MATH 770: HOMEWORK 6 (DUE NOV. 19, 2010)

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Problem 1. Let $\rho : G \to GL(V)$ be a linear representation and let $V^* := Hom_k(V, k)$ be the dual vector space of V (here k is the ground field). For $v \in V$ and $v^* \in V^*$ denote by $< v, v^* >$ the value $v^*(v)$. Show that there is a unique linear representation $\rho^* : G \to GL(V^*)$ such that $< \rho(g)v, \rho^*(g)v^* >=< v, v^* >$ for every $g \in G, v \in V, v^* \in V^*$. Identify the character of ρ^*.

Problem 2. Let G be a finite group. Show that if the character of a representation of G has value 0 for all $g \in G \setminus \{1\}$, then this character is an *integral* multiple of the character $\chi_{\rho_{reg}}$ of the regular representation.

Problem 3. Let G be a finite group. Let $\rho : G \to GL(V)$ be an irreducible complex linear representation. Show that if $dim_{\mathbb{C}} V > 1$, then G is not abelian.

Problem 4. Let G be a finite group and S a finite set on which G acts. Let $\rho_S : G \to GL(V)$ be the associated permutation representation. Namely V is a complex vector space of dimension $|S|$ with a basis $\{e_s | s \in S\}$, such that $\rho_S(g)e_s := e_{gs}$. Show that the value $\chi_{\rho_S}(g)$ of the character is equal the number of elements in S fixed by g.

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