Conditional Proofs

One of the most important ideas to understand is the method of conditional proof. It relies upon the definition of $P \Rightarrow Q$. Namely, that this implication is **always** true when P is false. The basic idea is to assume that P is true and deduce that Q must be true. If this can be done, then we know that this implication must in fact be a tautology. There is no need to assume that P is false as this case always leads to a true implication as mentioned above. We indicate to the reader of our proof that we don't need to consider the case where P is false by using the phrase: "Discharging ...". When your exercises ask you to use this method and tell you not to use cases, they mean that you do not consider the case where P is false. It does not mean that you can not use any case analysis in your proof. In particular, if a sentence involves an "or", you more than likely need to consider the truth values of each side of the "or" in separate cases.

One important rule of logical inference is that of **modus ponens**. This is basically the observation that if P is true and $P \Rightarrow Q$ is true, then Q must be true. This can be seen by looking at the truth table for $\Rightarrow$.

The structure of these proofs are generally the same. When you are just starting out, you should model your proofs on the examples from the textbook as well as the ones included here. In particular, you should label your assumptions and indicate when you are discharging an assumption. You should also indicate what needs to be shown after making an assumption. This is a good practice as it is easy to lose sight of what needs to be done in complicated proofs.

Example 1: Show that $P \Rightarrow (P \vee Q)$ is a tautology.

Proof:

(A1): Assume that P is true.

We need to show that $P \vee Q$ is true.

Since P is true by (A1), $P \vee Q$ is true under assumption (A1).

Discharging (A1), $P \Rightarrow (P \vee Q)$ is true under no assumptions. Thus $P \Rightarrow (P \vee Q)$ is a tautology.

Example 2: Show that $[P \wedge (Q \vee R)] \Rightarrow [(P \wedge Q) \vee (P \wedge R)]$ is a tautology.

Proof:

(A1): Assume that $P \wedge (Q \vee R)$ is true.

We need to show that $(P \wedge Q) \vee (P \wedge R)$ is true.

Since $P \wedge (Q \vee R)$ is true by (A1), P is true and $Q \vee R$ is true. This means that Q or R is true.

(Case 1): Suppose that Q is true. Then $P \wedge Q$ is true, thus $(P \wedge Q) \vee (P \wedge R)$ is true.

(Case 2): Suppose that R is true. Then $P \wedge R$ is true, thus $(P \wedge Q) \vee (P \wedge R)$ is true.

Therefore, $(P \wedge Q) \vee (P \wedge R)$ is true in both cases and under assumption (A1).

Discharging (A1), $[P \wedge (Q \vee R)] \Rightarrow [(P \wedge Q) \vee (P \wedge R)]$ is true under no assumptions. Thus $[P \wedge (Q \vee R)] \Rightarrow [(P \wedge Q) \vee (P \wedge R)]$ is a tautology.

Example 3: Show that $[(P \Rightarrow Q) \wedge (P \Rightarrow R)] \Rightarrow [P \Rightarrow (Q \wedge R)]$ is a tautology.

Proof:

(A1): Assume that $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ is true.

We need to show that $P \Rightarrow (Q \wedge R)$ is true.

(A2): Assume that P is true.

We need to show that $Q \wedge R$ is true.

Since $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ is true by (A1), $P \Rightarrow Q$ is true and $P \Rightarrow R$ is true. Since P is true by (A2), Q is true by modus ponens and R is true by modus ponens. Thus $Q \wedge R$ is true.

Discharging (A2), $P \Rightarrow (Q \wedge R)$ is true under only (A1).

Discharging (A1), $[(P \Rightarrow Q) \wedge (P \Rightarrow R)] \Rightarrow [P \Rightarrow (Q \wedge R)]$ is true under no assumptions. Thus $[(P \Rightarrow Q) \wedge (P \Rightarrow R)] \Rightarrow [P \Rightarrow (Q \wedge R)]$ is a tautology.

Example 4: Show that $[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)]$ is a tautology.

Proof:

(A1): Assume that $P \Rightarrow (Q \Rightarrow R)$ is true.

We need to show that $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ is true.

(A2): Assume that $P \Rightarrow Q$ is true.

We need to show that $P \Rightarrow R$ is true.

(A3): Assume that P is true.

We need to show that R is true.

Since P is true by (A3) and $P \Rightarrow Q$ is true by (A2), Q is true by modus ponens. By (A1) and (A3), $Q \Rightarrow R$ is true by modus ponens. Thus R must be true by modus ponens.

Discharging (A3), $P \Rightarrow R$ is true under only (A1) and (A2).

Discharging (A2), $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$ is true under only (A1).

Discharging (A1), $[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)]$ is true under no assumptions. Thus $[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)]$ is a tautology.