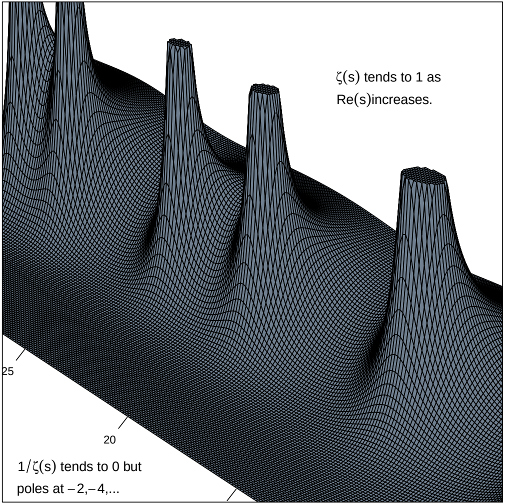
Large values and Random Matrices

Ghaith Hiary

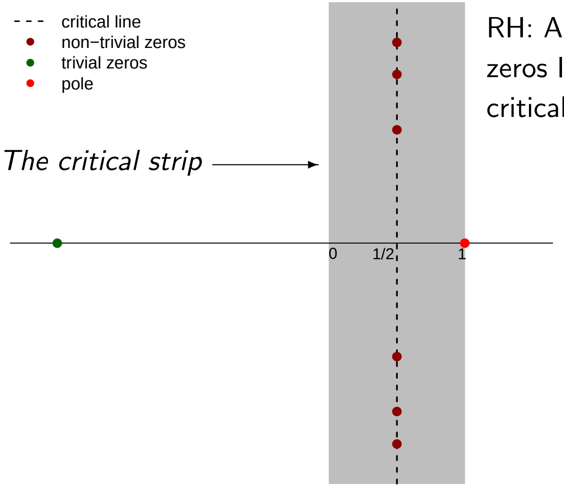
Warning: This presentation contains explicit material.

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \qquad \Re(s) > 1.$$

Initial zeros as poles of 1/ζ(s)

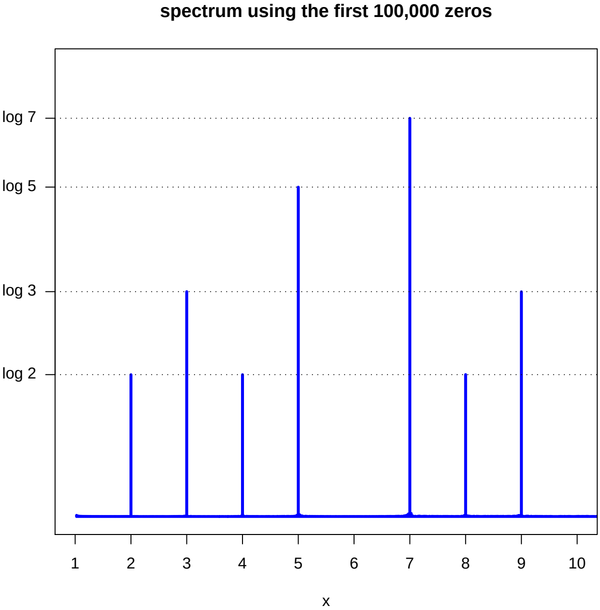


$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots , \qquad \Re(s) > 1 .$$



RH: All non-trivial zeros lie on the critical line.

Explicit/trace formula



Euler product:

$$\zeta(s) = \prod_{\text{primes } p} \frac{1}{1 - p^{-s}} .$$

Landau: $x > 1$, $T \to \infty$,

$$\frac{2\pi}{T} \sum_{0 < \Im(\rho) \leq T} x^{\rho} \sim -\Lambda(x)$$

where $\Lambda(x) = \log p$ if $x = p^m$, and 0 otherwise.

A model for the primes

$$\pi(x) = \{\#\ \text{primes} \leq x\ \} \sim x/\log x\,.$$

Primes deterministic.

Correlations via Hardy-Littlewood k-tuple conjecture.

Cramér's model: $z_2, z_3, \ldots,$ indp. Bernoulli var.

$$\text{Prob}(z_n = 1) = 1/\log n\,.$$

Consequence of Cramér's model:

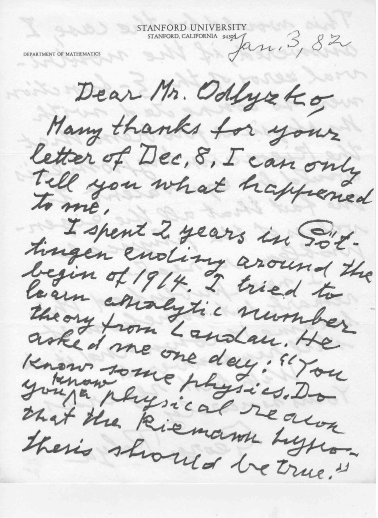
$$\pi(N+H) - \pi(N) \sim H/\log N \qquad \text{for } H = \log^k N,\ k > 2,$$

disproved by Maier 1985. Primes in short intervals: more subtle.

Complicated behvaior due to zeros.

The Hilbert-Pólya conjecture and spectral interpretation

1-st page of Pólya's response to Odlyzko's letter



STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

DEPARTMENT OF MATHEMATICS

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Dear Mr. Odlyzko,

Many thanks for your letter of Dec. 8, I can only tell you what happened to me.

I spent 2 years in Göttingen ending around the begin of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true."

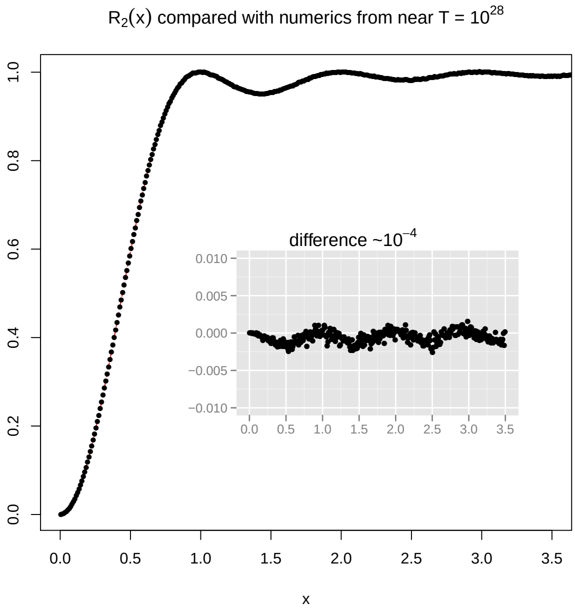
Hilbert-Pólya conjecture: Zeroes of zeta are the eigenvalues of a self-adjoint operator.

Dyson and Montgomery, IAS 1972.

" ... it looks like the differences between zeros of the zeta function have a distribution with a density $1 - \sin^2(\pi r)/(\pi r)^2$. Freeman immediately responded that this is the pair correlation density of random matrices."

Asymptotic behavior of the form factor still only proven in restricted ranges.

Pair correlation $R_2(x)$



GUE Pair-correlation:

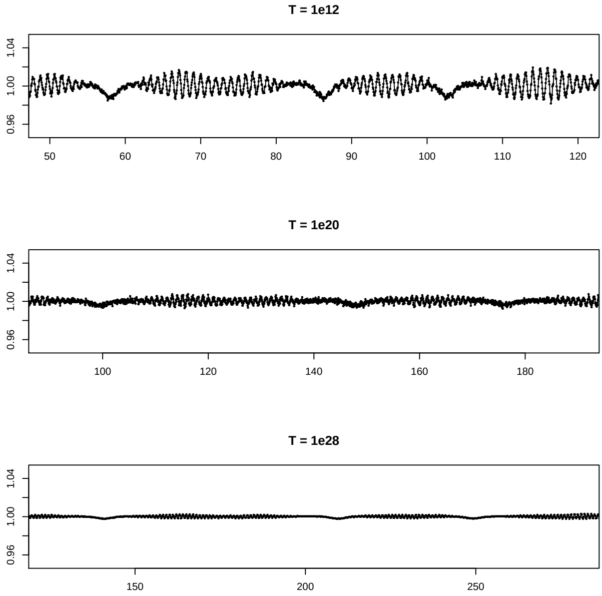
$1 - \sin^2(\pi x)/(\pi x)^2$

Plotted points:

$|\{T + H_1 \leq \gamma, \gamma' \leq T + H_2,$
$\tilde{\gamma} - \tilde{\gamma}' \in [r\delta, (r+1)\delta)\}| \,/\, \delta M$

$M = |\{\gamma - T \in [H_1, H_2]\}|$
$\delta = 0.01$, $T = 10^{28}$, 14 random $H_1 < H_2 < 10^{21}$ spanning $\approx 3.7 \times 10^8$ zeros.

Tail of $R_2(x)$


T = 1e12
1.04
1.00
0.96
50
60
70
80
90
100
110
120
T = 1e20
1.04
1.00
0.96
100
120
140
160
180
T = 1e28
1.04
1.00
0.96
150
200
250

Random Matrix Theory (RMT)

RMT developed to interpret the energy levels; doesn't describe an individual nucleus, but general appearance of levels in complicated ones.

Dyson:

> *we picture a complex nucleus as a black box in which a large number of particles are interacting according to unknown laws. The problem then is to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally probable.*

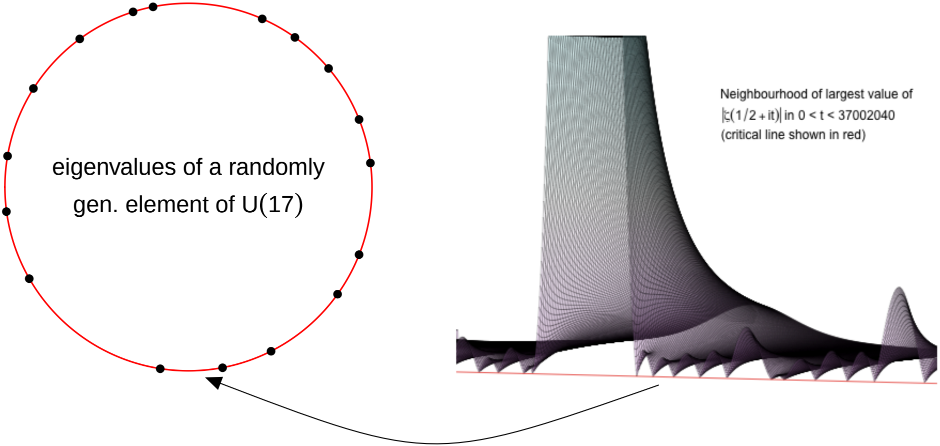
Appropriate ensemble depends on symmetries of the system.

Initially, zeta connected to GUE, but statistical equivalence with eigenphases of random unitary matrices ($U(N)$ or CUE).

Identify local zero densities: $N \longleftrightarrow \log(T/2\pi)$

Models large but finite heights.

Nearest-neighbour spacing functions: $p(0,s)$, $p(1,s)$,...



Value distribution

$$\frac{\log Z(A_N,\theta)}{\sqrt{(1/2)\log N}} \sim \mathcal{CN}(0,1) \qquad \frac{\log \zeta(1/2+iT+it)}{\sqrt{(1/2)\log\log T}} \sim \mathcal{CN}(0,1)$$

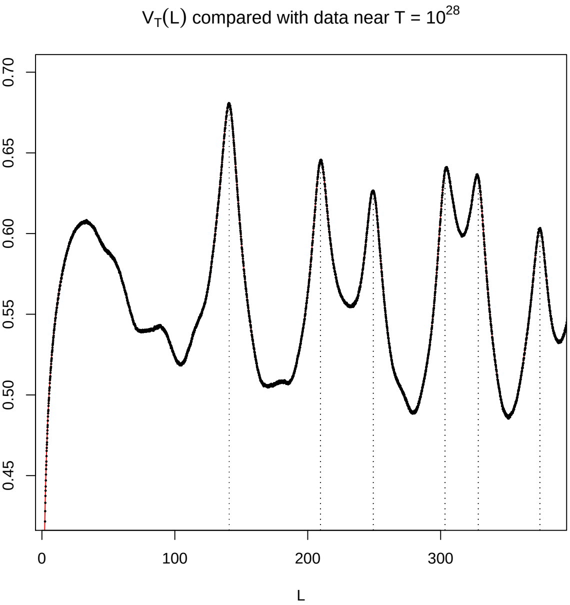
The Keating-Snaith constant:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}\,dt \sim \frac{g(k)a(k)}{k^2!}\,\log^{k^2} T$$

Conrey-Ghosh: $g(3) = 42$
Conrey-Gonek: $g(4) = 24024$
Keating-Snaith: $g(k) = \Gamma(k^2+1)G^2(k+1)/G(2k+1)$

Deviations from asypmtotic value distribution of $\Re \log \zeta(1/2+it)$ using data near $t = 10^{22}$ explained.

Another example: Number-variance



$$V_T(L) = \langle (N(T + L/d) - N(T) - L)^2 \rangle$$

Berry-Keating prediction computed with $\tau = 0.15$

Short and long range predictions: $V_T(L) = U(L) + B_T(L)$.

Low zeros appear in unfolded Gram scaling.

RMT and Maximum size of $\zeta(1/2+it)$

Typical size of $\log|\zeta(1/2+it)|$ on order of $\sqrt{\log\log t}$.

Maximum size? Difficult to control, large values rare.

- Lindelöf hypothesis: $\log|\zeta(1/2+it)| = o(\log t)$.
- Assuming the RH: $\log|\zeta(1/2+it)| = O(\log t/\log\log t)$.
- Farmer, Gonek, and Hughes' conjecture:

$$\max_{t\in[0,T]} \log|\zeta(1/2+it)| = (1+o(1))\sqrt{\frac{1}{2}\log T\log\log T}\,.$$

- Soundararajan's conjecture: For $T>1000$, without any further restriction,

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \le T(\log T)^{k^2}\,.$$

Large values via moments

Large values: Need uniformity in tail of value distribution.

The CFKRS conjecture:

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \sim \int_0^T P_k(\log t/2\pi)\,dt$$

$$P_k(x) = c_0(k)x^{k^2} + c_1(k)x^{k^2-1} + \cdots + c_{k^2}(k)\,.$$

$$P_k(x) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint \frac{G(z_1,\ldots,z_{2k})\Delta^2(z_1,\ldots,z_{2k})}{\prod_{i=1}^{2k} z_i^{2k}} \times e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{k+i}}\,dz_1\ldots dz_{2k}\,,$$

$$P_k(x) = c_0(k)x^{k^2} + \cdots + c_1(k)x^{k^2-1} + \cdots + c_{k^2}(k)\,.$$

$$c_0(k) = \frac{a(k)g(k)}{k^2!}\,,$$

where

$$a(k) := \prod_p (1-1/p)^{k^2} F(k,k;1;1/p)\,,$$

$$g(k) := k^2! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}\,.$$

Theorem (H-Rubinstein): $c_r(k) \sim (4k\log k)^r \binom{k^2}{r} c_0(k)$ uniformly in $r < k^{1-\epsilon}$.

Suggests an identical upper bound to Farmer, Gonek, and Hughes.

$$|\zeta(1/2+it)| \ll \exp\left(\left(\left(\tfrac{1}{2} + o(1)\right) \log T \log\log T \right)^{1/2} \right)$$

Another approach: Large values via computations

New algorithms enabled computation ~ 10 orders of magnitude higher than before. Used to compute near large values.

Will enable computing at several orders of magnitude more.

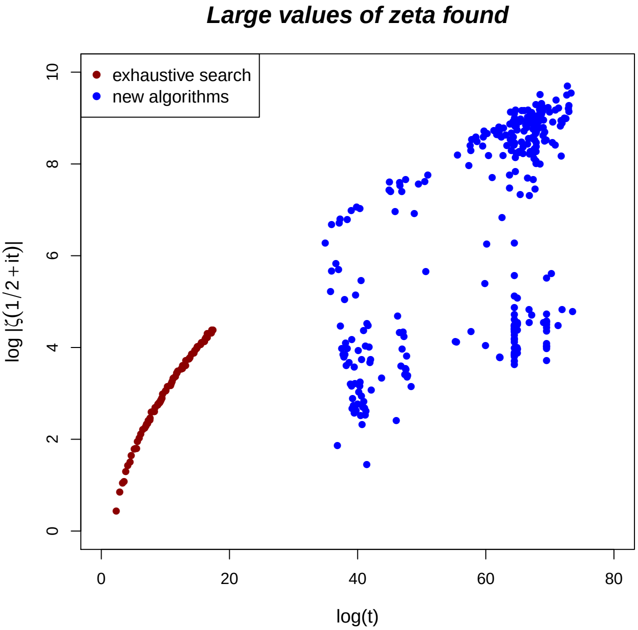
Generalized to Degree 1 L-functions in t and q aspects.

Interesting analogies between methods to compute zeta and methods to bound it.

Largest value found so far:

$|\zeta(1/2+i3.92467645898943091552511692841040506222 01\times 10^{31})|=16244.86526\ldots$

Empirical lower bound



If $\sqrt{}$, large values numerous enough, so LLL can capture a representative sample after a few hundred tries?

Only "*typical* large values"?

LLL application similar to proof under the RH of Ω-results.

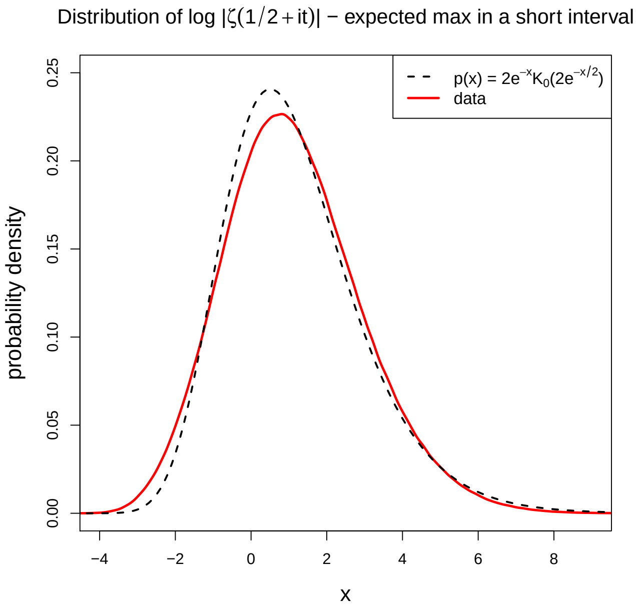
Third approach: Max over short ranges

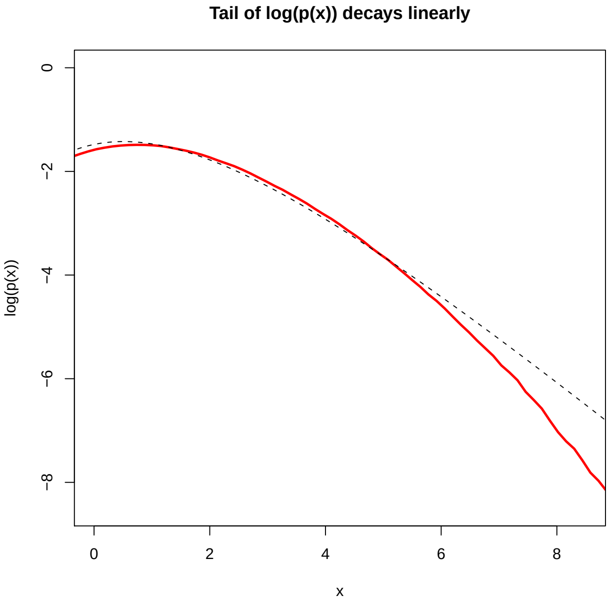
Local maxima of $|\zeta(1/2+it)|$ statistically independent? A good approximation. However, for suitable $H \ll T$

$$\int_{T-H}^{T+H} \log|\zeta(1/2+it+ix_1)| \, \log|\zeta(1/2+it+ix_2)\,dt =$$

$$\begin{cases} -(1/2)\log|x_1-x_2|, \text{ for } & \frac{1}{\log t} \ll |x_1-x_2| \ll 1 \\ 2\log\log t, \text{ for } & |x_1-x_2| \ll \frac{1}{\log t} \end{cases}$$

Logarithmic correlations exist.


Distribution of log |ζ(1/2 + it)| − expected max in a short interval
p(x) = 2e^{−x}K_0(2e^{−x/2})
data
probability density
0.00
0.05
0.10
0.15
0.20
0.25
−4
−2
0
2
4
6
8
x


Tail of log(p(x)) decays linearly
log(p(x))
0
−2
−4
−6
−8
0
2
4
6
8
x

Thank you!