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I. Tensorial Basis Expansion

16.2

The basic mathematical building blocks of tensors are vectors and covectors. Tensor products of these elements results in new multilinear maps, i.e. tensors. Having generated $\binom{n}{m}$ rank tensors one can take linear combinations as dictated physical and geometrical considerations. Of these the most fundamental is to represent a tensor relative to a given basis. This task is achieved by the following

Proposition ("Basis representation of a tensor")

Given:

(i) a basis $\{e_i\}$ for V and its corresponding dual basis $\{\omega^j\}$ for V^*

(ii) a tensor H of rank $\binom{n}{m}$.

Conclusion:

$$H = H^{j_1 \dots j_m}_{i_1 \dots i_n} e_{j_1} \otimes \dots \otimes e_{j_m} \otimes \omega^{i_1} \otimes \dots \otimes \omega^{i_n} \quad (16.1)$$

In this representation $\{e_{j_1} \otimes \dots \otimes e_{j_m} \otimes \omega^{i_1} \otimes \dots \otimes \omega^{i_n}\}$ is the set of basis elements for $\binom{n}{m}$ tensors.

The set $\{H^{j_1 \dots j_m}_{i_1 \dots i_n}\}$ refers to the coordinates of H relative to this basis.

The validation of this representation consist of that the value of the l.h.s. equals that of the r.h.s. for all $(n+m)$ -tuples of covectors and vectors.

To concretize this line of reasoning, apply it to an archetypical tensor, one of rank (1) :

$$H: V^* \times V \rightarrow \mathbb{R}$$

for which one must show that

$$H = H^j_i e_j \otimes \omega^i \quad (16.2)$$

The validation consists of showing that

$$H(\underline{\sigma}, \vec{v}) = H^j_i e_j \otimes \omega^i(\underline{\sigma}, \vec{v}) \quad \forall (\underline{\sigma}, \vec{v}) \in V^* \times V, \quad (16.3)$$

i.e., for all $\underline{\sigma} \in V^*$ and $\vec{v} \in V$.

\mathbb{H} is linear. Thus, it suffices to validate equality for all $(\omega^k, e_l) \in V^* \times V$. This is a two step process:

(i) Observe that as in the Definition on page 15.5,

$$\mathbb{H}(\omega^k, e_l) = H^k_l \tag{16.4}$$

is the $(k, l)^{th}$ component of \mathbb{H} relative to the given basis.

(ii) On the r. h. s. one has

$$\begin{aligned}
H^i_j e_i \otimes \omega^j(\omega^k, e_l) &= H^i_j e_i(\omega^k) \omega^j(e_l) \\
&= H^i_j \underbrace{\langle \omega^k | e_i \rangle}_{\delta^k_i} \underbrace{\langle \omega^j | e_l \rangle}_{\delta^j_l} \\
&= H^k_l
\end{aligned}
\tag{16.5}$$

Equations (16.4) and (16.5) hold for all pairs of basis elements (ω^k, e_l) , and, because of linearity of \mathbb{H} , Eq. (16.3) holds for all pairs $(\underline{\omega}, \underline{v})$. Thus Eq. (16.2) is a valid tensorial basis expansion indeed.

II. Examples

1. Metric tensor:

$$g = g_{ij} \omega^i \otimes \omega^j$$

2. Inverse metric tensor:

$$g^{-1} = g^{ij} e_i \otimes e_j$$

3. Cartan's unit tensor:

$$\begin{aligned}
dP &= \delta^i_j e_i \otimes \omega^j \\
&= e_i \otimes \omega^i
\end{aligned}$$

4. Totally antisymmetric (Levi-Civita) tensor (a. k. a. Volume tensor in n dimensions):

$$\epsilon = \epsilon_{i_1, \dots, i_n} \omega^{i_1} \otimes \dots \otimes \omega^{i_n}$$

where $\epsilon_{i_1, \dots, i_n}$ is the totally antisymmetric (Levi-Civita) symbol, (16.4)

$$\epsilon_{i_1, \dots, i_n} = \begin{cases} 0 & \text{if any pair of indices are the same} \\ +\epsilon_{1, \dots, n} & \text{if } i_1, \dots, i_n \text{ is an even permutation of } 1, \dots, n \\ -\epsilon_{1, \dots, n} & \text{if } i_1, \dots, i_n \text{ is an odd permutation of } 1, \dots, n \end{cases}$$

An equivalent, but basis ("frame") independent definition is

$$\epsilon(\vec{A}_1, \dots, \vec{A}_n) = \det \begin{vmatrix} \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \dots & \omega^n(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \dots & \omega^n(\vec{A}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \omega^1(\vec{A}_n) & \omega^2(\vec{A}_n) & \dots & \omega^n(\vec{A}_n) \end{vmatrix} = \text{volume of a parallelepiped subtended by } \{\vec{A}_i\}_{i=1}^n \text{ in } \mathbb{R}^n$$

III. Tensor Space

Tensors of rank $\binom{n}{m}$ can be added; they can be multiplied by scalars. This feature was already implicit in Eq. (16.1), the basis representation of a tensor and the examples on pages 16.3 and 16.4. More formally one has the following

Proposition ("Tensor Space")

Tensors of rank $\binom{n}{m}$ form a vector space. This space is denoted by the

tensor space $\underbrace{V \otimes \dots \otimes V}_n \otimes \underbrace{V^* \otimes \dots \otimes V^*}_m$.

A typical element \mathbb{H} , an $\binom{n}{m}$ rank tensor, is evaluated on any of the elements of $V^* \times \dots \times V^* \times V \times \dots \times V$, the set of $(n+m)$ -tuples.

IV. New Tensors

1. "Raising" and "lowering" tensor indices.

Consider a metric g .

$$V \xrightarrow{g} V^*$$

$$u = u^i e_i \xrightarrow{g} g(u) = \underline{u} = u_j \omega^j \in V^*$$

In transforming vectors into covectors, the effect of this metric on their expansion coefficients,

$$\{u^i\}_{i=1}^n \xrightarrow{g} \{u_j = g_{ji} u^i\}_{j=1}^n,$$

is to lower their indices for the creation of the covector \underline{u} .

This process is generalized to tensors by means of the following

Proposition ("Lowering of indices")

The metric g lowers the indices of tensors

$$g: \underbrace{V \otimes \dots \otimes V}_n \otimes \underbrace{V^* \otimes \dots \otimes V^*}_m \rightarrow \underbrace{V \otimes \dots \otimes V}_{(n-1)} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{(m+1)}$$

$$\binom{n}{m} \text{ tensors} \rightarrow \binom{n-1}{m+1} \text{ tensors}$$

Applied to an $\binom{n}{m}$ tensor this process consists of the following transformation

$$H = e_{j_1} \dots e_{j_{n-1}} \boxed{e_{j_n}} \omega^{i_1} \dots \omega^{i_m} \rightsquigarrow$$

$$H^{j_1 \dots j_{n-1}}_{j_n i_1 \dots i_m} e_{j_1} \dots e_{j_{n-1}} \boxed{\omega^{j_n}} \omega^{i_1} \dots \omega^{i_m}$$

The correspondingly coordinate components have their n^{th}

superscript (j_n) lowered:

$$\{H^{j_1 \dots j_{n-1} j_n}_{i_1 \dots i_m}\} \xrightarrow{g} \{H^{j_1 \dots j_{n-1} k}_{i_1 \dots i_m} g_{kj_n}\} \equiv \{H^{j_1 \dots j_{n-1} j_n}_{i_1 \dots i_m}\}$$

2. Contraction of a tensor

Regardless of the nature of the metric on V , one can lower the rank of a tensor by means of the contraction map C its properties are condensed into the following

Definition ("Contraction of a tensor")

The contraction map C is a transformation on a tensor in which the one of the superscripts of its components get equated to one of their subscripts before one sums over the pair of equated indices:

$$C: \binom{n}{m} \text{ tensors} \longrightarrow \binom{n-1}{m-1} \text{ tensors}$$

$$\{H^{j_1 \dots j_{n-1} j_n}_{i_1 \dots i_m}\} \rightsquigarrow \{H^{j_1 \dots j_{n-1} k}_{i_1 \dots i_m} g_{kj_n}\}$$