BASE CHANGE TRACE IDENTITY FOR $U(3)$

By

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Abstract. An identity of trace formulae which appears in the theory of base change for $U(3)$ is proven for arbitrary matching functions, under no restriction on any component. The method requires no detailed analysis of weighted orbital integrals, or of orbital integrals of singular classes.

Introduction

Let E/F be a quadratic extension of global fields. Put G' for $G(E) = \mathrm{GL}(3, E)$. Denote by $G = G(F)$ the quasi-split unitary group in three variables. It consists of all g in G' with $\sigma g = g$, where we write $\sigma x = J{}^t\bar{x}^{-1}J$ for x in $G' : \bar{x}$ is $(\bar{x}_{ij})$ if $x = (x_{ij})$, the bar indicating the action of the non-trivial element of the galois group $\mathrm{Gal}(E/F)$, and

$$J = \begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}.$$

Similarly we can introduce

$$H' = H(E) = \mathrm{GL}(2, E) \quad \text{and} \quad H = H(F) = \{g \text{ in } H'; \sigma g = g\}.$$

Here

$$\sigma x = w{}^t\bar{x}^{-1}w^{-1}, \quad w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \text{for } x \text{ in } H'.$$

Then $G = U(3)$, $H = U(2)$. Our notations are the same as in [U5], where the following smooth complex-valued functions are introduced.

(1) $'f = \otimes' f_v$ and $'\phi = \otimes' \phi_v$ are compactly supported on $H(\mathbf{A})$ ($\mathbf{A} = \mathbf{A}_F$ indicates the ring of adeles of F).

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(2) $f = \otimes f_v$ on $G(\mathbf{A})$ transforms under the center $Z(\mathbf{A})$ ($\simeq \mathbf{A}_E^1$: E-ideles of norm 1 in $\mathbf{A}_F^\times$) of $G(\mathbf{A})$ by a fixed character ω^{-1}, where ω is a character of $\mathbf{A}_E^1/E^1$ (as in [U5], $E^1 = \{x \in E;\, N_{E/F^\times} = 1\}$); f is compactly supported modulo $Z(\mathbf{A})$.

(3) $\phi = \otimes \phi_v$ is a function on $G'(\mathbf{A}) = G(\mathbf{A}_E)$ which transforms under the center $Z'(\mathbf{A}) = Z(\mathbf{A}_E)$ ($\simeq \mathbf{A}_E^\times$) of $G'(\mathbf{A})$ by ω'^{-1}, where $\omega'(x) = \omega(x/\bar{x})$, x in the group $A_E^\times$ of ideles.

The local components of $'f$, f, ϕ, $'\phi$ are taken to be matching, namely their orbital integrals are related in a certain way, specified in [U5].

Our purpose here is to prove the following:

Theorem. *Let $'f, f, \phi, '\phi$ be matching. Then we have the following identity of trace formulae:*

$$\sum_{\Pi} m(\Pi) \operatorname{tr} \Pi(\phi \times \sigma) - \frac{1}{2} \sum_{\{\rho\}} n(\rho) \operatorname{tr}\{\rho\}('\phi)$$

$$= \sum_{\pi} m(\pi) \operatorname{tr} \pi(f) - \frac{1}{2} \sum_{\{\rho\}} n(\rho) \operatorname{tr}\{\rho\}('f).$$

Some applications of this identity are discussed in [U5]. They concern a definition and description of packets of admissible [BZ] and automorphic [Av], [BJ] representations of $G = U(3)$, and their relations with those of $H = U(2)$ and $G' = \mathrm{GL}(3, E)$. This includes a proof of multiplicity one and rigidity theorems for packets of discrete series automorphic representations of $G = U(3)$. The work of [U5] is rather involved; it uses, for example, Arthur's explicit computations [Ar] of the trace formula, Kottwitz's base-change lemma [Ko2], Kazhdan's pseudo-coefficients [K] (and so the Bernstein center [BD]), the rigidity theorem of Piatetski-Shapiro and Jacquet–Shalika [JS], the Casselman–Deligne character formula [CD], and so on. In the present paper we isolate a single technical but essential tool, namely the identity of the Theorem, which is needed to establish the theory of [U5] in the greatest generality of *all* automorphic representations. Of course our work should be viewed in the context of the "Langlands program" [L1/2] and is likely to have interesting applications in the study of Shimura varieties [L2/3]. However, our method of proof is inspired by the use of the Iwahori algebra in [KL]; see also [W].

The present paper is a continuation, and completion of the global aspects, of the project [U]. The notations in this paper, and in particular in the statement of the Theorem, are the same as in [U3–5]. The sum over Π ranges over various automorphic σ-invariant $G'(\mathbf{A})$-modules (of [U3]; p. 136, *l*. -2, -3), and $m(\Pi)$ is 1 if Π is discrete-series, $\frac{1}{2}$ if $\Pi = I(\tau)$ and $\frac{1}{4}$, $-\frac{1}{8}$ or $-\frac{3}{8}$ if $\Pi = I(\eta)$. The π are automorphic $G(\mathbf{A})$-modules which may be discrete-series or induced and the $m(\pi)$ are integers, $\frac{1}{2}$ or $\frac{1}{4}$. The $\{\rho\}$ are automorphic $H(\mathbf{A})$-packets, and the

$n(\{\rho\}) = n(\rho)$ are again rational numbers, explicitly described in [U3] and [U5]. In [U3] we proved the Theorem under the additional assumption that two local components of (each of) $'f, f, \phi, '\phi$ are elliptic (= discrete). The proof which we give in [S; IV] (in the more difficult context of the symmetric square lifting) suffices to establish the Theorem under the assumption that (only) one component of $'f, f, \phi, '\phi$ is elliptic, i.e. discrete. Our purpose here is to prove the Theorem unconditionally, and by a simple technique.

Trace identity as in the Theorem, for general test functions $f, \phi, \ldots$ on two (or more) groups $G, G', \ldots$, appears already in (Chapter 16 of) [JL]. But attention to the problem was drawn by Langlands' study [L4] of the first non-trivial case, namely the comparison needed for the completion of the cyclic base-change theory for GL(2), initiated by Saito [Sa] and Shintani [Sh]. Langlands proved the required identity for GL(2) on (1) computing the weighted orbital integrals and orbital integrals of singular classes which appear in the trace formulae, (2) analyzing the asymptotic behavior of the weighted integrals, (3) applying the Poisson summation formula, and so on. The correction argument which is introduced in [G2], §2.3.1, and [U1], simplifies the behavior of the weighted integrals, but not sufficiently to our liking; in [U2] we found it to be too long and complicated to be worth formalizing in our situation.

The method presented in this paper is entirely different. The principle is that it suffices to check the identity of the Theorem only for a small class of convenient test functions, and then use the fact that we deal with characters of representations to conclude that the identity holds in general. It is not necessary to deal with arbitrary $f, \phi, \ldots$ at the initial stage. In fact, it is shown below that for a suitable choice of test functions (whose definitions we leave to the text itself), the weighted orbital integrals and the orbital integrals at the singular classes are zero. In particular they need not be further computed and transformed. The proof turns out to be rather simple, once the right track is found. This solves the problem raised in [U3], p. 121, *l*. 3–4, to "abstractly establish the unconditional comparison of trace formulae".

It is clear that the present method applies also in the case considered in [L4] to yield a simple and short proof of the trace identity needed for the comparison of base-change for GL(2); applying our method in the context of the (easier) case considered in [L4] would be a constructive exercise for the interested reader.

The observation underlying our approach is that the subgroup $F^\times$ of rationals is discrete in the group $\mathbb{A}_F^\times$ of ideles. That this simple fact can actually be used to annihilate the undesirable terms in the trace formula was suggested byDrinfeld's use in [D] of spherical functions related to powers of the Frobenius, in the course of the work [FK2], [FK3] with D. Kazhdan, on the Ramanujan conjecture for automorphic forms with a supercuspidal component of GL(n) over a function field.

In the present paper admissible spherical functions are used to establish the Theorem in our simple approach. This technique is developed in [FK1] to establish the metaplectic and simple algebra correspondences in the context of cusp forms with a single supercuspidal component. Since this paper was written, we have developed a different variant of the approach. This variant is based on the use of regular (or Iwahori) functions; cf. [KL], [W]. This different technique is applied in [G1] to give a simple proof of base change for GL(2), in [S; VI] to prove the absolute form of the symmetric square lifting from SL(2) to PGL(3), and in [G3] to establish by simple means base change for cusp forms with a supercuspidal component on GL(n).

To complete this introduction we now sketch the proof, which is given below. We deal with four trace formulae for test functions f, ϕ, $'f$, $'\phi$, on the groups $G = U(3)$, $G' = \mathrm{GL}(3, E)$, $H = U(2)$ and again H. Put $\boldsymbol{q}$ for the quadruple $(f, \phi, 'f, '\phi)$. Each trace formula is an equality of distributions in the test function. These distributions are as follows. OI involves "good" orbital integrals, on the set of rational regular elliptic elements. WI involves "bad" orbital integrals, on the set of rational elements which are not regular elliptic; these "bad" integrals are mostly weighted and non-invariant as distributions in the test function. RD is a (discrete) sum of traces of automorphic representations; these occur with coefficients which may be negative when the representation is not cuspidal. RC is an integral (continuous sum) of traces of induced representations; these traces are often weighted, and the distributions which make up RC are mostly non-invariant. The trace formula takes the form $I = R$, where $R = \mathrm{RD} + \mathrm{RC}$ is the representation theoretic side, and $I = \mathrm{OI} + \mathrm{WI}$ is the geometric side of the formula.

We shall be interested in a linear combination of the four formulae. Put

$$\mathrm{RD}(\boldsymbol{q}) = [\mathrm{RD}(\phi) - \tfrac{1}{2}\mathrm{RD}('\phi)] - [\mathrm{RD}(f) - \tfrac{1}{2}\mathrm{RD}('f)],$$

and introduce $\mathrm{OI}(\boldsymbol{q})$, $\mathrm{RC}(\boldsymbol{q})$ analogously. From now on *we always choose* the components of $\boldsymbol{q}$ to have matching orbital integrals. This choice implies the vanishing of $\mathrm{OI}(\boldsymbol{q})$. Hence

$$\mathrm{RD}(\boldsymbol{q}) = \mathrm{WI}(\boldsymbol{q}) - \mathrm{RC}(\boldsymbol{q}).$$

In these notations, the Theorem can be restated as follows.

Theorem. *For any quadruple $\boldsymbol{q}$ of matching functions we have* $\mathrm{RD}(\boldsymbol{q}) = 0$.

Fix a non-archimedean place u of F which splits in E. Then

$$G(F_u) = \mathrm{GL}(3, F_u), \quad G'(F_u) = \mathrm{GL}(3, F_u) \times \mathrm{GL}(3, F_u), \quad H(F_u) = \mathrm{GL}(2, F_u).$$

Fix a quadruple $\boldsymbol{q}^u = (f^u, \phi^u, 'f^u, '\phi^u)$ of the components outside u of $\boldsymbol{q}$. Put $\mathrm{RC}(\boldsymbol{q}_u)$ for $\mathrm{RC}(\boldsymbol{q}_u \otimes \boldsymbol{q}^u)$, where $\boldsymbol{q}_u = (f_u, \phi_u, 'f_u, '\phi_u)$. As the first step in the proof we explicitly construct for any f_u a quadruple $\boldsymbol{q}_u = \boldsymbol{q}(f_u)$ which has the property that $\mathrm{RC}(\boldsymbol{q}(f_u))$ depends only on the orbital integrals of f_u (see §2).

For the second step of the proof (see §1), we say that a function f'_u on $G(F_u)$ is *n_0-admissible* (for some $n_0 > 0$) if it is spherical and its orbital integrals on the split regular set vanish at a distance $\leqq n_0$ from the walls [namely, on the orbits with eigenvalues of valuations n_1, n_2, n_3 such that $|n_i - n_j|$ is at most n_0 for some $i \neq j\ (i, j = 1, 2, 3)$]. We prove: For any quadruple $\boldsymbol{q}^u$ of matching $f^u, \phi^u, 'f^u, '\phi^u$, which vanish on the adeles-outside-u orbits of the singular-rational elements, there exists an integer $n_0 = n_0(\boldsymbol{q}^u)$ such that $\mathrm{WI}(\boldsymbol{q}(f'_u)) = 0$ for every n_0-admissible f'_u. Note that in this case all of the components of $\boldsymbol{q}(f'_u)$ are spherical.

To prove this we show in the Proposition of §1 that, given f^u which vanishes on the $G(\mathbf{A}^u)$-orbits of the singular set in $G(F)$, there exists $n_0 = n_0(f^u) > 0$, such that for every n_0-admissible f'_u there exists a function f_u with the same orbital integrals as f'_u, with the property that $f^u \otimes f_u$ is zero on the $G(\mathbf{A})$-orbits of all "bad" rational elements. In particular $\mathrm{WI}(f^u \otimes f_u) = 0$. The function f_u is obtained by replacing f'_u by zero on a small neighborhood of finitely many split orbits where the orbital integral of f'_u is zero. Choosing n_0 sufficiently large, depending on $\boldsymbol{q}^u$, and noting that the construction of $\boldsymbol{q}(f_u)$ is such that its components are zero on the image of the split regular orbits where f_u is zero, we conclude that for every n_0-admissible f'_u there is f_u with orbital integrals equal to those of f'_u such that $\mathrm{WI}(\boldsymbol{q}(f_u)) = 0$. Consequently $\mathrm{WI}(\boldsymbol{q}(f'_u)) = 0$ for every n_0-admissible f'_u, since

$$\mathrm{WI}(\boldsymbol{q}(f_u)) = \mathrm{RD}(\boldsymbol{q}(f_u)) + \mathrm{RC}(\boldsymbol{q}(f_u))$$

and $\mathrm{RC}(\boldsymbol{q}(f_u))$ depends (by Step 1) only on the orbital integrals of f_u (which are equal to those of f'_u).

The third step asserts that since $\mathrm{RD}(\boldsymbol{q}(f'_u)) = -\mathrm{RC}(\boldsymbol{q}(f'_u))$ for every n_0-admissible f'_u, we have $\mathrm{RD}(\boldsymbol{q}(f'_u)) = \mathrm{RC}(\boldsymbol{q}(f'_u)) = 0$ for *every* spherical f'_u. This follows from the final Proposition in [FK1], where this claim is stated and proven in the context of an arbitrary p-adic group.

Fix a non-archimedean place u' of F. It follows from Step 3 that for any $\boldsymbol{q}_{u'}$ whose components vanish on the singular set, we have $\mathrm{RD}(\boldsymbol{q}^{u'} \otimes \boldsymbol{q}_{u'}) = 0$ for all $\boldsymbol{q}^{u'}$. The fourth step is to show that this holds for any spherical $\boldsymbol{q}_{u'}$. The proof is the same as in [U5], §1.2. Hence it will not be given in the text below, but we will recall the argument here.

Write $\mathrm{RD}(\boldsymbol{q})$ as a sum $\Sigma_\chi \mathrm{RD}(\boldsymbol{q}, \chi)$ over all infinitessimal characters χ, of the partial sums $\mathrm{RD}(\boldsymbol{q}, \chi)$ of $\mathrm{RD}(\boldsymbol{q})$ taken only over those automorphic representations whose infinitesimal character is χ. Since the archimedean components of $\boldsymbol{q}$ are arbitrary, a standard argument of "linear independence of characters" implies that since $\mathrm{RD}(\boldsymbol{q}) = 0$, for every χ we have $\mathrm{RD}(\boldsymbol{q}, \chi) = 0$ if $\boldsymbol{q}_{u'} = 0$ on the singular set. Fix $\boldsymbol{q}^{u'}$, and consider $\mathrm{RD}(\boldsymbol{q}, \chi)$ as a functional on the space of Iwahori quadruples $\boldsymbol{q}_{u'}$ (i.e., quadruples whose components are biinvariant under the standard Iwahori subgroups). There are only finitely many automorphic representations with a fixed infinitesimal character, fixed ramification at each finite place $\neq u'$, whose component at u' has a non-zero vector fixed under the action of an Iwahori subgroup. Hence as a functional in the Iwahori quadruple $\boldsymbol{q}_{u'}$, $\mathrm{RD}(\boldsymbol{q}, \chi)$ is a finite sum of characters. As it is zero on all $\boldsymbol{q}_{u'}$ which vanish on the singular set, it is easy to see that it is identically zero. In particular $\mathrm{RD}(\boldsymbol{q}, \chi)$ vanishes on the spherical quadruples $\boldsymbol{q}_{u'}$, from which the Theorem easily follows. This completes our outline of the proof of the Theorem.

§1. Conjugacy classes

Let v be a place of F. Denote by F_v the completion of F at v, and put $E_v = E \otimes_F F_v$. If v stays prime in E, then E_v/F_v is a quadratic field extension. If v splits into v', v'' in E, then $E_v = E_{v'} \times E_{v''}$, where $E_{v'} \simeq E_{v''} \simeq F_v$. In this case

$$G(E_v) = \mathrm{GL}(3, F_v) \times \mathrm{GL}(3, F_v),$$

and

$$G(F_v) = \{(g, \sigma g); g \text{ in } \mathrm{GL}(3, F_v)\} \simeq \mathrm{GL}(3, F_v).$$

Here $\sigma g = J{}^t g^{-1} J$, as $\mathrm{Gal}(E/F)$ maps $g = (g', g'')$ in $G(E_v)$ to $g = (g'', g')$. Let u be a fixed non-archimedean place of F which splits in E. Put $f^u = \otimes_{v \neq u} f_v$, where at each place $v \neq u$ of F we take the function f_v to be fixed. The component f_v is a locally constant function on $G_u = G(F_u) = \mathrm{GL}(3, F_u)$. We choose u such that the central character ω has an unramified component ω_u at u. Replacing ω by its product with an unramified (global) character we may assume that $\omega_u = 1$. Then $f_u(zg) = f_u(g)$ for g in G_u, z in the center Z_u of G_u, and f_u is compactly supported on $Z_u \backslash G_u$. Let $F(g, f_u) = \Delta(g)\Phi(g, f_u)$ be the normalized orbital integral of f_u in the notations of [U5]. Let R_u be the ring of integers in F_u. Put $K_u = G(R_u)$; it is a maximal compact subgroup of G_u. A spherical function is a K_u-biinvariant function. The theory of the Satake transform implies that a spherical f_u on G_u is determined by its orbital integral on the split set. Let $|\cdot|$ be the (normalized) valuation on F_u, put $q = q_u$ for the cardinality of the residue field of F_u, and val for the additive valuation, defined by $|a| = q^{-\mathrm{val}(a)}$ for a in $F_u^\times$. Let $\mathbf{n} = (n_1, n_2, n_3)$ be a triple of integers. Let f'_u be the spherical function on G_u for which $F(g, f'_u)$ is zero at the regular diagonal element $g = (a, b, c)$, unless up to conjugation and modulo the center we have $(\mathrm{val}\, a, \mathrm{val}\, b, \mathrm{val}\, c) = \mathbf{n}$, in which case we require $F(g, f'_u)$ to be equal to one. Embed $\mathbf{Z}$ in $\mathbf{Z}^3$ diagonally. The symmetric group S_3 on three letters acts on $\mathbf{Z}^3$. Denote by $\mathbf{Z}^3/S_3\mathbf{Z}$ the quotient space. Then f'_u depends only on the image of $\mathbf{n}$ in $\mathbf{Z}^3/S_3\mathbf{Z}$. We write $f'_u = f'_u(\mathbf{n})$ to indicate the dependence of f'_u on $\mathbf{n}$.

Definitions. (1) The function f_u on G_u is called *pseudo-spherical* if there exists a spherical function f'_u with $F(g, f_u) = F(g, f'_u)$ for all g in G_u. We write $f_u(\mathbf{n})$ for f_u if $f'_u = f'_u(\mathbf{n})$.

(2) Let n_0 be a non-negative integer. An element $\mathbf{n} = (n_1, n_2, n_3)$ of $\mathbf{Z}^3/S_3\mathbf{Z}$ is called *n_0-admissible* if $|n_i - n_j| \geqq n_0$ for all $i \neq j$; $i, j = 1, 2, 3$.

We also fix a place u' of F which stays prime in E such that $E_{u'}/F_{u'}$ is unramified, and a positive integer n'. Let $S = S(u', n')$ be the set of g in $G_{u'}$ which are conjugate to some diagonal matrix $(a, b, \bar{a}^{-1})$ with $|a|_{u'} = q_{u'}^{n'}$ (and $|b|_{u'} = 1$); a and b are elements of $E_{u'}^\times$. We shall assume from now on that the component $f_{u'}$ is a (compactly supported, locally constant) function on $G_{u'}$ such that $F(g, f_{u'})$ is the characteristic function of S. Since S is open and closed we may and do take $f_{u'}$ to be supported on S.

Proposition. *There exists an integer $n_0 \geqq 0$ depending on f^u, such that for any n_0-admissible $\mathbf{n}$ there is a pseudo-spherical $f_u = f_u(\mathbf{n})$ with the property that $f = f^u \otimes f_u$ satisfies the following. If γ lies in $G(F)$, x in $G(\mathbf{A})$, and $f(x^{-1}\gamma x) \neq 0$, then γ is elliptic regular.*

Proof. If $f_{u'}(x^{-1}\gamma x) \neq 0$ then γ lies in S, hence it is regular in $G_{u'}$, and also in G. If γ is not elliptic, then we may assume that it is the diagonal element $(a, b, \bar{a}^{-1})$ with a in $E^\times$ and b in $E^1 = \{b \text{ in } E^\times; b\bar{b} = 1\}$. Modulo the center we may assume that $b = 1$. Also we have $a\bar{a} \neq 1$. At u we have $a = (\alpha, \beta)$, with α, β in $F_u^\times$. Hence γ is $(\alpha, 1, \beta^{-1})$ in G_u. Since f^u is fixed, there are $C_v \geqq 1$ for all $v \neq u$, with $C_v = 1$ for almost all v, such that $C_v^{-1} \leqq |a|_v \leqq C_v$ for all $v \neq u$ if $f^u(x^{-1}\gamma x) \neq 0$ for some x in $G(\mathbf{A})$. Here $|a|_v = |N_{E/F}a|_v$. Since a lies in $E^\times$, and $N_{E/F}a$ in $F^\times$, the product formula on $F^\times$ implies that $|\alpha\beta|_u = |N_{E/F}a|_u = |a|_u$ lies between $C_u = \Pi_{v \neq u} C_v$ and C_u^{-1}. We take n_0 with $q_u^{n_0} > C_u$. Consider an n_0-admissible $\mathbf{n}$ and the spherical $f'_u = f'_u(\mathbf{n})$. If $f'_u(x^{-1}\gamma x) \neq 0$ for some x in G_u, then there is some $C'_u > 1$ such that $|\alpha|_u$ and $|\beta|_u$ are bounded between C'_u and ${C'}_u^{-1}$, so that a lies in the discrete set $E^\times$ and in a compact of $\mathbf{A}_E^\times$, hence in a finite set. Hence γ lies in finitely many conjugacy classes modulo the center; let $\gamma_1, \ldots, \gamma_t$ be a set of representatives. Put $\gamma_i = (\alpha_i, 1, \beta_i^{-1})$. By definition of f'_u, if $F(\gamma_i, f'_u) \neq 0$ then we have that $|\alpha_i\beta_i|$ or $|\alpha_i\beta_i|^{-1}$ is bigger than $q_u^{n_0}$, hence $f(x^{-1}\gamma_i x) = 0$ for all x and i. We conclude that $F(\gamma_i, f'_u) = 0$ for all i. Let S_i be the characteristic function of the complement of a small open closed neighborhood of the orbit of γ_i in G_u. Then the function $f_u = f'_u \Pi_i S_i$ on G_u has the required properties.

Let $L(G)$ denote the space of automorphic functions on $G(\mathbf{A})$; these are the slowly increasing functions on $G(F)\backslash G(\mathbf{A})$ which transform on $Z(\mathbf{A})$ by ω and are right invariant by some compact open subgroup; see [BJ] and [Av]. $G(\mathbf{A})$ acts on $L(G)$ by right translation: $(r(g)\psi)(h) = \psi(hg)$. Then r is an integral operator with kernel $K_f(x, y) = \Sigma_\gamma f(x^{-1}\gamma y)$, where γ ranges over $Z(F)\backslash G(F)$. In view of the Proposition, the integral of $K_f(x, y)$ on the diagonal $x = y$ in $G(\mathbf{A})/Z(\mathbf{A})$ is precisely the sum (2.2.1) of [U3], which is stabilized and analyzed in [U3], §2. The remarkable phenomenon to be noted is that for f with a component f_u as in the Proposition, the only conjugacy classes which contribute to the trace formula are elliptic regular. The weighted orbital integrals and the orbital integrals of the singular classes are zero, for our function f. Moreover, the truncation which is usually used to obtain the trace formula is trivial, for our f.

Each component ϕ_v of the function $\phi = \otimes \phi_v$ on $G'(\mathbf{A}) = G(\mathbf{A}_E)$ is taken to be matching f_v in the terminology of [U3], §3, and [U5]. In particular we take ϕ_u to be (f_u, f_u^0), where f_u^0 is as in [G2], p. 47, $l. -3$. Namely the pseudo-spherical f_u is biinvariant under some σ-invariant compact open subgroup I_u of G_u, where $\sigma(g) = J^t g^{-1} J$, and f_u^0 is taken to be the characteristic function of $Z_u I_u$, divided by the volume of $I_u Z_u / Z_u$. Then $f_u = f_u^0 * f^u = f^u * f_u^0$. An immediate twisted analogue of the proof of the Proposition establishes the following.

Proposition. *If* $\mathbf{n}$ *is* n_0*-admissible,* δ *lies in* $G(E)$*,* x *in* $G(\mathbf{A}_E)$ *and* $\phi(x^{-1}\delta\sigma(x)) \neq 0$*, then* $N\delta$ *is elliptic regular in* $G(F)$*.*

Here N denotes the norm map from the set of stable σ-conjugacy classes in $G(E)$ (and $G(\mathbf{A}_E)$) onto the set of stable conjugacy classes in $G(F)$ (and $G(\mathbf{A})$) (see [Ko1]). Again we can introduce the space $L(G')$ of automorphic functions on $G'(F) \backslash G'(\mathbf{A})$ which transform on $Z'(\mathbf{A})$ by ω' and the right action r' of $G(\mathbf{A}_E)$ on $L(G')$. Gal(E/F) acts on $L(G')$ by $(r'(\sigma)\psi)(g) = \psi(\sigma g)$. The operator $r'(\phi \times \sigma)$ is an integral operator with kernel $K_\phi(x, y) = \Sigma_\delta\, \phi(x^{-1}\delta\sigma(y))$ (δ in $Z(E) \backslash G(E)$). The Proposition shows that the integral of K_ϕ along the diagonal $x = y$ in $Z(\mathbf{A}_E) \backslash G(\mathbf{A}_E)$ is precisely the sum of [U3], p. 131, $l.$ 6, which is stabilized and discussed in [U3], §3.

The functions $'f$ and $'\phi$ on $H(\mathbf{A})$ are taken to be matching with f and ϕ, as defined in [U3], §§2–3. Their components at u can be taken to be pseudo-spherical, and the Proposition and its applications hold for $'f$ and $'\phi$ as well. There is no need to repeat the analogous discussion. It remains to consider the contribution to the trace formulae from the representation theoretic side.

§2. Intertwining operators

For brevity we denote by J the difference of the two sides in the equality of our theorem. Then J is the difference of the two sides in the equality of [U3], Proposition 4.4. The work of [U3], §§2–3, concerns the stabilization of the orbital integrals on the elliptic regular conjugacy classes which appear in the trace formulae. It implies that for arbitrary matching functions $'f, f, \phi, '\phi$ the difference J can be expressed as a sum of integrals of logarithmic derivatives of certain intertwining operators, which we momentarily describe. In [U3], (4.4), we conclude from this that $J = 0$ if the functions $f, \phi, \ldots$ have two elliptic (= discrete) components. To deal with the case of arbitrary $f, \phi, \ldots$ we now record an expression for J, as follows. The expression consists of four terms, one for each of $\phi, f, 'f, '\phi$. These are the terms involving integrals (over $i\mathbf{R}$) in the trace formulae. They are analogous to the terms (vi), (vii), (viii) of [JL], p. 517. We use the notations of [S; III], §3, which are the standard notations.

The term $J(\phi)$, from the twisted formula for G', is the sum of three expressions, equal to each other. As the coefficient $[W_0^M]/[W_0](\det(1 - s \times \sigma)|_{\mathscr{A}_M/\mathscr{A}})$ of [S; III], (3.1) (and [Ar], Thm 8.2, p. 1324) is $\frac{1}{12}$ (here $M = M_0$ is the diagonal subgroup A; the Lie algebra $\mathscr{A}$ is one-dimensional), we obtain

$$J(\phi) = \frac{1}{4} \sum_{\tau} \int_{i\mathbf{R}} \operatorname{tr}[\mathscr{M}(\lambda, 0, -\lambda) I_{P_0,\tau}((\lambda, 0, -\lambda); \phi \times \sigma)] d\lambda.$$

The sum is over all connected components (with representatives $\tau = (\mu_1, \mu_2, \mu_3)$) of characters of $A(\mathbf{A}_E)/A(E)$, with $\sigma\tau = \tau$. More precisely, let ν be the character $\nu(x) = |x|$ of $\mathbf{A}_E^\times$. Note that $A \simeq \mathbf{G}_m^3$. The connected component of τ consists of $\tau_\lambda = (\mu_1 \nu^\lambda, \mu_2, \mu_3 \nu^{-\lambda})$, λ in $i\mathbf{R}$. The μ_i are unitary characters of $\mathbf{A}_E^\times/E^\times$, and $\mu_1\mu_2\mu_3 = \omega'$. We put $I_{P_0,\tau}((\lambda, 0, -\lambda))$ for the $G(\mathbf{A}_E)$-module normalizedly induced from τ_λ; τ_λ is regarded as a character of the upper triangular subgroup P_0 which is trivial on the unipotent radical of P_0. The action of σ takes τ to $(\bar{\mu}_3^{-1}, \bar{\mu}_2^{-1}, \bar{\mu}_1^{-1})$, where $\bar{\mu}(x) = \mu(\bar{x})$. Hence $\sigma\tau = \tau$ implies $\tau = (\mu, \omega'\bar{\mu}/\mu, \bar{\mu}^{-1})$, where $\mu = \mu_1$.

The operator $\mathscr{M}$ is a logarithmic derivative of an operator $M = m \otimes_v R_v$, where R_v denotes a local normalized intertwining operator. The normalizing factor $m = m(\lambda) = m(\lambda, \tau)$ is an easily specified (see [S; III], (3.2)) quotient of L-functions, which has neither zeroes nor poles on the domain $i\mathbf{R}$ of integration. Then the logarithmic derivative $\mathscr{M}$ is

$$m'(\lambda)/m(\lambda) + (\otimes R_v^{-1}) \frac{d}{d\lambda} (\otimes R_v),$$

and we obtain $J(\phi) = J'(\phi) + \Sigma_v J_v(\phi)$, where

$$J'(\phi) = \frac{1}{4} \sum_{\tau} \int_{i\mathbf{R}} \frac{m'(\lambda)}{m(\lambda)} \left[\prod_v \operatorname{tr} I_{\tau_v}(\lambda; \phi_v \times \sigma) \right] d\lambda$$

and

$$J_v(\phi) = \frac{1}{4} \sum_{\tau} \int_{i\mathbf{R}} [\operatorname{tr} R_{\tau_v}(\lambda)^{-1} R_{\tau_v}(\lambda)' I_{\tau_v}(\lambda; \phi_v \times \sigma)] \cdot \prod_{w \neq v} \operatorname{tr} I_{\tau_w}(\lambda; \phi_w \times \sigma) d\lambda.$$

The abbreviated notations are standard. The sum over v is finite. It extends over the places v where ϕ_v is not spherical, since when ϕ_v is spherical the operator $I_{\tau_v}(\lambda; \phi_v \times \sigma)$ factors through the projection on the one-dimensional subspace (if τ_v is unramified) of $K_v = \mathrm{GL}(3, R_v)$-fixed vectors, on which $R_{\tau_v}(\lambda)$ acts as the scalar one, so that $R_{\tau_v}(\lambda)' = 0$.

Next we have to record the analogous term $J(f)$ of the trace formula for G. Again we use the notations of [S; III], (3.1), with $\sigma = 1$; this rank one non-twisted case is well-known (see [JL], pp. 516–517). We take $M = M_0$, and $\mathscr{A} = \mathscr{A}_M$ is one-dimensional. The element s of the Weyl group is $s = \mathrm{id}$; it lies in $W^{\mathscr{A}}(\mathscr{A}_M)$. The Weyl group W_0 has cardinality two, and $[W_0^M] = 1$, and $\mathscr{A}_M/\mathscr{A} = \{0\}$. Hence the coefficient of $J(f)$ is $\frac{1}{2}$, and

$$J(f) = \frac{1}{2} \sum_{\mu} \int_{i\mathbf{R}} \operatorname{tr} \mathscr{M}(\lambda) I(\mu \otimes \lambda; f) d\lambda.$$

The sum ranges over all connected components with representatives μ, where $\mu(a, b, \bar{a}^{-1}) = \mu(a/b)\omega(b)$. Here a lies in $\mathbf{A}_E^\times$, b in $\mathbf{A}_E^1$, μ is a character of $\mathbf{A}_E^\times/E^\times$, and the connected component of μ consists of $\mu \otimes \lambda$, where μ is replaced by $\mu\nu^\lambda$, for λ in $i\mathbf{R}$. The induced $G(\mathbf{A})$-module $I(\mu \otimes \lambda)$ lifts (see [U4], Lemma 1.4) to the induced $G(\mathbf{A}_E)$-module $I_\tau(\lambda)$, where $\tau = (\mu, \omega'\bar{\mu}/\mu, \bar{\mu}^{-1})$, and this relation defines a bijection $\mu \leftrightarrow \tau$ between the sets over which the sums of $J(\phi)$ and $J(f)$ are taken. Here $\mathscr{M}(\lambda)$ is again a logarithmic derivative of an operator $M = m \otimes_v R_v$, and $J(f)$ is the sum of $J'(f)$ and $\Sigma_v J_v(f)$, where

$$J'(f) = \frac{1}{2} \sum_{\mu} \int_{i\mathbf{R}} \frac{m'(\lambda)}{m(\lambda)} \left[\prod_v \operatorname{tr} I(\mu_v \otimes \lambda; f_v)\right] d\lambda$$

and

$$J_v(f) = \frac{1}{2} \sum_{\mu} \int_{i\mathbf{R}} \operatorname{tr}[R_{\mu_v}(\lambda)^{-1} R_{\mu_v}(\lambda)' I(\mu_v \otimes \lambda; f_v)] \cdot \prod_{w \neq v} \operatorname{tr} I(\mu_v \otimes \lambda; f_v) d\lambda.$$

Note that here the normalizing factors $m(\lambda)$ depend on μ, while those of $J'(\phi)$ depend on τ. It is clear (see [U4], Lemma 1.4) that for matching functions f_v and ϕ_v we have

$$\operatorname{tr} I(\mu_v \otimes \lambda; f_v) = \operatorname{tr} I_{\tau_v}(\lambda; \phi_v \times \sigma), \qquad \text{if } \tau_v = (\mu_v, \omega_u' \bar{\mu}_v/\mu_v, \bar{\mu}_v^{-1}).$$

It can be shown directly that $2m'(\lambda, \mu)/m(\lambda, \mu) = m'(\lambda, \tau)/m(\lambda, \tau)$, and hence that $J'(f) = J'(\phi)$, but we do not need this observation. The fundamental observation which we do require is the following.

Lemma. *For our choice of f_u and $\phi_u = (f_u, f_u^0)$ we have $J_u(\phi) = J_u(f)$.*

Proof. This is precisely Lemma 16, p. 47, of [G2], in the case $l = 2$. Note that the proof of this Lemma 16 is elementary and self-contained. To see that this Lemma 16 applies in our case, recall that we choose f_u^0 to be the characteristic function (up to a scalar multiple) of $Z_u I_u$, where I_u is a σ-invariant open compact subgroup of G_u. Then

$$f_u^0(\sigma g) = f_u^0(g), \quad {}^\sigma\pi_u(f_u^0) = \pi(f_u^0) \quad \text{and} \quad f_u = f_u * {}^\sigma f_u^0 = f_u * f_u^0$$

in the notations of [G2], (1.5.2), p. 42, l.7. In fact this Lemma 16 of [G2] asserts that

$$\operatorname{tr} R_{\tau_u}(\lambda)^{-1} R_{\tau_u}(\lambda)' I_{\tau_u}(\lambda; \phi_u \times \sigma) = l \operatorname{tr} R_{\mu_u}(\lambda)^{-1} R_{\mu_u}(\lambda)' I(\mu_u \otimes \lambda; f_u)$$

in our notations, where $l = 2$. This is precisely the factor needed to match the $\frac{1}{4}$ of $J_u(\phi)$ with the $\frac{1}{2}$ of $J_u(f)$. Our lemma follows.

It remains to deal with the terms of $J('f)$ and $J('\phi)$. Since this case of $U(2)$ is well-known (see [U1]) we do not write out the expressions here, but simply note the following.

(1) We may assume that the place u is such that the component κ_u of the character κ on $\mathbb{A}_E^\times / E^\times N \mathbb{A}_E^\times$ is unramified.

(2) We may and do multiply κ by an unramified (global) character to assume that $\kappa_u = 1$.

(3) If $'f_v$ and $'\phi_v$ are matching functions on H_v in the notations of [U3], and $\rho_v = 'I(\mu_v)$, $\rho_v' = 'I(\mu_v \kappa_v)$ in the same notations, then $\operatorname{tr} \rho_v('f_v) = \operatorname{tr} \rho_v'('\phi_v)$ by Lemma 1.4 in [U4].

(4) At the split place u we take the components $'f_u$ and $'\phi_u$ to be defined directly by the same formula ([U3], p. 130, $l. - 8$) in terms of f_u; they are equal to each other. We conclude:

Lemma. *In the above notations, we have* $J_u('f) = J_u('\phi)$.

Proof. This follows from (3) and (4). Note that the sets of μ parametrizing the sums which appear in $J('f)$ and $J('\phi)$ are isomorphic. The isomorphism $('I(\mu) \to 'I(\mu\kappa))$ is defined by the dual group diagram of [U4], (1.3), and [U4], Lemma 1.4, which is recorded also in [U5].

Remark. $J'('f)$ and $J'('\phi)$ are given by precisely the same formulae, hence they are equal to each other by (3). We do not use this remark below.

§3. Approximation

We conclude that for $f = f^u \otimes f_u$ with fixed f^u and $f_u = f_u(\mathbf{n})$ where $\mathbf{n}$ is n_0-admissible for some $n_0 = n_0(f^u)$, we have the identity

$$(3.1)\qquad J = J'(\phi) - J'(f) + J'('f) - J'('\phi) + \sum_v [J_v(\phi) - J_v(f) + J_v('f) - J_v('\phi)].$$

The sum over v is finite and ranges over $v \neq u$. On the left J represents a sum with complex coefficients (depending on f^u but not on f_u) of traces of the form $\operatorname{tr} \pi_u(f_u)$, $\operatorname{tr} \Pi_u(\phi_u \times \sigma)$, $\operatorname{tr}\{\rho_u\}('f_u)$ or $\operatorname{tr}\{\rho_u\}('\phi_u)$. This is an invariant distribution in f_u; it depends only on the orbital integrals of f_u. On the right we have a sum over the connected components (represented by μ_u) of the manifold of characters mentioned in §2, of integrals over $i\mathbf{R}$. The integrands are of the form $c(\lambda)\operatorname{tr} I(\mu_u \otimes \lambda; f_u)$. The right side of (3.1) is therefore also an invariant distribution in f_u, depending only on the orbital integrals of f_u. We conclude

Lemma. *The identity* (3.1) *holds with* $f_u = f_u(\mathbf{n})$ *replaced by the spherical* $f'_u = f'_u(\mathbf{n})$.

Proof. By definition $f_u(\mathbf{n})$ and $f'_u(\mathbf{n})$ have equal orbital integrals.

From now on we denote by f_u a spherical function of the form $f'_u(\mathbf{n})$ with n_0-admissible $\mathbf{n}$. The identity (3.1) holds for our $f = f^u \otimes f_u$. Since f_u is spherical, $\operatorname{tr} \pi_u(f_u) \neq 0$ only when π_u is unramified. The theory of the Satake transform establishes an isomorphism from the set of unramified irreducible G_u-modules π_u, to the variety $\mathbf{C}^{\times 3}/S_3$: The unordered triple $\mathbf{z} = (z_1, z_2, z_3)$ of non-zero complex numbers corresponds to the unramified subquotient $\pi_u(\mathbf{z})$ of the G_u-module $I_u(\mathbf{z})$ normalizedly induced from the unramified character $(a_{ij}) \to \Pi_i z_i^{\mathrm{val}(a_{ii})}$ of the upper triangular subgroup. The central character of $\pi_u(\mathbf{z})$ is trivial if and only if $z_1z_2z_3 = 1$. For z in $\mathbf{C}^\times$ and $\mathbf{z}$ in $\mathbf{C}^{\times 3}$ we write $\mathbf{z}Z$ for (z_1z, z_2, z_3z^{-1}). We conclude that there are $\mathbf{t}_i$ in $\mathbf{C}^{\times 3}/S_3$ $(i \geqq 0)$ and $\mathbf{z}_i$ in $\mathbf{C}^{\times 3}$ $(i \geqq 0)$ with $t_{i1}t_{i2}t_{i3} = 1$, $z_{i1}z_{i2}z_{i3} = 1$ and $|z_{ij}| = 1$; further there are complex numbers c_i, and integrable functions $c_i(z)$ on $|z| = 1$, such that (3.1) takes the form

$$(3.2)\qquad \sum_i c_i \operatorname{tr}(\pi_u(t_i))(f_u) = \sum_i \int_{|z|=1} c_i(z)\operatorname{tr}(\pi_u(\mathbf{z}_i z))(f_u)d^\times z.$$

The Satake transform $f_u \to f_u^\vee$, defined by $f_u^\vee(\mathbf{z}) = \operatorname{tr}(\pi_u(\mathbf{z}))(f_u)$, is an isomorphism from the convolution algebra of spherical functions f_u on G_u to the algebra of Laurent series $f_u^\vee$ of $\mathbf{z}$ in $\mathbf{C}^{\times 3}/S_3$ with $z_1z_2z_3 = 1$. Then (3.2) can be put in the form

$$(3.3)\qquad \sum_i c_i f_u^\vee(t_i) = \sum_i \int_{|z|=1} c_i(z) f_u^\vee(\mathbf{z}_i z)d^\times z.$$

Our aim is to show that $c_i = 0$ for all $i \geqq 0$. For that we note that all sums and products in the trace formula are absolutely convergent for any f_u, in particular for the function with $f_u^\vee = 1$. Hence $\Sigma_i |c_i|$ is finite, and $\Sigma_i \int |c_i(z)|\ |dz|$ is finite.

Moreover, let X be the set of $\mathbf{z}$ in $\mathbf{C}^{\times 3}/S_3$ with $z_1z_2z_3 = 1$, $\bar{\mathbf{z}}^{-1} = \mathbf{z}$, and $q^{-1} \leqq |z_i| \leqq q$ for each entry z_i of $\mathbf{z}$. Since all representations which contribute to the trace formula are unitary, the $\mathbf{t}_i$ and $\mathbf{z}_i z$ lie in X. But then the case where $n = 3$ of the final Proposition in [FK1], where the analogous problem is rephrased and solved for an arbitrary reductive group, implies that all c_i in (3.3) are zero, and the theorem follows.

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