Math 530

Autumn 2006

Solution to 3.4.5

(a.) Let X be the number of tosses until a head appears.

$$P(X > n) = 1 - P(X \leq n) = 1 - [1 - (1 - p_1)^n] = \mathbf{(1 - p_1)^n}$$

(b.) Let X_1, X_2 and X_3 be the number of tosses until a head occurs for Bill, Mary and Tom, respectively. Note that X_i are independent.

$$P(X_1 > n, X_2 > n, X_3 > n) = P(X_1 > n)P(X_2 > n)P(X_3 > n) = \mathbf{(1 - p_1)^n(1 - p_2)^n(1 - p_3)^n}$$

(c.) Let A be the event that everyone must make more than $n - 1$ flips. Let B be the event that everyone must make more than n flips. Then $B \subset A$. Moreover $B^c \cap A$ is the event that someone makes a head in n flips and everyone else takes n or more flips. Note that $P(B^c \cap A) = P(A) - P(B)$. Using the notation from (b.):

$$P(A) = P(X_1 > n - 1, X_2 > n - 1, X_3 > n - 1) = (1 - p_1)^{n-1}(1 - p_2)^{n-1}(1 - p_3)^{n-1}$$

and

$$P(B) = P(X_1 > n, X_2 > n, X_3 > n) = P(X_1 > n)P(X_2 > n)P(X_3 > n) = (1 - p_1)^n(1 - p_2)^n(1 - p_3)^n$$

so

$$P = \mathbf{(1 - p_1)^{n-1}(1 - p_2)^{n-1}(1 - p_3)^{n-1} - (1 - p_1)^n(1 - p_2)^n(1 - p_3)^n}$$

(d.) Using the notation from (b.), the probability that Bill gets a head before Mary is $P(X_1 < X_2)$ and the probability that Tom gets a head before Mary is $P(X_3 < X_2)$. We want:

$$P = 1 - P(X_1 < X_2) - P(X_3 < X_2)$$

We'll compute $P(X_1 < X_2)$ and then $P(X_3 < X_2)$ will be similar:

$$P(X_1 < X_2) = \sum_{j=1}^{\infty} P(X_2 = j)P(X_1 < j) = \sum_{j=1}^{\infty} P(X_2 = j)P(X_1 \leq j - 1)$$

$$P(X_1 < X_2) = \sum_{j=1}^{\infty} (1 - p_2)^{j-1} p_2 [1 - (1 - p_1)^{j-1}]$$

(See the notes on geometric series for $P(X_1 \leq j - 1)$)

$$P(X_1 < X_2) = \sum_{k=0}^{\infty} (1 - p_2)^k p_2 [1 - (1 - p_1)^k] = \sum_{k=0}^{\infty} (1 - p_2)^k p_2 - p_2 \sum_{k=0}^{\infty} (1 - p_2)^k (1 - p_1)^k$$

$$P(X_1 < X_2) = 1 - p_2 \sum_{k=0}^{\infty} [(1 - p_2)(1 - p_1)]^k = 1 - \frac{p_2}{p_1 + p_2 - p_1 p_2}$$

Similarly:

$$P(X_3 < X_2) = 1 - \frac{p_2}{p_3 + p_2 - p_3 p_2}$$

Therefore

$$P = 1 - 1 + \frac{p_2}{p_1 + p_2 - p_1 p_2} - 1 + \frac{p_2}{p_3 + p_2 - p_3 p_2} = \mathbf{\frac{p_2}{p_1 + p_2 - p_1 p_2} + \frac{p_2}{p_3 + p_2 - p_3 p_2} - 1}$$