SECOND MIDTERM REVIEW
TEST: February 18, 2008

Test covers 3.1, 3.2, 3.3, 3.4, 3.6, 4.1, 4.2, 4.3, 4.4.

DEFINITIONS and THEOREMS:

1. n is divisible by d, d|n, n is a multiple of d. etc.
2. Unique factorization theorem. Standard factored form.
3. Quotient Remainder Theorem. Parity.
4. Finite and infinite sequences.
5. Summation notation. Product notation. Factorial.
6. The method of ordinary induction - basis case, induction hypothesis, induction conclusion.
7. Geometric sequence and its sum.
8. The method of strong induction - basis case, strong induction hypothesis, strong induction conclusion.
9. Well-ordering principle for integers.
10. Binary integer representation theorem.

KNOW HOW TO

1. Use the quotient remainder theorem to divide into cases and give a proof by division into cases.
2. Properly define infinite and finite sequences, being careful about the indices. Know how to work with $(-1)^n$.
3. Evaluate simple finite sums in summation notation. Same with products.
4. Make change of variable as in class or on page 208.
5. Use properties of summations and products.
6. Work with factorials.
7. Know how to use and explain the template for induction proofs (see below). Basis, Induction Hypothesis, Induction Conclusion format.
8. Use ordinary induction to prove equalities, inequalities, existence statements, etc.
9. Use Have and Want statements in proofs; e.g., when simplifying inequalities.
10. Put mathematical statements in logical notation.
11. Argue by contraposition. Optional, since you can always replace this by an argument by contradiction.

METHODS OF PROOF:

1. Suppose you Want (to prove) a universal statement. Let the universal quantifier(s) be given and Want (to prove) the inside.

2. Suppose you Want (to prove) an existential statement. Give an example for the existential quantifier(s), and Want (to prove) the inside. The example may involve variables.
3. Suppose you Want (to prove) an implication. Assume the left side and Want (to prove) the right side.
4. Suppose you Want (to prove) an equivalence (↔, iff). Assume the left side and Want (to prove) the right side. Separately, assume the right side and Want (to prove) the left side.
5. Suppose you Want (to prove) a conjunction. Want (to prove) each conjunct.
6. Suppose you Want (to prove) a disjunction. Want (to prove) one of the disjuncts.
7. Suppose you Want (to prove) a negation. Assume the statement and Want (to prove) a contradiction.

8. Suppose you already Have a universal statement. To use it, give a value for the universal quantifier(s) and Have (use) the inside.
9. Suppose you already Have an existential statement. To use it, choose a new letter for the existential quantifier(s), and Have (use) the inside.
10. Suppose you already Have an implication. To use it, suppose also that you already Have the left side. Then you Have (use) the right side.
11. Suppose you already Have an equivalence (↔, iff). To use it, Have (use) the forward and backward implications.
12. Suppose you already Have a conjunction. To use it, Have (use) each conjunct.
13. Suppose you already Have a disjunction. To use it, divide into cases according to each disjunction. Have (use) each case separately (proof by cases).
14. Suppose you already Have a negation. To use it, suppose also that you also Have the inside. Then you Have a contradiction. Usually, you will have Want contradiction, and are done. In fact, you are allowed to conclude anything from a contradiction.

15. Proof by contradiction should be used when all else fails. Suppose you Want A. Assume ¬A, and obtain a contradiction.

16. Proof by contraposition. Suppose you Want (to prove) A → B. You can assume ¬B and instead Want ¬A.

17. Ordinary Induction template. Suppose you Want (∀n ≥ a)(P(n)), where n,a stand for integers.
Basis case. n = a. Want P(a).
Induction Hypothesis. Assume (have) P(n), n ≥ a.
Induction Conclusion. Want P(n+1).
Done!

18. Strong Induction template. Suppose you Want (∀n ≥ a)(P(n)), where n,a stand for integers.
Basis case. Want P(a),...,P(b).
Strong Induction Hypothesis. Assume (have) n > b, (∀k)(a ≤ k < n → P(k)).
Induction Conclusion. Want P(n).
Done!

SOME HOMEWORK
10/30/07

3.1. 5. True. Example: n = 1, m = -1.
12. False. Counterample: n = 1. Want (1-1)/2 is not odd. Want 0 is not odd. Algebra.
27. THEOREM. (∀n,m ∈ Z)(n odd ∧ m odd → n+m even).

Proof: Let n,m ∈ Z. Let n odd ∧ m odd. Want n+m even. Then n odd, m odd. Let n = 2p+1, p ∈ Z. Let m = 2q+1, q ∈ Z. Then n+m = 2p+1+2q+1 = 2p+2q+2 = 2(p+q+1), which is even. QED

54. THEOREM. (∀n ∈ Z)(n^2 - $(n+1)^2$ is odd).

Proof: Let n ∈ Z. Want n2 - (n+1)2 is odd. Want n2 - (n2 + 2n + 1) is odd. Want -2n -1 is odd. Want 2(-n-1)+1 is odd. 2(-n-1)+1 is odd by the definition of odd. QED

3.2. 10. Because the numerator is an integer and the denominator is an integer ≠ 0.

21. THEOREM. (∀a ∈ Z)(a odd → a2 + a even).

Proof: Let a ∈ Z. Assume a odd. Want a^2 + a even. Let a = 2b+1, b ∈ Z. Want $(2b+1)^2$ + 2b+1 even. Want $4b^2$ + 4b + 1 + 2b + 1 even. Want $4b^2$ + 6b + 2 even. Want 2(b^2 + 3b + 1) even. 2(b^2 + 3b +1) even by the definition of even. QED

24. Yes. (ax+b)/(cx+d) = 1. sx+b = cx+d. (a-c)x = d-b. x = (d-b)/(a-c), which makes sense because a ≠ c.

3.3. 16. All variables range over integers.
THEOREM. $(\forall a,b,c)(a|b \wedge a|c \rightarrow a|b-c)$.

Proof: Let $a,b,c \in Z$. Assume $a|b \wedge a|c$. Want $a|b-c$. Have $a|b$, $a|c$. Have $(\exists x)(b = ax)$, $(\exists x)(c = ax)$. Let $b = ax$, $c = ay$. Then $b-c = ax-ay = a(x-y)$. By definition of |, $a|b-c$. QED

25. False. $4|2\bullet 2$, but $4|2$ is false.

26. All variables range over integers

THEOREM. $(\forall a,b)(a|b \rightarrow a^2|b^2)$.

Proof: Let $a,b \in Z$. Assume $a|b$. Want $a^2|b^2$. Let $b = ax$, by definition of |. Then $b^2 = a^2\bullet x^2$ by algebra. Hence $a^2|b^2$ by the definition of |. QED

39. $6! = 6\bullet 5\bullet 4\bullet 3\bullet 2 = 2\bullet 3\bullet 5\bullet 2\bullet 2\bullet 3\bullet 2 = 2^4\bullet 3^2\bullet 5$.
$20! =$
$20(19)(18)(17)(16)(15)(14)(13)(12)(11)(10)(9)(8)(7)(6)(5)(4)(3)(2) =$
$(5)(2)(2)(19)(2)(3)(3)(17)(2^4)(5)(3)(7)(2)(13)(2)(2)(3)(11)(2)(5)(3)(3)(2^3)(7)(2)(3)(5)(2)(2)(3)(2) =$
$(2^{18})(3^8)(5^4)(7^2)(11)(13)(17)(19)$.
$(20!)^2 = (2^{36})(3^{16})(5^8)(7^4)(11^2)(13^2)(17^2)(19^2) =$
$(10^8)(2^{28})(3^{16})(7^4)(11^2)(13^2)(17^2)(19^2)$. 8 zeroes.

3.4. 6. $-27 = (-4)\bullet 8 + 5$.

26. All variables range over integers.
THEOREM. $(\forall n)(n^2 - n + 3$ odd).

Proof: Let $n \in Z$. Want $n^2 - n + 3$ odd. By QRT, let $n = 2q+r$, $r = 0 \vee r = 1$.

case 1. $r = 0$. Have $n = 2q$. Have $n^2 - n + 3 = 4q^2 - 2q + 3 = 2(2q^2 - q +1) + 1$. Have $n^2 - n + 3$ is odd by definition of odd.

case 2. $r = 1$. Have $n = 2q+1$. Have $n^2 - n + 3 = 4q^2 + 4q + 1 -(2q + 1) + 3 = 2(2q^2 + q + 1) + 1$. Have $n^2 - n + 3$ is odd by definition of odd.

QED

30. All variables range over integers.

THEOREM. $(\forall n)(\exists k)(n(n+1) = 3k \vee n(n+1) = 3k+2)$.

Proof: Let $n \in Z$. Want $(\exists k)(n(n+1) = 3k \vee n(n+1) = 3k+2)$. By QRT, let $n = 3q+r$, $r = 0 \vee r = 1 \vee r = 2$.

case 1. $r = 0$. Then $n = 3q$. Hence $n(n+1) = (3q)(3q+1) = 3(q(3q+1))$. Hence $(\exists k)(n(n+1) = 3k)$.

case 2. $r = 1$. Then $n = 3q+1$. Hence $n(n+1) = (3q+1)(3q+2) = 9q^2 + 9q + 2 = 3(3q^2 + 3q) + 2$. Hence $(\exists k)(n(n+1) = 3k+2)$.

case 3. $r = 2$. Then $n = 3q+2$. Hence $n(n+1) = (3q+2)(3q+3) = 9q^2 + 12q + 6 = 3(3q^2 + 4q + 2)$. Hence $(\exists k)(n(n+1) = 3k)$.

QED

3.6. 4. All variables range over Z.
THEOREM. $(\forall m)(\neg\ 7|4m+4)$.

Proof: Let $m \in Z$. Suppose $7|7m+4$. Want contradiction. Let $7m+4 = 7q$. Then $7(m-q) = -4$. Hence $-4/7 \in Z$. This is a contradiction by algebra. QED

9.
THEOREM. $(\forall x,y \in \mathfrak{R})(x \notin Q \wedge y \in Q \rightarrow x-y \notin Q)$.

Proof: Let $x,y \in \mathfrak{R}$. Assume $x \notin Q \wedge y \in Q$. Want $x-y \notin Q$. Assume $x - y \in Q$. Want contradiction. Have $x \notin Q$, $y \in Q$. Have $(x-y)+y \in Q$, since sum of two rationals is rational by book. Hence $x \in Q$. Contradiction, since we have $x \notin Q$. QED

18.
THEOREM. $(\forall x,y \in \mathfrak{R})(x+y < 50 \rightarrow x < 25 \vee y < 25)$.

Proof: Let $x,y \in \mathfrak{R}$. Assume $x + y < 50$. Want $x < 25 \vee y < 25$. Assume $\neg(x < 25 \vee y < 25)$. Then $\neg\ x < 25 \wedge \neg\ y < 25$ by Chapter 1. Hence $\neg\ x < 25$, $\neg\ y < 25$. Hence $x \geq 25$, $y \geq 25$ by algebra. Hence $x+y \geq 50$ by algebra. Contradiction, since we Have $x+y < 50$. QED

22.
THEOREM. $(\forall x \in \mathfrak{R})(x \notin Q \rightarrow 1/x \notin Q)$.

Proof: Let $x \in \mathfrak{R}$. Assume $x \notin Q$. Want $1/x \notin Q$. Assume $1/x \in Q$. Want contradiction. Have $1/(1/x) \in Q$ by "division of rationals is a rational assuming denominator is nonzero, book". Hence $x \in Q$ by algebra. Have contradiction, since have $x \notin Q$. QED

4.1. 22. $(-1)^0 \bullet (-1)^1 \bullet (-1)^2 \bullet (-1)^3 \bullet (-1)^4$ - 1•-1•1•-1 = 1.

39. ∑ from j = 1 through n of j/(j+1)!.

56. j = i-1. i = j+1. ∑ from j+1 = 1 through j+1 = n-1 of $(j+1)/(n-j-1)^2$. ∑ from j = 0 through j = n-2 of $(j+1)/(n-j-1)^2$.

4.2. 7. All variables range over Z.
THEOREM. (∀n ≥ 1)(1 + 6 + ... + (5n-4) = n(5n-3)/2).

Proof: By induction on n ≥ 1.

Basis. n = 1. Want 1 = 1(5-3)/2. By algebra.

IH. 1 + 6 + ... + (5n-4) = n(5n-3)/2, n ≥ 1.

IC. 1 + 6 + ... + (5n-4) + (5n+1) = (n+1)(5n+2)/2.

Have 1 + 6 + ... + (5n-4) + (5n+1) = n(5n-3)/2 + 5n+1 = $(5n^2 - 3n + 10n + 2)/2$ = $(5n^2 + 7n + 2)/2$ - (n+1)(5n+2)/2 by algebra. QED.

14. All variables range over integers.
THEOREM. For all n ≥ 0, ∑ from i = 1 through i = n+1 of $i \bullet 2^i = n \bullet 2^{n+2} + 2$.

Proof: By induction on n ≥ 0.

Basis. n = 0. Want $1 \bullet 2^1 = 0 \bullet 2^2 + 2$. By algebra.

IH. ∑ from i = 1 through i = n+1 of $i \bullet 2^i = n \bullet 2^{n+2} + 2$, n ≥ 0.

IC. ∑ from i = 1 through i = n+2 of $i \bullet 2^i = (n+1) \bullet 2^{n+3} + 2$.

Have ∑ from i = 1 through i = n+2 of $i \bullet 2^i$ = (∑ from i = 1 through i = n+1 of $i \bullet 2^i + (n+2) \bullet 2^{n+2} = n \bullet 2^{n+2} + 2 + (n+2) \bullet 2^{n+2} = (2n+2) \bullet 2^{n+2} + 2 = (n+1) \bullet 2^{n+3} + 2$ by algebra. QED

20. 5 + 10 + ... 300 = 5(1 + 2 + ... + 60) = 5(60(61)/2) = 5(30)(61).

28. $1 - 2 + 2^2 - 2^3 + ... + (-1)^n 2^n$, n a positive integer. This is the sum of the geometric series with base r = -2 from 0 through n. Hence by book this is $((-2)^{n+1} - 1)/(-2 - 1) =$
$((-1)^{n+1}2^{n+1} - 1)/-3 = (1-(-1)^{n+1}2^{n+1})/3$.

4.3. 7. a. P(2) is $2^2 < (2+1)!$. True.
b. P(k) is $2^k < (k+1)!$.
c. P(k+1) is $2^{k+1} < (k+2)!$.
d. c must be shown using b and k ≥ 2.

10. All variables range over integers. This could also be proved using QRT.
THEOREM. $(\forall n \geq 0)(3|n^3 - 7n + 3)$.

Proof: By induction on n ≥ 0.
Basis. n = 0. Want $3|0^3 - 7(0) + 3)$. By algebra.

IH. Assume $3|n^3 - 7n + 3$, n ≥ 0.

IC. Want $3|(n+1)^3 -7(n+1) + 3$.

Want $3|n^3 + 3n^2 + 3n + 3 -7n - 7 + 3$. By IH, want $3|n^3 - 7n + 3 - n^3 - 3n^2 - 3n - 3 + 7n + 7 - 3$. Want $3|-3n -3n^2 -3$. Want $3|3(-n - n^2 - 3)$. Have $3|3(-n - n^2 - 3)$ by the definition of |. QED

17. All variables range over integers.
THEOREM. $(\forall n \geq 0)(1+3n \leq 4^n)$.

Proof: By induction on n ≥ 0.

Basis. n = 0. Want $1+3(0) \leq 4^0$. By algebra.

IH. Assume $1 +3n \leq 4^n$, n ≥ 0.

IC. Want $1 + 3(n+1) \leq 4^{n+1}$.

Want $1 + 3n+3 \leq 4^{n+1}$. Want $3n + 4 \leq 4^{n+1}$. Have $3n+4 \leq 4^n + 3$, by algebra. Want $4^n+3 \leq 4^{n+1}$. Want $4^n+3 \leq 4(4^n)$. Want $3 \leq 3(4^n)$. Want $1 \leq 4^n$. By algebra, since n ≥ 0. QED

29. By induction on the number of business people, n ≥ 1. Basis case is n = 1. 1 person, no handshakes, and 1(1-1)/2 = 0. Induction hypothesis is that for n business people, n(n-1)/2 handshakes, n ≥ 1.

Now we have n+1 business people. We want to show that the number of handshakes is (n+1)(n)/2. The number of handshakes with n+1 business people is the number for n business people, plus n more handshakes. This is n(n-1)/2 + n = $(n^2-n+2n)/2 = (n^2+n)/2$ = (n+1)(n)/2. QED

4.4. 3.
THEOREM. (∀n ≥ 0)(c_n is even).

Proof: By strong induction on n ≥ 0.
1. Basis. Want c0,c1,c2 are even. Want 2,2,6 even. Algebra.
2. SIH. Assume (∀i)(0 ≤ i < n → c_i even), n > 2.
3. SIC. Want c_n even.
Since n ≥ 3, by definition we have $c_n = 3c_{n-3}$. Since 0 ≤ n-3 < n, by SIH we have cn-3 is even. Hence $3c_{n-3}$ is even. Have c_n even. QED

9.
THEOREM. (∀n ≥ 1)($a_n \leq (7/4)^n$).

Proof: By strong induction on n ≥ 1.
Basis. Want $1 \leq (7/4)^1$, $2 \leq (7/4)^2$. Want 1 ≤ 1, 2 ≤ 49/16. Algebra.
SIH. Assume (∀i)(1 ≤ i < k → $a_i \leq (7/4)^i$), k > 2.
SIC. Want $a_k \leq (7/4)^k$.
Since k ≥ 3, by definition we have $a_k = a_{k-1} + a_{k-2}$. Since 1 ≤ k-1,k-2 < k, by SIH we have ak-1 ≤ $(7/4)^{k-1}$, $a_{k-2} \leq (7/4)^{k-2}$. Have $a_k = a_{k-1} + a_{k-2} \leq (7/4)^{k-1} + (7/4)^{k-2}$. Want $(7/4)^{k-2} + (7/4)^{k-1} \leq (7/4)^k$. Want $1 + 7/4 \leq (7/4)^2$. Want 11/4 ≤ 49/16. Algebra. QED

19.
THEOREM. (∀n > 1)(n is prime or a product of primes).

Proof: By the well ordering principle (could use strong induction instead). Assume ~(∀n > 1)(n is prime or a product of primes). Want contradiction. Have (∃n > 1)(~(n is prime or a product of primes)). Let S = {n > 1| ~(n is a prime or a product of primes)}. Have S is nonempty. By the well ordering principle, let n be the least element of S. Then n > 1 and (∀i)(1 < i < n → i is prime or a product of primes).
case 1. n is prime. Nothing left to prove.
case 2. n is not prime. Let n = a•b, where 1 < a,b < n). Have a is prime or a product of primes, and b is prime or a product of primes. Hence n = a•b is a product of primes. QED

24. No. Define $P(n) \leftrightarrow n \le 3 \vee 3|n$.

5.
THEOREM. $(\forall n \ge 0)(e_n \le 3^n)$.

Proof: Basis. Want $e_0 \le 3^0$, $e_1 \le 3^1$, $e_2 \le 3^2$. Algebra.
SIH. Assume $n > 2$, $(\forall k)(0 \le k < n \rightarrow e_k \le 3^k)$.
SIC. $e_n \le 3^n$.
since $n \ge 3$, we have $e_n = e_{n-1} + e_{n-2} + e_{n-3}$. Hence $e_n \le 3^{n-1} + 3^{n-2} + 3^{n-3} \le 3^{n-1} + 3^{n-1} + 3^{n-1} = 3^n$.
QED