Characteristic Radiation Impedance of Free Space:

Calculation of $\eta^2 = \left(\frac{qE}{mc\omega}\right)^2 =$ " $\mathbf{a}_0^2$ "

in terms of the intensity I (power per unit area)

There are essentially two steps to the calculation. One is to express the radiation loss across a given area as a resistive dissipation of effective surface currents in this area. That process is ohmic and is characterized by a resistance of 337 Ohms.

The second step consists of relating the radiation field to the motion of a charged particle with a given mass.

STEP I. Relate I to E^2.
Starting with SI units one has

$$energyflux = \frac{(joules)}{(sec)(meter)^2} \tag{1}$$

$$= \frac{watts}{(meter)^2} \equiv I \tag{2}$$

This flux is related to to the

$$energy\ density = \epsilon_0 E^2 \quad \left[\frac{joules}{(meter)^3}\right]$$

by

$$I = c\,\epsilon_0 E^2$$

where

$$c = \sqrt{\frac{1}{\epsilon_0} \times \frac{1}{\mu_0}} \tag{3}$$

$$= \sqrt{4\pi\, 9 \cdot 10^9 \times \frac{1}{4\pi\, 10^7}} = 3 \cdot 10^8 \ \frac{(meters)}{(sec)} \tag{4}$$

so that

$$c\,\epsilon_0 = \sqrt{\frac{\epsilon_0}{\mu_0}} \ \left[\frac{(coulomb)^2}{(sec)\ (newton)\ (meter)}\right] \tag{5}$$

$$= \sqrt{\frac{1}{4\pi\, 9 \cdot 10^9} \times \frac{1}{4\pi\, 10^7}} \ \left[\frac{(coulomb)^2}{(sec)\ (newton)\ (meter)} \equiv \frac{1}{(Ohm)}\right] \tag{6}$$

$$= \frac{1}{120\pi} = \frac{1}{377} \ \left[\frac{1}{(Ohm)}\right] \quad \text{“}surface\ conductance\ of\ empty\ space\text{''} \tag{7}$$

(*Nota bene*: The concept "surface conductance" is defined in the Appendix.) Consequently, the intensity is related to the square of the electric field by

$$I\left[\frac{watts}{(meter)^2}\right] = \frac{1}{377}E^2 \tag{8}$$

or equivalently,

$$\boxed{I\left[\frac{watts}{(meter)^2}\right] = \frac{1}{120\pi}E^2 \;\left[\frac{1}{(Ohm)}\frac{(volts)^2}{(meter)^2}\right]}$$

STEP II. Relate I to η^2

$$I = \frac{1}{120\pi}\left(\frac{qE}{mc\omega}\right)^2\frac{\omega^2}{c^2}\times\frac{(mc^2)^2}{q^2} \tag{9}$$

$$= \frac{1}{120\pi}\qquad\eta^2\;\frac{(2\pi)^2}{\lambda^2}\times\left(\frac{mc^2}{q}\right)^2 \tag{10}$$

Introducing 10^{-6} meter $\equiv$ 1 micron $\equiv$ 10,000Åone obtains

$$I = \frac{\pi}{30}\qquad\eta^2\;\left(\frac{(10,000\text{Å}}{\lambda}\right)^2\left(\frac{1}{10^{-6}\text{meters}}\right)^2\times\left(\frac{mc^2}{q}\right)^2 \tag{11}$$

The rest mass energy and the charge of an electron are

$$mc^2 = 9.11\cdot 10^{-31}\text{ kg}\times 9\cdot 10^{16}(\text{m/sec})^2 = 81.99\cdot 10^{-15}\text{joules}\ , \tag{12}$$

$$q = 1.60\cdot 10{-}19\text{ coulomb} \tag{13}$$

respectively. Thus one has

$$I = \frac{\pi}{30}\qquad\eta^2\;\left(\frac{(10,000\text{Å}}{\lambda}\right)^2 10^{12}\left(\frac{81.99\cdot 10^{-15}}{1.60\cdot 10^{-19}}\right)^2\left[\frac{\text{watts}}{(\text{meter})^2}\right] \tag{14}$$

$$= 2.75\qquad\eta^2\;\left(\frac{(10,000\text{Å}}{\lambda}\right)^2 10^{22}\left[\frac{\text{watts}}{(\text{meter})^2}\right] \tag{15}$$

Thus one has

$$\boxed{I = 2.75\cdot\eta^2\left(\frac{(10,000\text{Å}}{\lambda}\right)^2 10^{18}\left[\frac{\text{watts}}{(\text{cm})^2}\right]}$$

The *average* intensity is only half of that,

$$I_{aver} = 1.37\cdot\eta^2\left(\frac{(10,000\text{Å}}{\lambda}\right)^2 10^{18}\left[\frac{\text{watts}}{(\text{cm})^2}\right]\ .$$

Appendix: Surface Conductivity of Empty Space

The concept "surface conductance" comes from the Poynting relation

$$I = \sqrt{\frac{\epsilon_0}{\mu_0}} E_x^2$$

for the energy flux of a linearly x-polarized plane wave propagating into the z-direction. Multiply this relation by the area element $\Delta(area) = \ell_x \ell_y$. This yields

$$\text{power} = I\Delta(area) = \underbrace{E_x \ell_x}_{\substack{voltage \\ along \\ x-direction}} \times \underbrace{\left(\sqrt{\frac{\epsilon_0}{\mu_0}}\right) E_x \ell_y}_{\substack{\text{current} \\ \text{along} \\ x-\text{direction}}} \qquad (16)$$

The current along the x-direction is the current $J_x \Delta z \ell_y$ in a thin slab of thickness Δz and $\Delta(area) = \ell_x \ell_y$. See Figure 1. If J_x refers to the volume current

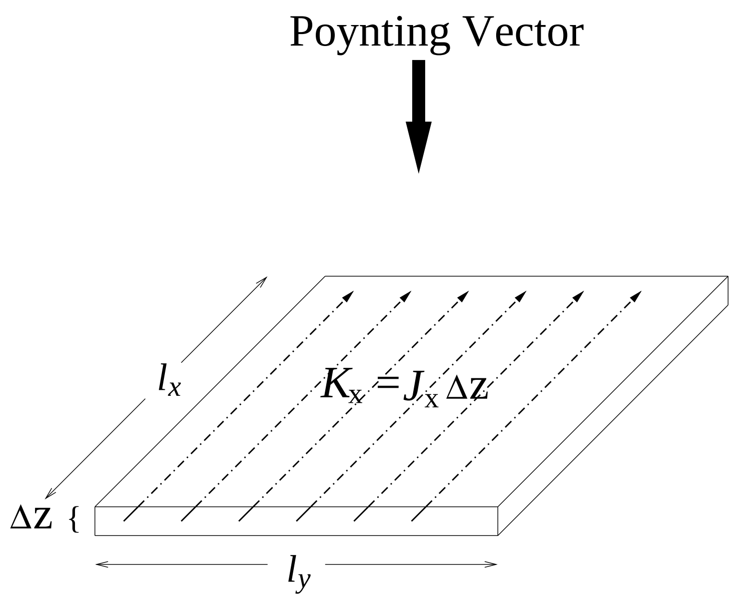

Figure 1: Volume current density and surface current density due to a Poynting vector along the z-direction

density, let us introduce the *surface current density* K_x by letting $J_x \Delta z \equiv K_x$ so that

$$J_x \Delta z \ell_y = K_x \ell_y = \text{current along the } x\text{-direction.} \qquad (17)$$

Ohm's law applied to the surface current reads

$$K_x = \begin{pmatrix} surface \\ conductivity \end{pmatrix} \times E_x \tag{18}$$

as compared to a volume current for which it reads

$$J_x = \begin{pmatrix} volume \\ conductivity \end{pmatrix} \times E_x$$

Multiply Ohm's law by ℓ_y. According to Eq. (17) this is the current along the x-direction. Comparing this quantity with the second factor on the right hand side of Eq. (16) one obtains

$$\begin{pmatrix} surface \\ conductivity \end{pmatrix} \times E_x \ell_y = \left(\sqrt{\frac{\epsilon_0}{\mu_0}} \right) E_x \ell_y$$

Thus one finds that the effective surface conductivity of empty space is

$$\begin{pmatrix} surface \\ conductivity \end{pmatrix} = \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{1}{120\pi} = \frac{1}{377} \quad \left[\frac{1}{(Ohms)} \right]$$

One also says that the characteristic radiation impedance of empty space has a universal value, namely 377 Ohms.