

## A Rigorous Introduction to Limits

### 1. DEFINITIONS

The intuitive idea of limit is this: We say  $\lim_{x \rightarrow c} f(x) = L$  if  $f(x)$  is *near*  $L$  when  $x$  is *suitably near*  $c$ . Mathematically, the notion of *near* is too vague to be useful. The rigorous definition of limit is the following:

**Definition 1.1.**  $\lim_{x \rightarrow c} f(x) = L$  means that for each given  $\epsilon > 0$ , there is a corresponding  $\delta > 0$  such that  $|f(x) - L| < \epsilon$  whenever  $0 < |x - c| < \delta$ .

There are a corresponding rigorous definitions for one-sided limits as well:

**Definition 1.2.**  $\lim_{x \rightarrow c^+} f(x) = L$  means that for each given  $\epsilon > 0$ , there is a corresponding  $\delta > 0$  such that  $|f(x) - L| < \epsilon$  whenever  $0 < x - c < \delta$ .

**Definition 1.3.**  $\lim_{x \rightarrow c^-} f(x) = L$  means that for each given  $\epsilon > 0$ , there is a corresponding  $\delta > 0$  such that  $|f(x) - L| < \epsilon$  whenever  $0 < c - x < \delta$ .

Notice in the last two definitions, we do not have  $|x - c|$ , but rather  $x - c$  or  $c - x$ . What definition 1.1 is saying is that we can make the error in  $f(x)$  small (within  $\epsilon$  units of  $L$ ) provided that we make sure that the distance between  $x$  and  $c$  is small (within  $\delta$  units). It is important to point out that in all three definitions, we are assuming that  $x \neq c$ , so that a limit is an indication of what is happening near a value (in this case near  $c$ ) and not at a value.

### 2. PROVING LIMITS USING THE RIGOROUS DEFINITION ( $\epsilon - \delta$ PROOFS)

It is generally hard to prove a given limit using the above definitions, but there are some standard types of limits which be explored below. The method of proof that we will use is referred to as an  $\epsilon - \delta$  proof. Each proof will consist of two parts. The first part is some preliminary analysis. In this step, which is usually the hardest, we try to bound  $|f(x) - L|$  above by some expression involving  $|x - c|$ . Most of the time this is just a matter of algebra, though sometimes, more advanced work must be done. This bound above will lead to a choice of  $\delta$ , which will generally be some function of  $\epsilon$ . With this we proceed to the second step, which is the actual proof. In this step we verify that our choice of  $\delta$ , given  $\epsilon$ , leads to  $|f(x) - L| < \epsilon$ . In many cases, this second step looks identical to the first step.

### 3. EXAMPLES

**Example 1:** Show that  $\lim_{x \rightarrow -21} (3x - 1) = -64$  using an  $\epsilon - \delta$  proof.

**Preliminary Analysis:** We want a bound for  $|f(x) - L|$  in terms of  $|x - c|$ :

$$\begin{aligned} |(3x - 1) - (-64)| &= |3x + 63| \\ &= 3|x + 21| \\ &= 3|x - (-21)| \end{aligned}$$

Thus we see that  $|(3x - 1) - (-64)| = 3|x - (-21)|$ . In this case we take  $\delta = \frac{\epsilon}{3}$ . Now we proceed to a formal proof:

**Formal Proof:** Let  $\epsilon > 0$  be given, choose  $\delta = \frac{\epsilon}{3}$  and assume that  $0 < |x - (-21)| < \delta$ . Then

$$\begin{aligned} |(3x - 1) - (-64)| &= |3x + 63| \\ &= 3|x + 21| \\ &= 3|x - (-21)| \\ &< 3\delta = 3\frac{\epsilon}{3} = \epsilon \end{aligned}$$

So, with our assumptions, we have  $|(3x - 1) - (-64)| < \epsilon$ .  $\square$

**Example 2:** Show that  $\lim_{x \rightarrow 0} \frac{2x^2 - x}{x} = -1$  using an  $\epsilon - \delta$  proof.

**Preliminary Analysis:** We want a bound for  $|f(x) - L|$  in terms of  $|x - c|$ :

$$\begin{aligned} \left| \left( \frac{2x^2 - x}{x} \right) - (-1) \right| &= |(2x - 1) - (-1)| \text{ (because } x \neq 0\text{)} \\ &= |2x - 0| \\ &= 2|x - 0| \end{aligned}$$

Thus we see that  $\left| \left( \frac{2x^2 - x}{x} \right) - (-1) \right| = 2|x - 0|$ . In this case we take  $\delta = \frac{\epsilon}{2}$ . Now we proceed to a formal proof:

**Formal Proof:** Let  $\epsilon > 0$  be given, choose  $\delta = \frac{\epsilon}{2}$  and assume that  $0 < |x - 0| < \delta$ . Then

$$\begin{aligned} \left| \left( \frac{2x^2 - x}{x} \right) - (-1) \right| &= |(2x - 1) - (-1)| \text{ (because } x \neq 0\text{)} \\ &= |2x - 0| \\ &= 2|x - 0| \\ &< 2\delta = 2\frac{\epsilon}{2} = \epsilon \end{aligned}$$

So, with our assumptions, we have  $\left| \left( \frac{2x^2 - x}{x} \right) - (-1) \right| < \epsilon$ .  $\square$

**Example 3:** Show that  $\lim_{x \rightarrow 1} \sqrt{2x} = \sqrt{2}$  using an  $\epsilon - \delta$  proof.

We want a bound for  $|f(x) - L|$  in terms of  $|x - c|$ :

$$\begin{aligned} |\sqrt{2x} - \sqrt{2}| &= \left| \frac{\sqrt{2x} - \sqrt{2}}{1} \cdot \frac{\sqrt{2x} + \sqrt{2}}{\sqrt{2x} + \sqrt{2}} \right| \\ &= \left| \frac{2x - 2}{\sqrt{2x} + \sqrt{2}} \right| \\ &= 2 \left| \frac{x - 1}{\sqrt{2x} + \sqrt{2}} \right| \\ &\leq 2 \left| \frac{x - 1}{\sqrt{2}} \right| \text{ (because } \sqrt{2x} \geq 0\text{)} \\ &= \frac{2}{\sqrt{2}} |x - 1| \end{aligned}$$

Thus we see that  $|\sqrt{2x} - \sqrt{2}| \leq \frac{2}{\sqrt{2}}|x - 1|$ . Note that we have an upper bound on  $|f(x) - L|$  given by an inequality, rather than by an equality as in the first two examples. In this case we take  $\delta = \frac{\sqrt{2}}{2}\epsilon$ . Now we proceed to a formal proof:

**Formal Proof:** Let  $\epsilon > 0$  be given, choose  $\delta = \frac{\sqrt{2}}{2}\epsilon$  and assume that  $0 < |x - 1| < \delta$ . Then

$$\begin{aligned} |\sqrt{2x} - \sqrt{2}| &= \left| \frac{\sqrt{2x} - \sqrt{2}}{1} \cdot \frac{\sqrt{2x} + \sqrt{2}}{\sqrt{2x} + \sqrt{2}} \right| \\ &= \left| \frac{2x - 2}{\sqrt{2x} + \sqrt{2}} \right| \\ &= 2 \left| \frac{x - 1}{\sqrt{2x} + \sqrt{2}} \right| \\ &\leq 2 \left| \frac{x - 1}{\sqrt{2}} \right| \quad (\text{because } \sqrt{2x} \geq 0) \\ &= \frac{2}{\sqrt{2}}|x - 1| \\ &< \frac{2}{\sqrt{2}}\delta = \frac{2}{\sqrt{2}}\frac{\sqrt{2}}{2}\epsilon = \epsilon \end{aligned}$$

So, with our assumptions, we have  $|\sqrt{2x} - \sqrt{2}| < \epsilon$ .  $\square$

**Example 4:** Show that  $\lim_{x \rightarrow 1} 2x^2 - 1 = 1$  using an  $\epsilon - \delta$  proof.

**Preliminary Analysis:** We want a bound for  $|f(x) - L|$  in terms of  $|x - c|$ :

$$\begin{aligned} |(2x^2 - 1) - 1| &= |2x^2 - 2| \\ &= 2|x^2 - 1| \\ &= 2|(x + 1)(x - 1)| \\ &= 2|(x + 1)| \cdot |(x - 1)| \end{aligned}$$

Thus we see that  $|(2x^2 - 1) - 1| = 2|(x + 1)| \cdot |(x - 1)|$ . Unlike the previous cases, this bound on  $|f(x) - L|$  not only involves  $|x - c|$  but also an addition function of  $x$ , namely  $|x + 1|$ . We will now use the fact that we can always assume  $x$  and  $c$  are already close, in other words that  $\delta$  is small. We can choose any initial upper bound for  $\delta$ , here we will assume that  $\delta \leq 1$ . Note that we could have chosen any other positive value as an initial upper bound for  $\delta$ , our choice is for convenience only. By assuming  $\delta \leq 1$ , we are assuming that  $x$  is at most one unit away from 1. If we assume that  $|x - 1| \leq 1$ , then

$$\begin{aligned} |x + 1| &= |(x - 1) + 2| \\ &= |x - 1| + 2 \quad (\text{by the triangle inequality}) \\ &\leq 1 + 2 = 3 \end{aligned}$$

What this says is that if  $|x - 1| \leq 1$  then  $|x + 1| \leq 3$ . Putting this together with our earlier analysis, we find that  $|(2x^2 - 1) - 1| = 2|(x + 1)| \cdot |(x - 1)| \leq 6|x - 1|$  if we assume that  $\delta \leq 1$ . Thus we must have both  $\delta \leq 1$  and  $\delta = \frac{\epsilon}{6}$ . We can accomplish this by assuming that  $\delta = \min\{1, \frac{\epsilon}{6}\}$

**Formal Proof:** Let  $\epsilon > 0$  be given, choose  $\delta = \min\{1, \frac{\epsilon}{6}\}$  and assume that  $0 < |x - 1| < \delta$ . Then

$$\begin{aligned} |(2x^2 - 1) - 1| &= |2x^2 - 2| \\ &= 2|x^2 - 1| \\ &= 2|(x + 1)(x - 1)| \\ &= 2|(x + 1)| \cdot |x - 1| \\ &\leq 6|x - 1| \text{ (because } \delta \leq 1) \\ &< 6\delta = \frac{\epsilon}{6} = \epsilon \end{aligned}$$

So, with our assumptions, we have  $|(2x^2 - 1) - 1| < \epsilon$ .  $\square$