Math 530

Autumn 2006

Solution to 3.2.13

Let $X_1, \ldots, X_{10}$ be the result of roll i. Note that

$$E(X_i) = \sum_{k=1}^{6} k * P(X_i = k) = \frac{21}{6} = 3.5$$

(a.) Let $S_{10} = X_1 + \cdots + X_{10}$.

$$E(S_{10}) = \sum_{i=1}^{10} E(X_i) = 10(3.5) = \mathbf{35}$$

(b.) Let $S_3 = X_1 + X_2 + X_3$ and $M = \min(X_1, X_2, X_3)$ then the sum of the two largest of the first three numbers is $S_3 - M$. Now,

$$E(M) = P(M \geq 1) + P(M \geq 2) + \cdots + P(M \geq 6)$$

and

$$P(M \geq m) = \left(\frac{6 - m + 1}{6}\right)^3$$

So

$$E(S_3 - M) = \mathrm{E}(S_3) - \mathrm{E}(M) = 10.5 - \frac{216 + 125 + 64 + 27 + 8 + 1}{216} = \mathbf{8.458}$$

(c.) Let $M = \max(X_1, \ldots, X_5)$. Then, as above,

$$E(M) = P(M \geq 1) + \cdots + P(M \geq 6)$$

and

$$P(M \geq m) = 1 - P(M < m) = 1 - \left(\frac{m - 1}{6}\right)^5$$

So

$$E(M) = 6 - \frac{1^5 + 2^5 + 3^5 + 4^5 + 5^5}{6^5} = \mathbf{5.435}$$

(d.) Let A_i be the event that roll i is 3 or 6 and let I_{A_i} be the indicator random variable of Y_i. Note that $P(A_i) = \frac{1}{3}$. Then $S_{10} = \sum_{i=1}^{10} I_{A_i}$ is the number of multiples of 3 rolled in the 10 rolls.

$$E(S_{10}) = \sum_{i=1}^{10} E(I_{A_i}) = \sum_{i=1}^{10} P(A_i) = \mathbf{3.333}$$

(e.) Let A_i be the event that the face i does not appear in any of the 10 rolls and let I_{A_i} be the indicator random variable of i. Note that $P(A_i) = \left(\frac{5}{6}\right)^{10}$. Then $S_6 = \sum_{i=1}^{6} I_{Y_i}$ is the number of distinct faces that do not occur in any of the 10 rolls.

$$E(S_6) = \sum_{i=1}^{6} E(I_{A_i}) = \sum_{i=1}^{6} P(A_i) = \mathbf{6\left(\frac{5}{6}\right)^{10}} = \mathbf{0.9690}$$

(f.) Let S_6 be as above, then $6 - S_6$ is the number of different faces that occur in 10 rolls.

$$E(6 - S_6) = E(6) - E(S_6) = 6 - 0.9690 = \mathbf{5.0310}$$