Math 530

Autumn 2006

Solutions to Exam 1 Review

(1.) Let A_i be the event that box i is chosen and let B be the event that a red ball is drawn

(a.)

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) = \frac{1}{5}\frac{1}{2} + \frac{1}{3}\frac{1}{2} = \mathbf{\frac{4}{15}}$$

(b.)

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)} = \frac{\frac{1}{5}\frac{1}{2}}{\frac{4}{15}} = \mathbf{\frac{3}{8}}$$

(2.) The set of good elements is $\{0, \dots 7\}$, so $N = 10, G = 8, B = 2$ and $n = 5$.

(a.) $g = 5, b = 0,$

$$P = \binom{5}{5}\frac{(8)^5(2)^0}{(10)^5} = \mathbf{\frac{1024}{3125}}$$

(b.) $g = 5, b = 0,$

$$P = \binom{5}{5}\frac{(8)_5(2)_0}{(10)_5} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} = \mathbf{\frac{2}{9}}$$

(3.) The probability of rolling a 5 is $p = \frac{1}{8} = 0.125$, so $q = \frac{7}{8} = 0.875$.

(a.) The probability of rolling exactly k 5's in 25 trials is:

$$P(k) = \binom{25}{k}\left(\frac{1}{8}\right)^k\left(\frac{7}{8}\right)^{25-k}$$

There are two equivalent expressions for the probability that at least eight 5's were rolled:

$$\mathbf{P} = \sum_{\mathbf{k=8}}^{\mathbf{25}} \mathbf{P(k)} \qquad \text{or} \qquad \mathbf{P} = \mathbf{1} - \sum_{\mathbf{k=0}}^{\mathbf{7}} \mathbf{P(k)}$$

(b.) $\mu = (25)(0.125) = 3.125$ and $\sigma = \sqrt{\mu q} = \sqrt{(3.125)(0.875)} = 1.654$ The probability that fewer than eight 5's were rolled is given by

$$P(\leq 7) = \Phi\left(\frac{7 + 0.5 - 3.125}{1.654}\right) = \Phi(2.65) = 0.9960$$

Thus the probability that at least eight 5's was rolled is given by:

$$P = 1 - \Phi(2.65) = \mathbf{0.0040}$$

(c.) The probability that exactly k 5's were rolled is given by:

$$P(k) = e^{-3.125}\frac{(3.125)^k}{k!}$$

So the probability that at most three 5's were rolled is given by:

$$P = \sum_{k=0}^{3} P(k) = e^{-3.125}\left(1 + 3.125 + \frac{(3.125)^2}{2} + \frac{(3.125)^3}{6}\right) = (0.044)(1 + 3.125 + 4.883 + 5.086) = \mathbf{0.619}$$

(4.)

(a.)

$$P = \frac{\mathbf{10!}}{\mathbf{10^{10}}} \approx \mathbf{0.0004}$$

(b.)

$$P = \binom{10}{3} \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^7 = \mathbf{0.0574}$$

(5.) We are concerned with the probability that there is between $0.99np$ and $1.01np$ successes, that is, we are within 1% of $\mu = np$. We want:

$$P(0.99\mu \text{ to } 1.01\mu) = \Phi\left(\frac{1.01\mu + 0.5 - \mu}{\sqrt{\mu q}}\right) - \Phi\left(\frac{0.99\mu - 0.5 - np}{\sqrt{\mu q}}\right) = 2\Phi\left(\frac{0.01\mu + 0.5}{\sqrt{\mu q}}\right) - 1 \geq 0.90$$

$$\Phi\left(\frac{0.01\mu + 0.5}{\sqrt{\mu q}}\right) \geq 0.95$$

so

$$\frac{0.01\mu + 0.5}{\sqrt{\mu q}} \geq 1.65$$

Solving this quadratic gives:

$$n \geq \frac{\mathbf{1}}{\mathbf{p}} \left(\frac{\mathbf{165\sqrt{q} + \sqrt{27,225q - 200}}}{\mathbf{2}}\right)^{\mathbf{2}}$$

Note that ignoring the continuity adjustment of $\frac{1}{2}$ we get

$$n \geq 27,225\frac{q}{p}$$

(6.) We'll use a normal approximation to determine the probability that one student gets no more than 219 heads: $n = 400, p = 0.5$ so $\mu = np = 200$ and $\sigma = \sqrt{npq} = \sqrt{(200)(.5)} = 10$

$$P(k \leq 219) = \Phi\left(\frac{219 + 0.5 - 200}{10}\right) = \Phi(1.95) = 0.9744$$

Since the trials of each student are independent from another students:

$$P = [P(k \leq 219)]^{31} = (0.9744)^{31} = \mathbf{0.4476}$$

(7.) Note that $A \cap B = (A \cap B \cap C) \cup (A \cap B \cap C^c)$ and this union is disjoint.

(a.)

$$P = P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$$

$$P = [P(A \cap B) - P(A \cap B \cap C)] + [P(A \cap C) - P(A \cap B \cap C)] + [P(B \cap C) - P(A \cap B \cap C)]$$

$$P = [0.13 - 0.07] + [0.12 - 0.07] + [0.1 - 0.07] = \mathbf{0.14}$$

(c.) Note, it is easier to compute (c.) before (b.),

$$P = P(A^c \cap B^c \cap C^c) = P[(A \cup B \cup C)^c] = 1 - P(A \cup B \cup C)$$

Now

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

So

$$P(A \cup B \cup C) = 0.22 + 0.18 + 0.16 - 0.13 - 0.12 - 0.1 + 0.07 = 0.28$$

and

$$P(A^c \cap B^c \cap C^c) = 1 - 0.28 = \mathbf{0.72}$$

(b.) Note that this probability is just 1-part (a.)-part(c.)-$P(A \cap B \cap C)$, so

$$P = 1 - 0.72 - 0.14 - 0.07 = \mathbf{0.07}$$

(8.) We have 13 choices for the value of x and 12 choices for the value of y. Once we have chosen a value for x, there are $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ combinations for x, x. Once the value of y has been chosen, there are $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ combinations for y, y, y. Thus the probability of a full house is:

$$P = \frac{(13)\begin{pmatrix} 4 \\ 2 \end{pmatrix}(12)\begin{pmatrix} 4 \\ 3 \end{pmatrix}}{\begin{pmatrix} 52 \\ 5 \end{pmatrix}} = \mathbf{\frac{6}{4165}}$$