The First Fundamental Theorem of Calculus

In class, we learned the first fundamental theorem of calculus which can be used to find derivatives of the form

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right].$$

This theorem is powerful and very easy to use, but it does not apply directly to problems like

$$\frac{d}{dx}\left[\int_{x^3}^{x^4} (2t-1)^3\,dt\right].$$

In this handout, we prove a more general theorem which can be used to easily find above derivative.

Theorem 1 (The "Generalized" First Fundamental Theorem of Calculus) *Let f be continuous on $(-\infty,\infty)$ and let $a(x), b(x)$ be differentiable on $(-\infty,\infty)$. Then*

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x).$$

Remark. In particular, we have

$$\frac{d}{dx}\left[\int_a^{b(x)} f(t)\,dt\right] = f(b(x))\cdot b'(x), \qquad \frac{d}{dx}\left[\int_{a(x)}^{b} f(t)\,dt\right] = -f(a(x))\cdot a'(x).$$

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Proof. The main idea of the proof is to use the chain rule and the first fundamental theorem of calculus. First, by the interval additive property,

$$\int_{a(x)}^{b(x)} f(t)\,dt = \int_{a(x)}^{c} f(t)\,dt + \int_c^{b(x)} f(t)\,dt$$

for any fixed real number c. To apply the first fundamental theorem of calculus, we also rewrite the first integral on the right side so that

$$\int_{a(x)}^{b(x)} f(t)\,dt = -\int_c^{a(x)} f(t)\,dt + \int_c^{b(x)} f(t)\,dt = \int_c^{b(x)} f(t)\,dt - \int_c^{a(x)} f(t)\,dt.$$

Now

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = \frac{d}{dx}\left[\int_c^{b(x)} f(t)\,dt\right] - \frac{d}{dx}\left[\int_c^{a(x)} f(t)\,dt\right].$$

To find the derivative

$$\frac{d}{dx}\left[\int_c^{b(x)} f(t)\,dt\right],$$

we define

$$F(x) = \int_c^{b(x)} f(t)\,dt, \qquad G(u) = \int_c^{u} f(t)\,dt$$

and introduce the intermediate variable

$$u = b(x).$$

Then clearly $F(x) = G(b(x)) = G(u)$, so by the chain rule

$$\frac{d}{dx}\left[\int_c^{b(x)} f(t)\,dt\right] = \frac{d}{dx}[F(x)] = \frac{d}{dx}[G(b(x))] = \frac{dG(u)}{du}\cdot\frac{du}{dx}.$$

By the first fundamental theorem of calculus,

$$\frac{dG(u)}{du} = \frac{d}{du}\left[\int_c^{u} f(t)\,dt\right] = f(u)$$

(remember that c is a fixed real number), so

$$\frac{d}{dx}\left[\int_c^{b(x)} f(t)\,dt\right] = \frac{dG(u)}{du}\cdot\frac{du}{dx} = f(u)\cdot\frac{du}{dx}.$$

After substituting $u = b(x)$ into above equation we obtain

$$\frac{d}{dx}\left[\int_c^{b(x)} f(t)\,dt\right] = f(b(x))\cdot b'(x).$$

Similarly we can show that

$$\frac{d}{dx}\left[\int_c^{a(x)} f(t)\,dt\right] = f(a(x))\cdot a'(x),$$

so

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x).$$

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Example. Find $\frac{d}{dx}\left[\int_{x^3}^{x^7} (2t-1)^3\,dt\right]$.

Solution. By the theorem,

$$\begin{aligned}\frac{d}{dx}\left[\int_{x^3}^{x^7} (2t-1)^3\,dt\right] &= (2x^7-1)^3\cdot(x^7)' - (2x^3-1)^3\cdot(x^3)' \\ &= (2x^7-1)^3\cdot(7x^6) - (2x^3-1)^3\cdot(3x^2).\end{aligned}$$

(Just leave the answer in that form).