Math 571
Spring 2003

Introduction to Graph Theory

1. Introduction

The intuitive notion of a **graph** is a figure consisting of points and lines adjoining these points. More precisely, we have the following precise definition: A graph is a set of objects called **vertices** along with a set of **unordered** pairs of vertices called **edges**. Note that each edge in a graph has no direction associated with it. If we wish to specify a direction, then we us the notion of a **directed graph** or **digraph**. The definition of a digraph is the same as that of a graph, except the edges are **ordered** pairs of edges. If (u, v) is an edge in a digraph, then we say that (u, v) is an edge from u to v. We also say that if (u, v) is an edge in a graph or digraph then u is **adjacent** to v (and v is adjacent from u in a digraph). Below are some examples of graphs and digraphs:

A **path** in a graph or digraph is a sequence of vertices $v_1, v_2, \ldots, v_k$, not necessarily distinct, such that (v_i, v_{i+1}) is an edge in the graph or digraph. The **length** of a path is number of edges in the path, equivalently it is equal to $k - 1$. We will call a path **reduced** if there are no repeated edges. A **cycle** is a reduced path with $v_1 = v_k$. A graph is called **connected** if for each pair of vertices u and v, there is a path in G containing u and v. A digraph is called connected if the underlying graph is connected.

For example, in Fig. 1, v_1, v_2, v_3, v_7, v_5 is a path of length 4 from v_1 to v_5. In Fig. 2, v_1, v_2, v_3, v_4, v_1 is a cycle of length 4. In Fig. 3, v_2, v_5, v_7, v_6 is a path of length 3, but v_1, v_2, v_3 is not a path because (v_1, v_2) is not an edge.

2. Adjacency Matrix

Given a graph or digraph G with vertices $\{v_1, v_2, \ldots, v_n\}$, we define the **adjacency matrix** of G to be the matrix:

$$A = (a_{ij}) \text{ with } a_{ij} = \begin{cases} 1 \text{ if } (v_i, v_j) \text{ is an edge in } G \\ 0 \text{ otherwise} \end{cases}$$

Some examples of graphs and their adjacency matrices:

Note that if A is the adjacency matrix of a graph then $A^T = A$. This is not necessarily the case for digraphs. The main application of adjacency matrices is to determine the connectivity of a graph and the number of paths in a graph or digraph. In particular, we have the following results:

Theorem 1. *If A is the adjacency matrix of a graph or digraph G with vertices $\{v_1, \ldots v_n\}$, then the i, j entry of A^k is the number of paths of length k from v_i to v_j.*

Proof: The result proceeds by induction on k. Clearly, the case when $k = 1$ is true. Now suppose that the result is true for some $k > 1$, so that the entries of A^k are as claimed. Consider any path of length $k+1$ from v_i to v_j. Then there must be a vertex v_l on this path such that v_l is adjacent to v_j. If we delete v_j from this path, then the remaining path is a path of length k from v_i to v_l. The number of such paths is given by i, l entry of A^k by induction. Now each such v_l corresponds to a 1 for the l, j entry of A. The result follows by considering the i, j entry of $A^{k+1} = A^k A$. $\square$

Theorem 2. *If A is the adjacency matrix of a graph G with vertices $\{v_1, \ldots v_n\}$, then G is connected if and only if there is an integer k such that all the entries of $A + A^2 + \cdots + A^k$ are non-zero.*

Proof: Just note that the i, j entry of $A + A^2 + \cdots + A^k$ is the number of paths of length at most k from v_i to v_j. $\square$

Now let's put these results to use:

example 1:

example 2:

3. Problems

(1.)

Suppose that $A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$ is the adjacency matrix of a graph G.

(a.) Draw G.

(b.) Find the number of paths of length 5 from v_3 to v_6 in G.

(c.) Find the number of paths of length at most 4 from v_2 to v_5 in G.

(d.) Determine if G is connected.

(2.) Given the graph G:

(a.) Find the adjacency matrix of G.

(b.) Use the matrix from (a.) to find the number of paths from v_2 to v_4 of length 6.