FINAL REVIEW SHEET AND SOME HOMEWORK
03/07/08

FINAL is on Wed, 3/12/08, at 1:30PM in the classroom.

You already have the previous two review sheets. This review sheet covers material since second midterm. Sections 5.1, 5.2, 5.3, 10.1, 10.2, 10.3, 7.1, 7.2, 7.4.

DEFINITIONS:

1. $\in$, $\subseteq$, $\emptyset$, $\{x\}$, $\{x_1,...,x_n\}$, $\{x|P(x)\}$, $\{x \in A|P(x)\}$.
2. Proper subset, A = B, $\cup$, $\cap$, $-$, ^{c}.
3. Disjoint, mutually disjoint, partition.
4. Power set, ordered pairs, ordered tuples, Cartesian products.
5. Relation from A to B. x R y. Arrow diagram for relation.
6. Reflexive, symmetric, transitive.
7. Equivalence relation, partition, transitive closure.
8. Inverse of a relation, R^{-1}. Functions as relations. Directed graph of a relation.
9. $f:A \to B$. domain of a function. range of a function. co-domain of a function. Inverse image of a point under a function.
10. f = g. Sequences as functions.
11. f is one-one. $f:A \to B$ is onto. $f:A \to B$ is a bijection. Inverse of a bijection. gof (when well defined).

THEOREMS (know <u>***full***</u> statements):

1. Cardinality of power set of finite sets.
2. Correspondence between equivalence relations and partitions.
3. Composition of one-one functions is one-one. Composition of onto functions is onto. Composition of bijections is a bijection.
4. Composition of bijection with inverse is the identity (composition on left and right sides).

METHODS OF PROOF:

1. To show that A = B, show $A \subseteq B$ and $B \subseteq A$.
2. To show $A \subseteq B$, let $x \in A$ and want $x \in B$.
3. Use display on page 270. E.g., from $x \in A \cup B$, derive $x \in A \vee x \in B$. From $x \in A$, $x \in B$, derive $x \in A \cap B$. Etc.

KNOW HOW TO:

1. Determine whether a relation is reflexive, symmetric, or transitive.
2. Prove equations and inclusions, etc., about sets, using the element method.
3. Read definitions of relations and functions, paying particular attention to domains of functions.
4. Read/write arrow diagrams of relations and functions.
5. Read/write directed graphs of relations.
6. Tell whether a function is well defined. Determine inverses of functions. Determine compositions of functions.
7. Prove that a given function is one-one, or onto, or both.
8. Determine the transitive closure of a given relation.

SOME HOMEWORK

5.2. 13.
THEOREM. For all A,B,C, $A \subseteq B \rightarrow A \cup C \subseteq B \cup C$.

Proof: Let A,B,C be sets. Assume $A \subseteq B$. Want $A \cup C \subseteq B \cup C$.
Let $x \in A \cup C$. Want $x \in B \cup C$. Have $x \in A \vee x \in C$.
case 1. $x \in A$. Have $x \in B$. Have $x \in B \cup C$.
case 2. $x \in C$. Have $x \in B \cup C$.
QED

24.
THEOREM. For all A,B,C, $(A-C) \cap (B-C) \cap (A-B) = \emptyset$.

Proof: Let A,B,C be sets. Let $x \in (A-C) \cap (B-C) \cap (A-B)$. Want contradiction. Have $x \in A-C$, $x \in B-C$, $x \in A-B$. Have $x \in B$, $x \notin B$. Have contradiction. QED

33.
THEOREM. For all A,B,C, $B \cap C \subseteq A \rightarrow (C-A) \cap (B-A) = \emptyset$.

Proof: Let A,B,C be sets. Assume $B \cap C \subseteq A$. Want $(C-A) \cap (B-A) = \emptyset$. Let $x \in (C-A) \cap (B-A)$. Want contradiction. Have $x \in C$, $x \notin A$, $x \in B$. Have $x \in B \cap C$. Have $x \in A$. Have contradiction. QED

5.3. 16.
THEOREM. For all A,B, $\wp(A \cap B) = \wp(A) \cap \wp(B)$.

Proof: Let A,B be sets. Want $\wp(A \cap B) \subseteq \wp(A) \cap \wp(B)$. Let $x \in \wp(A \cap B)$. Want $x \in \wp(A) \cap \wp(B)$. Have $x \subseteq A \cap B$.

Since $A \cap B \subseteq A$, we have $x \subseteq A$. Since $A \cap B \subseteq B$, we have $x \subseteq B$. Have $x \in \wp(A)$, $x \in \wp(B)$. Have $x \in \wp(A) \cap \wp(B)$.

Want $\wp(A) \cap \wp(B) \subseteq \wp(A \cap B)$. Let $x \in \wp(A) \cap \wp(B)$. Want $x \in \wp(A \cap B)$. Have $x \in \wp(A)$, $x \in \wp(B)$. Have $x \subseteq A$, $x \subseteq B$. Have $x \subseteq A \cap B$ (easy to derive). Have $x \in \wp(A \cap B)$.

QED

31. Algebraic proof of $A-(A \cap B) = A-B$.
$A-(A \cap B) = A \cap (A \cap B)^c = A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) = \varnothing \cup (A \cap B^c) = A \cap B^c = A-B$.

10.1. 2a. 2 S 4 false, 4S3 true, $(4,4) \in S$, $(3,2) \notin X$. b. $\{(3,3),(4,3),(4,4),(5,3),(5,4)\}$.

10. a. true. b. false. c. false.

18. $S = \{(3,6), (4,4), (5,5)\}$. $S^{-1} = \{(6,3), (4,4), (5,5)\}$.

10.2. 4. not reflexive, since 1 R 1 is false. symmetric. not transitive, since 1 R 2, 2 R 1, yet 1 R 1 is false.

10. First step yields the new elements (0,2), (2,3), (3,0). Second step yields new elements (1,3), (2,2), (3,3). There are no new yields at the third step. So the transitive closure is {(0,0), (0,3), (1,0), (1,2), (2,0), (3,2), (0,2), (2,3), (3,0), (1,3), (2,2), (3,3)}.

22. Not reflexive because (0) G {0} is false. Not symmetric because (0,0) G (0), yet (0) G (0,0) is false. Transitive, because of transitivity of > on the natural numbers.

36. Not reflexive. No line is perp to itself. Not transitive. The x-axis is perp to the y-axis, the y-axis is perp to the x-axis, yet the x-axis is not perp to the x-axis. Symmetric, since if l_1 is perp to l_2, then l_2 is perp to l_1.

7.1. 2. a. dom(g) = {1,3,5}. co-dom(g) = {a,b,c,d}.
b. g(1) = b, g(3) = b. g(5) = b.
c. rng(g) = {6}.
d. No. Yes.
e. {1,3,5}. $\varnothing$.
f. {(1,b),(3,b),(5,b)}.

3. a. yes. b. no. c. no. d. yes. e. no.

10. There exist F,G such that f-G G-F. Take F:$\Re \to \Re$ given by f(x) = 0. Take G:$\Re \to \Re$ given by G(x) = 1.

34. There are four cases.
case 1. u ∈ A, u ∈ B. Then u ∈ A ∩ B, u ∈ A ∪ B. $X_{A\cap B}(u)$ = 1, $X_A(u)$ = 1, $X_B(u)$ = 1, $X_{A\cup B}(u)$ = 1. Equations check.
case 2. u ∈ A, u ∉ B. Then u ∉ A ∩ B, u ∈ A ∪ B. $X_{A\cap B}$ = 0, $X_A(u)$ = 1, $X_B(u)$ = 0, $X_{A\cup B}(u)$ = 1. Equations check.
case 3. u ∉ A, u ∈ B. Then u ∉ A ∩ B, u ∈ A ∪ B. $X_{A\cap B}(u)$ = 0, $X_A(u)$ = 0, $X_B(u)$ = 1, $X_{A\cup B}(u)$ = 1. Equations check.
case 4. u ∉ A, u ∉ B. Then u ∉ A ∩ B, u ∉ A ∪ B. $X_{A\cap B}(u)$ = 0, $X_A(u)$ = 0, $X_B(u)$ = 0, $X_{A\cup B}(u)$ = 0. Equations check.

7.2. 8. a. H not one-one. H(b) = H(c). H not onto. x not hit.
b. K one-one. K(a),K(b),K(c) all different. K not onto. z not hit.

13. a. (i). g one-one. Let 4n-5 = 4m-5. Then 4(n-m) = 0. Hence n = m. (ii) g not onto. 0 not hit. Let 4n-5 = 0. Then n = 5/4 ∉ Z. b. yes. Let y ∈ $\Re$. Then 4x-5 = y, where x = x/4 ∈ $\Re$.

23. a. No. D(10) = D(1100) = 0. b. Yes. Let n ∈ Z. If n = 0, let x = 10. If n > 0, let x = 1...1, where there are n 1's. If n < 0, let s = 0...0, where there are n 0's.

26. a. C is one-one. Let ax = at. Then s = t, by peeling off a from the front of both sides. b. C not onto. The string b is not hit.

7.4. 2. gof:X → X defined by
gof(1) = g(f(1)) = g(3) = 1.
gof(3) = g(f(3)) = g(1) = 1.
gof(5) = g(f(5)) = g(5) = 1.

17. No. Set f:**Z** → **Z**, g:**Z** → **Z**, where f(x) = 2x, and g(x) = if x even x/2 else 0. Then gof = $i_{\mathbf{Z}}$, which is onto, but f is not onto.

20.
THEOREM. Let f:W → X, g:X → Y, h:Y → Z. Then ho(gof) = (hog)of.

Proof: Let f:W → X, g:X → Y, h:Y → Z. Want ho(gof) = (hog)of. Want (∀x ∈ X)((ho(gof))(x) = ((hog)of)(x)). Let x

$\in$ X. Want $((h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$. Have $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Have $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. Have $((h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$. QED

24. $g \circ f:\Re \to \Re$ is defined by $(g \circ f)(x) = g(f(x)) = g(x+3) = -x-3$. $(g \circ f)^{-1}:\Re \to \Re$ is defined by $(g \circ f)^{-1}(x)$ is the unique y $\in \Re$ such that $-y-3 = x$, which is $-x-3$. $g^{-1}:\Re \to \Re$ is defined by $g^{-1}(x)$ is the unique $y \in \Re$ such that $-y = x$, which is $-x$. $f^{-1}:\Re \to \Re$ is defined by $f^{-1}(x)$ is the unique y $\in \Re$ such that $y+3 = x$, which is $x-3$. $f^{-1} \circ g^{-1}:\Re \to \Re$ is defined by $f^{-1} \circ g^{-1}(x) = f^{-1}(g^{-1}(x)) = f^{-1}(-x) = -x-3$. In this case, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

26.
THEOREM. Let $f:X \to Y$ and $g:Y \to Z$ be bijections. Then $(g \circ f)^{-1}:Z \to X$ is a bijection, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: Let $f:X \to Y$ and $g:Y \to Z$ be bijections. Then $g \circ f:X \to Z$ is a bijection by book. Hence $(g \circ f)^{-1}:Z \to X$ is a bijection by book. $f^{-1}:Y \to X$ and $g^{-1}:Z \to Y$ are bijections by book. Hence $f^{-1} \circ g^{-1}:Z \to X$ is a bijection by book. To verify the last claim, let $x \in Z$. Want $(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$. Have $(g \circ f)^{-1}(x)$ = the unique $y \in X$ such that $(g \circ f)(y) = x$, which is the unique $y \in X$ such that $g(f(y)) = x$. Have $f^{-1} \circ g^{-1}(x) = f^{-1}(g^{-1}(x))$. We claim that $f^{-1}(g^{-1}(x))$ is also a $y \in X$ such that $g(f(y)) = x$. This is because $g(f(f^{-1}(g^{-1}(x))) = g(g^{-1}(x)) = x$. Have $(g \circ f)^{-1}(y) = f^{-1}(g^{-1}(x))$. QED