Math 530

Autumn 2006

Solutions to Quiz 6

(1a.)

$$P(Z \le z) = P(X - 2Y \le z) = P\left(Y \ge \frac{x-z}{2}\right) = 1 - P\left(Y \le \frac{x-z}{2}\right)$$

So

$$P(Z \le z) = 1 - P\left(Y \le \frac{x-z}{2}\right) = 1 - \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{x-z}{2}} f(x,y)\, dy\, dx$$

(1b.) Let $t = 2y - x$ then

$$1 - \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{x-z}{2}} f(x,y)\, dy\, dx = 1 - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{-z} f\left(x, \frac{t+x}{2}\right)\, dt\, dx = 1 - \frac{1}{2}\int_{-\infty}^{-z}\int_{-\infty}^{\infty} f\left(x, \frac{t+x}{2}\right)\, dx\, dt$$

By differentiating with respect to z we get the density function:

$$f_Z(z) = \frac{1}{2}\int_{-\infty}^{\infty} f\left(x, \frac{x-z}{2}\right)\, dx$$

(2a.) Consider the region R bounded by $x = 0, y = 1$ and $y = x^{3/2}$. Then $X^{3/2} \le Y$ if and only if (X, Y) are in R. As

$$\int_0^1 1 - x^{3/2}\, dx = \frac{3}{5} = \text{ Area of } R$$

$$P(X^{3/2} \le Y) = \mathbf{\frac{3}{5}}$$

(2b.) Consider the region R bounded by $x = 0, y = 0$ and $y = \frac{0.25 - 2x}{3}$. Then $|2X + 3Y| \le 0.25$ if and only if (X, Y) are in R. As

$$\text{Area of } R = \frac{1}{192}$$

$$P(|2X + 3Y| \le 0.25) = \mathbf{\frac{1}{192}}$$

(3.) Let $g(x) = x^3 - 3x^2 + 2x$. Then $g'(x) = 3x^2 - 6x + 2$ and $g'(x)$ has roots at $r_1 = 1 - \frac{\sqrt{3}}{3}$ and $r_2 = 1 + \frac{\sqrt{3}}{3}$. In particular $g(x)$ is one-to-one on $(-\infty, r_1)$, (r_1, r_2) or (r_2, ∞). Let $g_i^{-1}(y)$ be the inverse of $g(x)$ on each of these intervals. and let $y_i = g(r_i)$. Note that the density function of X is $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Therefore

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2}|g'(g_1^{-1}(y))|} e^{-\frac{[g_1^{-1}(y)]^2}{2}} & y \le g(r_2) \\ \sum_{i=1}^{3} \frac{1}{\sqrt{2}|g'(g_i^{-1}(y))|} e^{-\frac{[g_i^{-1}(y)]^2}{2}} & g(r_2) < y \le g(r_1) \\ \frac{1}{\sqrt{2}|g'(g_3^{-1}(y))|} e^{-\frac{[g_3^{-1}(y)]^2}{2}} & y > g(r_1) \end{cases}$$

(4.)

$$P(X^2 - 5X \le t) = P\left(X^2 - 5X + \frac{25}{4} \le t + \frac{25}{4}\right) = P\left[\left(X - \frac{5}{2}\right)^2 \le t + \frac{25}{4}\right]$$

so

$$P(X^2 - 5X \le t) = P\left(-\sqrt{t + \frac{25}{4}} \le X - \frac{5}{2} \le \sqrt{t + \frac{25}{4}}\right) = P\left(\frac{5}{2} - \sqrt{t + \frac{25}{4}} \le X \le \frac{5}{2} + \sqrt{t + \frac{25}{4}}\right)$$

Since X is exponentially distributed, $P(X \leq x) = 1 - e^{-\lambda x}$, so

$$P(X^2 - 5X \leq t) = 1 - e^{-\lambda\left(\frac{5}{2}+\sqrt{t+\frac{25}{4}}\right)} - 1 - e^{-\lambda\left(\frac{5}{2}-\sqrt{t+\frac{25}{4}}\right)} = \mathbf{e}^{-\lambda\left(\frac{\mathbf{5}}{\mathbf{2}}+\sqrt{\mathbf{t}+\frac{\mathbf{25}}{\mathbf{4}}}\right)} - \mathbf{e}^{-\lambda\left(\frac{\mathbf{5}}{\mathbf{2}}-\sqrt{\mathbf{t}+\frac{\mathbf{25}}{\mathbf{4}}}\right)}$$

Note: $t \geq -\frac{25}{4}$