Converses, Contrapositives and Proof by the Contrapositive

The **converse** of the implication $P \Rightarrow Q$ is the reverse implication $Q \Rightarrow P$. It is very important to realize that these two implications are **not** logically equivalent.

Example 1: From calculus, if $f(x)$ is continuous on $[a, b]$ then the Riemann integral $\int_a^b f(x)\,dx$ exists. But the converse of this statement: if the Riemann integral $\int_a^b f(x)\,dx$ exists then $f(x)$ is continuous on $[a, b]$ is not true. There are functions which are not continuous on $[a, b]$ which are integrable over $[a, b]$. In particular, if $f(x)$ is a function with a finite number of discontinuities on $[a, b]$ it will be integrable over $[a, b]$.

Sometimes replacing an implication by its contrapositive leads to an easier implication to prove. We can also form the contrapositive of a biconditional: if $P \Leftrightarrow Q$ then $\neg Q \Leftrightarrow \neg P$. These two biconditionals are also logically equivalent.

Example 2: Another example from calculus: if $f(x)$ is differentiable at a then $f(x)$ is continuous at a.

(a.) The converse of this statement is: if $f(x)$ is continuous at a then it is differentiable at a. This statement is false, the classic example being $f(x) = |x|$ at $a = 0$.

(b.) The contrapositive of this statement is: if $f(x)$ is not continuous at $x = a$ then $f(x)$ is not differentiable at a. This statement is true as it is the contrapositive of a true statement.

Example 3: Show that if $x \neq 5$ then $x^2 - 10x + 25 \neq 0$ is always true.

Proof: Let P be the statement $x \neq 5$ and Q be the statement $x^2 - 10x + 25 \neq 0$, then we wish to show that $P \Rightarrow Q$ is always true. We will do this by showing that the contrapositive is always true. Namely, $\neg Q \Rightarrow \neg P$. The contrapositive is: if $x^2 - 10x + 25 = 0$ then $x = 5$. Using the chain of biconditionals:

$$x^2 - 10x + 25 = 0 \Leftrightarrow (x-5)^2 = 0 \Leftrightarrow x - 5 = 0 \Leftrightarrow x = 5$$

we see that $\neg Q \Rightarrow \neg P$ is always true. We actually have proved a stronger statement, that $\neg Q \Leftrightarrow \neg P$.

As we see in this example, sometimes it is easier to prove a stronger statement then what is being asked.