MATH 770: HOMEWORK 7 (DUE DEC. 01, 2010)

HSIAN-HUA TSENG

Convention: All representations are finite dimensional and over the complex numbers.

Problem 1. Let G be a finite group. List all the conjugacy classes of G as

$$(g_1), (g_2), ..., (g_n).$$

Let

$$\rho_1, \rho_2, ..., \rho_n$$

be a collection of irreducible representations of G which are pairwise non-isomorphic. (Note that $\rho_1, ..., \rho_n$ form a complete set of representatives of isomorphism classes of irreducible representations of G.) Let M be the square matrix whose (i, j)-th entry is $\chi_{\rho_i}(g_j)$. Is M invertible? Justify your answer.

Problem 2. Let G be a finite group and $H \subset G$ a subgroup. Show that each irreducible representation of G is contained in a representation of G induced from an irreducible representation of H.

Problem 3. Let G be a finite group, and $\rho : G \to GL(V)$ an irreducible representation.

(a) Prove that if s is contained in the center $Z(G)$ of G, then $\rho(s)$ is a scalar multiple of the identity map $Id_V : V \to V$.
(b) Prove that $(dim_{\mathbb{C}} V)^2 \leq |G|/|Z(G)|$.

Problem 4. Let G be a finite group. Observe that if $\rho : G \to GL(V)$ is a representation and $\phi : G \to G$ is an automorphism of G, then $\rho \circ \phi^{-1}$ is also a representation. Check that the assignment

$$(\phi, \rho) \mapsto \rho \circ \phi^{-1}$$

defines an action of $Out(G)$ on $\widehat{G}$, where $Out(G) = Aut(G)/Inn(G)$ is the group of outer automorphisms of G and $\widehat{G}$ is the set of isomorphism classes of irreducible representations of G.

Problem 5. Let $\pi : H \to Q$ be a surjective homomorphism of finite groups. Let $(q) \subset Q$ be a conjugacy class of Q.

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(a) Show that the preimage $\pi^{-1}((q)) \subset H$ is a disjoint union of conjugacy classes of H.

(b) Let

$$H = D_n := \langle r, s | r^n = 1, s^2 = 1, srs = r^{-1} \rangle$$

be the dihedral group of order $2n$. Let

$$\pi : H = D_n \to Q := D_n / \langle r \rangle \simeq \mathbb{Z}_2$$

be the quotient map. Denote the two conjugacy classes of Q by 1 and b. What is the number of distinct conjugacy classes of H that are contained in $\pi^{-1}(1)$ (respectively $\pi^{-1}(b)$)?

Remark to Problem 5: It is interesting to find out, for a given conjugacy class (q) of Q, the number of distinct conjugacy classes of H contained in the preimage $\pi^{-1}((q))$. I describe here an answer to this question assuming that $G := Ker(\pi)$ is contained in the center $Z(H)$.

Let $s : Q \to H$ be a set-theoretic map such that $\pi \circ s$ is the identity map on Q. Define a set-theoretic map

$$\sigma : Q \times Q \to G$$

by

$$\sigma(q_1, q_2) = s(q_1)s(q_2)s(q_1 q_2)^{-1}.$$

The map σ measures the failure of s being a group homomorphism. For $q \in Q$ let $C_Q(q) := \{q_1 \in Q | q_1 q = q q_1\}$ be the centralizer subgroup.

Claim: Let $(q) \subset Q$ be a conjugacy class of Q. The number of distinct conjugacy classes of H contained in the preimage $\pi^{-1}((q))$ is equal to the cardinality of the following set:

$$\{\rho = \text{character of irreducible representation of } G | \rho(\sigma(q_1, q)\sigma(q, q_1)^{-1}) = 1, \forall q_1 \in C_Q(q)\}.$$

This Claim can be proved by standard (but non-trivial) group theoretic arguments. Indeed such a proof was communicated to me by Professor I. Martin Isaacs of University of Wisconsin-Madison. However the Claim and its generalization was originally discovered by Xiang Tang (Washington University- St. Louis) and myself using a indirect geometric consideration.