1.5 Bayes' Rule

1. Introduction

Suppose that we have two coins. One is fair and has an equal chance of landing on either side when flipped, the other is biased in that it has a $\frac{2}{3}$ chance of landing on heads. Suppose that a coin is chosen at random and flipped with the result being a head. What is the probability that the coin is fair?

Consider this question by letting A be the event that a head occurs when flipped, B_1 be the event that the fair coin is picked and B_2 be the event that the biased coin is flipped. We are then trying to find $P(B_1|A)$, that is we are trying to find the probability of the event B_1 happening prior to the occurrence of A. Notice that we do know that $P(B_1) = P(B_2) = \frac{1}{2}$, $P(A|B_1) = \frac{1}{2}$ and $P(A|B_2) = \frac{2}{3}$. We have enough previous results to solve this, but we will first derive a very useful formula.

2. Definitions

Suppose that $B_1, \ldots, B_n$ is a partition of Ω. Recall that

$$P(A|B_i) = \frac{P(A \cap B_i)}{P(B_i)} \text{ and } P(B_i|A) = \frac{P(A \cap B_i)}{P(A)}$$

and

$$P(A) = P(A|B_1)P(B_1) + \cdots + P(A|B_n)P(B_n)$$

By combining these two observations, we get a formula which is known as **Bayes' Rule**:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + \cdots + P(A|B_n)P(B_n)}$$

Example 1: Consider the first example concerning fair and biased coins. We know that $P(B_1) = P(B_2) = \frac{1}{2}$, $P(A|B_1) = \frac{1}{2}$ and $P(A|B_2) = \frac{2}{3}$, so using Bayes' Rule to find $P(B_1|A)$ we get:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} = \frac{3}{7}$$

Example 2: Suppose there a three chests each having two drawers. The first chest has a gold coin in each drawer, the second has a gold coin in one drawer and a silver coin in the other, and the third chest has a silver coin in each drawer. A chest is chosen at random and a drawer opened. If the drawer contains a gold coin, what is the probability that the first chest was chosen?

Let A be the event that a gold coin is in the opened drawer and let B_i be the event that chest i was chosen. We are interested in finding $P(B_1|A)$ and we will use Bayes' Rule to find this probability. First note that $P(B_i) = \frac{1}{3}$ for all i. Secondly, note that $P(A|B_1) = 1, P(A|B_2) = \frac{1}{2}$ and $P(A|B_3) = 0$. Therefore, by Bayes' Rule:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)} = \frac{2}{3}$$