Equinumerous Intervals

The goal of these notes is to answer questions 5-9 in section 15 of the textbook. Rather than answer each one individually, we will show that any non-degenerate interval in $\mathbb{R}$ is equinumerous to any other non-degenerate interval. We will do this by showing that each one of these intervals is equinumerous to the interval $(0, 1)$. Then we get the other results by composing a suitable chain of bijections.

First, there are nine different types of non-degenerate intervals in $\mathbb{R}$. Let a, b be real numbers with $a \neq b$. There are four bounded intervals:

$$(a, b), (a, b], [a, b) \text{ and } [a, b]$$

and five unbounded intervals:

$$(-\infty, a), (-\infty, a], (a, \infty), [a, \infty) \text{ and } (-\infty, \infty)$$

The last of these intervals being equal to $\mathbb{R}$.

1. Bounded Intervals - Part One

In this section we show that any bounded interval is equinumerous to an interval with endpoints 0 and 1. Note that these endpoints may or may not be included in that interval.

Let

$$f_1(x) = \frac{1}{b-a}(x-a)$$

Then $f_1(x)$ is a bijection between the following pairs of intervals:

$$(a, b) \approx (0, 1)$$

$$(a, b] \approx (0, 1]$$

$$[a, b) \approx [0, 1)$$

$$[a, b] \approx [0, 1]$$

2. Bounded Intervals - Part Two

In this section we will show that every bounded interval is equinumerous to $(0, 1)$.

First, let $f_2(x)$ be the following function:

$$f_2(x) = \begin{cases} x & \text{if } x \neq \frac{1}{2^n} \text{ for any } n \in \omega \\ \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for } n \in \omega \end{cases}$$

Then $f_2(x)$ is a bijection from $(0, 1]$ to $(0, 1)$ and from $[0, 1]$ to $[0, 1)$, so we have

$$(0, 1] \approx (0, 1)$$

$$[0,1] \approx [0,1)$$

Now let $f_3(x) = 1 - x$ then $f_3(x)$ is a bijection from $[0,1)$ to $(0,1]$, so we have

$$[0,1) \approx (0,1]$$

Combining this with the above gives us:

$$[0,1] \approx [0,1) \approx (0,1] \approx (0,1)$$

Given the results of the previous section, we see that any bounded interval must be equinumerous to $(0,1)$.

3. Unbounded Intervals - Part One

Let

$$f_4(x) = x - a$$

Then $f_4(x)$ is a bijection between the following pairs of intervals:

$$(a,\infty) \approx (0,\infty)$$

$$[a,\infty) \approx [0,\infty)$$

$$(-\infty,a) \approx (-\infty,0)$$

$$(-\infty,a] \approx (-\infty,0]$$

4. Unbounded Intervals - Part Two

In this section we will show that all of the unbounded intervals are equinumerous to $(-\infty,\infty)$, in other words, to $\mathbb{R}$.

First, let

$$f_5(x) = \begin{cases} x & \text{if } x \notin \omega \\ x+1 & \text{if } x \in \omega \end{cases}$$

The $f_5(x)$ is a bijection from $[0,\infty)$ to $(0,\infty)$.

Now let

$$f_6(x) = -x$$

Then $f_6(x)$ is a bijection from $(-\infty,0)$ to $(0,\infty)$ and from $(-\infty,0]$ to $[0,\infty)$.

Using these bijections, we see that

$$(-\infty,0) \approx (0,\infty)$$

$$(-\infty,0] \approx [0,\infty) \approx (0,\infty)$$

Finally, let

$$f_7(x) = x - \frac{1}{x}$$

Then $f_7(x)$ is a bijection from $(0, \infty)$ to $(-\infty, \infty)$. So

$$(0, \infty) \approx (-\infty, \infty) = \mathbb{R}$$

Thus

$$(-\infty, 0] \approx (-\infty, 0) \approx [0, \infty) \approx (0, \infty) \approx (-\infty, \infty) = \mathbb{R}$$

5. Conclusion

At this stage we have that every bounded interval is equinumerous to $(0, 1)$ and that every unbounded interval is equinumerous to $(-\infty, \infty)$. It remains to show that these two intervals are equinumerous.

Let

$$f_8(x) = \frac{\frac{1}{2} - x}{x^2 - x}$$

Then f_8 is a bijection from $(0, 1)$ to $(-\infty, \infty)$. Thus

$$(0, 1) \approx (-\infty, \infty) = \mathbb{R}$$

6. A Note On Bijections

Most of the functions above are differentiable functions over the domains listed. By checking that the sign of the derivative does not change over the given domain, it can be verified that the function is injective. They can be seen to be surjective by checking the values at the endpoints (possibly using limits) and appealing to the intermediate value theorem for the interior points of the intervals in question. (Recall that a differentiable function is continuous.) Only f_2 and f_5 are not differentiable (or continuous) but they can be seen to be bijections by direct calculations.