

Solutions to Exam 3 Review

(1a.) $f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$

(i) Critical points: $x = -3, 1$

(ii) $f(x)$ is inc. on $(-\infty, -3]$ and $[1, \infty)$ and $f(x)$ is dec. on $[-3, 1]$

(iii.) $f(x)$ has a rel. max. at $x = -3$ and $f(x)$ has a rel. max. at $x = 1$ (by first derivative test)

(iv.) $f(x)$ has no maximum or minimum on $(-\infty, \infty)$

(1b.) $f'(x) = \frac{-3x^2}{(x^3+1)^2}$

(i) Critical points: $x = 0, -1$

(ii) $f(x)$ is dec. on $(-\infty, \infty)$

(iii.) $f(x)$ has no rel. max or rel. min. on $(-\infty, \infty)$

(iv.) $f(x)$ has no minimum or maximum on $(-\infty, \infty)$

(1c.) $f'(x) = 3x^2 - 2x = 3x(x - \frac{2}{3})$

(i) Critical points: $x = -1, 0, \frac{2}{3}, 1$

(ii) $f(x)$ is inc. on $(-1, 0]$ and $[\frac{2}{3}, 1]$ and $f(x)$ is dec. on $[0, \frac{2}{3}]$

(iii.) $f(x)$ has a rel. max. at $x = 0, 1$ and $f(x)$ has a rel. min. at $x = -1, \frac{2}{3}$ (by first der. test)

(iv.) $f(0) = f(1) = 0$ is the max. and $f(-1) = -2$ is the min. of $f(x)$ on $[-1, 1]$

(1d.) $f'(x) = 4x^3 + 6x^2 = 4x^2(x + \frac{3}{2})$

(i) Critical points: $x = 0, 4$

(ii) $f(x)$ is inc. on $(-1, 4]$

(iii.) $f(x)$ has a rel. max. at $x = 4$ and $f(x)$ has no rel. min. (by first derivative test)

(iv.) $f(4) = 384$ is the max. of $f(x)$ on $(-1, 4]$. $f(x)$ has no min. on $(-1, 4]$

(1e.) $f'(x) = \frac{(x^2+1)-2x(x-1)}{(x^2+1)^2} = \frac{-(x^2-2x+1)}{(x^2+1)^2} = \frac{-[x-(1-\sqrt{2})][x-(1+\sqrt{2})]}{(x^2+1)^2}$

(i) Critical points: $x = -5, 1 - \sqrt{2}, 1 + \sqrt{2}, 5$

(ii) $f(x)$ is inc. on $[1 - \sqrt{2}, 1 + \sqrt{2}]$ and $f(x)$ is dec. on $[-5, 1 - \sqrt{2}]$ and $[1 + \sqrt{2}, 5]$

(iii.) $f(x)$ has a rel. max at $x = -5, 1 + \sqrt{2}$ and $f(x)$ has a rel. min. at $x = 1 - \sqrt{2}, 5$ (by first derivative test)

(iv.) $f(1 + \sqrt{2}) = \frac{\sqrt{2}}{4+2\sqrt{2}} \approx 0.207$ is the max and $f(1 - \sqrt{2}) = \frac{-\sqrt{2}}{4-2\sqrt{2}} \approx -1.207$ is the min.

(1e.) $f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

(i) Critical points: $x = -1, 1, 3$

(ii) $f(x)$ is inc. on $[-1, 1]$ and $f(x)$ is dec. on $(-3, -1]$ and $[1, 3]$.

(iii.) $f(x)$ has a rel. max at $x = 1$ and a rel. min at $x = -1, 3$ (by first derivative test)

(iv.) $f(1) = \frac{1}{2}$ is the max. and $f(-1) = -\frac{1}{2}$ is the min.

(2a.) $f'(x) = 3x^2 - 2x \Rightarrow f''(x) = 6x - 2$. $f(x)$ is concave down on $(-\infty, \frac{1}{3})$ and concave up on $(\frac{1}{3}, \infty)$. There is an inflection point at $x = \frac{1}{3}$.

(2b.) $f'(x) = \frac{-6x}{(1+3x^2)^2} \Rightarrow f''(x) = \frac{(1+3x^2)^2(-6)+6x(2)(1+3x^2)(6x)}{(1+3x^2)^4} = \frac{6(1+3x^2)(9x^2-1)}{(1+3x^2)^4}$. $f(x)$ is concave up on $(-\infty, -\frac{1}{3})$ and on $(\frac{1}{3}, \infty)$ and $f(x)$ is concave down on $(-\frac{1}{3}, \frac{1}{3})$. There are inflection points at $x = -\frac{1}{3}, \frac{1}{3}$.

(3.) $S = x + y$ and $36 = P = xy \Rightarrow y = \frac{36}{x}$ and $S = x + \frac{36}{x}$. Note that the interval for x is $(-\infty, 0)$.
 $S'(x) = 1 - \frac{36}{x^2} = \frac{x^2 - 36}{x^2}$. Setting $S'(x) = 0$ gives $\frac{x^2 - 36}{x^2} = 0 \Rightarrow x^2 - 36 = 0$ so $x = -6$ is the only critical point. By the first derivative test, $x = -6$ is a relative maximum, hence $S(6) = 12$ is the largest possible sum. There is no smallest sum.

(4.) See Fig. 1. The total distance is $D(x) = 2\sqrt{x^2 + 1} + (1 - x)$ and the interval for x is $[0, 1]$.
 $D'(x) = \frac{2x}{\sqrt{x^2 + 1}} - 1$. Setting $D'(x) = 0$ gives $\frac{2x}{\sqrt{x^2 + 1}} - 1 = 0 \Rightarrow \frac{2x}{\sqrt{x^2 + 1}} = 1 \Rightarrow 2x = \sqrt{x^2 + 1}$
 Squaring both sides gives: $4x^2 = x^2 + 1 \Rightarrow 3x^2 = 1 \Rightarrow x^2 = \frac{1}{3}$. So the only critical point is $x = \frac{1}{\sqrt{3}}$.
 Now $D(0) = 3$, $D(1) = 2\sqrt{2}$ and $D(1/\sqrt{3}) = \sqrt{3} + 1$. As the last value is the smallest, the distance is minimal when $x = \frac{1}{\sqrt{3}}$.

(5.) See Fig. 2. The cost of the container is given by $C = 1\pi r^2 + 2(2\pi r h) + 5\pi r^2$. Note also $100 = V = \pi r^2 h \Rightarrow h = \frac{100}{\pi r^2}$, so $C(r) = 6\pi r^2 + \frac{400\pi r}{\pi r^2} = 6\pi r^2 + \frac{400}{r}$. Note that the interval for r is $(0, \infty)$.
 $C'(r) = 12\pi r - \frac{400}{r^2} = \frac{12\pi r^3 - 400}{r^2}$. Setting $C'(r) = 0$ gives $\frac{12\pi r^3 - 400}{r^2} = 0 \Rightarrow 12\pi r^3 - 400 = 0$. So the only critical point is $r = \sqrt[3]{\frac{100}{3\pi}}$. By the first derivative test, this is a local minimum. Therefore the least expensive container to produce has radius $r = \sqrt[3]{\frac{100}{3\pi}}$.

(6.) First suppose all 200 items are shipped by plane. In this case the cost is 120,001 dollars. Next suppose that all 200 items are shipped by truck. In this case the cost is 40,005 dollars. Now we may assume that at least one item is shipped by each method. If x items are sent by plane, then $x - 200$ items are sent by truck. We may also assume that x is on the interval $(0, 200)$. Here the cost is $C(x) = [1 + 3x^2] + [5 + (200 - x)^2]$.

$C'(x) = 6x + 2(200 - x)(-1) = 8x - 400$. Setting $C'(x) = 0$ gives $8x - 400 = 0$, so the only critical number is $x = 50$. By the first derivative test, this is a relative minimum. Note that we need to check $C(50) = 30,006$. Thus the cost is minimal if 50 items are sent by plane and 150 are sent by truck.

(7a.) $\int (x + 1)^3 dx = \frac{(x + 1)^4}{4} + C$

(7b.) $\int (x - 1)(x + 2) dx = \int x^2 + x - 2 dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x + C$

(8a.) $f'(x) = \frac{x^2}{2} + C_1 \Rightarrow f(x) = \frac{x^3}{6} + C_1 x + C_2$. Now $1 = f(1) = \frac{1}{6} + C_1 + C_2$ and $2 = f(-1) = -\frac{1}{6} - C_1 + C_2$. Solving for C_1 and C_2 gives: $C_1 = -\frac{2}{3}$ and $C_2 = \frac{3}{2}$, so $f(x) = \frac{x^3}{6} - \frac{2x}{3} + \frac{3}{2}$

(8b.) $f'(x) = 2x + C_1 \Rightarrow f(x) = x^2 + C_1 x + C_2$. Now $2 = f(0) = C_2$ and $0 = f(2) = 4 + 2C_1 + C_2$. Solving for C_1 and C_2 gives: $C_1 = -3$ and $C_2 = 2$, so $f(x) = x^2 - 3x + 2$.