Solutions to Review 1

(1a.) Rational function, $(1,-1,2)$ is in its domain $\Rightarrow \displaystyle\lim_{(x,y,z)\to(1,-1,2)} \frac{x^2yz-2xyz}{x^2+y^2-z^2} = \frac{-2+4}{1+1-4} = -1$

(b.) Note: $\dfrac{-x^2\sin x}{x^2} \le \dfrac{x^2 \sin x}{x^2+y^2} \le \dfrac{x^2\sin x}{x^2} \Rightarrow -\sin x \le \dfrac{x^2\sin x}{x^2+y^2} \le \sin x$. As $\sin x \to 0$ as $x \to 0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin x}{x^2+y^2} = 0$ by the squeeze theorem.

(c.)DNE: the limit along $y = x$ is $1/2$ and the limit along $y=-x$ is $-1/2$.

(2a.) $f_x = 2\ln(x^2+2xy)\frac{1}{x^2+2xy}(2x+2y)$ $\quad f_y = 2\ln(x^2+2xy)\frac{1}{x^2+2xy}(2x)$

(b.) $f_x = \sqrt{xyz}(2x) + \frac{1}{2\sqrt{xyz}}(yz)(x^2-2z)$ $\quad f_y = \sqrt{xyz}(0) + \frac{1}{2\sqrt{xyz}}(xz)(x^2-2z)$

$f_z = \sqrt{xyz}(2) + \frac{1}{2\sqrt{xyz}}(xy)(x^2-2z)$

(c.) $f_x = \frac{\cos(x^2y)(2xy)(x^2-y)-\sin(x^2y)(2x)}{(x^2-y)^2}$ $\quad f_x = \frac{\cos(x^2y)(x^2)(x^2-y)-\sin(x^2y)(-1)}{(x^2-y)^2}$

(3a.) $\frac{dw}{dt} = \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt} = (2u+2v^2)(3t^2)+(4uv)(2+1/t) = [2t^3+2(2t+\ln t)^2](3t^2)+4t^3(2t+\ln t)(2+1/t)$

(b.) $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = [\cos(uv)(v)+\sin(2uv)(2v)](2x+y) + [\cos(uv)(u)+\sin(2uv)(2u)](2y)$

$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = [\cos(uv)(v)+\sin(2uv)(2v)](x) + [\cos(uv)(u)+\sin(2uv)(2u)](2y+x)$

Note: substitute $u = x^2+xy$, $v = y^2+2xy$ in the above to get the complete answer.

(c.) $f_{xx} = 2$ $\quad f_{yy} = 2$ $\quad f_{xy} = f_{yx} = -2$

(4.) Let $r = x-y$, $s = y-z$ and $t = z-x$. Then $w = f(r,s,t)$ and $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$, $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$ and $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z}$. Now note that $\frac{\partial r}{\partial x} = \frac{\partial s}{\partial y} = \frac{\partial t}{\partial z} = 1$, $\frac{\partial r}{\partial y} = \frac{\partial s}{\partial z} = \frac{\partial t}{\partial x} = -1$ and $\frac{\partial r}{\partial z} = \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = 0$ and then just substitute and add.

(5.) $D_{\mathbf{u}}f(-1,0,0) = \nabla f(-1,0,0)\cdot\frac{\mathbf{u}}{||\mathbf{u}||} = \frac{3f_x(-1,0,0)-f_y(-1,0,0)-2f_z(-1,0,0)}{\sqrt{14}}$. Since $f_x = -2x$, $f_y = 6y+1$ and $f_z = 1 \Rightarrow D_{\mathbf{u}}f(-1,0,0) = \frac{3(2)-1-2}{\sqrt{14}} = \frac{3}{\sqrt{14}}$

(6.) $\nabla f(2,3,0) = f_x(2,3,0)\mathbf{i} + f_y(2,3,0)\mathbf{j} + f_z(2,3,0)\mathbf{k}$ and $f_x = ye^z, f_y = xe^z, f_z = xye^z \Rightarrow \nabla f(3,2,0) = 3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$.

(7.) Let $f(x,y) = x^2-3x^2y+2y^3$. Then $\nabla f(-1,2)$ will be normal to the curve $f(x,y) = 11$ at the point $(-1,2)$ where $\nabla f(-1,2) = f_x(-1,2)\mathbf{i} + f_y(-1,2)\mathbf{j}$. As $f_x = 2x-6xy$ and $f_y = -3x^2+6y^2 \Rightarrow \nabla f(-1,2) = 10\mathbf{i}+93\mathbf{j}$

(8.) The equation of the tangent plane at $(0,1,1)$ is given by $f_x(0,1)(x-0)+f_y(0,1)(y-1)+f(0,1)=z \Rightarrow f_x(0,1)(x-0)+f_y(0,1)(y-1)-1(z-1)=0$. So normal vector is $f_x(0,1)\mathbf{i}+f_y(0,1)\mathbf{j}-\mathbf{k}$ where $f_x = ye^x$ and $f_y = e^x$. Therefore $\mathbf{i}+bj-\mathbf{k}$ is a normal vector.

(9.) Let $w = x^2-y^2-z^2$, then eq. of tangent space at (3,-2,-2) is $w_x(3,-2,-2)(x-3)+w_y(3,-2,-2)(y+2)+w_z(3,-2,-2)(z+2)+w(3,-2,-2)=w$. So the eq. of the tangent plane to the level surface $w=1$ is $w_x(3,-2,-2)(x-3)+w_y(3,-2,-2)(y+2)+w_z(3,-2,-2)(z+2)=0$. Now $w_x = 2x, w_y = -2y$ and $w_z = -2z$ so eq. of tangent plane is $6(x-3)+4(y+2)+4(z+2)=0$.

(10a.) $f_x = 2x$ and $f_y = 1+1/y^3$. Note that f_y is undefined at $(x,0)$, but this point is not in the domain. So the only critical points occur where $f_x = f_y = 0$. In particular $x=0$ and $y=-1$. Now check the discriminant: $D(0,-1) = f_{xx}(0,-1)f_{yy}(0,-1)-[f_{xy}(0,-1)]^2$. As $f_{xx}=2, f_{yy}=-3/y^4, f_{xy}=0 \Rightarrow D(0,-1) = -6 > 0$. Therefore there is a saddle point at $(0,-1)$.

(b.) $f_x = -4x+y-4$ and $f_y = x+2y+3$. Solving for $f_x = f_y = 0$, we get $x=-11/9$ and $y=-8/9$. Since $f_{xx}=-4, f_{yy}=2$ and $f_{xy}=1$, $D(-11/9,-8/9) = -7 < 0$, so $(-11/9,-8/9)$ is a saddle point.

(11.) Since $f_x = 2x-2$ and $f_y = 2y-4$, the only critical point in the given disk is $(1,2)$. Note that $f(1,2) = -11$. Now we need to check $f(x,y)$ on the boundary of the disk, which is $x^2+y^2=16$. Parameterizing this curve, we get $x(t)=4\cos t$ and $y(t)=4\sin t$. So, on the boundary, $f(x,y) = 16-8\cos t-16\sin t$. Since $\frac{df}{dt} = 8\sin t - 16\cos t$, the maximum and minimum of $f(x,y)$ will occur where $\frac{df}{dt} = 8\sin t-16\cos t = 0$. In particular $\tan t = 2$. In terms of x,y, we have possible max/min's at $(4/\sqrt{5}, 8/\sqrt{5})$ and $(-4/\sqrt{5},-8/\sqrt{5})$. Now check $f(4/\sqrt{5},8/\sqrt{5}) = -7.9$ and $f(-4/\sqrt{5},-8/\sqrt{5}) = 20.73$. Therefore the minimum is at $(1,2)$ and the maximum is at $(-4/\sqrt{5},-8/\sqrt{5})$

(12.) From the picture, the equation of the line connecting $(1,0)$ and $(0,1)$ is $y=1-x$. In particular $b=1-a$ and $d=1-c$. The area of box 1 is $ab = a(1-a) = a-a^2$ and the area of box 2 is $c(d-b) = c(1-c-1+a) = c^2+ac$. Therefore the total area of the boxes is given by $A(a,c) = a-a^2+c^2+ac$. Note that it is clear that $0 < c < a < 1$. We now maximize $A(a,c)$:

$A_a = 1-2a+c$ and $A_c = 2c+a$. Solving for $A_a = 0$ and $A_c = 0$ simultaneously gives $(a,c) = (2/3,1/3)$ as the only critical point. So we have $A(2/3,1/3) = 1/3$. Since $0 < c < a < 1$, there is no boundary to consider, so the maximal total area is $1/3$.