Proving Logical Equivalencies and Biconditionals

Suppose that we want to show that P is logically equivalent to Q. We need to show that these two sentences have the same truth values. One method that we can use is to assume P is true and show that Q must be true under this assumption and then to assume Q is true and show that P must be true under this assumption. An equivalent method relies on the following:

$$P \text{ is logically equivalent to } Q \text{ is the same as } P \Leftrightarrow Q \text{ being a tautology}$$

Now recall that there is the following logical equivalence:

$$P \Leftrightarrow Q \text{ is logically equivalent to } (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

So to show that $P \Leftrightarrow Q$ is a tautology we show both $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$ are tautologies.

Example 1: Show that $[(P \wedge Q) \Rightarrow R] \Leftrightarrow [P \Rightarrow (Q \Rightarrow R)]$ is a tautology. Note that this question could have been rephrased as: "Show that $(P \wedge Q) \Rightarrow R$ is logically equivalent to $P \Rightarrow (Q \Rightarrow R)$". We will break the proof into two parts which we label $(\Rightarrow)$ and $(\Leftarrow)$.

Proof:

$(\Rightarrow)$**:** We wish to show $[(P \wedge Q) \Rightarrow R] \Rightarrow [P \Rightarrow (Q \Rightarrow R)]$ is a tautology

(A1): Assume that $(P \wedge Q) \Rightarrow R$ is true.

We need to show that $P \Rightarrow (Q \Rightarrow R)$ is true.

(A2): Assume P is true.

We need to show $Q \Rightarrow R$ is true.

(A3): Assume Q is true.

We need to show R is true.

Since P is true by (A2) and Q is true by (A3), $P \wedge Q$ is true. As $(P \wedge Q) \Rightarrow R$ is true by (A1), R is true by modus ponens.

Discharging (A3), $Q \Rightarrow R$ is true under only (A1) and (A2).

Discharging (A2), $P \Rightarrow (Q \Rightarrow R)$ is true under only (A1).

Discharging (A1), $[(P \wedge Q) \Rightarrow R] \Rightarrow [P \Rightarrow (Q \Rightarrow R)]$ is true under no assumptions, thus $[(P \wedge Q) \Rightarrow R] \Rightarrow [P \Rightarrow (Q \Rightarrow R)]$ is a tautology.

$(\Leftarrow)$**:** We wish to show $[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \wedge Q) \Rightarrow R]$ is a tautology

(A1): Assume that $P \Rightarrow (Q \Rightarrow R)$ is true.

We need to show that $(P \wedge Q) \Rightarrow R$ is true.

(A2): Assume $P \wedge Q$ is true.

We need to show R is true.

By (A2), P and Q are true. Since $P \Rightarrow (Q \Rightarrow R)$ is true by (A1), $Q \Rightarrow R$ is true by modus ponens. Therefore R is true by modus ponens.

Discharging (A2), $(P \wedge Q) \Rightarrow R$ is true under only (A1).

Discharging (A1), $[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \wedge Q) \Rightarrow R]$ is true under no assumptions, thus $[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \wedge Q) \Rightarrow R]$ is a tautology.

Therefore we have show that $[(P \wedge Q) \Rightarrow R] \Rightarrow [P \Rightarrow (Q \Rightarrow R)]$ is a tautology and that $[P \Rightarrow (Q \Rightarrow R)] \Rightarrow [(P \wedge Q) \Rightarrow R]$ is a tautology. Thus $[(P \wedge Q) \Rightarrow R] \Leftrightarrow [P \Rightarrow (Q \Rightarrow R)]$ is a tautology.

Notice that in this example, the forward implication ($\Rightarrow$) was harder than the reverse implication ($\Leftarrow$). This is a common occurrence in proving biconditionals.