Math 151
Spring 2004

Solutions to Exam 1 Review

(1a.)

$$\begin{aligned}
\lim_{x\to 2}\frac{2x+1}{5-3x} &= \frac{\lim_{x\to 2}(2x+1)}{\lim_{x\to 2}(5-3x)} \text{ (limit of quotient=quotient of limit)}\\
&= \frac{\lim_{x\to 2}2x+\lim_{x\to 2}1}{\lim_{x\to 2}5-\lim_{x\to 2}3x} \text{ (limit of sum=sum of limit)}\\
&= \frac{2\lim_{x\to 2}x+\lim_{x\to 2}1}{\lim_{x\to 2}5-3\lim_{x\to 2}x} \text{ (limit of constant times } f\text{=constant times limit of } f)\\
&= \tfrac{(2)(2)+1}{5-(3)(2)} = -5
\end{aligned}$$

(1b.)

$$\begin{aligned}
\lim_{w\to -2}\sqrt{-3w^3+7w^2} &= \sqrt{\lim_{w\to -2}(-3w^3+7w^2)} \text{ (limit of root=root of limit)}\\
&= \sqrt{\lim_{w\to -2}(-3w^3)+\lim_{w\to -2}(7w^2)} \text{ (limit of sum=sum of limit)}\\
&= \sqrt{-3\lim_{w\to -2}(w^3)+7\lim_{w\to -2}(w^2)} \text{ (limit of constant times } f\text{=constant times limit of } f)\\
&= \sqrt{-3(\lim_{w\to -2}w)^3+7(\lim_{w\to -2}w)^2} \text{ (limit of power =power of limit)}\\
&= \sqrt{(-3)(-2)^3+(7)(-2)^2} = \sqrt{52}
\end{aligned}$$

(2a.)

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos(2+h)-\cos(2)}{h} &= \lim_{h\to 0}\frac{\cos(2)\cos(h)-\sin(2)\sin(h)-\cos(2)}{h} = \lim_{h\to 0}\frac{\cos(2)[\cos(h)-1]-\sin(2)\sin(h)}{h}\\
&= \lim_{h\to 0}\frac{\cos(2)[\cos(h)-1]}{h} - \lim_{h\to 0}\frac{\sin(2)\sin(h)}{h} = \cos(2)(0)-\sin(2)(1) = -\sin(2)
\end{aligned}$$

(2b.) Let $t = 3\theta$, then $\theta \to 0 \Rightarrow t \to 0$. So $\lim_{\theta\to 0}\frac{\sin 3\theta}{2\theta} = \lim_{t\to 0}\frac{\sin t}{2t/3} = \frac{3}{2}\lim_{t\to 0}\frac{\sin t}{t} = \frac{3}{2}$

(2c.) $\lim_{t\to 3^-}\frac{t^2}{9-t^2} = \lim_{t\to 3^-}\frac{t^2}{(3-t)(3+t)}$. As $\lim_{t\to 3^-}\frac{1}{3-t} = \infty$ and $3^2, (3+3) > 0$. Thus $\lim_{t\to 3^-}\frac{t^2}{9-t^2} = \infty$

(2d.) Use squeeze theorem: $-1 \le \sin x \le 1$. We may assume that $x > 0$, so that $-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$. As $\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty}\frac{1}{x} = 0$, the squeeze theorem implies that $\lim_{x\to\infty}\frac{\sin x}{x} = 0$

(2e.) $\lim_{x\to\infty} \sqrt{3x^2+5} - \sqrt{3x^2-2} = \lim_{x\to\infty}\left[\sqrt{3x^2+5}-\sqrt{3x^2-2}\,\frac{\sqrt{3x^2+5}+\sqrt{3x^2-2}}{\sqrt{3x^2+5}+\sqrt{3x^2-2}}\right] = \lim_{x\to\infty}\left[\frac{3x^2+5-(3x^2-2)}{\sqrt{3x^2+5}+\sqrt{3x^2-2}}\right]$
$= \lim_{x\to\infty}\left[\frac{7}{\sqrt{3x^2+5}+\sqrt{3x^2-2}}\right] = 0$

(2f.) $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4}\left[\frac{x-4}{\sqrt{x}-2}\,\frac{\sqrt{x}+2}{\sqrt{x}+2}\right] = \lim_{x\to 4}\frac{(x-4)(\sqrt{x}+2)}{x-4} = \lim_{x\to 4}\sqrt{x}+2 = 2+2 = 4$

(3a.) Let $f(t) = t^4+t^2-7t+4$. Then the equation has a real solution if and only if $f(t)$ has a real zero. Since $f(t)$ is a polynomial in t, it is continuous at every real number. Now:

$$f(0) = 0+0-0+4 = 4 > 0 \text{ and } f(1) = 1+1-7+4 = -1 < 0$$

So, by the Intermediate Value Theorem, there is a c in $[0,1]$ such that $f(c) = 0$.

(3b.) Let $f(\theta) = \sin\theta+\cos\theta-\theta^2$. Then the equation has a solution if and only if $f(\theta)$ has a real zero. Since $f(\theta)$ is the sum of three functions which are continuous everywhere, $f(\theta)$ is continuous at each real number. Now:

$$f(0) = 0+1-0 = 1 > 0 \text{ and } f(\pi/2) = 1+0-(\pi/2)^2 < 0$$

So, by the Intermediate Value Theorem, there is a c in $[0,\pi/2]$ such that $f(c) = 0$.

(4a.) $\lim_{x\to 1^-}\frac{x+1}{x^2-1} = \lim_{x\to 1^-}\frac{x+1}{(x-1)(x+1)} = \lim_{x\to 1^-}\frac{1}{x-1} = -\infty$

(4b.) $\lim_{x\to 1^+}\frac{x+1}{x^2-1} = \lim_{x\to 1^+}\frac{x+1}{(x-1)(x+1)} = \lim_{x\to 1^+}\frac{1}{x-1} = \infty$

(4c.) $\lim_{x\to \infty}\frac{x+1}{x^2-1} = \lim_{x\to \infty}\frac{x+1}{(x-1)(x+1)} = \lim_{x\to \infty}\frac{1}{x-1} = 0$

(4d.) $\lim_{x\to -1}\frac{x+1}{x^2-1} = \lim_{x\to -1}\frac{x+1}{(x-1)(x+1)} = \lim_{x\to -1}\frac{1}{x-1} = -1/2$

(5a.) For $f(x)$ to be continuous at 1, we need $f(1) = 2(1)-1 = 1 = a(1)+b$. Similarly for $f(x)$ to be continuous at 2, we need $f(2) = -2 = a(2)+b$. Solving these equations for a and b gives $a = -3$ and $b = 4$. There are no other values to check as $f(x)$ is continuous at all other real numbers.

(5b.) For $f(x)$ to be continuous at -1, we need $f(-1) = 2(-1)^2+1 = 3 = a(-1)+b$. Similarly for $f(x)$ to be continuous at 1, we need $f(1) = 1+1 = 2 = a(1)+b$. Solving these equations for a and b gives $a = -1/2$ and $b = 5/2$. There are no other values to check as $f(x)$ is continuous at all other real numbers.

(6a.)Preliminary Analysis:

$$\begin{aligned}
|\sqrt{3x+1}-\sqrt{7}| &= \left|\frac{\sqrt{3x+1}-\sqrt{7}}{1}\cdot\frac{\sqrt{3x+1}+\sqrt{7}}{\sqrt{3x+1}+\sqrt{7}}\right| \\
&= \left|\frac{3x-6}{\sqrt{3x+1}+\sqrt{7}}\right| \\
&= 3\left|\frac{x-2}{\sqrt{3x+1}+\sqrt{7}}\right| \\
&\leq 3\left|\frac{x-2}{\sqrt{7}}\right| \text{ (because } \sqrt{3x+1}\geq 0) \\
&= \frac{3}{\sqrt{7}}|x-2|
\end{aligned}$$

In this case we take $\delta = \frac{\sqrt{7}}{3}\epsilon$.

Formal Proof: Let $\epsilon > 0$ be given, choose $\delta = \frac{\sqrt{7}}{3}\epsilon$ and assume that $0 < |x-2| < \delta$. Then

$$\begin{aligned}
|\sqrt{3x+1}-\sqrt{7}| &= \left|\frac{\sqrt{3x+1}-\sqrt{7}}{1}\cdot\frac{\sqrt{3x+1}+\sqrt{7}}{\sqrt{3x+1}+\sqrt{7}}\right| \\
&= \left|\frac{3x-6}{\sqrt{3x+1}+\sqrt{7}}\right| \\
&= 3\left|\frac{x-2}{\sqrt{3x+1}+\sqrt{7}}\right| \\
&\leq 3\left|\frac{x-2}{\sqrt{7}}\right| \\
&= \frac{3}{\sqrt{7}}|x-2| \\
&< \frac{3}{\sqrt{7}}\,\delta = \frac{3}{\sqrt{7}}\frac{\sqrt{7}}{3}\,\epsilon = \epsilon
\end{aligned}$$

So, with our assumptions, we have $|\sqrt{3x+1}-\sqrt{7}| < \epsilon$. $\square$

(6b.) Preliminary Analysis:

$$\begin{aligned}
|x^2-3x+2| &= |(x-2)(x-1)| \\
&= |(x-2)|\cdot|(x-1)|
\end{aligned}$$

If we assume that $|x-2| \leq 1$, then

$$\begin{aligned}
|x-1| &= |(x-2)+1| \\
&= |x-2|+1 \text{ (by the triangle inequality)} \\
&\leq 1+1 = 2
\end{aligned}$$

We can take $\delta = \min\{1, \frac{\epsilon}{2}\}$

Formal Proof: Let $\epsilon > 0$ be given, choose $\delta = \min\{1, \frac{\epsilon}{2}\}$ and assume that $0 < |x-2| < \delta$. Then

$$\begin{aligned}
|x^2-3x+2| &= |(x-2)(x-1)| \\
&= |(x-2)|\cdot|(x-1)| \\
&\leq 2|(x-2)| \text{ (because } \delta \leq 1) \\
&< 2\delta = 2\frac{\epsilon}{2} = \epsilon
\end{aligned}$$

So, with our assumptions, we have $|x^2-3x+2| < \epsilon$. $\square$