

Adjacent Ramsey Theory
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This concerns Special FOM e-Mail Postings number 36 and 37.

Let $k \geq 2$ and $f: N^k \rightarrow [1, k]$ and $n \geq 1$ be such that there is no $x_1 < \dots < x_{k+1} \in n$ such that $f(x_1, \dots, x_k) = f(x_1, \dots, x_{k+1})$. Then we want to find $g: N^{k+1} \rightarrow [1, 3]$ such that there is no $x_1 < \dots < x_{k+2} \in n$ such that $g(x_1, \dots, x_{k+1}) = g(x_2, \dots, x_{k+2})$. This reduces adjacent Ramsey in k dimensions with k colors to adjacent Ramsey in $k+1$ dimensions with 3 colors.

Let $x_1 < \dots < x_{k+1} \in n$ be given. Look at t_1, \dots, t_{k+1} , where $t_i = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+1})$.

case 1. t_1, \dots, t_{k+1} all = or strictly monotone. Equality violates the hypotheses, whereas strict monotonicity is impossible because f goes into $[1, k]$.

case 2. first i such that t_i, t_{i+1}, t_{i+2} not strictly monotone is even. Set $g(x_1, \dots, x_{k+1}) = 1$.

case 3. first i such that t_i, t_{i+1}, t_{i+2} is odd and $t_i < t_{i+1}$. Set $g(x_1, \dots, x_{k+1}) = 2$.

case 4. first i such that t_i, t_{i+1}, t_{i+1} is odd and $t_i \geq t_{i+1}$. Set $g(x_1, \dots, x_{k+1}) = 3$.

Now let $x_1 < \dots < x_{k+2} \in n$ be such that $g(x_1, \dots, x_{k+1}) = g(x_2, \dots, x_{k+2})$.

All cases easily lead to contradictions. QED

$s(k, r) =$ least n such that for all $F: N^k \rightarrow \{1, \dots, r\}$ there exists $x_1 < \dots < x_{k+1} \in n$ such that F is constant on all subsequences of x_1, \dots, x_{k+1} .

$\text{Adj}(k, r) =$ least n such that for all $F: N^k \rightarrow \{1, \dots, r\}$ there exists $x_1 < \dots < x_{k+1} \in n$ such that $F(x_1, \dots, x_k) = F(x_2, \dots, x_{k+1})$.

THEOREM (Duffus, Lefmann, Rodl 1995). For $k \geq 1$ and $r \geq 4$, $s(k, r) >$ exponential stack of $k-1$ 1.5's topped off with r .

THEOREM. For $k \geq 2$, $\text{Adj}(k+1, 3) > s(k, k)$.

THEOREM. $\text{Adj}(k, 2) = 2k+1$.

THEOREM. $s(k, r) \square$ a stack of $k-1$ 2's topped with a reasonable function of k and r . So $\text{Adj}(k, 3) \square$ a stack of k 2's topped with a reasonable function of k .

Now do $\text{Adj}(k, 2) = 2k+1$.

Let $f: N^k \square \{1, 2\}$ be given. $f(2, \dots, k+1) \neq f(3, \dots, k+2) \neq \dots \neq f(k+2, \dots, 2k+1)$, alternating with length $k+1$. But also $f(1, \dots, k) \neq f(2, \dots, k, k+2) \neq \dots \neq f(k, k+2, \dots, 2k) \neq f(k+2, \dots, 2k+1)$, also alternating with length $k+1$. Hence $f(1, \dots, k) = f(2, \dots, k+1)$.

This shows that $\text{Adj}(k, 2) \square 2k+1$.

Now we need to get an appropriate function on $\{1, \dots, 2k\}^k$.

For $k = 1$ this is trivial. Suppose f is an appropriate function on $\{2, \dots, 2k+1\}^k$. We now want to construct an appropriate function g on $\{1, \dots, 2k+2\}^{k+1}$.

case 1. $x_1 \geq 2$. Set $g(x_1, \dots, x_{k+1}) = f(x_1, \dots, x_k)$.

case 2. $x_1 = 1$, $x_{k+1} \square k+1$. Set $g(x_1, \dots, x_{k+1}) = 1 - f(x_2, \dots, x_{k+1})$.

case 3. $x_1 = 1$, $x_{k+1} = k+2$. Set $g(x_1, \dots, x_{k+1}) = 1$.

Suppose $g(x_1, \dots, x_{k+1}) = g(x_2, \dots, x_{k+2})$.

case 1. $x_1 \geq 2$. $g(x_1, \dots, x_{k+1}) = f(x_1, \dots, x_k) = f(x_2, \dots, x_{k+1})$. Contradiction.

case 2. $x_1 = 1$. Since $x_{k+1} \square k+1$, we have $g(x_1, \dots, x_{k+1}) = 1 - f(x_2, \dots, x_{k+1})$. But $g(x_2, \dots, x_{k+1}) = f(x_2, \dots, x_{k+1})$. Contradiction. QED

