS] H. Jacquet, J. Shalika, On Euler products and the classification of automorphic forms II, Amer. J. Math. 103 (1981), 777-815.

[Ke] D. Keys, Principal series representations of special unitary groups over local fields, Compo. Math. 51 (1984), 115-130.

[Ko] R. E. Kottwitz, Rational conjugacy classes in reductive groups, Duke Math. J. 49 (1982), 785-806.

[Ku] S. Kudla, On certain Euler products for SU(2, 1), Compo. Math. 42 (1981), 321-344.

[L] R. P. Langlands, Les débuts d'une formule des traces stables, Publ. Math. Univ. Paris VII, 13 (1984).

[Sh] F. Shahidi, Local coefficients and normalization of intertwining operators for GL(n), Compo. Math. 48 (1983), 271-295.

[T] J. Tate, The cohomology groups of tori in finite galois extensions of number fields, Nagoya Math. J. 27 (1966), 709-719.

FINE HALL - WASHINGTON ROAD
PRINCETON, NJ 08544

Current address:
HARVARD UNIVERSITY
SCIENCE CENTER, 1 OXFORD ST.
CAMBRIDGE, MA 02138