Contradictions and Proving Negatives

Proof by contradiction is a very powerful method of proof. Basically, if we wish to show that a given sentence is always true, we assume that it is false and produce a contradiction. This means that our assumption can never happen, thus the statement is always true.

If we want to show that a sentence of the form $\neg P$ is always true, then we can assume that P is true and attempt to produce a contradiction.

Example: Show that $[(P \Rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$ is a tautology.

Proof:

> **(A1):** Assume that $(P \Rightarrow Q) \wedge \neg Q$ is true.
>
> We need to show that $\neg P$ is true.
>
> > **(A2):** Suppose that P is true. We wish to produce a contradiction. By (A1), $(P \Rightarrow Q) \wedge \neg Q$ is true. Thus $P \Rightarrow Q$ is true and $\neg Q$ is true. Since P is true by (A2), Q is true by modus ponens. Thus $Q \wedge \neg Q$ is true. This is a contradiction so (A2) is not possible and we must have P false and $\neg P$ true.

Discharging (A1), $[(P \Rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$ is true under no assumptions. Thus $[(P \Rightarrow Q) \wedge \neg Q] \Rightarrow \neg P$ is a tautology.