1.6 Sequences of Events

1. Definitions

Suppose that $A_1, A_2, \dots, A_n$ are events. The the probability that all of these events occur is given by:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

Example 1: Suppose that a coin is biased so that heads occurs with a probability of $\frac{2}{3}$. The coin is flipped until a tail occurs

(a.) What is the probability that a this takes at most 3 flips?

Let H_i, for $1 \le i \le 3$ be the event that a head shows up on flip i and let T_i, for $1 \le i \le 3$ be the event that a head shows up on flip i. Note that H_i is independent of T_j for $i \ne j$. We want the probability that either T_1, $H_1 \cap T_2$ or $H_1 \cap H_2 \cap T_3$ occurs:

$$P[(T_1) \cup (H_1 \cap T_2) \cup (H_1 \cap H_2 \cap T_3)] = P(T_1) + P(H_1 \cap T_2) + P(H_1 \cap H_2 \cap T_3) = \frac{1}{3} + \frac{2}{3}\frac{1}{3} + \frac{2}{3}\frac{2}{3}\frac{1}{3} = \frac{19}{27}$$

(b.) What is the probability that exactly 3 flips are required for a tail to occur?

Using the notation above, this is just $P(H_1 \cap H_2 \cap T_3) = \frac{6}{27}$.

Example 2: Suppose that a deck of 52 playing cards is shuffled and a cards are drawn until there is a repeated face value. What is the probability that at most 4 draws are needed?

Let P_i be the probability that exactly i draws are needed. Then $P = P_1 + P_2 + P_3 + P_4$ is the required probability. Clearly $P_1 = 0$. Now $P_2 = \frac{3}{51}$ because the second card drawn must be one of the three remaining of the value of the first card. $P_3 = \frac{48}{51}\frac{6}{50}$ because the second card can't match the first card, so there are 48 choices for this card. The third card can either be the value of the first or the second, which gives 6 choices for this card. $P_4 = \frac{48}{51}\frac{42}{50}\frac{9}{49}$. Again, 48 choices for the second because of no match, and 42 choices for the third because it can not be one the two possible values for the first two cards. Finally the last card can be one of nine possible cards that have the same value as one of the first three.

$$P = \frac{3}{51} + \frac{48}{51}\frac{6}{50} + \frac{48}{51}\frac{42}{50}\frac{9}{49} = \frac{39606}{124950} = \frac{943}{2975} \approx 0.317$$

2. Independence

We say events A, B and C are (mutually) independent if

$$P(A \cap B \cap C = P(A)P(B)P(C)$$

It is important to note that if A, B and C are pairwise independent then there are not necessarily mutually dependent.