Review 2 Solutions

(1.) Row reduce A to $U = \begin{pmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(1a.) A basis for the rowspace of A is given by the non-zero rows of U:

$$\{(1,2,0,0,-1,0),(0,0,1,0,-2,0),(0,0,0,1,5,0),(0,0,0,0,0,1)\}$$

(1b.) A basis for the columnspace of A is given by the columns of A which correspond to pivot columns in U:

$$\left\{ \begin{pmatrix} 3 \\ 5 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 7 \\ 2 \end{pmatrix} \right\}$$

(1c.) The matrix U corresponds to the linear system:

$$\begin{aligned} x_1 + 2x_2 - x_5 &= 0 \\ x_3 - 2x_5 &= 0 \\ x_4 + 5x_5 &= 0 \\ x_6 &= 0 \end{aligned}$$

which can be rewritten as:

$$\begin{aligned} x_1 &= -2x_2 + x_5 \\ x_2 &= x_2 \\ x_3 &= 2x_5 \\ x_4 &= -5x_5 \\ x_5 &= x_5 \\ x_6 &= 0 \end{aligned}$$

The parametric solution is then $\mathbf{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \\ 2 \\ -5 \\ 1 \\ 0 \end{pmatrix} x_5$ which means a basis for $N(A)$ is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\}$

(1d.) The rank of A is the dimension of its rowspace, which is 4.

(2.) Form the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 2 \\ 1 & 3 & 4 & 8 \end{pmatrix}$ and row reduce to $U = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Note that $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is equal to the columnspace of A. Thus a basis is given by the columns of A which correspond to pivot columns in U. In particular:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}$$

is a basis for $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

(3a.) NO: Just note that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are in S, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not in S.

(3b.) YES: A typical vector is: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

(o.) $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in S.

(i.) $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ 0 & c+f \end{pmatrix}$, so the sum of any two vectors in S remains in S.

(ii.) $\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ 0 & \alpha c \end{pmatrix}$, so the scalar multiples of any vector in S remains in S.

(3c.) NO: Just note that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are in S, but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is not in S.

(3d.) YES:

(o.) $\mathbf{0}(1) = 0$ so $\mathbf{0}$ is in S.

(i.) Let $p(x)$ and $q(x)$ be vectors in S. Then $p(1) = 0$ and $q(1) = 0$. Now consider $(p+q)(x) = p(x) + q(x)$:

$$\begin{aligned} (p+q)(1) &= p(1) + q(1) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

which shows that $p(x) + q(x)$ is in S.

(ii.) Let α be a scalar and consider $\alpha p(x)$:

$$\begin{aligned} \alpha p(1) &= \alpha(0) \\ &= 0 \end{aligned}$$

which shows that $\alpha p(x)$ is in S.

(3e.) YES: note that $f(x)$ is an odd function if and only if $f(-x) = -f(x)$.

(o.) $\mathbf{0}(-x) = 0 = \mathbf{0}(x)$ for all x, so $\mathbf{0}$ is in S.

(i.) Let $f(x)$ and $g(x)$ be vectors in S. Then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now consider $(f+g)(x) = f(x) + g(x)$:

$$\begin{aligned} (f+g)(-x) &= f(-x) + g(-x) \\ &= -f(x) - g(x) \\ &= -[f(x) + g(x)] \\ &= -(f+g)(x) \end{aligned}$$

which shows that $f(x) + g(x)$ is in S.

(ii.) Let α be a scalar and consider $\alpha f(x)$:

$$\begin{aligned} \alpha f(-x) &= \alpha[-f(x)] \\ &= -[\alpha f(x)] \end{aligned}$$

which shows that $\alpha f(x)$ is in S.

(4.) Compute the Wronskian of these functions:

$$W = \det \begin{pmatrix} 1 & e^x & \cos x \\ 0 & e^x & -\sin x \\ 0 & e^x & -\cos x \end{pmatrix} = e^x(\sin x - \cos x)$$

Since this determinant is not zero everywhere, $1, e^x$ and $\cos x$ are linearly independent in $C[0,1]$.

(5a.) A typical vector in S is $\mathbf{x} = \begin{pmatrix} a-b+c \\ a+c \\ a+2b-c \\ b-3c \end{pmatrix}$ which can be written as $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} a + \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} b + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} c$. Thus a basis is given by

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\}$$

Note, we do have to check that these vectors are linearly independent, which we can do by inspection.

(5b.) Note that $\sin 2x = 2\sin x \cos x$, so these vectors are linearly dependent. In particular, $\sin x \cos x$ depends on $\sin 2x$. We remove one of these vectors and check if the remaining vectors are linearly independent. We can check if 1 and $\sin 2x$ are linearly independent by computing a Wronskian:

$$W = \det \begin{pmatrix} 1 & \sin 2x \\ 0 & 2\cos 2x \end{pmatrix} = 2\cos 2x$$

As this function is not zero everywhere, 1 and $\sin 2x$ are linearly independent and form a basis for S.

(5c.) A typical vector is $p(x) = (x)(x-1)(ax+b) = (x)(x-1)(x)(a) + (x)(x-1)(b)$. So a basis is given by $\{(x^2)(x-1), (x)(x-1)\}$. Again, we do need to check if these are linearly independent, but we can see that they are by inspection.

(6.) Since we need to find a basis for S as well as show that it is a subspace, we will start by finding a typical vector in S:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 2a+b \\ c & 2c+d \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix}$$

Since these matrices are equal we get the linear system:

$$\begin{aligned} a &= a+2c \\ 2a+b &= b+2d \\ c &= c \\ 2c+d &= d \end{aligned}$$

Thus $a = d$ and $c = 0$ which means a typical vector has the form: $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

(6a.)

(o.) $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in S.

(i.) $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & a+c \end{pmatrix}$, so the sum of any two vectors in S remains in S.

(ii.) $\alpha\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ 0 & \alpha a \end{pmatrix}$, so the scalar multiples of any vector in S remains in S.

Thus S is a subspace of $\mathbb{R}^{2\times 2}$.

(6b.) We can write the typical vector $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} a + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b$. Thus a basis is given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

(7a.) TRUE: can always take the coefficient of $\mathbf{0}$ to be non-zero.

(7b.) TRUE: every vector space must contain $\mathbf{0}$.

(7c.) FALSE: $\dim(\mathrm{Col}(A))+\dim(\mathrm{N}(A))=n$

(7d.) TRUE: $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the span of the columns of A, which is the same as the column space of A.

(7e.) FALSE: If A is singular, then the columns of A are linearly dependent and can not be a basis.

(7f.) FALSE: A basis is a set of vectors which is both linearly independent and a spanning set.

(8a.) (o.) $\mathbf{0}^T = \mathbf{0}$, so $\mathbf{0}$ is in S.

(i.) Let A and B be vectors in S. Then $A^T = A$ and $B^T = B$. Now consider $A + B$:

$$\begin{aligned}(A+B)^T &= A^T + B^T \\ &= A + B\end{aligned}$$

which shows that $A + B$ is in S.

(ii.) Let α be a scalar and consider αA:

$$\begin{aligned}(\alpha A)^T &= \alpha(A^T) \\ &= \alpha A\end{aligned}$$

which shows that αA is in S.

(8b.) Suppose that $\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ do span V, then $\mathbf{v}_1$ can be written as a linear combination of these vectors. In particular

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

for some $c_2, \ldots, c_n$. Subtracting $\mathbf{v}_1$ from both sides yields:

$$c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n - \mathbf{v}_1 = \mathbf{0}$$

Note that the coefficient of $\mathbf{v}_1$ is not zero. This contradicts the fact that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. Thus $\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ can not span V.

(8c.) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ do not span V, then there is a vector $\mathbf{v}_0$ in V which is not a linear combination of these vectors. Now consider

$$c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

Subtracting $c_0\mathbf{v}_0$ from both sides gives:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = -c_0\mathbf{v}_0$$

If $c_0 \neq 0$ then we have just written $\mathbf{v}_0$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. As this is not the case, $c_0 = 0$. This leaves the equation:

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

As $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, $c_1 = \cdots = c_n = 0$. Thus $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set.