Review 2

(1.) Let $A = \begin{pmatrix} 1 & 2 & -1 & 2 & 2 & 4 & 4 \\ 3 & 6 & -5 & -8 & 7 & 6 & -7 \\ 2 & 4 & 1 & 1 & 0 & 4 & 13 \\ -2 & -4 & -3 & 1 & 3 & 1 & -12 \\ 2 & 4 & -2 & 4 & 4 & 8 & 8 \end{pmatrix}$ and find:

(a.) A basis for the row space of A.
(b.) A basis for the column space of A.
(c.) A basis for the nullspace of A.
(d.) The rank and nullity of A.

(2.) Suppose that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{v}_4 = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}$. Find a basis for $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$. Note: there is more than one answer.

(3.) Determine if S is a subspace of V where:
(a.) $V = \mathbb{R}^{2\times 2}$ and S is the set of 2×2 matrices A with $\det(A)$=0.
(b.) $V = \mathbb{R}^{2\times 2}$ and S is the set of 2×2 upper triangular matrices.
(c.) $V = \mathbb{R}^2$ and $S = \{(x_1, x_2)^T \mid |x_1| = |x_2|\}$.
(d.) $V = \mathbb{P}_5$ and S is the set of all polynomials $p(x)$ in V such that $p(1) = 0$.
(e.) $V = C[-1, 1]$ and S is the set of odd functions in V.

(4.) Determine if $1, e^x$ and $\cos x$ are linearly independent in $C[0, 1]$.

(5.) Find a basis for the subspace S of V where:
(a.) $V = \mathbb{R}^4$ and $S = \{(a - b + c, a + c, a + 2b - c, b - 3c)^T \mid a, b, c, d$ are real numbers $\}$.
(b.) $V = C[0, 1]$ and S=Span$(1, \sin 2x, \sin x \cos x)$.
(c.) $V = \mathbb{P}_4$ and S is the set of all polynomials $p(x)$ in V with $p(0) = 0$ and $p(1) = 0$.

(6.) True or False?
(a.) A linearly independent set can not contain $\mathbf{0}$.
(b.) A subspace of a vector space must contain $\mathbf{0}$.
(c.) If A is an $m \times n$ matrix, then $\dim(\text{Col}(A))+\dim(\text{N}(A))=m$.
(d.) If A is an $m \times n$ matrix, then $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.
(e.) If A is a singular $n \times n$ matrix, then the columns of A form a basis for $\mathbb{R}^n$.
(f.) A spanning set can never be linearly independent.

(7.) Suppose that a molecule has three excited states which are denoted by A, B and C. Each second, the probability that it transitions from one state to the other is as follows: From A to B: 0.2, from A to C: 0.3, from B to A: 0.4, from B to C: 0.2, from C to A: 0.5 and from C to B: 0.2. Note that these transitions are a Markov process.
(a.) Find the transition matrix of this process.
(b.) Suppose that the initial distributions of states is 100 in state A, 75 in state B and 25 in state C. Find the resulting distributions after 10 seconds.
(c.) Find the steady state probability vector for this process.

(8.) Prove the following:
(a.) Let S be the subset of $\mathbb{R}^{n\times n}$ consisting of matrices A such that $A^T = -A$. Show that S is a subspace of $\mathbb{R}^{n\times n}$. (The matrices in S are called skew-symmetric matrices.)
(b.) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent vectors in a vector space V, then $\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ does not span V.
(c.) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent vectors in a vector space V which do not span V, then there is a vector $\mathbf{v}_{n+1}$ in V such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$ is also linearly independent.