Elementary Matrices

1. Preliminaries

Consider the following situation: A is a matrix, possible augmented, and U is the reduced row echelon form of A. The U is obtained from A by a series of elementary row operations. However, these operations are, in some sense, external to the matrix A. It turns out that we can accomplish this row reduction by multiplying A by a sequence of matrices $\mathcal{E}_i$ called elementary matrices. In other words, $U = \mathcal{E}_k \cdots \mathcal{E}_1 A$. There are three types of elementary matrices, each corresponding to one of the types of elementary row operations. Note that elementary matrices are necessarily $n \times n$.

2. Definitions and Notation

We will assume that all of our matrices are of size $n \times n$ and our notation will not refer to their size. Also all specific examples will be 3×3 matrices. For convenience let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

1. Permutation Matrices

Notation: $P_{ij}, i \neq j$

Definition: $P_{ij} =$ the $n \times n$ identity matrix I_n with row i and row j interchanged

Examples: $P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad P_{12}A = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$ $\quad AP_{12} \begin{pmatrix} b & a & c \\ e & d & f \\ h & g & i \end{pmatrix}$

Multiplication on Left: Interchanges row i and row j of A.

Multiplication on Right: Interchanges column i and column j of A.

Inverse: $P_{ij}^{-1} = P_{ij}$

Determinant: $\det(P_{ij}) = -1$

2. Diagonal Matrices

Notation: $D_i(t), t \neq 0$

Definition: $D_i(t) =$ the $n \times n$ identity matrix I_n with t in row i, column i

Examples: $D_3(5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ $\quad D_3(5)A = \begin{pmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{pmatrix}$ $\quad AD_3(5) \begin{pmatrix} a & b & 5c \\ d & e & 5f \\ g & h & 5i \end{pmatrix}$

Multiplication on Left: Multiplies row i of A by t.

Multiplication on Right: Multiplies column i of A by t

Inverse: $D_i(t)^{-1} = D_i(1/t)$

Determinant: $\det(D_i(t)) = t$

3. Unipotent Matrices

Notation: $E_{ij}(t), i \neq j$

Definition: $E_{ij}(t) =$ the $n \times n$ identity matrix I_n with t in row i, column j

Examples: $E_{13}(4) = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad E_{13}(4)A = \begin{pmatrix} a+4g & b+4h & c+4i \\ d & e & f \\ g & h & i \end{pmatrix}$ $\quad AE_{13}(4) \begin{pmatrix} a & b & 4a+c \\ d & e & 4d+f \\ g & h & 4g+i \end{pmatrix}$

Multiplication on Left: Adds t times row j of A to row i of A.

Multiplication on Right: Adds t times column i of A to column j of A.

Inverse: $E_{ij}(t)^{-1} = E_{ij}(-t)$

Determinant: $\det(E_{ij}(t)) = 1$

3. Main Theorem

Theorem 3.1. *Let A be an $n \times n$ matrix, then the following are equivalent:*

(1) *A is non-singular.*

(2) *$A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*

(3) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(4) *A can be row reduced to the identity matrix.*

Proof: The way this theorem is proved is by assuming each statement is true, and showing that the next statement is also true. Then we assume statement (4.) is true, and show that (1.) must be true. Logically we get (1.) $\Rightarrow$ (2.) $\Rightarrow$ (3.) $\Rightarrow$ (4.) $\Rightarrow$ (1.), so that the truth of any one statement implies the truth of all statements.

(**1**.) $\Rightarrow$ (**2**.) Assume that A is non-singular so that A has an inverse A^{-1}. If $\mathbf{b}$ is any vector in $\mathbb{R}^n$, then $A(A^{-1}\mathbf{b}) = \mathbf{b}$. In particular the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution, namely $A^{-1}\mathbf{b}$. As $\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$, this solution must be unique.

(**2**.) $\Rightarrow$ (**3**.) Assume that $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$. By taking $\mathbf{b} = \mathbf{0}$ and considering the uniqueness of the solution, $\mathbf{x}$ must be the trivial solution.

(**3**.) $\Rightarrow$ (**4**.) Suppose that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Let U be the reduced row echelon form of A. Since there is only one solution to the above equation, the corresponding linear system has no free variables. This means that every column of U must be a piviot column. In particular, U has n leading ones as U has n columns. But, U has n rows as well. There is exactly one $n \times n$ matrix with this property, namely I_n. Thus $U = I_n$ and A can be row reduced to I_n.

(**4**.) $\Rightarrow$ (**1**.) Suppose that A can be row reduced to I. Then there is a sequence of elementary matrices $\mathcal{E}_1, \ldots, \mathcal{E}_k$ such that $\mathcal{E}_k \cdots \mathcal{E}_1 A = I$. By definition, $\mathcal{E}_k \cdots \mathcal{E}_1 = A^{-1}$ implying that A is invertible. $\square$

Notice that in the last part of the proof above $A^{-1} = \mathcal{E}_k \cdots \mathcal{E}_1$. By taking inverses, we get $A = (\mathcal{E}_k \cdots \mathcal{E}_1)^{-1}$. Thus $A = \mathcal{E}_1^{-1} \cdots \mathcal{E}_k^{-1}$. This implies the following important corollary:

Corollary 3.1. *If A is an $n \times n$ matrix then A is non-singular if and only if A is the product of elementary matrices.*