PACKETS AND LIFTINGS FOR $U(3)$

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0. Introduction

Let E/F be a quadratic extension of non-archimedean local fields of characteristic 0, put $G' = \mathrm{GL}(3, E)$, and denote by G the unitary group in three variables over F which splits over E. We realize G as the group of g in G' with $\sigma(g) = g$, where $\sigma(g) = J{}^t\bar{g}^{-1}J$, and

$$J = \begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}.$$

Similarly, we realize the unitary group H in two variables over E/F as a subgroup of $H' = \mathrm{GL}(2, E)$, where J is replaced by

$$w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in the definition of σ.

Our aim here is to give a complete description of the (equivalence classes of) irreducible admissible G-modules π, in terms of the (equivalence classes of) σ-invariant σ-stable admissible irreducible G'-modules Π. This is done by means of character relations with respect to the base-change lift b from G to G', and the endo-lift e from H to G, defined in [U]. Namely we now complete the second step, local lifting, of the program initiated in [U], where the global quasi-lifting is dealt with. Consequently, the third step, global lifting, of [U], can be carried out for most automorphic representations. This is done in chapter II below.

We use the notations and definitions of [U]. Thus ϕ, f, $'f$ denote matching ([U], Lemmas 2.7 and 3.3) complex-valued locally-constant functions on G', G, H. $'f$ is compactly supported; ϕ, f transform under the center Z', Z of G', G by

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matching characters ω'^{-1}, ω^{-1} ($\omega'(z) = \omega(z/\bar{z})$), and are compactly- supported modulo the center. Thus the stable orbital integrals of f match the twisted stable orbital integrals of ϕ, and the unstable orbital integrals of f match the stable orbital integrals of $'f$. We denote by χ_π the character [H] of π. It is a locally-integrable function on G with $\operatorname{tr} \pi(fdg) = \int \chi_\pi(g)f(g)dg$ (g in G) for all measures fdg; it is locally constant on the regular set.

Definition. A G-module Π is called *σ-invariant* if ${}^\sigma\Pi \simeq \Pi$, where ${}^\sigma\Pi(g) = \Pi(\sigma(g))$.

The twisted character χ_Π of such Π is a locally-integrable function on G', which satisfies

$$\operatorname{tr} \Pi(\phi dg \times \sigma) = \int \chi_\Pi(g)\phi(g)dg \qquad (g \text{ in } G')$$

for all ϕdg. χ_Π depends on the σ-conjugacy classes; it is locally constant on the σ-regular set.

Definition. A σ-invariant Π is called *σ-stable* if its character depends only on the stable σ-conjugacy classes in G', namely $\operatorname{tr} \Pi(\phi \times \sigma)$ depends only on f.

Our purpose in this paper is to refine the following base-change result, which is our initial assumption here. It is proven in [U″]. All of our modules are admissible and irreducible.

Local Base-Change. *For every σ-stable tempered G'-module Π there exist non-negative integers $m'(\pi)$ which are non-zero for finitely many tempered G-modules π, so that for all matching ϕ, f we have*

$$(*) \qquad \operatorname{tr} \Pi(\phi \times \sigma) = \sum_\pi m'(\pi) \operatorname{tr} \pi(f).$$

In fact, this relation defines a partition of the set of (equivalence classes of) tempered irreducible G-modules into disjoint finite sets.

Definition. (1) This finite set of π which appear in the sum on the right of (*) is called a *packet*, and denoted by $\{\pi\}$, or $\{\pi(\Pi)\}$. It consists of tempered G-modules.

(2) Π is called the *base-change lift* of (each element π in) the packet $\{\pi(\Pi)\}$.

To refine the identity (*) we prove here that the multiplicities $m'(\pi)$ are equal to 1 (using [GP]), and count the π which appear in the sum. The result depends on the σ-stable Π. It is clear from the work of [U], §4, that:

The σ-stable Π are: *the σ-invariant Π which are square-integrable, one dimensional, or induced $I(\tau \otimes \kappa)$ from a maximal parabolic subgroup, where on* *the* 2×2 *factor the* H'*-module* $\tau \otimes \kappa$ *is the tensor product of an* H'*-module* τ *obtained by the stable base change map of* [U(2)] (b″ in [U′], §1.3), *and the fixed character* κ *of* C_E/NC_E *from* [U], *which is non-trivial on* C_F.

In the local case $C_E = E^\times$ and N is the norm from E to F. Namely $\tau \otimes \kappa$ is obtained by the unstable map of [U(2)] (b′ in [U′], §1.3), from a packet $\{\rho\}$ of H-modules (defined in [U(2)]). From now on the Π are assumed to be σ-stable. Our main local results are as follows:

Local Results. (1) *If* Π *is square-integrable, the packet* $\{\pi(\Pi)\}$ *consists of a single square-integrable G-module* π. *If* Π *is of the form* $I(\tau \otimes \kappa)$, *and* τ *is the stable base-change lift of a square-integrable H-packet* $\{\rho\}$, *then the cardinality of* $\{\pi(\Pi)\}$ *is twice that of* $\{\rho\}$.

Remark. In the last case we denote $\{\pi(\Pi)\}$ also by $\{\pi(\rho)\}$, and say that $\{\pi\}$ *endo-lifts* to $\{\pi(\rho)\} = \{\pi(I(\rho \otimes \kappa))\}$.

Let $\{\rho\}$ be a square-integrable H-packet. It consists of one or two elements.

Local Results. (2) *If* $\{\rho\}$ *consists of a single element then* $\{\pi\}$ *consists of two elements,* π^+ *and* π^-, *and we have the character relation*

$$\operatorname{tr} \rho('f) = \operatorname{tr} \pi^+(f) - \operatorname{tr} \pi^-(f)$$

for all matching $'f, f$. *If* $\{\rho\}$ *consists of two elements, then there are four members in* $\{\pi(\rho)\}$, *and three distinct square-integrable H-packets* $\{\rho_i\}$ $(i = 1, 2, 3)$, *with* $\{\pi(\rho_i)\} = \{\pi(\rho)\}$. *With this order, the four members of* $\{\pi_i\}$ *can be indexed so that we have the relations*

$$\operatorname{tr}\{\rho_i\}('f) = \operatorname{tr} \pi_1(f) + \operatorname{tr} \pi_{i+1}(f) - \operatorname{tr} \pi_{i'}(f) - \operatorname{tr} \pi_{i''}(f)$$

for all matching f, $'f$. *Here* i', i'' *are so that* $\{i+1, i', i''\} = \{2, 3, 4\}$. *A single element in the packet has a Whittaker vector. It is* π^+ *if* $[\{\rho\}] = 1$, *and* π_1 *if* $[\{\rho\}] = 2$.

Remark. The proof that a packet contains no more than one non-degenerate member is only sketched, in the case of odd residual characteristic, as it depends on a twisted analogue of [Ro], which is not yet available in print.

In the case of the special H-module $s(\beta)$ and its one-dimensional complement $1(\beta)$, we denote their stable base-change lifts by $s'(\beta')$ and $1'(\beta')$. β is a character of $C_E^1 = E^1$ (norm-one subgroup in $E^\times$), and $\beta'(a) = \beta(a/\bar{a})$ is a character of C_E.

Local Results. (3) *The packet* $\{\pi(s(\beta))\}$ *consists of a supercuspidal* $\pi^- = \pi_\beta^-$, *and the square-integrable subquotient* $\pi^+ = \pi_\beta^+$, *of the induced G-module* $I = I(\beta'\kappa\nu^{1/2})$. *I is reducible* ([*U′*], *§3.2*), *of length two, and its non-tempered subquotient is denoted by* $\pi^\times = \pi_\beta^\times$. *The character relations are*

$$\operatorname{tr}(s(\beta))('f) = \operatorname{tr} \pi^+(f) - \operatorname{tr} \pi^-(f),$$

$$\operatorname{tr}(1(\beta))('f) = \operatorname{tr} \pi^\times(f) + \operatorname{tr} \pi^-(f),$$

$$\operatorname{tr} I(s'(\beta') \otimes \kappa; \phi \times \sigma) = \operatorname{tr} \pi^+(f) + \operatorname{tr} \pi^-(f),$$

$$\operatorname{tr} I(1'(\beta') \otimes \kappa; \phi \times \sigma) = \operatorname{tr} \pi^\times(f) - \operatorname{tr} \pi^-(f).$$

As the base-change character relations for induced modules are easy ([U], Lemma 1.4), we obtained the character relations for all (not necessarily tempered) σ-stable G'-modules.

It will be useful to record here in a diagram our standard notations.

Liftings

$$\begin{array}{ccccc} G = U(3) & \xrightarrow{\ b\ } & G' = \mathrm{GL}(3, E) & & \\ e\uparrow & & i\uparrow & \nwarrow e' & \\ H = U(2) & \xrightarrow{\ b'\ } & H' = \mathrm{GL}(2, E) & \xleftarrow{\ b''\ } & H \end{array}$$

The notations are explained in [U′], §1.3. The diagram is commutative since we put here the unstable base change map b' on the left, and the stable base change map b'' on the right.

Functions

$$\begin{array}{ccc} f & \xleftarrow{\ b\ } & \phi \\ \downarrow & & \searrow e' \\ 'f & & '\phi \end{array}$$

Packets

$$\begin{array}{ccccccc} \pi & \xleftrightarrow{\ b\ } & \Pi & I(\tau \otimes \kappa) & I(\tau) & & \\ \uparrow & & & \uparrow i & \uparrow i & \nwarrow e' & \\ \rho & \longrightarrow & & \tau \otimes \kappa & \tau & \xleftarrow{\ b'\ } & \rho \end{array}$$

In addition to an identity of trace formulae ([U′], Proposition 4.4), we use the recent fundamental study by Kazhdan [K] which yields in particular the orthogonality relations for characters conjectured in [U(3)]. Hence the present work is a modified reproduction of most of chapter 7 of [U(3)]. Other parts of chapter 7, as well as chapter 8, of [U(3)], appear in [U″]. The transfer of orbital integrals used below is stated in [U], Lemma 3.3, for the case $\phi \to f$ of base-change (for spherical functions the proof is given in [Sph]). For the case $f \to 'f$ of endo-lifting, it is stated in [U], Lemma 2.7. Here, when E/F is unramified, the case of the unit element of the Hecke algebra is due to Rogawski (thesis, unpublished), the extension to other spherical functions is as in [Sph], and homogeneity of germs ([H]) implies the transfer $f \to {}'f$ for general functions. When E/F is ramified, the transfer $f \to {}'f$ has been proven by Langlands and/or Shelstad (in preparation). We envisage another proof, based on the ideas of [Sym; V], but have not carried it out as yet. To obtain the above local results for an unramified E/F, no knowledge of any ramified place is needed.

These precise local results can be used together with the identity of trace formulae to obtain complete results about the global liftings. In this paper we prove these global results only for automorphic representations with two elliptic components. As we explain below, the global results can also be proven by means of simple methods for *all* automorphic representations, but this we delay to another paper (see [TF]). For the global results of the present work we need the identity [U′], Proposition 4.4, of trace formulae, only for matching test functions $\phi, f, {}'f$ with two local components whose orbital integrals vanish on the regular split set (these components will be called *discrete* or *elliptic* below). For such global functions, all weighted orbital integrals and integrals involving logarithmic derivatives of intertwining operators in the trace formulae of the rank one groups G', G, H, are zero; ($G' = \mathrm{GL}(3, E)$ has *twisted* rank one, and its twisted trace formula is similar to that of a group of rank one (such as G and H)). Using regular functions it is shown in [Sym; III (3.7.2)] and [Sym; IV (1.6.3)] that the required equality of trace formulae of G', G, H holds for matching $\phi, f, {}'f$ with a single discrete component (it is clear that the computations of [Sym] in the context of the symmetric square hold also in the present easier situation). Consequently, our global results hold also for automorphic representations with one elliptic component.

It is important to note that working with a global function $\phi = \otimes\, \phi_v$ which has at least one discrete component ϕ_u at a place which stays prime in E, we may choose the unstable transfer ${}'\phi_u$ of ϕ_u to be zero (namely we require the (twisted) orbital integrals of ϕ_u to be stable). In this case the function ${}'\phi$ on H is zero, and we do not need to establish the unstable base-change transfer $\phi_v \to {}'\phi_v$ from G'_v to H_v for any place v. This unstable transfer (stated in [U] as Lemma 3.4) can be proven using combinatorics on the Bruhat–Tits building of PGL(3). A more abstract proof of this unstable transfer can be given along the lines of [Sym; V], where the analogous unstable transfer is proven in the context of the symmetric-square lifting. But we have not yet carried out his "more abstract" proof. Here the crucial case is that of the transfer of the unit element ϕ_v^0 in the Hecke algebra of G'_v to the unit element ${}'\phi_v^0$ in the Hecke algebra of H_v. Assuming this transfer ($\phi_v^0 \to {}'\phi_v^0$) we prove in [TF] the equality of the trace formulae for G' (twisted), G and H, for arbitrary matching test functions (${}'\phi, \phi, f, {}'f$). The argument in [TF] is simple, and relies on properties of pseudo-spherical functions (introduced in [TF]). Consequently, our global lifting results are valid for all automorphic representations with no restriction at any place.

Another proof for the identity of the trace formulae of G', G, H for arbitrary matching $('\phi, \phi, f, 'f)$ is given in [Sym; VI] (in the context of the more difficult comparison of the symmetric square), and in [BC] in the context of the easier comparison of base change for GL(2). It relies on properties of regular functions. The proofs of [TF] and [Sym; VI] are simple. An optimal choice of a component of the test functions annihilates *a priori* the complicated terms (weighted and singular orbital integrals) in the trace formulae. These "simple" proofs can be carried out also in the context of cuspidal G-modules with a supercuspidal component, where G is any reductive group (see [FK]). They are analogous to Deligne's conjecture on the fixed point formula in étale cohomology of correspondences on a separated scheme of finite type over a finite field (see [FK′]). In [U(3)] we proposed yet another technique, based on computing all terms in the trace formulae and "correcting" the weighted orbital integrals as in [U(2)] and [GL(3)]. However these computations seemed (and still do seem) to me to be too long and complicated to be worth formalizing. The (two different) proofs of [TF] and [Sym; VI] seem to me to be satisfactorily short and abstract.

Having made these comments, we shall now state our global results, in the context of automorphic forms with "two elliptic components" (these results are based on a straightforward comparison of trace formulae, and do not use the unstable transfer $\phi_v \to '\phi_v$). Fix two places u, u' of F, such that u is non-archimedean and stays prime in E. Fix G_u and $G_{u'}$-modules π_u^0 and $\pi_{u'}^0$ which are one-dimensional, Steinberg or not contained in any modules induced from the Borel subgroup. Let Π_u^0 (and $\Pi_{u'}^0$) be the G_u'-(and $G_{u'}'$-)modules which are the base-change lifts of the packets $\{\pi_u^0\}$ (and $\{\pi_{u'}^0\}$), and $\{\rho_u^0\}$ (and $\{\rho_{u'}^0\}$) the H_u (and $H_{u'}$) packets which endo-lift to $\{\pi_u^0\}$ (and $\{\pi_{u'}^0\}$); note that these ρ's do not always exist (ρ_u^0 does not exist if u splits in E/F and π_u^0 is elliptic, and if E_u is a field and π_u^0 is one-dimensional or Steinberg). Denote by $A(\pi_u^0, \pi_{u'}^0)$ the family (set with multiplicities) of discrete-series automorphic G-modules $\pi = \otimes \pi_v$ whose components at u and u' are π_u^0 and $\pi_{u'}^0$. Similarly, we introduce $A(\Pi_u^0, \Pi_{u'}^0)$ and $A(\{\rho_u^0\}, \{\rho_{u'}^0\})$. Our global results concern only members of these three families. To start with, we have:

Multiplicity One Theorem. *Each discrete-series automorphic G-module in $A(\pi_u^0, \pi_{u'}^0)$ occurs in the discrete-spectrum of $L^2(G, \omega)$ with multiplicity one.*

In particular, $A(\pi_u^0, \pi_{u'}^0)$ consists of inequivalent representations.

Our main global results consist of a definition and description of the packets of discrete-series G-modules. To introduce the definition, recall that we defined above G_v-packets of tempered G_v-modules at each v (if v splits then $G_v = \mathrm{GL}(3, F_v)$ and a packet consists of a single irreducible). If π_v is a non-tempered irreducible G_v-module then its packet $\{\pi_v\}$ consists of π_v alone. For example, the packet of $\pi_v^\times$ consists only of $\pi_v^\times$. Also we make the following

Definition. The *quasi-packet* $\pi(\beta_v)$ of the non-tempered subquotient $\pi_v^\times = \pi_{\beta_v}^\times$ of $I(\beta_v'\kappa_v\nu_v^{1/2})$, where β_v is a character of $C_{E_v}^1 = E_v^1$, consists of $\pi_v^\times$ and $\pi_v^- = \pi_{\beta_v}^-$.

This local definition is made for global purposes. Thus a packet consists of tempered G_v-modules, or of a single non-tempered element. A quasi-packet consists of a non-tempered $\pi_v^\times$ and a supercuspidal π_v^-. The packet of π_v^- consists of π_v^- and π_v^+, where π_v^+ is the square-integrable constituent of $I(\beta_v'\kappa_v\nu_v^{1/2})$.

Definition. (1) Given a local packet P_v for all v such that P_v contains an unramified member π_v^0 for almost all v, we define the *global packet P* to be the set of products $\otimes\,\pi_v$ over all v, where π_v lies in P_v for all v, and $\pi_v = \pi_v^0$ for almost all v.

(2) Given a character μ of $C_E^1 = \mathbf{A}_E^1/E^1$, the quasi-packet $\pi(\mu)$ is defined as in the case of packets, where P_v is replaced by the quasi-packet $\pi(\mu_v)$ for all v.

(3) The $H(\mathbf{A})$-module $\rho = \otimes\,\rho_v$ *endo-lift* to the $G(\mathbf{A})$-module $\pi = \otimes\,\pi_v$ if ρ_v endo-lifts to π_v (i.e. $\{\rho_v\}$ endo-lifts to $\{\pi_v\}$) for all v. Similarly, $\pi = \otimes\,\pi_v$ *base-change* lifts to the $G'(\mathbf{A})$-module $\Pi = \otimes\,\Pi_v$ if π_v base-change lifts to Π_v for all v.

A complete description of the packets is as follows.

Global Lifting. *The base change lifting is a one-to-one correspondence from the set of packets and quasi-packets which contain an automorphic G-module, to the set of σ-invariant automorphic G′-modules* Π *whose components are σ-stable (these are described by the sums* Φ_1, Φ_2, Φ_3 *in* §I.1 *below).*

As usual, we write $\pi(\rho)$ for a packet which base-changes to $\Pi = I(\tau\otimes\kappa)$, where τ is the stable base-change lift of the H-packet ρ. We conclude:

Description of packets. *Each discrete-series G-module* π *lies in one of the following.*

(1) *A packet* $\pi(\Pi)$ *associated with a discrete series σ-invariant G′-module* Π.

(2) *A packet* $\pi(\rho)$ *associated with a discrete series automorphic H-module* ρ *which is not of the form* $\rho(\theta, \omega/\theta^2)$.

(3) *A quasi-packet* $\pi(\mu)$ *associated with an automorphic one-dimensional H-module* $\rho = \mu(\det)$.

Multiplicities. (1) *The multiplicity of a* $G(\mathbf{A})$*-module* $\pi = \otimes\,\pi_v$ *from a packet* $\pi(\Pi)$ *of type* (1) *in the discrete-spectrum of G is one. Namely each element* π *of* $\pi(\Pi)$ *is automorphic, in the discrete-series.*

(2) *The multiplicity of* π *from a packet* $\pi(\rho)$ *or a quasi-packet* $\pi(\mu)$ *in the discrete-spectrum of G is equal to one or zero; this multiplicity is not constant over* $\pi(\rho)$ *and* $\pi(\mu)$*. It is given by*

$$m(\mu, \pi) = \left(1 + \prod_v \varepsilon_v(\mu_v, \pi_v)\right) \Big/ 2$$

with $\varepsilon_v(\mu_v, \pi_v) = 1$ *if* $\pi_v = \pi_{\mu_v}^{\times}$ *and* $\varepsilon_v(\mu_v, \pi_v) = -1$ *if* $\pi_v = \pi_v^-$, *if* π *lies in* $\pi(\mu)$. *If* π *lies in* $\pi(\rho)$, *and there is a single* ρ *which endo-lifts to* π, *then the multiplicity is*

$$m(\rho, \pi) = \left(1 + \prod_v \varepsilon_v(\rho_v, \pi_v)\right) \Big/ 2,$$

where $\varepsilon_v(\rho_v, \pi_v) = 1$ *if* π_v *lies in* $\pi(\rho_v)^+$, *and* $\varepsilon_v(\rho_v, \pi_v) = -1$ *if* π_v *lies in* $\pi(\rho_v)^-$. *If* π *lies in* $\pi(\rho_1) = \pi(\rho_2) = \pi(\rho_3)$ *where* ρ_1, ρ_2, ρ_3 *are distinct H-packets, then the multiplicity of* π *is* $(1 + \Sigma_{i=1}^3 < \varepsilon_i, \pi >)/4$, *where the signs* $\langle \varepsilon_i, \pi \rangle = \Pi_v \langle \varepsilon_i, \pi_v \rangle$ *are defined in* II.1.2.

In particular we have the following

Rigidity Theorem. *If* π *and* π' *are discrete series G-modules whose components* π_v *and* π_v' *are equivalent for almost all* v, *then they lie in the same packet, or quasi-packet.*

Corollary. (1) *Suppose that* π *is a discrete-series G-module which has a component of the form* $\pi_w^{\times}$. *Then* π *lies in a quasi-packet* $\pi(\mu)$. *In particular its components are of the form* $\pi_v^{\times}$ *for almost all* v, *and of the form* π_v^- *for the remaining finite set of even cardinality of places of F which stay prime in E.*

(2) *If* π *is a discrete series G-module with an elliptic component at a place of F which splits in E, or a one-dimensional or Steinberg component at a place of F which stays prime in E, then* π *lies in a packet* $\pi(\Pi)$, *where* Π *is a discrete-series G'-module.*

The discrete series G-modules with an elliptic component at a place v of F which splits in E can easily be transferred to discrete-series $'G$-modules, where $'G$ is the inner form of G which is ramified at v.

Our local results hold for every local non-archimedean field, of any characteristic, since by the Theorem of [K′] our results can be transferred from the case of characteristic zero to the case of positive characteristic. Consequently (once the twisted trace formula for G and σ is made available in the function field case) our global results hold for every global field, in particular function fields, not only number fields.

Finally, we note that in this paper we concentrate on the description of G-packets in the p-adic case, and we work with a global extension E/F which splits at each archimedean place. The analogous results in the real case are well-known. They are described in the Appendix to this paper. The main difference is that in the p-adic case $G = U(3)$ does not have inner forms non-isomorphic to itself, while in the real case the quasi-split $U(2, 1)$ has a compact inner form $U(3)$. Consequently, a discrete-series $U(2, 1)$-packet in the real case consists of three G-

modules, while in the p-adic case the G-packet consists of one, two or four elements. Of course in the real case there are no supercuspidal G-modules. In particular, in the Appendix we use our global results to determine those automorphic G-modules which make a non-zero contribution to the cohomology outside the middle dimension. These are the automorphic elements in the quasi-packets $\pi(\mu)$.

I. Local Lifting

§1.1. Trace formulae

Our aim here is to study the local liftings. Thus we fix a quadratic extension of local non-archimedean fields. We start with the identity of trace formulae of [U′], Proposition 4.4. We denote by E/F a quadratic extension of number fields such that F has no real places and at the place w of F we obtain that E_w/F_w is our quadratic extension. Denote by V a finite set of places of F including the archimedean and those which ramify in E. The products below range over V; at each v in V we choose matching functions $\phi_v, f_v, {}'f_v$, as in [U], Lemmas 2.7, 3.3. We fix an unramified G_v-module π_v^0 at each v outside V. The sums below range over the automorphic G', G or H-modules with component matching π_v^0 at all v outside V. Proposition 4.4. of [U′] asserts the following

Proposition. *The identity of trace formulae takes the form*

$$\Phi_1 + \tfrac{1}{2}\Phi_2 + \tfrac{1}{4}\Phi_3 = F_1 - \tfrac{1}{2}F_2 - \tfrac{1}{4}F_3 + \tfrac{1}{4}F_6.$$

By the rigidity theorem for G' at most one of the terms Φ_i is non-zero, and consists of a single contribution, where

$$\Phi_1 = \sum_{\Pi} \prod \operatorname{tr} \Pi_v(\phi_v \times \sigma),$$

the sum being over the σ-invariant discrete-series (automorphic) G'-modules Π; these are the (σ-invariant) cuspidal or one-dimensional G'-modules;

$$\Phi_2 = \sum_{\tau} \prod \operatorname{tr} I(\tau_v \otimes \kappa_v; \phi_v \times \sigma),$$

the sum being over the σ-invariant discrete-series (i.e. cuspidal or one-dimensional) H'-modules τ which are obtained by the stable base-change map b'' in [U];

$$\Phi_3 = \sum \prod \operatorname{tr} I((\mu, \mu', \mu''); \phi_v \times \sigma),$$

where the sum is over the distinct unordered triples μ, μ', μ'' of characters of C_E/C_F.

On the right,

$$F_1 = \sum_{\pi} m(\pi) \prod \operatorname{tr} \pi_v(f_v);$$

the sum is over the equivalence classes of discrete-series (automorphic) G-modules π; they occur with finite multiplicities $m(\pi)$.

$$F_2 = \sum_{\rho \neq \rho(\theta,'\theta)} \prod \operatorname{tr}\{\rho_v\}('f_v),$$

the sum ranges over the (automorphic) discrete-series packets ρ of H which are not of the form $\rho(\theta, '\theta)$ (see [U(2)]). These packets ρ are cuspidal or one-dimensional (see [U(2)]).

$$F_3 = \sum_{\rho = \rho(\theta,'\theta)} \prod \operatorname{tr}\{\rho_v\}('f_v),$$

where the sum ranges over the packets $\rho = \rho(\theta, '\theta)$, where θ, $'\theta$ and $\omega/\theta \cdot '\theta$ are distinct.

$$F_6 = \sum_{\mu} \prod \operatorname{tr} R(\mu_v) I(\mu_v, f_v) - \sum_{\rho} \prod \operatorname{tr}\{\rho_v\}('f_v);$$

the first sum is over the characters μ of C_E/C_F with $\mu^3 \neq \omega'$. The second is over the packets $\rho = \rho(\theta, \omega/\theta^2)$, where $\theta^3 \neq \omega$.

§1.2. Coinvariants

Some of our proofs below are inductive on the rank, and depend on reduction to the elliptic set of a smaller Levi subgroup.

In our rank one case there is only one induction step, and here we set up the required notations. Let E/F be a quadratic extension of local fields.

Denote by A the diagonal subgroup, by N the unipotent upper-triangular subgroup of G, and by K the maximal compact subgroup $G(R)$ of G, so that $G = ANK$; R is the ring of integers in F. We use the analogous notations $'A$, $'N$, $'K$ in the case of H, and A', N', K' in the case of G', the even drop the primes if no confusion is likely to occur. Put

$$f_N(a) = \delta(a)^{1/2} \int_K \int_N f(k^{-1}ank)\,dn\,dk \qquad (a = (\alpha, \beta, \bar{\alpha}^{-1}),\ |\alpha| < 1),$$

where $\delta(\alpha, \beta, \bar{\alpha}^{-1}) = |\alpha|^2$ is the modulus function on G. If (π, V) is a G-module, then the quotient V_N of V by the span of the vectors $\pi(n)v - v$ (n in N, v in v) is an A-module $\tilde{\pi}_N$, whose tensor product with $\delta^{-1/2}$ is the normalized A-module (π_N, V_N) of N-coinvariants of π. The theorem of Deligne–Casselman [C] asserts that at $a = (\alpha, \beta, \bar{\alpha}^{-1})$ with $|\alpha| < 1$, we have $\chi_\pi(a) = \chi_{\pi_N}(a)$, hence $\Delta\chi_\pi(a) =$

$\chi_{\pi_N}(a)$, where $\Delta(a) = |(\alpha - \beta)(\beta - \bar{\alpha}^{-1})|$ $(= |\alpha|^{-1}$ if $|\alpha| < 1)$. Consequently, if f is supported on the conjugacy classes of the a with $|\alpha| < 1$, we conclude from the Weyl integration formula that

$$\operatorname{tr} \pi(f) = \operatorname{tr} \pi_N(f_N).$$

Similar definitions can be introduced in the cases of H and G'-modules.

Definition. A G-module π is called *supercuspidal* if π_N is $\{0\}$. In our case π_N can have up to two central exponents (characters of A). π is called *tempered* if they are bounded, and *square-integrable* if it is strictly less than 1 on the a with $|\alpha| < 1$. In particular, a square-integrable π has at most one central exponent in π_N.

We shall use these results to study the following identity. Suppose that $\{\rho\}$ is a square-integrable H-module, and $m(\rho, \pi)$, c and c' are complex numbers, where π are (equivalence classes of) unitarizable G-modules, and the sum $\Sigma_\pi m(\rho, \pi) \operatorname{tr} \pi(f)$ is absolutely convergent. Moreover, suppose that this sum ranges over a countable set S which has the following property. For every open compact subgroup K_1 of G there is a finite set $S(K_1)$ such that $\operatorname{tr} \pi(f) = 0$ for every π in $S - S(K_1)$ and every K_1-biinvariant f. Suppose that for all matching $(\phi, f, {}'f)$ we have

$$c \operatorname{tr} I(\tau \otimes \kappa; \phi \times \sigma) + c' \operatorname{tr}\{\rho\}({}'f) = \sum_\pi m(\rho, \pi) \operatorname{tr} \pi(f), \tag{1.2.1}$$

where τ is the stable base-change lift of $\{\rho\}$. In this case we have

Proposition. (i) *The set S consists of* (1) *square-integrable but not Steinberg G-modules, and* (2) *proper submodules of G-modules induced from a unitary character of A.*

(ii) *If $\{\rho\}$ is supercuspidal then the π of* (1) *are supercuspidal.*

(iii) *If $\{\rho\}$ is Steinberg then precisely one π of* (1) *is not supercuspidal; it is a subquotient of an induced G-module $I(\mu\kappa\nu^{1/2})$.*

(iv) *If the $m(\rho, \pi)$ are all non-negative then the π are all square integrable.*

Remark. (a) Then π mentioned in (2) above are not square-integrable, since their central exponents do not decay. They exist, and are described in [U], (1) of §3.2, but we need not use this fact. (b) In (iii), $\nu(x) = |x|$ and μ is a (unitary) character of $E^\times$ trivial on $F^\times$. Our proof implies that if the identity (1.2.1) exists, then $I(\theta\kappa\nu^{1/2})$ is reducible. In this way, we recover a result of Keys, recorded in [U], (3) of §3.2. In [U], §3.2, we give a complete list of reducible induced G-modules. There we quote the work of Keys. Our work here gives an alternative proof that the list describes all reducible induced G-modules.

Proof. Let β be a character of $E^\times$. For every $n \geqq 1$ let f_n be a function which is supported on the conjugacy classes of $(\alpha, \beta, \bar{\alpha}^{-1})$ with $|\alpha| = q^n$, with $F(a, f_n) = \beta(\alpha) + \beta(\bar{\alpha}^{-1})$ if $a = (\alpha, 1, \bar{\alpha}^{-1})$ with $|\alpha| = q^{-n}$. If $\{\rho\}$ is supercuspidal then $\{\rho_N\}$ is zero and so is $I(\tau \otimes \kappa)_N$. If ρ is Steinberg then $I(\tau \otimes \kappa)_N$ has a single σ-invariant exponent, which satisfies $\mathrm{tr}[I(\tau \otimes \kappa)_N](\phi_N \times \sigma) = \mathrm{tr}\{\rho\}_N('f_N)$ for any triple $(\phi, f, 'f)$ of matching functions, where f is in the span of the f_n, $n \geqq 1$. In particular, (1.2.1) takes the form

$$(1.2.2) \qquad (c' + c)\mathrm{tr}\{\rho\}_N('f_N) = \sum_\pi m(\rho, \pi)\mathrm{tr}\, \pi_N(f_N)$$

for f as above. It is clear that there exists a compact open subgroup K_1 of G, depending only on the restriction of β to the group $R_E^\times$ of units in $E^\times$, such that f can be chosen to be K_1-biinvariant. Hence the sum in (1.2.2) is finite. Applying linear independence of finitely many characters of the form $n \to z^n$, the proposition follows once we make the following observation. If π and π' are irreducible inequivalent G-modules which have equal central exponent, then they are the (only) constituents of a reducible G-module $I(\eta)$ induced from a character η of A with $\eta(a) = \eta(JaJ^{-1})$; namely the composition series of $I(\eta)_N$ consists of two equal characters, necessarily unitary. Then $\mathrm{tr}\, \pi_N(f_N) = \mathrm{tr}\, \pi'_N(f_N)$, and $m(\rho, \pi)\mathrm{tr}\, \pi_N(f_N) + m(\rho, \pi')\mathrm{tr}\, \pi'_N(f_N)$ is zero if $m(\rho, \pi) + m(\rho, \pi')$ is zero. If $m(\rho, \pi)$ and $m(\rho, \pi')$ are both non-negative their central exponents cannot cancel each other, and (iv) follows.

Remark. We have $m(\rho, \pi) = c + c'$ for the π of (iii).

§1.3. Local identity

As in (1.2), let E/F be a quadratic extension of local non-archimedean fields. Let $\{\rho\}$ be a square-integrable H-packet, and τ its stable base chane lift. In this section our aim is to prove the following

Proposition. *For every square-integrable G-module π there exists a non-negative integer $m(\rho, \pi)$ such that for every triple $(\phi, f, 'f)$ of matching functions we have the identity*

$$(1.3.1) \qquad \mathrm{tr}\{\rho\}('f) + \mathrm{tr}\, I(\tau \otimes \kappa; \phi \times \sigma) = 2 \sum_\pi m(\rho, \pi)\mathrm{tr}\, \pi(f).$$

Proof. We use the identity of Proposition 1.1. Thus we fix a quadratic extension of global fields where F has no real places, such that for some place w of F the completion E_w/F_w is the local quadratic extension of the proposition. Denote by $Z(E)$ the center of $\mathrm{GL}(2, E)$. Let H_1 be the group of g in $\mathrm{GL}(2, F)$ with determinant in $N_{E/F}E^\times$. Using the relation $Z(E)H_1 = Z(E)H$, and the Deligne–

Kazhdan simple trace formula for H_1, it is easy to prove the existence of a cuspidal H-packet $\{\rho\}$ whose component at w is the H_w-packet $\{\rho\}$ of the proposition, which has the following properties. At some place $w' \neq w$ of F which is unramified (in particular non-split) in E/F the component $\rho_{w'}$ is the Steinberg $H_{w'}$-module. At each place v of F which ramifies in E the component $\{\rho_v\}$ is properly induced. At each $v \neq w,\ w'$ which is unramified in E/F the component $\{\rho_v\}$ is unramified. At some v which splits in E/F the component is supercuspidal.

The packet $\{\rho\}$ lifts to a σ-invariant cuspidal H'-module τ via the stable base change map of [U(2)], and to $\tau \otimes \kappa$ via the unstable map. There are contributions only in Φ_2, F_2 and possibly F_1 of Proposition 1.1, as we now choose the π_v^0 there so that the packet $\{\rho\}$ is the only term in F_2. *A priori* the identity $\Phi_2 + F_2 = 2F_1$ holds where V is a "sufficiently large" set containing w and w'. However, for each $v \neq w$, w' we may choose π_v^0 to be defined by the component $\{\rho_v\}$ of $\{\rho\}$ in the natural way, since the endo-lifting $e: H_v \to G_v$ is already defined for split places v, for unramified v if $\{\rho\}$ is unramified, and for properly induced H_v-modules (at the finitely many v which ramify in E/F). Then we can apply a standard argument of "generalized linear independence of characters in an absolutely convergent sum of unitary characters" at each $v \neq w,\ w'$ in V, to conclude that $\Phi_2 + F_2 = 2F_1$ where V consists of $\{w, w'\}$ only (and π_v^0 are fixed for all $v \neq w,\ w'$).

To write the identity $\Phi_2 + F_2 = 2F_1$ with $V = \{w, w'\}$ in a convenient form, any object (such as a function or representation) x_w with a subscript w will be written simply as x (without a subscript), while an object with subscript w', such as $x_{w'}$, will be written as $\tilde{x}$. Then we have

$$\text{(i)} \qquad \begin{aligned} &\operatorname{tr} I(\tilde{\tau} \otimes \tilde{\kappa}; \tilde{\phi} \times \sigma) \cdot \operatorname{tr} I(\tau \otimes \kappa; \phi \times \sigma) + \operatorname{tr}\{'\tilde{\rho}\}('\tilde{f}) \cdot \operatorname{tr}\{\rho\}('f) \\ &\quad = 2 \sum m(\tilde{\pi}, \pi) \cdot \operatorname{tr} \tilde{\pi}(\tilde{f}) \cdot \operatorname{tr} \pi(f). \end{aligned}$$

The sum ranges over a set of equivalence classes of unitary G-modules π and $\tilde{G}$-modules $\tilde{\pi}$. The multiplicities $m(\tilde{\pi}, \pi)$ are non-negative integers.

The identity (i) holds for any pairs $(\tilde{\phi}, \tilde{f}, '\tilde{f})$ and $(\phi, f, 'f)$ of matching triples, such that the unstable orbital integrals of either $\tilde{\phi}$ or ϕ (or both) are zero, so that we need not use the twisted unstable transfer $\phi \to '\phi$ from G' to H of [U], Lemma 3.4. Had we used this transfer we would construct $\{\rho\}$ such that $\{\rho_{w'}\}$ is unramified, and use "linear independence" of characters also at w' to derive the identity (1.3.1) as the case $V = \{w\}$ of the identity $\Phi_2 + F_2 = 2F_1$. As the $m(\rho, \pi)$ would be non-negative multiplicities, the π in (1.3.1) are square-integrable by Proposition 1.2, (iv).

To derive (1.3.1) from the identity (i), we use Kazhdan's result [K] concerning the existence of pseudo-coefficients. Namely, let π' be a tempered elliptic G-module. If π' is not supercuspidal then it is a constituent of a reducible properly induced G-module I of length two, and we denote the other constituent by π''.

This π'' is non-tempered if π' is square-integrable. Then [K] proves that there exists a function f' whose orbital integrals vanish on the regular split set, such that $\operatorname{tr}\pi'(f')=1$, $\operatorname{tr}\pi''(f)=-1$ (if π'' exists), and $\operatorname{tr}\pi(f')=0$ for any other irreducible equivalence class of G-modules. This f' is called a *pseudo-coefficient* of π'. Since it is a discrete function (its orbital integrals vanish on the regular split set), there exists ϕ' matching f' whose unstable twisted orbital integrals are zero. We prove the following

Lemma. *Denote by S the set of pairs $(\tilde{\pi}, \pi)$ which appear in the sum on the right of* (i) *such that both $\tilde{\pi}$ and π are square-integrable. Then for any pair $(\tilde{f}, f)$ such that at least one of $\tilde{f}$ or f is discrete, the right side of* (i) *is equal to $\Sigma' m(\tilde{\pi}, \pi)\operatorname{tr}\tilde{\pi}(\tilde{f})\operatorname{tr}\pi(f)$, where the sum Σ' ranges over S.*

Proof. We first note that if neither $\tilde{\pi}$ nor π is elliptic, then $\operatorname{tr}\tilde{\pi}(\tilde{f})\operatorname{tr}\pi(f)$ vanishes if one of $\tilde{f}$ and f is discrete. Now if π'' is non-square-integrable with $m(\tilde{\pi}, \pi'')\neq 0$ for some $\tilde{\pi}$, then we have the following possibilities:

(1) Both π'' and $\tilde{\pi}$ are not elliptic.

(2) π'' is not elliptic and $\tilde{\pi}$ is supercuspidal. Evaluating (i) with $\tilde{f}$ being a pseudo-coefficient of $\tilde{\pi}$ we derive a contradiction from (iv) of Proposition 1.2.

(3) π'' is not elliptic and $\tilde{\pi}$ is one of the two constituents $\tilde{\pi}'$ and $\tilde{\pi}''$ of a reducible induced $\tilde{G}$-module. At least one of the two, say $\tilde{\pi}'$, is tempered. Evaluate (i) at a pseudo-coefficient of $\tilde{\pi}'$, to deduce from Proposition 1.2 that $m(\tilde{\pi}', \pi'')=m(\tilde{\pi}'', \pi'')$; we are reduced to the situation of (1) on writing $\tilde{\pi}$ for the sum of $\tilde{\pi}'$ and $\tilde{\pi}''$.

(4) π'' is elliptic, and π' is the tempered G-module such that $\{\pi', \pi''\}$ are the constituents of a reducible induced G-module. If $\tilde{\pi}$ is supercuspidal, we are done as in (2). If $\tilde{\pi}$ is not elliptic, we are done as in (3). Then we assume that $\tilde{\pi}$ is one of the two constituents, $\tilde{\pi}'$ (tempered) and $\tilde{\pi}''$, of a reducible induced $\tilde{G}$-module. Put $a=m(\tilde{\pi}', \pi')$, $b=m(\tilde{\pi}', \pi'')$, $c=m(\tilde{\pi}'', \pi')$, $d=m(\tilde{\pi}'', \pi'')$. We claim that $a=b=c=d$. If we prove this, we are reduced to the case of (1), and the lemma follows. To prove the claim, we evaluate (i) at $f=f'$, where f' is a pseudo-coefficient of π'. Proposition 1.2 and the following remark imply that $(\alpha) c=d$, and (β) either $a=b$ or $|a-b|=1$. Next, we evaluate (i) at $\tilde{f}=\tilde{f}'$, where $\tilde{f}'$ is a pseudo-coefficient of $\tilde{\pi}'$, and deduce from Proposition 1.2 that $a+b=c+d=2c$. If $|a-b|=1$ then $a+b$ is both odd and even, a contradiction. If $a=b$ then $a=c$, and the lemma follows.

By virtue of the Lemma we may assume that π and $\tilde{\pi}$ in (i) are square integrable. Let $\tilde{\pi}'$ be the square integrable constituent in the composition series of the reducible induced $\tilde{G}$-module $\tilde{I}(\kappa\nu^{1/2})$. It is the square-integrable $\tilde{G}$-module assigned to $\tilde{\rho}$ by Proposition 1.2, (iii). Evaluating (i) at $\tilde{f}=\tilde{f}'$, where $\tilde{f}'$ is a pseudo-coefficient of $\tilde{\pi}'$, we obtain

(ii) $$c\,\mathrm{tr}\, I(\tau\otimes\kappa;\phi\times\sigma)+c'\,\mathrm{tr}\{\rho\}('f)=2\sum_{\pi} m(\tilde{\pi}',\pi)\mathrm{tr}\,\pi(f)$$

for all matching $(\phi, f, 'f)$. Here c, c' are complex numbers. If f is a discrete function then using the argument of Proposition 1.2, we obtain

(iii) $$\mathrm{tr}\, I(\tau\otimes\kappa;\phi\times\sigma)+\mathrm{tr}\{\rho\}('f)=2\sum_{\pi} m(\tilde{\pi}',\pi)\mathrm{tr}\,\pi(f).$$

The sums on the right of (ii), (iii) range over the same sets. Since $\mathrm{tr}\{\rho\}('f)$ depends only on the unstable orbital integrals of the discrete f, and $\mathrm{tr}\, I(\tau\otimes\kappa;\phi\times\sigma)$ on the stable integrals only, it follows that $c=1$ and $c'=1$. But then (ii) is (1.3.1), and the proposition follows.

Remark. We are permitted to use Proposition 1.2 in the proof of the above proposition since the sums Φ_i, F_i consist of automorphic forms, and a well-known result of Harish-Chandra asserts that there exists only finitely many automorphic G-modules with a given infinitesimal character and a non-zero vector fixed by the action of a given compact open subgroup of $G(\mathbf{A}_f)$. Here $\mathbf{A}_f$ denotes the ring of adeles with no archimedean components.

Our next aim will be to show that the sum of (1.3.1) is finite. We repeat the base-change result (*) of the introduction as follows:

(1.3.2) $$\mathrm{tr}\, I(\tau\otimes k;\phi\times\sigma)=\sum_{\pi} m'(\rho,\pi)\mathrm{tr}\,\pi(f).$$

The sum is finite, the π are square-integrable, the $m'(\rho,\pi)$ are non-negative integers. Putting (1.3.1) and (1.3.2) together we obtain

(1.3.3) $$\mathrm{tr}\{\rho\}('f)=\sum_{\pi} m''(\rho,\pi)\mathrm{tr}\,\pi(f),$$

where $m''(\rho,\pi)=2m(\rho,\pi)-m'(\rho,\pi)$ is an integer, which need not be positive. Note that the right side of (1.3.3) is independent of the orbital integrals of f on the cubic tori of G.

§2.1. Conjugacy

In this section we recall results of [U], §1, to be used below to prove that the sum of (1.3.3) (hence of (1.3.1)) is finite. Two regular elements g, g' of G, and two tori T, T' of G, are called *stably conjugate* if they are conjugate in $G(\bar{F})$ where $\bar{F}$ is an algebraic closure of F. Let $A(T/F)$ be the set of x in $G(\bar{F})$ such that $T'={}^xT=xTx^{-1}$ is defined over F. Then the set $B(T/F)=G\backslash A(T/F)/\mathbf{T}$ parametrizes the morphisms of T into G over F, up to inner automorphisms by elements of G. If T is the centralizer of g in G then $B(T/F)$ parametrizes the set of conjugacy classes within the stable conjugacy class of g in G. The map

$$x \mapsto \{\tau \mapsto x_\tau = x^{-1}\tau(x);\ \tau \text{ in } \mathrm{Gal}(\bar{F}/F)\}$$

defines a bijection

$$B(T/F) \simeq \ker[H^1(F, T) \to H^1(F, G)].$$

Since F is non-archimedean, $H^1(F, G_{sc}) = \{0\}$ and

$$\ker[H^1(F, T) \to H^1(F, G)] = \mathrm{Im}[H^1(F, T_{sc}) \to H^1(F, G)]$$

is a group. By the Tate–Nakayama theory this group is isomorphic to

$$C(T/F) = \mathrm{Im}[H^{-1}(X_*(T_{sc})) \to H^{-1}(X_*(T))].$$

We denote by $W(T)$ the Weyl group of T in G, and by $W'(T)$ the Weyl group of T in $A(T/F)$. We write σ for the non-trivial element in $\mathrm{Gal}(E/F)$. There are four types of tori in G, denoted (0), (1), (2), (3), which we now describe.

A torus of type (0) is one which is stably conjugate to the diagonal subgroup A of G. The stable conjugacy class of a regular element in such a torus consists of a single conjugacy class, and $[W(A)] = [W'(A)] = 2$. Here A is isomorphic to $E^\times \times E^1$. We shall also denote below A by S.

A torus T of type (3) is associated with a cubic extension K of E and an automorphism σ' of order two of K whose restriction to E is σ. Then T is isomorphic to the kernel of the norm map $x \mapsto x\sigma'x$ on K. It is easy to check that the galois closure of K over E is galois over F. If K/E is not galois or if $\mathrm{Gal}(K/F) = \mathbb{Z}/6$ then $W'(T)$ is trivial. If $\mathrm{Gal}(K/F) = S_3$ then $W'(T) = \mathbb{Z}/3$. The stable conjugacy class of any regular element in a torus of type (3) consists of a single conjugacy class.

A torus T of type (2) is associated with a quadratic extension K of E which is biquadratic over F, and an element σ' of $\mathrm{Gal}(K/F)$ whose restriction to E is σ. Then T is isomorphic to $K^1 \times E^1$, where K^1 is the kernel of the norm map $x \mapsto x\sigma'x$ on $K^\times$. Here $W'(T) = \mathbb{Z}/2$. Moreover, the stable conjugacy class of any element of G whose centralizer is a torus of type (2) consists of two conjugacy classes in G. We may and we do choose a representative S for this stable conjugacy class in the subgroup $\tilde{H} = \{(a_{ij});\ a_{ij} = 0 \text{ if } i + j \text{ is odd}\}$ of G. We also choose a torus S' which is stably conjugate but not conjugate to S. It does not lie in $\tilde{H}$.

A torus T of type (1) is compact and splits over E. It is isomorphic to $E^1 \times E^1 \times E^1$. The group $C(T/F)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, hence there are four conjugacy classes within the stable conjugacy class of a regular element of G whose centralizer is a torus of type (1). It is clear that $W'(T) = S_3$, and that this group acts transitively on the set of non-trivial characters of the group $C(T/F)$. Moreover, there are four conjugacy classes, say S, S', $'S$, $'S'$, in the stable conjugacy class of tori of type (1), and one of these classes, say S, is distinguished, in the following sense.

All unitary groups $G(\mathbf{J}) = \{g \text{ in } GL(3, E); g\mathbf{J}{}^t\bar{g} = \mathbf{J}\}$, where $\mathbf{J}$ is any form (matrix in $GL(3, F)$), are isomorphic over F. We normally work with $\mathbf{J} = J$ since then the proper parabolic subgroup of $G = G(J)$ is the upper triangular subgroup. Suppose now that $\mathbf{J} = \text{diag}(1, 1, j)$, where j lies in $F^\times$, and put $G(j)$ for $G(\mathbf{J})$. Denote the diagonal subgroup of $G(j)$ by $T(j)$ It is clear that (a) if j lies in $NE^\times$ the $W(T(j)) = S_3$; (b) If j lies in $F - NE$ then $W(T(j))$ contains the transposition represented by

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $W(T(j)) = \mathbf{Z}/2$. Since $W'(T) = S_3$, there is a torus S in G with $W(S) = S_3$ and three non-conjugate tori S', $'S$, $'S'$ in G with F-Weyl group isomorphic to $\mathbf{Z}/2$. We may and we do choose representatives S and S' in the subgroup $\tilde{H}$ of G which, modulo Z, is isomorphic to H over F. We also write below S_H for the intersection of S with H.

To recall the definition of transfer of orbital integrals of functions f on G to functions $'f$ on $H = \tilde{H}/Z$, we embed H in $\tilde{H} \subset G$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}.$$

A regular stable conjugacy class in H with eigenvalues α, β determines a semi-simple stable conjugacy class with eigenvalues α, 1, β in G/Z. This map is neither injective nor surjective, as we now show on listing to tori in H.

There are three types of tori in H. A torus T_H of H is of type (0) if it is stably conjugate to the diagonal subgroup A_H of H. The stable conjugacy class of a regular element in such a torus consists of a single conjugacy class. If $(a, 1, \bar{a}^{-1})$ is the set of eigenvalues of a regular stable conjugacy class γ of type (0) in G, then γ is the image of a unique stable conjugacy class γ_H in H. The eigenvalues of γ_H are $(a, \bar{a}^{-1})$. Note that $W(A_H) = W'(A_H) = \mathbf{Z}/2$.

A torus T_H of H is of type (2) if its splitting fields is a quadratic extension K of E which is biquadratic over F. It is associated with an extension σ' to K of σ on E, and isomorphic to $K^1 = \{x \text{ in } K^\times; x\sigma'x = 1\}$. Here there is no difference between conjugacy and stable conjugacy as $C(T_H/F) = \{0\}$. If $(\alpha, 1, \beta)$ are the eigenvalues of a stable conjugacy class γ of type (2) which is regular in G, then γ is the image of a unique stable conjugacy class γ_H in H. The eigenvalues of γ_H are (α, β). Note that $W'(T_H) = \mathbf{Z}/2$.

A torus T_H of H is of type (1) if it is compact and it splits over E. It is isomorphic to $E^1 \times E^1$, and $C(T_H/F) = \mathbf{Z}/2$. We have $W'(T_H) = \mathbf{Z}/2$. Let γ be a regular stable conjugacy class of type (1) in G with eigenvalues $(\alpha, 1, \beta)$ modulo Z. It is also represented by $(1/\alpha, 1, \beta/\alpha)$ and $(1/\beta, 1, \alpha/\beta)$ modulo Z. Hence the stable conjugacy class in G/Z defined by γ is obtained by three distinct stable conjugacy classes in H, with eigenvalues (α, β), $(1/\alpha, \beta/\alpha)$, $(1/\beta, \alpha/\beta)$.

Regular conjugacy classes of type (3) in G are not obtained from H, since each torus of H splits over a quadratic extension of E.

The homogeneous space $B(T/F)$ is acted upon by the group $C(T/F)$. We identify the two on choosing the base point S (out of S, $'S$, S', $'S'$ if the type is (1), and out of S, S', if the type is (2)). The group $C(T_H/F)$ naturally embeds as a subgroup of $C(T/F)$ if T is the image ZT_H of T_H in G. If T is a torus of type (1) or (2), let ε be the non-trivial character on $C(T/F)$ which is trivial on $C(T_H/F)$. By the choice of base point above we view ε as a character on $B(S/F)/B(S_H/F)$, and define for each regular γ in S

$$\Phi(\gamma, f, \varepsilon) = \sum_{\delta} \varepsilon(\delta)\Phi(\gamma^{\delta}, f),$$

where δ ranges over a set of representatives for $B(S/F)$. Explicitly, if the stable conjugacy class γ is regular of type (1) and contains the conjugacy classes represented by x in S, x' in S', $'x$ in $'S$, $'x'$ in $'S'$, then we put

$$\Phi(\gamma, f, \varepsilon) = \Phi(x, f) + \Phi(x', f) - \Phi('x, f) - \Phi('x', f).$$

If γ is a regular stable conjugacy class of type (2), it contains the conjugacy classes represented by x in S and x' in S', and we put

$$\Phi(\gamma, f, \varepsilon) = \Phi(x, f) - \Phi(x', f).$$

In addition, if γ is regular of type (0) we put $\Phi(\gamma, f, \varepsilon) = \Phi(\gamma, f)$.

We also need transfer factors, as follows. Let $|\cdot|$ be the valuation on F normalized by $|\boldsymbol{\pi}| = q^{-1}$, where $\boldsymbol{\pi}$ is a generator of the maximal ideal in the ring R of integers in F, and q is the cardinality of the residue field $R/(\boldsymbol{\pi})$. If a is algebraic over F we put $|a| = |N_{K/F}a|^{1/[K:F]}$, where $K = F(a)$. Now if γ_H is a stable conjugacy class in H with eigenvalues α, β, put $\Delta'(\gamma_H) = |\alpha - \beta|$. If γ is a stable conjugacy class in G with eigenvalues α, $1, \beta$ modulo Z then we put

$$\Delta(\gamma) = |(\alpha - 1)(\beta - 1)(\alpha - \beta)|.$$

Suppose that γ lies in the subgroup $\tilde{H}$ of G; then modulo Z it is of the form

$$\begin{bmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{bmatrix},$$

and we write

$$\kappa(\gamma) = \kappa(-(\alpha - 1)(\beta - 1)),$$

where κ is the character of $E^\times/NE^\times$ which is non-trivial on $F^\times$, fixed in [U].

Denote by $\Phi'(\gamma_H, 'f)$ the stable orbital integral of the function $'f$ on H at the regular stable conjugacy class γ_H.

Definition. The functions f on G and $'f$ on H are called *matching* if for every regular stable conjugacy class in G of type (0), (1) or (2) which is represented by γ in $S_H = S \cap H$, we have

$$\kappa(\gamma)\Delta(\gamma)\Phi(\gamma, f, \varepsilon) = \Delta'(\gamma)\Phi'(\gamma', f).$$

In the sequel we assume the following

Lemma. *For every f there exists a matching $'f$.*

Remark. This is also stated as Lemma 2.7 in [U]. If E/F is unramified and $f = f^0$, $'f = 'f^0$ are the unit elements of the Hecke algebras of G and H, this Lemma is due to Rogawski (thesis, unpublished). The case of locally constant f and $'f$ follows by homogeneity of germs. If E/F is ramified the Lemma is due to Langlands, who used his theory of "Igusa data" (in preparation). We envisage another proof but this will not be given here.

§2.2. Orthogonality

Here we study a transfer $'D \to 'D_G$ of distributions which is dual to the transfer $f \to 'f$ of orbital integrals from G to H. This study is used to conclude that the sum of (1.3.3) is finite.

Definition. (1) A distribution $'D$ on H is called *stable* if $'D('f)$ depends only on the stable orbital integrals of $'f$.

(2) A function $'f$ on H extends uniquely to a function $'\tilde{f}$ on $\tilde{H}$ with $'\tilde{f}(zh) = \omega^{-1}(z)'\tilde{f}(h)$ (z in Z, h in $\tilde{H}$). A distribution $'D$ on H extends to $'\tilde{D}$ on $\tilde{H}$ by $'\tilde{D}(\tilde{f}) = 'D(f)$.

(3) Given a stable distribution $'D$ on H, let $'D_G$ be the distribution on G with $'D_G(f) = 'D('f)(= '\tilde{D}('\tilde{f}))$, where $'f$ is a function on H matching f.

Remark. (1) The map $w \mapsto \mathbf{w} = \{\tau \mapsto w_\tau = \tau(w)w^{-1}; \tau \text{ in } \mathrm{Gal}(\bar{F}/F)\}$ embeds $W'(T)/W(T)$ as a subset of $C(T/F)$.

(2) $W'(T)$ acts on $C(T/F)$. If w lies in $W'(T)$, and δ in $C(T/F)$ is represented by $\{g_\tau = \tau(g)g^{-1}\}$ with g in $A(T/F)$, then

$$w(\delta) = \mathbf{w}^{-1}\cdot\{(wg)_\tau\}\ (=\{w\tau(w)^{-1}\cdot\tau(wg)(wg)^{-1}\}$$
$$=\{w\tau(g)g^{-1}w^{-1}\} = w\delta w^{-1})\in C(T/F).$$

(3) Let d be a locally integrable conjugacy invariant complex valued function on G with $d(zg) = \omega(z)d(g)$ (z in Z). The Weyl integration formula asserts

$$\int_{G/Z} f(g)d(g)dg = \sum_{\{T\}} [W(T)]^{-1} \int_{T/Z} \Delta(t)^2\Phi(t,f)d(t)dt.$$

The sum ranges over a set of representatives T for the conjugacy classes of tori in G. Suppose that t is a regular element of G which lies in T. Then the number of δ in $B(T/F)$ such that t^δ is conjugate to an element of T is $[W'(T)]/[W(T)]$. Hence if the function d is invariant under stable conjugacy, then we have

$$\int_{G/Z} f(g)d(g)dg = \sum_{\{T\}_s} [W'(T)]^{-1} \int_{T/Z} \Delta(t)^2\Phi'(t,f)d(t)dt.$$

Here $\{T\}_s$ is a set of representatives for the stable conjugacy classes of tori in G. If $'\tilde{d}$ is a locally integrable stable function on $\tilde{H}$ then

$$\int_{\tilde{H}/Z} {'\tilde{f}}(h)'\tilde{d}(h)dh = \sum_{\{T_H\}_s} [W'(T_H)]^{-1} \int_{T_H} \Delta'(t)^2\Phi'(t',\tilde{f})'\tilde{d}(t)dt.$$

As in [U], $\Phi'(t,'\tilde{f})$ denotes the stable orbital integral of $'\tilde{f}$, and $\Phi'(t,f)$ is that of f. $\{T_H\}_s$ is a set of representatives for the stable conjugacy classes of tori in H. $W'(T_H)$ indicates the Weyl group in $A(T_H/F)$. It consists of two elements.

2.2.1. Proposition. *Suppose that $'\tilde{D}$ is a stable distribution on $\tilde{H}$ represented by the locally integrable (stable) function $'\tilde{d}$. Then the corresponding distribution $'D_G$ on G is given by a locally integrable function $'d_G$ defined on the regular set of G by $'d_G(t) = 0$ if t lies in a torus of type* (3), *and by*

$$\Delta(t)'d_G(t^\delta) = \sum_{w} \kappa(w(t))\Delta'(w(t))\varepsilon(\mathbf{w})\varepsilon(w(\delta))'\tilde{d}(w(t)) \tag{2.2.1}$$

if t lies in the chosen torus S of type (0), (1) *or* (2), *and δ lies in $C(S/F)$ ($= B(S/F)$). Here $w(t) = wtw^{-1}$, and the sum ranges over all w in $W'(S_H)\setminus W'(S)$.*

Proof. Fix $i = 0$, 1 or 2, and let S be the distinguished torus of type (i). Let δ be an element of $B(S/F)$, g a representative of δ in $A(S/F)$, and $T = S^\delta = g^{-1}Sg$

the associated torus. Let f be a function on the regular set of G such that $\Phi(t,f)$ is supported on the conjugacy class of T. Then

$$'D_G(f) = '\tilde{D}(\tilde{f}) = [W'(S_H)]^{-1} \int_{S/Z} \Delta'(t)^2 \Phi'(t',\tilde{f})'d(t)\,dt$$

$$= [W'(S_H)]^{-1} \int_{S/Z} \Delta'(t)[\kappa(t)\Delta(t) \sum_{\delta'} \varepsilon(\delta')\Phi(t^{\delta'},f)]'d(t)\,dt.$$

The sum ranges over all δ' in $B(S/F)$ such that $S^{\delta'} = T$. Thus δ' is represented by wg (i.e. $\delta' = \{(wg)_\tau = \tau(wg)(wg)^{-1}\}$), where w ranges over $W'(S)/gW(T)g^{-1}$. Since ε is trivial on the image of $B(S_H/F)$ in $B(S/F)$, we obtain

$$[W(T)]^{-1} \int_{S/Z} \Delta(t)\kappa(t)\Delta'(t) \left[\sum_w \varepsilon(\mathbf{w}\cdot w(\delta))\Phi((w^{-1}tw)^\delta,f)\right]'\bar{d}(t)\,dt$$

$$= [W(T)]^{-1} \int_{S/Z} \left[\sum_w \kappa(w(t))\Delta'(w(t))\varepsilon(\mathbf{w})\varepsilon(w(\delta))'\bar{d}(w(t))\right]\Delta(t)\Phi(t^\delta,f)\,dt.$$

Here w ranges over $W'(S_H)\backslash W'(S)$. By definition of $'d_G$ this is equal to

$$= [W(T)]^{-1} \int_{T/Z} \Delta(t)^2\Phi(t,f)'d_G(t)\,dt = \int_{G/Z} f(g)'d_G(g)\,dg;$$

hence the proposition follows.

Definition. (1) Let d, d' be conjugacy invariant functions on the elliptic set of G. Put

$$\langle d,d'\rangle = \sum_{\{T\}_e} [W(T)]^{-1} \int_{T/Z} \Delta(t)^2 d(t)\bar{d}'(t)\,dt$$

$$= \sum_{\{T\}_{e,s}} [W'(T)]^{-1} \sum_{\delta\in D(T/F)} \int_{T/Z} \Delta(t)^2 d(t^\delta)\bar{d}'(t^\delta)\,dt.$$

Here $\{T\}_e$ (resp. $\{T\}_{e,s}$) is a set of representatives for the (resp. stable) conjugacy classes of elliptic tori T in G.

(2) Let $'d$, $'d'$ be stable conjugacy invariant functions of the elliptic set of H. Put

$$'\langle 'd,'d'\rangle = \sum_{\{T_H\}_{e,s}} [W'(T_H)]^{-1}[D(T_H/F)] \int_{T_H} \Delta'(t)^2\,'d(t)'\bar{d}(t)\,dt.$$

Here $\{T_H\}_{e,s}$ is a set of representatives for the stable conjugacy classes of elliptic tori in H.

2.2.2. Proposition. *Let $'d$, $'d'$ be stable functions on (the elliptic set of) $\tilde{H}$, and $'d_G$, $'d'_G$ the associated class functions on (the elliptic set of) G. Then*

$$\langle 'd_G, 'd'_G\rangle = 2\cdot '\langle 'd, 'd'\rangle.$$

Proof. By (2.2.1) we have

$$\langle 'd_G, 'd'_G\rangle = \sum_{\{S\}} \sum_{\delta\in C(S/F)} [W'(S)]^{-1} \int_{S/Z} \sum_{w,w'\in W'(S_H)\backslash W'(S)} \kappa(w(t))\kappa(w'(t))$$

$$\Delta'(w(t))\Delta'(w'(t))\varepsilon(\mathbf{w})\varepsilon(\mathbf{w}')'\bar{d}(w(t))'\bar{d}'(w'(t))\varepsilon(w(\delta))\varepsilon(w'(\delta)).$$

Note that ε is a character of order 2. Here S ranges over the set of (conjugacy classes of) distinguished tori in G of type (1) and (2). The group $W'(S_H)\backslash W'(S)$ acts simply transitively on the set of non-trivial characters of $C(S/F)$. Hence $\Sigma_\delta\, \varepsilon(w(\delta))\varepsilon(w'(\delta)) \neq 0$ implies that $\varepsilon(w(\delta)) = \varepsilon(w'(\delta))$ for all δ and that $w = w'$. Changing variables we conclude that

$$\langle 'd_G, 'd'_G\rangle = \sum_{\{S\}} ([C(S/F)]/(W'(S_H)]) \int_{S/Z} \Delta'(t)^2\, '\bar{d}(t)'\bar{d}'(t)\,dt$$

$$= 2 \sum_{\{T_H\}_e} ([C(T_H/F)]/[W'(T_H)]) \int_{T_H} \Delta'(t)^2\, 'd(t)'\bar{d}'(t)\,dt$$

$$= 2\cdot '\langle 'd, 'd'\rangle.$$

Here we used the relation $[C(T/F)] = 2[C(T_H/F)]$ for tori T of type (1) or (2). The proposition follows.

Definition. (1) Let d be a conjugacy invariant function on the elliptic set G_e of G. Define d_H to be the stable function on the elliptic set $\tilde{H}_e$ of $\tilde{H}$ with

$$\Delta'(t)d_H(t) = \Delta(t)\kappa(t) \sum_{\delta\in B(S/F)} \varepsilon(\delta)d(t^\delta)$$

on the t in S, where S is any distinguished torus of type (1) or (2) in $\tilde{H}$.

2.2.3. Proposition. (1) *If d is a conjugacy invariant function on G_e and $'d$ is a stable function on H_e, both locally integrable, then $\langle d, 'd_G\rangle = '\langle d_H, 'd\rangle$.*

(2) *The locally integrable class function d on G_e is stable if and only if $d_H = 0$, and if and only if $\langle d, \chi(\{\rho\})_G\rangle$ vanishes for every square-integrable H-packet $\{\rho\}$. Here $\chi(\{\rho\})$ is the sum of the characters of the (one or two) irreducible H-modules in $\{\rho\}$.*

Proof. By (2.2.1) the inner product $\langle d, 'd_G\rangle$ is equal to

$$\sum_{\{S\}} \sum_{\delta\in B(S/F)} [W'(S)]^{-1} \int_{S/Z} \Delta(t)d(t^\delta) \sum_w \bar{\kappa}(w(t))\Delta'(w(t))\varepsilon(\mathbf{w})\varepsilon(w(\delta))'\bar{d}(w(t))$$

$$= \sum_{\{S\}} \sum_{\delta} [W'(S)]^{-1} \int_{S/Z} \Delta(t)\Delta'(t)\bar{\kappa}(t)\left[\sum_w \varepsilon(\{(wg)_\tau\})d((w^{-1}tw)^\delta)\right]'\bar{d}(t)dt$$

$$= \sum_{\{S\}} [W'(S_H)]^{-1} \int_{S/Z} \Delta(t)\Delta'(t)\bar{\kappa}(t)\left[\sum_\delta \varepsilon(\delta)d(t^\delta)\right]'\bar{d}(t)dt$$

$$= \sum_{\{T_H\}_e} [W'(T_H)]^{-1} \int_{T_H} \Delta(t)^2 d_H(t)'\bar{d}(t)dt = '\langle d_H, 'd\rangle,$$

where w ranges over $W'(S_H)\setminus W'(S)$, and (1) follows. For (2), note that $d_H = 0$ if and only if $d_H(w^{-1}tw) = 0$ for every T, t in T and w in $W'(T)$, and $W'(T)$ acts transitively on the set of non-trivial characters of $C(T/F)$. Hence d is stable if and only if $d_H = 0$. Now the $\chi(\{\rho\})$ make a basis for the space of stable functions on the elliptic set of H, hence $d_H = 0$ if and only if $'\langle d_H, \chi(\{\rho\})\rangle = 0$ for all square-integrable H-packets $\{\rho\}$, as required.

2.2.4. Proposition. *The sum of* (1.3.3) *is finite.*

Proof. Numbering the countable set of π in (1.3.3) with $m''(\rho,\pi) \neq 0$ we rewrite (1.3.3) in the form $\operatorname{tr}\{\rho\}('f) = \Sigma_{i=1}^b m_i \operatorname{tr} \pi_i(f)$, where $1 \leqq b \leqq \infty$. The m_i are non-zero integers, and the π_i are square integrable. For each i in the sum let f_i be the product of a pseudo-coefficient of π_i with $m_i/|m_i|$. For any finite a $(1 \leqq a \leqq b)$ put $f^a = \Sigma^a f_i$, where Σ^a indicates the sum over i $(1 \leqq i \leqq a)$. Then

$$\begin{aligned} a^2 \leqq \left(\sum^a |m_i|\right)^2 &= \left(\sum^a m_i \operatorname{tr} \pi_i(f^a)\right)^2 \\ &= (\operatorname{tr}\{\rho\}('f^a))^2 \\ &= \langle \chi(\{\rho\})_G, \sum^a \chi_i m_i/|m_i|\rangle^2 \\ &\leqq \langle \chi(\{\rho\})_G, \chi(\{\rho\})_G\rangle \left\langle \sum^a \chi_i, \sum^a \chi_i\right\rangle \\ &= 2a \cdot '\langle \chi(\{\rho\}), \chi(\{\rho\})\rangle \\ &= 2a[\{\rho\}], \end{aligned}$$

where $[\{\rho\}]$ is the number of irreducibles in the H-packet $\{\rho\}$, and χ_i is the character of π_i. Then $a \leqq 2[\{\rho\}]$, and the proposition follows. In fact, we also proved the

Corollary. *The sum of* (1.3.3) *extends over at most two* π *if* $[\{\rho\}] = 1$ *and four* π *if* $[\{\rho\}] = 2$. *The coefficients* $m''(\rho, \pi)$ *are bounded by two in absolute value, and they are equal to one in absolute value if there are at least two* π *in the sum.*

§2.3. Evaluation

Let E/F be a quadratic extension of non-archimedean local fields.

Our next aim is to evaluate the integers $m''(\rho, \pi)$ and $m'(\rho, \pi)$ which appear in (1.3.2) and (1.3.3), and describe the π which occur in these sums. Recall ([U(2)]) that a packet $\{\rho\}$ of square-integrable H-modules consists of a single element, unless it is associated with two distinct characters $\theta, {}'\theta$ of E^1. In the last case $\{\rho\}$ is denoted by $\rho(\theta, {}'\theta)$. It consists of two supercuspidal H-modules. In Corollary 2.2.4 it is shown that the sum of (1.3.3) consists of a most $2[\{\rho\}]$ elements.

Proposition. *The sum in* (1.3.3) *consists of* $2[\{\rho\}]$ *terms. The coefficients* $m''(\rho, \pi)$ *are equal to* 1 *or* -1, *and both values occur for each* ρ.

Proof. Put $\theta_\rho = \chi(\{\rho\})_G$. Put θ_τ for the (twisted) character of $I(\tau \otimes \kappa)$ (of (1.3.2)), viewed as a stable (conjugacy) function on G. Consider the inner product

$$\langle \theta_\rho, \theta_\tau \rangle = \left\langle \sum_\pi m''(\rho, \pi)\chi_\pi, \sum_{\pi'} m'(\rho, \pi')\chi_{\pi'} \right\rangle = \sum_\pi m''(\rho, \pi) m'(\rho, \pi).$$

By (2.2.1), since θ_τ is a stable function $\langle \theta_\rho, \theta_\tau \rangle$ is equal to

$$\sum_{\{S\}} [W(S)]^{-1} \sum_{\delta \in C(S/F)} \int_{S/Z} (\Delta\tilde{\theta}_\tau)(t) \sum_{w \in W'(S_H)\backslash W'(S)}$$

$$\kappa(w(t))\Delta'(w(t))\varepsilon(\mathbf{w})\varepsilon(w(\delta))\tilde{\chi}(\{\rho\})(w(t))\,dt.$$

Since ε is a non-trivial character of the group $C(S/F)$, we have

$$\sum_{\delta \in C(S/F)} \varepsilon(w(\delta)) = 0.$$

Hence $\langle \theta_\rho, \theta_\tau \rangle = 0$; the point is that θ_τ is stable and θ_ρ is an anti-stable function. Since the $m'(\rho, \pi)$ are non-negative integers, we conclude that the integers $m''(\rho, \pi)$ do not all have the same sign. In particular, there are at least two π in (1.3.3). Corollary 2.2.4 then implies that $|m''(\rho, \pi)|$ is one (if it is non-zero). Moreover, if $\{\rho'\}$ is also a square-integrable H-packet, then

$$2\cdot{}'\langle\chi(\{\rho\}),\chi(\{\rho'\})\rangle=\langle\theta_\rho,\theta_{\rho'}\rangle$$
$$=\left\langle\sum_\pi m''(\rho,\pi)\chi_\pi,\ \sum_{\pi'} m''(\rho,\pi')\chi_{\pi'}\right\rangle$$
$$=\sum_\pi m''(\rho,\pi)m''(\rho',\pi)$$

by (2.2.2) and the orthonormality relations (of [K], Theorem K) for characters χ_π of square-integrable G-modules π. Taking $\rho=\rho'$ we conclude that $\Sigma_\pi m''(\rho,\pi)^2=2[\{\rho\}]$, and the proposition follows.

Corollary. *For each square-integerable H-packet* $\{\rho\}$ *there exist* $2[\{\rho\}]$ *inequivalent square-integrable G-modules which we gather in two non-empty disjoint sets* $\pi^+(\rho)$ *and* $\pi^-(\rho)$, *such that*

$$\operatorname{tr}\{\rho\}('f)=\operatorname{tr}(\pi^+(\rho))(f)-\operatorname{tr}(\pi^-(\rho))(f).$$

Here $\operatorname{tr}(\pi^+(\rho))(f)$ is the sum of $\operatorname{tr}\pi(f)$ over the π in the set $\pi^+(\rho)$. In particular, if $\{\rho\}$ consists of a single term, then $\pi^+(\rho)$ and $\pi^-(\rho)$ are irreducible G-modules.

§2.4. Stability

We shall now show that if $m'(\rho,\pi)\neq 0$, namely if π contributes to (1.3.2), then it lies either in $\pi^+(\rho)$ or in $\pi^-(\rho)$. We begin with rewriting (1.3.2). For each (irreducible) π^+ in $\pi^+(\rho)$ there is a non-negative integer $m(\pi^+)$, and for each π^- in $\pi^-(\rho)$ there is such $m(\pi^-)$, with the following property. Put

$$\Sigma^+(f)=\sum(2m(\pi^+)+1)\operatorname{tr}\pi^+(f)\qquad(\pi^+\text{ in }\pi^+(\rho)),$$
$$\Sigma^-(f)=\sum(2m(\pi^-)+1)\operatorname{tr}\pi^-(f)\qquad(\pi^-\text{ in }\pi^-(\rho)),$$

and

$$\Sigma^0(f)=\sum 2m(\rho,\pi)\operatorname{tr}\pi(f)\qquad(\pi\text{ not in }\pi^+(\rho),\pi^-(\rho)).$$

Then

$$\sum_\pi m'(\rho,\pi)\operatorname{tr}\pi(f)=\Sigma^+(f)+\Sigma^-(f)+\Sigma^0(f)$$

(this relation defines $m(\pi^+)$ and $m(\pi^-)$). Also we write χ^+, χ^-, χ^0 for the corresponding (finite) sums of characters:

$$\chi^+=\sum_{\pi^+\in\pi^+(\rho)}(2m(\pi^+)+1)\chi(\pi^+),$$
$$\chi^-=\sum_{\pi^-\in\pi^-(\rho)}(2m(\pi^-)+1)\chi(\pi^-),$$

$$\chi^0 = \sum_\pi m(\rho, \pi)\chi(\pi) \qquad (\pi \notin \pi^+(\rho) \cup \pi^-(\rho)).$$

Proposition. *$\Sigma^0(f) = 0$ for every f on G; equivalently, $m(\rho, \pi) = 0$ for every π not in $\pi^+(\rho)$ and $\pi^-(\rho)$.*

Proof. By virtue of the (elementary) Proposition 5 of [U″], it suffices to prove the followng.

Lemma. *The class function $\chi^+ + \chi^-$ on G is stable.*

Proof. In view of Proposition 2.2.3 (2) it suffices to show that $\langle \chi^+ + \chi^-, \theta_{\rho'}\rangle$ vanishes for every square-integrable H-packet $\{\rho'\}$. We distinguish between two cases, when $\rho' \neq \rho$ and when $\rho' = \rho$. In the first case we note that if the irreducible π occurs in $\pi^+(\rho)$ or $\pi^-(\rho)$, then it occurs in $I(\tau \otimes \kappa)$ with $m'(\rho, \pi) \neq 0$. But then $m'(\rho', \pi) = 0$ since the characters of $I(\tau \otimes \kappa)$ and $I(\tau' \otimes \kappa)$ are orthogonal (by the twisted analogue of [K]), and π does not occur in $\pi^+(\rho')$ or $\pi^-(\rho')$. Consequently

$$\langle \chi^+ + \chi^-, \theta_{\rho'}\rangle = \langle \chi^+ + \chi^-, \chi(\pi^+(\rho')) - \chi(\pi^-(\rho'))\rangle = 0.$$

If $\rho' = \rho$, as in the proof of Proposition 2.3 we have

$$0 = \langle \theta_\tau, \theta_\rho\rangle = \sum_{\pi^+ \in \pi^+(\rho)} (2m(\pi^+) + 1) - \sum_{\pi^- \in \pi^-(\rho)} (2m(\pi^-) + 1) = \langle \chi^+ + \chi^-, \theta_\rho\rangle.$$

This completes the proof of the lemma, hence also of the proposition.

Corollary. *For every square-integrable H-packet $\{\rho\}$ we have*

$$\sum_{\pi^+ \in \pi^+(\rho)} (2m(\pi^+) + 1) = \sum_{\pi^- \in \pi^-(\rho)} (2m(\pi^-) + 1).$$

In particular if the packet $\{\rho\}$ consists of one element then $m(\pi^+) = m(\pi^-)$.

In the next §3, we deal with each H-module ρ to show that $m(\pi^+) = m(\pi^-) = 0$. Thus we obtain a precise form of (1.3.2) and (1.3.3).

§3.1. Specials

There are several special cases which we now discuss. Let β be a character of E^1, and β' the character of $E^\times$ given by $\beta'(a) = \beta(a/\bar{a})$. Let ρ be the special (namely square-integrable) subquotient sp(β) of the unitarily induced H-module $'I = I(\beta'\nu^{1/2})$ from the character $(a, \bar{a}^{-1}) \to \beta'(a)|a|^{1/2}$. The image τ of ρ by the stable base-change map of [U(2)] is the special H'-module sp(β'), which lies in the induced module $'I' = I(\beta'\nu^{1/2}, \beta'\nu^{-1/2})$. As the packet of the square-integrable ρ consists of a single element, we conclude that there exist two tempered irreducible G-modules denoted $\pi^+ = \pi^+(\beta)$ and $\pi^- = \pi^-(\beta)$, and a non-negative integer m, so that

(3.1.1) $$\operatorname{tr} \rho('f) = \operatorname{tr} \pi^+(f) - \operatorname{tr} \pi^-(f)$$

and

(3.1.2) $$\operatorname{tr} I(\tau \otimes \kappa; \phi \times \sigma) = (2m+1)[\operatorname{tr} \pi^+(f) + \operatorname{tr} \pi^-(f)],$$

for all matching $\phi, f, 'f$.

Proposition. *$m = 0$, π^- is supercuspidal, and π^+ is the square-integrable subquotient $\pi_{\beta'}^+$ in the composition series of the induced G-module $I(\beta'\kappa\nu^{1/2})$.*

Proof. On the set of $x = (a, 1, \bar{a}^{-1})$ in G with $|a| < 1$, since $'f_N(x) = \boldsymbol{\kappa}(x) f_N(x)$ and $\boldsymbol{\kappa}(x) = \kappa(a)$, the Deligne–Casselman theorem [C] and the relation (3.1.1) imply that

$$\begin{aligned}\kappa(a)\beta'(a)|a|^{1/2} &= \kappa(a)(\Delta'\chi(\{\rho\}))((a, \bar{a}^{-1})) \\ &= (\Delta\chi(\pi^+))((a, 1, \bar{a}^{-1})) - (\Delta\chi(\pi^-))((a, 1, \bar{a}^{-1})) \\ &= (\chi(\pi_N^+))((a, 1, \bar{a}^{-1})) - (\chi(\pi_N^-))((a, 1, \bar{a}^{-1})).\end{aligned}$$

Since the composition series of an induced G-module has length at most two, and at most one of its constituents is square integrable, and since $\pi^+(\rho)$ and $\pi^-(\rho)$ consist of square-integrable G-modules, it follows from linear independence of characters on A that (1) $\chi(\pi_N^-) = 0$, hence π^- is supercuspidal, and (2) $(\chi(\pi_N^+))((a, 1, \bar{a}^{-1})) = \beta'(a)\kappa(a)|a|^{1/2}$.

By Frobenius reciprocity π^+ is a constituent of $I(\beta'\kappa\nu^{1/2})$. Since π^+ is square-integrable we conclude that $I(\beta'\kappa\nu^{1/2})$ is reducible, and $\pi^+ = \pi_{\beta'}^+$.

To show that $2m + 1 = 1$ (and $m = 0$) we use again the theorem of [C] to conclude from (3.1.2) that since the A'-module $I(\tau \otimes \kappa)_N$ of N'-coinvariants has a single decreasing σ-invariant component, and so does π^+, they are equal, and the proposition follows.

§3.2. Trivial

Let $1(\beta)$ be the one-dimensional complement of $\operatorname{sp}(\beta)$ in $'I$; $1'(\beta)$ its base-change lift, namely the one-dimensional constituent in $'I'$; and $\pi^\times = \pi_{\beta'}^\times$ the non-tempered subquotient of $I = I(\beta'\kappa\nu^{1/2})$.

Corollary. *For every matching $\phi, f, 'f$, we have*

$$\operatorname{tr}(1(\beta))('f) = \operatorname{tr} \pi^\times(f) + \operatorname{tr} \pi^-(f),$$

$$\operatorname{tr} I(1'(\beta) \otimes \kappa; \phi \otimes \sigma) = \operatorname{tr} \pi^\times(f) - \operatorname{tr} \pi^-(f).$$

Proof. This follows since the composition series of I consists of $\pi^\times$, π^+.

§3.3. Twins

The next special case to be studied is that of $[\{\rho\}] = 2$. Then in the notations of [U(2)], $\{\rho\}$ is of the form $\rho(\theta, '\theta)$, associated with an unordered pair θ, $'\theta$ of characters of E^1. $\{\rho\}$ consists of two supercuspidals when $\theta \neq '\theta$. It lifts to the induced H'-module $\tau \otimes \kappa^{-1} = I(\theta'\kappa^{-1}, '\theta'\kappa^{-1})$, where $\theta'(x) = \theta(x/\bar{x})$, $'\theta'(x) = '\theta(x/\bar{x})$ (x in $E^\times$), via the stable base-change map of [U(2)], and to $I(\theta', '\theta') = \tau$ via the unstable map. The σ-invariant G'-module $I(\tau)$ is $I(\theta', '\theta', \omega'/\theta' \cdot '\theta')$. It is also obtained, by the same process, from the H-module $\rho' = \rho(\theta, \omega/\theta \cdot '\theta)$, and the H-module $\rho'' = \rho('\theta, \omega/\theta \cdot '\theta)$. We now assume that θ, $'\theta$, $\omega/\theta \cdot '\theta$ are all distinct, so that $\{\rho\}$, $\{\rho'\}$ and $\{\rho''\}$ are disjoint packets consisting of two supercuspidals each.

We also write $\rho_1 = \rho$, $\rho_2 = \rho'$, $\rho_3 = \rho''$. If $\tau = I(\theta', '\theta')$, we conclude that there are four inequivalent irreducible supercuspidal G-modules π_j ($1 \leqq j \leqq 4$), and non-negative integers m_j, so that

$$\operatorname{tr} I(\tau; \phi \times \sigma) = \sum_j (2m_j + 1) \operatorname{tr} \pi_j(f).$$

Moreover, there are numbers ε_{ij} ($1 \leqq i \leqq 3$; $1 \leqq j \leqq 4$), equal to 1 or -1, such that for any $i = 1, 2, 3$, the set $\{\varepsilon_{ij}\ (1 \leqq j \leqq 4)\}$ is equal to the set $\{1, -1\}$, and they satisfy

$$\operatorname{tr}.\rho_i('f) = \sum_{j=1}^{4} \varepsilon_{ij} \operatorname{tr} \pi_j(f) \qquad (1 \leqq i \leqq 3).$$

§3.4. Proposition. (1) *For each i there are exactly two j with $\varepsilon_{ij} = 1$.* (2) *m_j is independent of j; put $m = m_j$.* (3) *The product $\varepsilon_{1j}\varepsilon_{2j}\varepsilon_{3j}$ is independent of j.*

Proof. Note that (1) asserts that $\pi^+ = \pi^+(\rho)$ and π^- consist of two elements each. To prove (1), note that the orthogonality relations on H imply that if there exists an i for which exactly two ε_{ij} are 1, then this is valid for all i. Thus, if (1) does not hold, then there are two i for which the number of j with $\varepsilon_{ij} = 1$ is (without loss of generality) one (otherwise this number is three, and this case is dealt with in exactly the same way). Hence we may assume that $i = 1$ and 2, and $\varepsilon_{11} = 1$, $\varepsilon_{22} = 1$ (we cannot have $\varepsilon_{21} = \varepsilon_{11} = 1$ since ρ, ρ' are inequivalent). Since the stable character θ_τ is orthogonal to the unstable character θ_{ρ_i} (all i), we conclude that

$$2m_1 + 1 = 2m_2 + 1 + 2m_3 + 1 + 2m_4 + 1,$$

$$2m_2 + 1 = 2m_1 + 1 + 2m_3 + 1 + 2m_4 + 1.$$

Hence $m_3 + m_4 + 1 = 0$, contradicting the assumption that m_j are non-negative. (1) follows.

To establish (2), we first claim that there exists j so that ε_{ij} is independent of i. If this claim is false, we may assume that $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{23}, \varepsilon_{32}, \varepsilon_{34}$ are equal. But then the characters of $\{\rho'\}$ and $\{\rho''\}$ are not orthogonal. This contradicts the orthogonality relations on H, hence the claim. Up to reordering indices, the claim implies that $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{34}$, are equal. As $\langle \theta_\tau, \theta_{\rho_i} \rangle = 0$, we conclude that

$$m_1 + m_2 = m_3 + m_4, \quad m_1 + m_3 = m_2 + m_4, \quad m_1 + m_4 = m_2 + m_3.$$

Hence m_j is independent of j, and (2) follows. Also it follows that $\varepsilon_{1j}\varepsilon_{2j}\varepsilon_{3j}$ is independent of j, hence (3).

§3.5. Proposition. *Let ρ be any square-integrable H-module, so that we have* $\operatorname{tr}\Pi(\phi \times \sigma) = \Sigma\,(2m(\pi) + 1)\operatorname{tr}\pi(f)$, *where $\Pi = I(\tau \otimes \kappa)$, the sum ranges over* $2[\{\rho\}]$ *inequivalent square-integrable π, and $m(\pi)$ are non-negative integers. Then there exists a π in the sum with $m(\pi) = 0$. This π has a Whittaker vector; the other* $2[\{\rho\}] - 1$ *G-modules do not have a Whittaker vector.*

Proof. We give two proofs, local and global. The local proof is based on a theorem of Rodier [Ro], and Harish–Chandra's theory [H] of characters. Theorem 5 of [H] asserts that the character χ of the admissible irreducible G-module π, at $x = \exp X$ near 1 (X near 0 in the Lie algebra $L(G)$ of G), is given by

$$\chi(x) = \sum_{\xi} c(\xi)\hat{\nu}_\xi(X).$$

The sum extends over all nilpotent orbits in $L(G)$ (equivalently, unipotent orbits in G). ν_ξ is a Haar measure on the orbit ξ, $\hat{\nu}_\xi$ is its Fourier transform, and $c(\xi)$ are complex coefficients. The theorem of [Ro] asserts that there is a regular orbit ξ' so that $c(\xi')$ is equal to the number of Whittaker vectors of π. In fact the proof of [Ro] is given only under some restriction on the residual characteristic, which amounts in our case to: the residual characteristic of F is odd.

The same statement holds in the twisted case (but this does not appear in print; hence our local proof is merely a sketch). $L(G)$ has to be replaced by the Lie algebra of the σ-centralizer G_1' of 1 in G'; but G_1' is G, and the same sum is obtained.

In the case of $G = U(3)$, there is a single regular unipotent conjugacy class ξ'. The $\hat{\nu}_\xi(tX)$ are homogeneous [H] in t in $F^\times$ near 0, and the degree of homogeneity of $\hat{\nu}_{\xi'}$ differs from the degree of $\hat{\nu}_\xi$ for the other three orbits. Since $\Pi = I(\tau \otimes \kappa)$ is tempered, it is non-degenerate, and has a unique Whittaker vector. We obtain the identity

$$1 = \sum_{\pi} (2m(\pi) + 1)c(\xi', \pi).$$

We put $c(\xi', \pi)$ for the coefficient $c(\xi')$ in the expression for the character χ of π. Hence there is π' with $m(\pi') = 0$ and $c(\xi', \pi') = 1$, while $c(\xi', \pi) = 0$ for the $\pi \neq \pi'$. The proposition follows (when the residual characteristic is odd).

The second proof is global. It relies on a result of Gelbart and Piatetski-Shapiro [GP]. Thus let ρ_u be a supercuspidal H_u-module, where E_u/F_u is a non-archimedean quadratic extension. Choose a quadratic extension E/F of global fields whose completion at a finite place u is the above E_u/F_u, such that at some $u' \neq u$ we have $F_{u'} \simeq F_u$ and $F_{u'} \simeq E_u$. Construct a cuspidal H-module ρ such that (1) its components at u and u' are (equivalent to) ρ_u, (2) at two finite places v' and v'' which split E/F, the components are supercuspidal. We may take a totally imaginary F. The trace formula yields the identity

$$2 \sum m(\pi) \prod \operatorname{tr} \pi_v(f_v) = \prod \operatorname{tr} I(\tau_v \otimes \kappa_v; \phi_v) + \prod \operatorname{tr} \rho_v('f_v)$$

$$= \prod m_v[\operatorname{tr} \pi_v^+(f_v) + \operatorname{tr} \pi_v^-(f_v)] + \prod [\operatorname{tr} \pi_v^+(f_v) - \operatorname{tr} \pi_v^-(f_v)]$$

$$= \sum \varepsilon(\pi) \prod \operatorname{tr} \pi_v(f_v).$$

The products range over the finite set V of the non-split finite places v where ρ_v is supercuspidal. $m(\pi)$ is the multiplicity of the discrete-series π whose component at v in V is π_v, and at v outside V it is determined by ρ_v. $\varepsilon(\pi)$ is the sum of $\Pi\, m_v$ and $\langle \varepsilon, \pi \rangle = \Pi \langle \varepsilon, \pi_v \rangle$ (v in V). Here $\langle \varepsilon, \pi_v \rangle$ is 1 if π_v lies in π_v^+, and -1 if π_v lies in π_v^-. Linear independence of characters implies that $m(\pi) = 0$ unless the component π_v of π at v in V is in π_v^+ or π_v^-, and then $2m(\pi)$ is equal to $\varepsilon(\pi)$.

Proposition 8.5(iii) (p. 172) and 2.4(i) of [GP] imply that for some π with $m(\pi) \neq 0$ above, we have $m(\pi) = 1$. Since $\langle \varepsilon, \pi \rangle$ is 1 or -1, we conclude that $\Pi\, m_v$ is 1 or 3. Since $m_{u'} = m_u$, and the m_v are all positive integers, we deduce that $m_u = 1$, as required.

§3.6. Proposition. *In the notations of Proposition* 3.4, $\varepsilon_{1j}\varepsilon_{2j}\varepsilon_{3j} = 1$.

Proof. Again we use the trace formula, and global notations. We study the situation at a place w. We may assume that E/F are totally imaginary. At three finite places $v = v_m$ ($\neq w$; $m = 1, 2, 3$) which do not split (and do not ramify) we choose θ_v, $'\theta_v$ so that $\rho_v, \rho_v', \rho_v''$ are supercuspidal. Since $\varepsilon_{1jv}\varepsilon_{2jv}\varepsilon_{3jv}$ is independent of j, then for each v there exists $j = j(v)$ so that ε_{ijv} is independent of i. Since ε_{ijv} can attain only two values, and we have three v at our disposal, we can assume that $\varepsilon_{i_1 j_1 v_1} = \varepsilon_{i_2 j_2 v_2}$, where $j_m = j(v_m)$, and both sides are independent of i_1, i_2.

We now construct global characters θ, $'\theta$ with the chosen components at v_1, v_2 and our place w, which are unramified at each place which does not split in E/F (we can take $\theta_v = {}'\theta_v$ at the v which ramify). It is clear that $\rho_1 = \rho(\theta, '\theta)$, $\rho_2 = \rho(\theta, \omega/\theta \cdot '\theta)$, $\rho_3 = \rho('\theta, \omega/\theta \cdot '\theta)$ are cuspidal and distinct. All three appear in the trace formula together with $I(\tau \otimes \kappa) = I(\theta', '\theta', \omega'/\theta' \cdot '\theta')$, and with coefficients $n(\rho) = \frac{1}{2}$ (see [U(2)]). Namely, we obtain

$$\prod \left[\sum_j \operatorname{tr} \pi_{jv}(f_v)\right] + \sum_j \prod \left[\sum_j \varepsilon_{ijv} \operatorname{tr} \pi_{jv}(f_v)\right] = 4 \sum m(\pi) \prod \operatorname{tr} \pi_v(f_v).$$

The product ranges over $v = w, v_1, v_2$. At $v = v_m$ ($m = 1, 2$) we take f_v to be a coefficient of π_{jv}, where $j = j(v)$ was chosen above. Then the product Π can be taken only over our place w. Hence, for every j, we have

$$1 + \sum_i \varepsilon_{ijw} \equiv 0 \quad \text{(modulo 4).}$$

This holds only if $\varepsilon_{ijw} = 1$ for an odd number of i, and the proposition follows.

§3.7. To sum up our case (3.3), fix θ, $'\theta$ so that $\rho_1 = \rho(\theta, '\theta)$, $\rho_2 = \rho(\theta, \omega/\theta \cdot '\theta)$ are disjoint supercuspidal H-packets. Denote by Π the induced G'-module $I(\theta', '\theta', \omega'/\theta' \cdot '\theta')$.

Corollary. *There exist four supercuspidal G-modules π_j ($1 \leqq j \leqq 4$), so that π_1 has a Whittaker vector but π_j ($j \neq 1$) do not, so that*

$$\operatorname{tr} \Pi(\phi \times \sigma) = \sum_j \operatorname{tr} \pi_j(f),$$

and

$$\operatorname{tr} \rho_i('f) = \operatorname{tr} \pi_1(f) + \operatorname{tr} \pi_{i+1}(f) - \operatorname{tr} \pi_{i'}(f) - \operatorname{tr} \pi_{i''}(f).$$

i', i'' are so that $\{i + 1, i', i''\} = \{2, 3, 4\}$.

We write $\pi^+(\rho_i)$ for $\{\pi_1, \pi_{i+1}\}$, and $\pi^-(\rho_i)$ for $\{\pi_{i'}, \pi_{i''}\}$.

§3.8. $\rho(\theta, \omega/\theta^2)$

The next special case of interest is that of the packet associated with $\rho = \rho(\theta, \omega/\theta^2)$, where $\theta^3 \neq \omega$, so that $\{\rho\}$ consists of supercuspidals; in fact $\{\rho\}$ consists of a single element, and this is clear also from the comments below. The associated G'-module is the σ-invariant tempered induced $\Pi = I(\theta', \omega'/\theta'^2, \theta')$. It is the base-change lift of the reducible G-module $\pi = I(\theta')$. π is the direct sum of the tempered irreducible π^+ and π^-. Then we have

$$\operatorname{tr} \Pi(\phi \times \sigma) = \operatorname{tr} \pi(f) = \operatorname{tr} \pi^+(f) + \operatorname{tr} \pi^-(f),$$

and also

$$\operatorname{tr} \rho('f) = \operatorname{tr} \pi^+(f) - \operatorname{tr} \pi^-(f),$$

for a suitable choice of π^+. Namely π^+ has a Whittaker vector, while π^- does not. In particular $2[\{\rho\}] = [\{\pi^+, \pi^-\}] = 2$, so that $\{\rho\}$ consists of a single element, as asserted.

§3.9. Packets

With this we have completed the description of all tempered packets $\{\pi\}$ of G. The packets are in a bijection with the tempered σ-stable G'-modules Π. It has been shown already in [U″] that if Π is a square-integrable σ-invariant G'-module, then it is σ-stable, and the packet $\{\pi\}$ consists of a single element. If Π is induced from a square-integrable H'-module, and it is σ-stable, then it is of the form $I(\tau \otimes \kappa)$, where τ is the stable base-change lift of a square-integrable packet $\{\rho\}$ of H. The associated G-packet $\{\pi\}$ consists of $2 = 2[\{\rho\}]$ elements, each occurring with multiplicity one. If Π is induced from the diagonal subgroup, and it is not simply the base-change lift of an induced G-module $I(\mu)$ (in which case the packet $\{\pi\}$ consists of the irreducible constituents of $I(\mu)$), then Π is of the form $I(\theta', '\theta', \omega'/\theta' \cdot '\theta')$, where the three characters are distinct, and trivial on $F^\times$. In this case the packet $\{\pi\}$ consists of $4 = 2[\{\rho\}]$ elements, where $\rho = \rho(\theta, '\theta)$.

Using this, and the related character identities between ρ and the difference of members of $\{\pi\}$, we can use the trace formula to describe the discrete spectrum of G.

II. Global Lifting

§1.1. Trace formula

First we recall Proposition I.1.1.

Proposition. *We have* $F_1 = \Phi_1 + \frac{1}{2}(\Phi_2 + F_2) + \frac{1}{4}(\Phi_3 + F_3)$.

Proof. We have to show that F_6 is 0, in the notation of (I.1.1). If μ and θ are related by $\mu(z) = \theta(z/\bar{z})$, and $\rho = \rho(\theta, \omega/\theta^2)$, then the G-module $I(\mu_v)$ is the direct sum of $\pi_{\mu_v}^+$ and $\pi_{\mu_v}^-$, and by (I.3.8) we have

$$\operatorname{tr} \rho_v('f) = \operatorname{tr} \pi_{\mu_v}^+(f_v) - \operatorname{tr} \pi_{\mu_v}^-(f_v).$$

Shahidi [Sh] used work of Keys to show that

$$\operatorname{tr} R(\mu_v) I(\mu_v, f_v) = (-1, E_v/F_v)[\operatorname{tr} \pi_{\mu_v}^+(f_v) - \operatorname{tr} \pi_{\mu_v}^-(f_v)],$$

where the Hilbert symbol $(-1, E_v/F_v)$ is equal to 1 if -1 lies in $N_{E/F}E_v^\times$, and to -1 otherwise. It is 1 for almost all v, and the product of $(-1, E_v/F_v)$ over all v is 1. Hence $F_6 = 0$, as required.

In view of the local liftings results, this gives an explicit description of the discrete spectrum of G.

§1.2. To write out the three terms in the expression for the discrete spectrum F_1, we introduce some notations. If Π_v is a tempered σ-stable G'_v-module, we write $\{\pi_v(\Pi_v)\}$ for the associated packet of G_v-modules. We apply this terminology also when Π_v is one-dimensional, where $\{\pi_v(\Pi_v)\}$ consists of a single one-dimensional G_v-module; and also when Π_v is the lift of an induced G_v-module $I(\mu_v)$. If $\{\rho_v\}$ is a packet of H_v which lifts by stable base-change to the H'_v-module τ_v, we put $\{\pi_v(\rho_v)\}$ for $\{\pi_v(I(\tau_v \otimes \kappa_v))\}$. It consists of $2[\{\rho_v\}]$ elements; it is the disjoint union of the set $\pi^+(\rho_v)$ and $\pi^-(\rho_v)$, whose cardinalities are equal if E_v is a field; $\pi^-(\rho_v)$ is empty if $E_v = F_v \oplus F_v$. Given ρ_v, we write $\varepsilon(\pi_v) = 1$ for π_v in $\pi^+(\rho_v)$, and $\varepsilon(\pi_v) = -1$ for π_v in $\pi_v^-(\rho_v)$. In particular, if $[\{\rho_v\}] = 2$, we defined in Proposition I.3.4 the sign ε_{ijv} as a coefficient of π_{jv} in $\{\pi_v(\rho_v)\}$, and we put $\varepsilon_i(\pi_{jv}) = \varepsilon_{ijv}$. We have $\{\pi_v(\rho_{1v})\} = \{\pi_v(\rho_{2v})\} = \{\pi_v(\rho_{3v})\}$, and ε_i depends on ρ_i.

§1.3. Using these notations we can write

$$\Phi_1 = \sum_{\Pi} \prod \operatorname{tr}\{\pi_v(\Pi_v)\}(f_v).$$

The sum ranges over all discrete-series automorphic σ-invariant G'-modules Π. Note that we use here the rigidity theorem, and the multiplicity one theorem for the discrete spectrum of $\mathrm{GL}(3, \mathbf{A}_E)$.

§1.4. $\frac{1}{2}(\Phi_2 + F_2)$ is the sum of two terms. The first is

$$\frac{1}{2} \sum_{\rho \neq \rho(\theta, '\theta)} \Big\{ \prod \, [\operatorname{tr}(\pi_v^+(\rho_v))(f_v) + \operatorname{tr}(\pi_v^-(\rho_v))(f_v)]$$

$$+ \prod \, [\operatorname{tr}(\pi_v^+(\rho_v))(f_v) - \operatorname{tr}(\pi_v^-(\rho_v))(f_v)] \Big\}$$

$$= \sum_{\pi} m(\rho, \pi) \prod \operatorname{tr} \pi_v(f_v).$$

The first sum is over the discrete-series automorphic H'-packets ρ which are neither one-dimensional, nor of the form $\rho(\theta, '\theta)$. The multiplicity $m(\rho, \pi)$ is $[1 + \varepsilon(\pi)]/2$, where $\varepsilon(\pi) = \Pi\, \varepsilon(\pi_v)$; it is 0 or 1. The sum over π is taken over all products $\otimes\, \pi_v$, such that there exists ρ as above, and π_v is in $\{\pi_v(\rho_v)\}$ for all v, and π_v is unramified (so that $\varepsilon(\pi_v) = 1$) for almost all v. Thus $m(\rho, \pi) = 1$ exactly when the number of components π_v in $\pi_v^-(\rho_v)$ is even. Otherwise the product $\otimes\, \pi_v$ is not automorphic.

The other term in $\frac{1}{2}(\Phi_2 + F_2)$ is

$$\frac{1}{2}\sum_{\mu}\left\{\prod\ \{\operatorname{tr}\pi_v^{\times}(f_v)-\operatorname{tr}\pi_v^{-}(f_v)]+\prod\ [\operatorname{tr}\pi_v^{\times}(f_v)+\operatorname{tr}\pi_v^{-}(f_v)]\right\}$$

$$=\sum_{\mu} m(\mu,\pi)\ \prod\ \operatorname{tr}\pi_v(f_v).$$

The first sum is over all characters μ of C_E^1, or one-dimensional automorphic H-modules. We put $m(\mu,\pi)=[1+\varepsilon(\pi)]/2$, where $\varepsilon(\pi)$ is $\prod\varepsilon(\pi_v)$, and $\varepsilon(\pi_v^{\times})=1$, $\varepsilon(\pi_v^{-})=-1$. The sum over π ranges over the products $\otimes\pi_v$, such that there exists a μ as above, with $\pi_v=\pi_v^{\times}$ for almost all v, and $\pi_v=\pi_v^{-}$ at the other places. $m(\mu,\pi)$ is 0 unless there is an even number of places v where π_v is π_v^{-}.

§1.5. There remains the sum $\frac{1}{4}(\Phi_3+F_3)$. It is equal to

$$\frac{1}{4}\sum_{\rho}\left[\prod\ \sum_{j=1}^{4}\operatorname{tr}(\pi_{jv}(\rho_v))(f_v)+\sum_{i=1}^{3}\ \prod\ \sum_{j=1}^{4}\ \varepsilon_{ijv}\operatorname{tr}(\pi_{jv}(\rho_v))(f_v)\right]$$

$$=\sum_{\pi} m(\rho,\pi)\ \prod\ \operatorname{tr}\pi_v(f_v).$$

The first sum ranges over the discrete-series automorphic H-packets of the form $\rho=\rho(\theta,{}'\theta)$, where $\theta,{}'\theta,\omega/\theta\cdot{}'\theta$ are distinct. They are taken modulo the equivalence relation $\rho(\theta,{}'\theta)\sim\rho(\theta,\omega/\theta\cdot{}'\theta)\sim\rho({}'\theta,\omega/\theta\cdot{}'\theta)$. The multiplicity $m(\rho,\pi)=[1+\sum_{i=1}^{3}\varepsilon_i(\pi)]/4$ is equal to 0 or 1. The sum ranges over the products $\otimes\pi_v$, such that there exists ρ as above so that π_v lies in $\{\pi_v(\rho_v)\}$ for all v, and it is unramified at almost all v (namely it is π_{1v}), so that $\varepsilon_i(\pi_v)$ is 1 at almost all v.

§1.6. This gives a complete description of the discrete-series of G. We introduce some more terminology. The local packets $\{\pi_v\}$ have been defined in all cases, except for $\pi_v=\pi_v^{\times}$. This is a non-tempered G_v-module. We define the packet of $\pi_v^{\times}$ to consist of $\pi_v^{\times}$ alone. The *quasi-packet* $\pi(\mu_v)$ of $\pi_v^{\times}$ will be the set $\{\pi_v^{\times},\pi_v^{-}\}$, consisting of a non-tempered and a supercuspidal. Thus a packet consists of tempered G_v-modules, or of a single non-tempered element; a quasi-packet is defined for global purposes. Given a local packet P_v at all v, so that it contains an unramified member π_v^0 for almost all v, we define the *global packet* P to be the set of products $\otimes\pi_v$ over all v, so that $\pi_v=\pi_v^0$ for almost all v, and $\{\pi_v\}=P_v$ for all v. Given a character μ of C_E^1, we define the *quasi-packet* $\pi(\mu)$ as in the case of the packets, where P_v is replaced by the quasi-packet $\pi(\mu_v)$ at all v.

A standard argument, based on the absolute convergence of the sums, and the unitarizability of all representations which occur in the trace formula, implies:

§2.1. Theorem. *The base-change lifting is a one-to-one correspondence from the set of packets and quasi-packets which contain a discrete-series automorphic G-module, to the set of σ-invariant automorphic G'-modules which appear in Φ_1, Φ_2 or Φ_3. Namely, a discrete-series G-module π lies in one of the following.*

(1) *A packet* $\pi(\Pi)$ *associated with a discrete-series* σ*-invariant* G'*-module* Π. (2) *A packet* $\pi(\rho)$ *associated with a discrete-series automorphic* H'*-module* ρ *which is not of the form* $\rho(\theta, \omega/\theta^2)$. (3) *A quasi-packet* $\pi(\mu)$, *associated with an automorphic one-dimensional* H*-module* $\rho = \mu(\det)$.

The multiplicity of π from a packet $\pi(\Pi)$ in the discrete spectrum of G is 1. Namely each member π of $\pi(\Pi)$ is automorphic, in the discrete series. The multiplicity of a member π of a packet $\pi(\rho)$ or a quasi-packet $\pi(\mu)$ in the discrete spectrum of G is equal to $m(\rho, \pi)$ or $m(\mu, \pi)$, respectively. This number is 1 or 0, but it is not constant over $\pi(\rho)$ or $\pi(\mu)$. Namely, in cases (2) and (3) not each member of $\pi(\rho)$ or $\pi(\mu)$ is automorphic.

§2.2. Corollary. *The multiplicity of an automorphic G-module in the discrete spectrum is* 1. *If* π *and* π' *are discrete-series G-modules whose components* π_v *and* π'_v *are equivalent at almost all* v, *then they lie in the same packet, or quasi-packet.*

The first statement is called *multiplicity one theorem* for the discrete spectrum of G. The second is the *rigidity theorem*.

The automorphic members π of the quasi-packet $\pi(\mu)$ have components π_v^- at the remaining finite set of places, which do not split in E/F. Each such π is a counter-example to the gèneralized Ramanujan Conjecture, which suggests that all components π_v of a cuspidal G-module π are tempered. However, we expect this Conjecture to be valid for the members π of the packets $\pi(\Pi)$, $\pi(\rho)$.

§2.3. Proposition. *Suppose that* π *is a discrete series G-module which has a component of the form* $\pi_w^\times$. *Then almost all components of* π *are of the form* $\pi_v^\times$, *and* π *lies in a quasi-packet* $\pi(\mu)$.

Proof. π defines a member Π of Φ_1, Φ_2 or Φ_3 whose component at w is of the form $I(\tau_w)$, where τ_w is a one-dimensional H'_w-module. But then Π must be of the form $I(\tau)$, where τ is a one-dimensional H'-module, and the claim follows.

The Theorem has the following obvious

§2.4. Corollary. *There is a bijection from the set of automorphic discrete-series H-packets* ρ *which are not of the form* $\rho(\theta, \omega/\theta^2)$, *to the set of automorphic discrete-series G-packets of the form* $\pi(\rho)$.

This generalizes a result of Kudla [Ku], and sharpens Theorem 4.4 of [U′]. Also we deduce

§2.5. Corollary. *Suppose that* π *is a discrete-series G-module whose component* π_v *at a place* v *which splits* E/F *is elliptic. Then* π *lies in a packet* $\pi(\Pi)$, *where* Π *is in the discrete-series.*

Let $'G'$ be the multiplicative group of a division algebra of dimension 9 central over E, which is unramified outside the places u', u'' of E above a finite place u of F which splits in E, and which is anisotropic at u' and u''. Suppose σ is an involution of the second kind, namely its restriction to the center $E^\times$ is $\sigma(z) = \bar{z}$. Denote by $'G$ the associated unitary group, namely the group of x in $'G'$ with $x\sigma(x) = 1$. It is not hard to compare the trace formulae in the compact case and deduce from our local lifting that we have

§2.6. Proposition. *The base-change lifting defines a bijection between the set of automorphic packets of $'G$-modules, and the set of σ-invariant automorphic $'G'$-modules.*

This sharpens the result of [U″], §5.

The Deligne–Kazhdan correspondence, from the set of automorphic $'G'$-modules, to the set of discrete-series automorphic G'-modules with an elliptic component at u and u', implies

§2.7. Corollary. *The relation $'\pi_v \simeq \pi_v$ for all $v \neq u$ defines a bijection between the set of automorphic packets of $'G$-modules $'\pi$, and the set of automorphic packets of G-modules of the form $\pi = \pi(\Pi)$, whose component at u is elliptic.*

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Appendix

Here we record well-known results concerning the representation theories of the groups of this paper in the case of the archimedean quadratic extension **C**/**R**. For proofs we refer to [BW], Ch. VI; [Kr], and [W], §7, and to [Cl], [Sd]. This is then used in conjuction with Theorem II.2.1 and its corollaries to determine all automorphic G-modules with non-trivial cohomology outside of the middle dimension.

We first recall some notations. Denote by σ the non-trivial element of Gal(**C**/**R**). Put $\bar{z} = \sigma z$ for z in **C**, and $\mathbf{C}^1 = \{z/\bar{z};\ z \text{ in } \mathbf{C}^\times\}$. Put $H' = \mathrm{GL}(2, \mathbf{C})$, $G' = \mathrm{GL}(3, \mathbf{C})$,

$$H = U(1,1) = \left\{h \text{ in } H';\ hw{}^t\bar{h} = w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$$

and

$$G = U(2,1) = \left\{g \text{ in } G';\ gJ{}^t\bar{g} = J = \begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}\right\},$$

The center Z of G is isomorphic to $\mathbf{C}^1$; so is that of H. Fix an integer $\mathbf{w}$ and a character $\omega(z/\bar{z}) = (z/\bar{z})^{\mathbf{w}}$ of $\mathbf{C}^1$. Put $\omega'(z) = \omega(z/\bar{z})$. Any representation of any subgroup of G which contains Z will be assumed below to transform under Z by ω. Choosing the positive square-root of a positive number, write $(z/\bar{z})^n$ for $z^{2n}/(z\bar{z})^n$, for n in $\frac{1}{2}\mathbf{Z}$ and z in $\mathbf{C}^\times$.

The diagonal subgroup A_H of H will be identified with the subgroup of the diagonal subgroup A of G consisting of $\mathrm{diag}(z, z', \bar{z}^{-1})$ with $z' = 1$. For any character χ_H of A_H there are complex a, c with $a + c$ in $\mathbf{Z}$ such that

$$\chi_H(\mathrm{diag}(z, \bar{z}^{-1})) = (z\bar{z})^{(a-c)/2}(z/\bar{z})^{(a+c)/2}.$$

This χ_H extends uniquely to a character χ of A whose restriction to Z is ω. In fact $b = \mathbf{w} - a - c$ is integral, and $\chi = \chi(a, b, c)$ is defined by

$$\chi(\mathrm{diag}(z, z', \bar{z}^{-1})) = (z\bar{z})^{(a-c)/2}(z/\bar{z})^{(a+c)/2}z'^b.$$

A character κ of $\mathbf{C}^\times$ which is trivial on the multiplicative group $\mathbf{R}_+^\times$ of positive real numbers but non-trivial on $\mathbf{R}^\times$ is of the form $\kappa(z) = (z/\bar{z})^{k+1/2}$, where k is integral.

The H-module $I(\chi_H) = I(\chi_H; B_H, H) = \mathrm{Ind}(\delta_H^{1/2}\chi_H; B_H, H)$ normalizedly induced from the character χ_H of A_H extended trivially to the upper triangular subgroup B_H of H, is irreducible unless a, c are distinct integers. In the latter case the sequence $JH(I(\chi_H))$ of constituents, repeated with their multiplicities, in the composition series of $I(\chi_H)$, consists of an irreducible finite-demensional H-module $F_H = F_H(\chi_H) = F_H(a, c)$ of dimension $|a - c|$ (and central character $z \mapsto z^{a+c}$), and the two irreducible square-integrable constituents of the packet $\rho = \rho(a, c)$ (of highest weight $|a - c| + 1$) on which the center of the universal enveloping algebra of H acts by the same character as on F_H. The Langlands classification of [BW], Ch. IV, defines a bijection between the set of packets and the set of $\hat{H}^0$-conjugacy classes of homomorphisms from the Weil group

$$W_{\mathbf{C}/\mathbf{R}} = \langle z, \sigma; z \text{ in } \mathbf{C}^\times, \sigma z = \bar{z}\sigma, \sigma^2 = -1 \rangle$$

to the dual group $\hat{H} = \hat{H}^0 \times W_{\mathbf{C}/\mathbf{R}}$, $\hat{H}^0 = \mathrm{GL}(2, \mathbf{C})$, whose composition with the second projection is the identity. Such homomorphism is called *discrete* if its image is not conjugate by $\hat{H}^0$ to a subgroup of $\hat{B}_H = B_H \times W_{\mathbf{C}/\mathbf{R}}$. The packet $\rho(a, c) = \rho(c, a)$ corresponds to the homomorphism $h(\chi_H) = h(a, c)$ defined by

$$z \mapsto \begin{pmatrix} (z/\bar{z})^a & 0 \\ 0 & (z/\bar{z})^c \end{pmatrix} \times z, \qquad \sigma \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \sigma.$$

It is discrete if and only if $a \neq c$.

The composition $h(a, b, c)$ of $h(\chi_H \otimes \kappa^{-1}) = h(a - k - \frac{1}{2}, c - k - \frac{1}{2})$ with the endo-lift $e: \hat{H} \to \hat{G}$ of $[U']$ is the homomorphism $W_{\mathbf{C}/\mathbf{R}} \to \hat{G}$ defined by

$$z \to \begin{bmatrix} (z/\bar{z})^a & & 0 \\ & (z/\bar{z})^b & \\ 0 & & (z/\bar{z})^c \end{bmatrix} \times z, \qquad \sigma \to J \times \sigma.$$

The corresponding G-packet $\pi = \pi(a, b, c)$ depends only on the set $\{a, b, c\}$. It consists of square-integrables if and only if a, b, c are distinct.

Suppose that $a \geqq b \geqq c$ are integers, and put

$$\chi = \chi(a, b, c), \quad \chi^+ = \chi(b, a, c), \quad \chi^- = \chi(a, c, b).$$

If $b > c$ (resp. $a > b$) then the normalizedly induced G-module $I(\chi^+)$ (resp. $I(\chi^-)$) has a unique irreducible non-tempered infinite-dimensional quotient $J^+ = J^+(a, b, c)$ (resp. $J^- = J^-(a, b, c)$). This J^+ (resp. J^-) is unitary if and only if $b - c = 1$ (resp. $a - b = 1$); see [Kr] or [W].

If $a > b = c$ (resp. $a = b > c$) then the character χ^+ (resp. χ^-) of A is unitary and the packet π consists of the two constituents of $I(\chi^+)$ (resp. $I(\chi^-)$). There is a unique element $\pi^+(a, b, c)$ in the packet $\pi = \pi(a, b, c)$ with a Whittaker model. The other element in π is denoted by $\pi^- = \pi^-(a, b, c)$. The composition series of $I(\chi^-)$ consists of J^- and π^+ if $a > b = c$, and that of $I(\chi^+)$ consists of J^+ and π^+ if $a = b > c$.

If $a > b > c$ then $I(\chi)$ has a unique finite-dimensional quotient $F = F(\chi)$; its highest weight, with respect to B, is the character $\mathrm{diag}(x, y, z) \to x^{a-1}y^b z^{c+1}$. The packet π consists of three square-integrable G-modules D, D^+, D^-. For a suitable choice (see [BW] or [W]) the composition series of $I(\chi)$ consists of F, J^+, J^-, D, that of $I(\chi^+)$ consists of J^+, D^+, D, and that of $I(\chi^-)$ consists of J^-, D^-, D. There is a unique choice of complex structure on G/K (see below) which makes J^+, D^+ holomorphic, J^-, D^- anti-holomorphic, and D neither holomorphic nor anti-holomorphic.

To fix notations in a manner consistent with the non-archimedean case, note that if β is a one-dimensional H-module then there are unique integers $a \geqq b \geqq c$ with $a + b + c = \mathrm{w}$ and either (i) $b = c + 1$, $\beta = \beta(b, c)$, or (ii) $a = b + 1$, $\beta = \beta(a, b)$. If, in addition, $a > b > c$, put $\pi_\beta^\times = J^+$ and $\pi_\beta^- = D^-$ in case (i), and $\pi_\beta^\times = J^-$, $\pi_\beta^- = D^+$ in case (ii). The motivation for this choice of notations is the following character identities. Put

$$\rho = \rho(a, c) \otimes \kappa^{-1}, \quad \rho^+ = \rho(b, c) \otimes \kappa^{-1}, \quad \rho^- = \rho(a, b) \otimes \kappa^{-1}.$$

Then $\{\rho, \rho^+, \rho^-\}$ is the set of H-packets which lift to the G-packet $\pi = \pi(a, b, c)$ via the endo-lifting e. As noted above, ρ, ρ^+ and ρ^- are distinct if and only if $a > b > c$, equivalently π consists of three square-integrable G-modules. Moreover, every square-integrable H-packet is of the form ρ, ρ^+ or ρ^- for unique $a \geqq b \geqq c$, $a > c$.

If $a = b = c$ then $\rho = \rho^+ = \rho^-$ is the H-packet which consists of the constituents of $I(\chi_H(a, c) \otimes \kappa^{-1})$, and $\pi = I(\chi(a, b, c))$ is irreducible.

If $a > b = c$ put $\langle \rho, \pi^+ \rangle = 1$, $\langle \rho, \pi^- \rangle = -1$.

If $a = b > c$ put $\langle \rho, \pi^+ \rangle = 1$, $\langle \rho, \pi^- \rangle = -1$.

If $a > b > c$ put $\langle \rho', D \rangle = 1$ for $\rho' = \rho, \rho^+, \rho^-$, and: $\langle \rho, D^+ \rangle = -1$, $\langle \rho, D^- \rangle = -1$; $\langle \rho^+, D^+ \rangle = 1$, $\langle \rho^+, D^- \rangle = -1$; $\langle \rho^-, D^+ \rangle = -1$, $\langle \rho^-, D^- \rangle = 1$.

A.1 Proposition ([Sd]). *For all matching functions f on G and f_H on H, we have*

$$\mathrm{tr}\, \sigma'(f_H) = \sum_{\pi' \in \pi} \langle \rho', \pi \rangle \mathrm{tr}\, \pi'(f) \qquad (\rho' = \rho, \rho^+ \textit{ or } \rho^-).$$

From this it is easy to conclude the following

A.2. Corollary. *For every one-dimensional H-module* β *we have* $\operatorname{tr}\beta(f_H) = \operatorname{tr}\pi_\beta^\times(f) + \operatorname{tr}\pi_\beta^-(f)$ *for all matching functions* f *on* G *and* f_H *on* H.

Further, if ρ is a tempered H-module, π the endo-lift of ρ (then π is a G-packet), ρ' is the base-change lift of ρ (thus ρ' is a σ-invariant H'-module), and $\pi' = I(\rho')$ is the G'-module normalizedly induced from ρ' (we regard H' as a Levi subgroup of a maximal parabolic subgroup of G'), then we have

A.3. Proposition ([Cl]). *For all matching* f *on* G *and* ϕ *on* G' *we have* $\operatorname{tr}\pi(f) = \operatorname{tr}\pi'(\phi\times\sigma)$.

From this it is easy to conclude the following

A.4. Corollary. *For every one-dimensional H-module* β *we have* $\operatorname{tr}(I(\beta'))(\phi\times\sigma) = \operatorname{tr}\pi_\beta^\times(f) - \operatorname{tr}\pi_\beta^-(f)$, *for all matching functions* f *on* G *and* ϕ *on* G'.

Our next aim is to determine the $(L(G), K)$-cohomology of the G-modules described above, where $L(G)$ denotes the complexified Lie algebra of G. For that we describe the K-types of these G-modules, following [W], §7, and [BW], Ch. VI. Note that $G = U(2,1)$ can be defined by means of the form

$$J' = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix}$$

whose signature is also (2,1) and it is conjugate to

$$J = \begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}$$

by

$$\mathbf{B} = \begin{bmatrix} 2^{-1/2} & 0 & 2^{-1/2} \\ 0 & 1 & 0 \\ 2^{-1/2} & 0 & -2^{-1/2} \end{bmatrix}$$

of [W], p. 181. To ease the comparison with [W] we now take G to be defined using J'. In particular we now take A to be the maximal torus of G whose conjugate by $\mathbf{B}$ is the diagonal subgroup of $G(J)$. A character χ of A is again associated with (a, b, c) in $\mathbf{C}^3$ such that $a + c$ and b are integral, and $I(\chi)$ denotes the G-module normalizedly induced from χ extended to the standard Borel subgroup B. The maximal compact subgroup K of G is isomorphic to $U(2) \times U(1)$; it consists of the matrices $\left(\begin{smallmatrix} \alpha u & 0 \\ 0 & \beta \end{smallmatrix}\right)$; u in SU(2); α, β in $U(1) = \mathbf{C}^1$. Write $d(x, y, z)$ for the diagonal matrix diag(x, y, z). Note that $A \cap K$ consists of $\gamma d(\alpha, \alpha^{-2}, \alpha)$, and the center of K consists of $\gamma d(\alpha, \alpha, \alpha^{-2})$.

Let π_K denote the space of K-finite vectors of the admissible G-module π. By Frobenius reciprocity, as a K-module $I(\chi)_K$ is the direct sum of the irreducible K-modules h, each occurring with multiplicity $\dim[\mathrm{Hom}_{A \cap K}(\chi, h)]$. The h are parametrized by (a', b', c') in $\mathbf{Z}^3$, such that $\dim h = a' + 1$, and the central character of h is $\gamma d(\beta, \beta, \beta^{-2}) \mapsto \beta^{b'} \gamma^{c'}$; hence $b' \equiv c'(3)$ and $a' \equiv b'(2)$. In this case we write $h = h(a', b', c')$. For any integers a, b, c, p, q with $p, q \geqq 0$ we also write

$$h_{p,q} = h(p + q, 3(p - q) - 2(a + c - 2b), a + b + c).$$

A.5. Lemma. *The K-module $I(\chi)_K$, $\chi = \chi(a, b, c)$, is isomorphic to $\bigoplus_{p,q \geqq 0} h_{p,q}$.*

Proof. The restriction of $h = h(a', b', c')$ to the diagonal subgroup

$$D = \{\gamma d(\beta\alpha, \beta/\alpha, \beta^{-2})\}$$

of K is the direct sum of the characters $\alpha^n \beta^{b'} \gamma^{c'}$ over the integral n with $-a' \leqq n \leqq a'$ and $n \equiv a'(2)$. Hence the restriction of h to $A \cap K$ is the direct sum of the characters $\gamma d(\alpha, \alpha^{-2}, \alpha) \mapsto \alpha^{(3n - b')/2} \gamma^{c'}$. On the other hand, the restriction of $\chi = \chi(a, b, c)$ to $A \cap K$ is the character $\gamma d(\alpha, \alpha^{-2}, \alpha) \mapsto \alpha^{a + c - 2b} \gamma^{a + b + c}$. If $-a \leqq n \leqq a'$ and $n \equiv a'(2)$, there are unique $p, q \geqq 0$ with $a' = p + q$, and $n = p - q$. Then $h(a', b', c') \mid (A \cap K)$ contains $\chi(a, b, c) \mid (A \cap K)$ if and only if there are $p, q \geqq 0$ with $a' = p + q$, $b' = 3(p - q) - 2(a + c - 2b)$ and $c' = a + b + c$, as required.

For integral a, b, c put $\chi = \chi(a, b, c)$, $\chi^+ = \chi(b, a, c)$, $\chi^- = \chi(a, c, b)$. Also write

$$h^+_{p,q} = h(p + q, 3(p - q) - 2(b + c - 2a), a + b + c),$$

and

$$h^-_{p,q} = h(p + q, 3(p - q) - 2(a + b - 2c), a + b + c).$$

The Lemma implies that

$$I(\chi)_K = \oplus\, h_{p,q}, \quad I(\chi^+)_K = \oplus\, h^+_{p,q}, \quad I(\chi^-)_K = \oplus\, h^-_{p,q}$$

(sum over $p, q \geqq 0$). Write $JH(\pi)$ for the unordered sequence of constituents of the G-module π, repeated with their multiplicities.

If $a > b > c$ then $JH(I(\chi)) = \{F, J^+, J^-, D\}$ and by [W], §7, the K-type decomposition of the constituents is of the form $\oplus\, h_{p,q}$, where the sums range over: (1) $p<a-b$, $q<b-c$ for F; (2) $p \geqq a-b$, $q<d-c$ for J^+; (3) $p<a-b$, $q \geqq b-c$ for J^-; (4) $p \geqq a-b$, $q \geqq b-c$ for D. Further, $JH(I(\chi^+)) = \{J^+, D^+, D\}$. The K-types are of the form $\oplus h^+_{p,q}$, with sums over: (1) $p \geqq 0$, $a-b \leqq q < a-c$ for J^+; (2) $p \geqq 0$, $q<a-b$ for D^+; (3) $p \geqq 0$, $q \geqq a-c$ for D. Finally, $JH(I(\chi^-)) = \{J^-, D^-, D\}$. The K-types are of the form $\oplus h^-_{p,q}$, with sums over: (1) $b-c \leqq p < a-c$, $q \geqq 0$ for J^-; (2) $p<b-c$, $q \geqq 0$ for D^-; (3) $p \geqq a-c$, $q \geqq 0$ for D^-. Recall that J^+ is unitary if and only if $b-c=1$, and J^- is unitary if and only if $a-b=1$.

If $a>b=c$ (resp. $a=b>c$) then χ^+ (resp. χ^-) is unitary, and $I(\chi^+)$ (resp. $I(\chi^-)$) is the direct sum of the unitary G-modules π^+ and π^-. The K-type decomposition is $\pi^+_K = \oplus h^+_{p,q}$ $(p \geqq 0, q \geqq a-b)$, $\pi^-_K = \oplus h^+_{p,q}$ $(p \geqq 0, q<a-b)$ if $a>b=c$, and $\pi^+_K = \oplus h^-_{p,q}$ $(p \geqq b-c, q \geqq 0)$, $\pi^-_K = \oplus h^-_{p,q}$ $(p<b-c, q \geqq 0)$ if $a=b>c$. Moreover, $JH(I(\chi))$ is $\{\pi^\times = J^-, \pi^+\}$ if $a>b=c$, and $\{\pi^\times = J^+, \pi^+\}$ if $a=b>c$. The corresponding K-type decompositions are $J^- = \oplus h_{p,q}$ $(p<a-b, q \geqq 0)$, $J^+ = \oplus h_{p,q}$ $(p \geqq 0, q<b-c)$. As noted above, J^- is unitary if and only if $a-1=b=c$; J^+ is unitary if and only if $a=b=c+1$.

Next we define holomorphic and anti-holomorphic vectors, and describe those G-modules which contain such vectors. Write P^+ for the vector space of matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{bmatrix}$$

in the complexified Lie algebra $L(G) = M(3, \mathbf{C})$, and

$$P^- = \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

Then P^+, P^- are K-modules under the adjoint action of K, clearly isomorphic to $h(1,3,0)$ and $h(1,-3,0)$.

Definition. A vector in the space π_K of K-finite vectors in a G-module π is called *holomorphic* if it is annihilated by P^-, and *anti-holomorphic* if it is annihilated by P^+.

A.6. Lemma. *If $I(\chi)$ is irreducible then $I(\chi)_K$ contains neither holomorphic nor anti-holomorphic vectors.*

Proof. The K-modules $P^+ = h(1,3,0)$ and $P^- = h(1,-3,0)$ act by

$$h(1,3,0) \otimes h(a,b,c) = h(a+1, b+3, c) \oplus h(a-1, b+3, c)$$

and

$$h(1,\ -3,0)\otimes h(a,b,c)=h(a+1,b-3,c)\oplus h(a-1,b-3,c).$$

Hence the action of P^+ on $I(\chi)_K$ maps $h_{p,q}$ to $h_{p+1,q}\oplus h_{p,q-1}$, and that of P^- maps $h_{p,q}$ to $h_{p,q+1}\oplus h_{p-1,q}$. Consequently if $h_{p',q'}$ is annihilated by P^+, then $\oplus\, h_{p,q}$ ($p\geqq p'$, $q\leqq q'$) is an $(L(G),K)$-submodule of $I(\chi)$, and if P^- annihilates $h_{p',q'}$ then $\oplus\, h_{p,q}$ ($p\leqq p'$, $q\geqq q'$) is an $(L(G),K)$-submodule of $I(\chi)$. The lemma follows.

Definition. Denote by $\pi_{K,\mathrm{hol}}$ the space of holomorphic vectors in π_K, and by $\pi_{K,\mathrm{ah}}$ the space of anti-holomorphic vectors.

The above proof implies also the following

A.7. Lemma. (i) *The following is a complete list of irreducible unitary G-modules with holomorphic vectors*:

(1) $\pi=D^+(a,b,c)$, *where* $a>b>c$; *then* $\pi_{K,\mathrm{hol}}=h(a-b-1,\ a+b-2c+3,\ a+b+c)$;

(2) $\pi=J^+(a,b,b-1)$, *with* $a\geqq b$; *then* $\pi_{K,\mathrm{hol}}=h(a-b,\ a-b+2,\ a+2b-1)$;

(3) $\pi=\pi^-(a,b,b)$, $a>b$; *then* $\pi_{K,\mathrm{hol}}=h(a-b-1,a-b+3,a+2b)$.

(ii) *The following is a complete list of irreducible unitary G-modules with anti-holomorphic vectors*:

(1) $\pi=D^-(a,b,c)$, $a>b>c$; *then* $\pi_{K,\mathrm{ah}}=h(b-c-1,\ b+c-2a-3,\ a+b+c)$;

(2) $\pi=J^-(b+1,b,c)$, $b\geqq c$; *then* $\pi_{K,\mathrm{ah}}=h(b-c,c-b-2,2b+c+1)$;

(3) $\pi=\pi^-(a,a,c)$, $a>c$; *then* $\pi_{K,\mathrm{ah}}=h(a-c-1,c-a-3,2a+c)$.

Let $F=F(a,b,c)$ be the irreducible finite-dimensional G-module with highest weight $d(x,y,z)\mapsto x^{a-1}y^bz^{c+1}$; it is the unique finite-dimensional quotient of $I(\chi)$, $\chi=\chi(a,b,c)$. Let $\tilde F$ denote the contragradient of F, let π be an irreducible unitary G-module, and denote by $H^j(L(G),K;\ \pi\otimes\tilde F)$ the $(L(G),K)$-cohomology of $\pi\otimes\tilde F$. This cohomology vanishes, by [BW], Theorem 5.3, p. 29, unless π and F have equal infinitesimal characters, namely π is associated with the triple (a,b,c) of F. It follows from the K-type computations above that one has (cf. [BW], Theorem VI.4.11, p. 201) the following

A.8. Proposition. *If* $H^j(\pi\otimes\tilde F)\neq 0$ *for some* j *then* π *is one of the following*.

(1) *If* π *is* $D(a,b,c)$, $D^+(a,b,c)$ *or* $D^-(a,b,c)$ *then* $H^j(\pi\otimes\tilde F)$ *is* 0 *if* $j\neq 2$ *and* $\mathbf{C}$ *if* $j=2$.

(2) *If* π *is* $J^-(a,b,c)$ *with* $a-b=1$ *or* $J^+(a,b,c)$ *with* $b-c=1$ *then* $H^j(\pi\otimes\tilde F)$ *is* 0 *if* $j\neq 1,3$ *and* $\mathbf{C}$ *if* $j=1,3$.

(3) *$H^j(F \otimes \tilde{F})$ is 0 unless $j = 0, 2, 4$, when it is* $\mathbf{C}$.

Finally we use the results of Theorem II.2.1 and its corollaries to describe the cohomology of automorphic forms. Thus let F be a totally real number field, E a totally imaginary quadratic extension of F, G' an inner form of G which is defined using the multiplicative group $'G'$ of a division algebra of dimension 9 central over E and an involution of the second kind. The set S of archimedean places of F is the disjoint union of the set S' where $'G$ is quasi-split ($\simeq U(2, 1)$), and the set S'' where $'G$ is anisotropic ($\simeq U(3)$). Put $'G_\infty = \Pi_{v\in S}\, 'G_v$, $'K_\infty = \Pi_{v\in S}\, 'K_v$, and write $'G'_\infty$, $'G''_\infty$, $'K'_\infty$, $'K''_\infty$ for the corresponding products over S' and S''. Here $'K_v = 'G_v$ for v in S'', $'K_v \simeq U(2) \times U(1)$ for v in S'. Fix an irreducible finite-dimensional $'G_v$-module F_v for all v in S. Put $\tilde{F} = \otimes \tilde{F}_v$ (v in S). Then $F_v = F_v(a_v, b_v, c_v)$ for integral $a_v > b_v > c_v$ if v is in S'. Let $\pi = \otimes \pi_v$ be a discrete-series infinite-dimensional automorphic $'G$-module. Then π_v is unitary for all v and π_v is infinite-dimensional for all v outside S''. Put $\pi_\infty = \otimes \pi_v$ (v in S). If $H^*(L('G_\infty), 'K_\infty; \pi_\infty \otimes \tilde{F}) \neq 0$, then $\pi_v = F_v$ for all v in S'', and

$$H^*(L('G_\infty), 'K_\infty; \pi_\infty \otimes \tilde{F}) = \prod_{v\in S'} H^*(L('G_v), 'K_v; \pi_v \otimes \tilde{F}_v).$$

A.9. Proposition. *Let π be an automorphic discrete series $'G$-module. Put $d = \dim[L('G_\infty)/L('K_\infty)]$. If $H^j(L('G_\infty), 'K_\infty; \pi_\infty \otimes \tilde{F}) \neq 0$ for $j \neq d$ then either π is one-dimensional or π lies in a quasi-packet $\pi(\mu)$ of Theorem* II.2.1, *associated with an automorphic one-dimensional H-module $\rho = \mu \circ \det$. In the last case we have (1) $a_v - b_v = 1$ or $b_v - c_v = 1$ for all v in S', (2) π_v is of the form $\pi_v^\times$ or π_v^- for all v outside S'' (it is $\pi_v^\times$ for almost all v), and (3) $'G$ is quasi-split at each finite place of the totally real field F (thus $'G' = \mathrm{GL}(3, E)$ is split).*

Proof. If π is infinite-dimensional and $H^j \neq 0$ for $j \neq d$, then there is v in S' such that π_v is of the form $\pi_v^\times$. Theorem II.2.1 then implies that π is of the form $\pi(\mu)$, and (2) follows. Since π_v is unitary (for v in S'), (1) follows from (2). Finally (3) results from Corollary II.2.7 of Theorem II.2.1, which asserts that if $'G$ has automorphic representations of the form $\pi(\mu)$ where μ is a character of H, then $'G' = \mathrm{GL}(3, E)$ is the multiplicative group of the split simple algebra of dimension 9 over E.

The last assertion of the Proposition can be rephrased as follows.

A.10. Corollary. *If $'G'$ is the multiplicative group of a division algebra, then any discrete-series automorphic $'G$-module with cohomology outside the middle dimension is necessarily one-dimensional.*

This sharpens results of [K], §4, in the case of $n = 3$.

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