Review 3

(1.) If $A = (a_{ij})$ is a real $n \times n$ matrix, then the trace of A is the real valued function $\text{tr}(A) = a_{11} + a_{22} + \ldots + a_{nn}$. Show that $\langle A, B\rangle = \text{tr}(A^T B)$ defines an inner product on $\mathbb{R}^{2\times 2}$.

(2.) Let $\mathbf{x} = (3, 1, -1, 5)^T$ and $\mathbf{y} = (2, 6, -4, 5)^T$ be vectors in $\mathbb{R}^4$ with the usual inner product. Find:

(a.) $\langle \mathbf{x}, \mathbf{y}\rangle$, $||\mathbf{x}||$, $||\mathbf{y}||$,

(b.) the orthogonal projection of $\mathbf{x}$ onto the line spanned by $\mathbf{y}$ and

(c.) the angle between $\mathbf{x}$ and $\mathbf{y}$.

(3.) Let $\mathbf{f} = x$ and $\mathbf{g} = \ln(x)$ be vectors in $\text{C}[1, e]$ with the usual inner product. Find:

(a.) $\langle \mathbf{f}, \mathbf{g}\rangle$, $||\mathbf{f}||$, $||\mathbf{g}||$,

(b.) the orthogonal projection of $\mathbf{f}$ onto the line spanned by $\mathbf{g}$ and

(c.) the angle between $\mathbf{f}$ and $\mathbf{g}$.

(4.) Suppose that $A = \begin{pmatrix} -6 & 3 & -27 & -33 & -13 \\ 6 & -5 & 25 & 28 & 14 \\ 8 & -6 & 34 & 38 & 18 \\ 12 & -10 & 50 & 41 & 23 \\ 14 & -21 & 49 & 29 & 33 \end{pmatrix}$.

(a.) Find a basis for $\text{Col}(A)$.

(b.) Find a basis for $\text{Col}(A)^{\perp}$.

(5.) Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 2 \\ -2 & 0 & -1 \end{pmatrix}$ be vectors in $\mathbb{R}^{3\times 3}$ with the usual inner product.

(a.) Verify that $\{A, B, C\}$ form an orthogonal basis for $S = \text{Span}(A, B, C)$ and

(b.) find the orthogonal projection of $D = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 4 & 5 \\ 0 & 1 & -2 \end{pmatrix}$ onto S.

(6.) Let $\mathbf{f}_1 = 1$, $\mathbf{f}_2 = 2x - 1$ and $\mathbf{f}_3 = 12x^2 - 12x + 2$ be vectors in $\text{C}[0, 1]$ with the usual inner product.

(a.) Verify that $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ form an orthogonal basis for $S = \text{Span}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ and

(b.) find the orthogonal projection of x^4 onto S.

(7.) Suppose that $f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$. Find the nth order Fourier series for $f(x)$ over $[-\pi, \pi]$.

(8.) To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second from $t = 0$ to $t = 5$. The positions in meters were: $0, 29.9, 104.7, 222.0, 380.4$ and 571.7

(a.) Find the least squares cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

(b.) Use the result of (a.) to estimate the velocity of the plane when $t = 4.5$ seconds.

(9.) Let $A = \begin{pmatrix} 7 & -6 & -4 & 1 \\ -5 & 1 & 0 & -2 \\ 10 & 11 & 7 & -3 \\ 19 & 9 & 7 & 1 \end{pmatrix}$

(a.) Find the condition number of A with respect to the 1-norm on A.

(b.) Find the condition number of A with respect to the ∞-norm on A.

(c.) Is this matrix well-conditioned or ill-conditioned? Explain your answer.

(10.) Let $A = \begin{pmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0.217 \\ 0.254 \end{pmatrix}$

(a.) Find the condition number of A with respect to the ∞-norm.

(b.) A solution to $A\mathbf{x} = \mathbf{b}$ is calculated by first rounding the entries in A to the nearest 0.01 and then row reducing. This calculated solution is $\mathbf{x}_c = (0.1885, 0.1250)^T$. Find an upper bound on the relative error with this calculation with respect to the ∞-norm.

(11.) True or False:

(a.) If A is an $m \times n$ matrix, then $\dim(\text{Col}(A)) + \dim(\text{N}(A^T)) = n$.

(b.) If $\mathbf{p}$ is the orthogonal projection of $\mathbf{x}$ onto the subspace S, then $\mathbf{p}$ and $\mathbf{p} + \mathbf{x}$ are orthogonal.

(c.) The matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution $\hat{\mathbf{x}}$.

(d.) If V is a normed linear space and $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, then $||\mathbf{x} + \mathbf{y}|| \geq ||\mathbf{x}|| + ||\mathbf{y}||$.

(e.) If V is a inner product space and S is a finite dimensional subspace of V, then S has an orthonormal basis.

(12.) Prove:

(a.) If S is subspace of the inner product space V, then the orthogonal complement $S^\perp$ of S is also a subspace.

(b.) If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors in the inner product space V, then $||\mathbf{x} - \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$.

(c.) If $\mathbf{p}$ is the projection of $\mathbf{x}$ onto the line spanned by $\mathbf{y}$, then $\mathbf{p}$ is orthogonal to $\mathbf{x} - \mathbf{p}$.