Math 572

$\boxed{4.2}$

(3a.) $L\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ $\quad L\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad L\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\Rightarrow \quad A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

(3b.) $L\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad L\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\quad L\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\Rightarrow \quad A=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$$

(3c.) $L\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\3\\2\end{pmatrix}$ $\quad L\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad L\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}$

$$\Rightarrow \quad A=\begin{pmatrix}0&0&2\\3&1&0\\2&0&-1\end{pmatrix}$$

Math 572

$\boxed{4.2}$

(14.) $L(x^2) = 2x \qquad L(x) = 1 \qquad L(1) = 1$

$\Rightarrow$ L wrt to $[x^2, x, 1]$ and $[x, 1]$ is given by the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

T from $[2, 1-x]$ to $[x, 1]$ is given by the matrix

$$T = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1 & 0 \end{pmatrix}$$ goes from $[x,1]$ to $[2, 1-x]$

$\Rightarrow$ $B = T^{-1}A = \begin{pmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{pmatrix}$ is the matrix of L wrt

$[x^2, x, 1]$ and $[2, 1-x]$

(14a.) $B\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -2 \end{pmatrix}$

(14b.) $B\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -2 \end{pmatrix}$

(14c.) $B\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}$

(14d) $B\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$

4.2

(18a.) T is matrix of $[\bar{b}_1, \bar{b}_2]$ to $[\bar{e}_1, \bar{e}_2]$ $\Rightarrow$

$$T = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \Rightarrow T^{-1} \text{ goes from } [\bar{e}_2, \bar{e}_1] \text{ to } [b_1, b_2]$$

$$\Rightarrow T^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

(18a.) $L(\bar{u}_1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $L(\bar{u}_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $L(\bar{u}_3) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ the matrix of L wrt $[\bar{u}_1, \bar{u}_2, \bar{u}_3]$ and $[\bar{e}_1, \bar{e}_2]$ is

$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ $\Rightarrow$ the matrix of L wrt $[\bar{b}_1, \bar{b}_2]$ is

$$B = T^{-1}A = \begin{pmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

(18b.) $L(\bar{u}_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $L(\bar{u}_2) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $L(\bar{u}_3) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$\Rightarrow$ the matrix of L wrt to $[\bar{e}_1, \bar{e}_2]$ is

$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & -2 \end{pmatrix}$ $\Rightarrow$ the matrix of L wrt $[\bar{b}_1, \bar{b}_2]$ is

$$B = T^{-1}A = \begin{pmatrix} -5 & -3 & 4 \\ 3 & 3 & -2 \end{pmatrix}$$

(18c.) $L(\bar{u}_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $L(\bar{u}_2) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $L(\bar{u}_3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow$ the matrix of L wrt $[\bar{e}_1, \bar{e}_2]$ is

$A = \begin{pmatrix} 0 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$ $\Rightarrow$ the matrixof L wrt $[\bar{b}_1, \bar{b}_2]$ is

$$B = T^{-1}A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{pmatrix}$$

(2.) $L(\bar{e}_1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\quad L(\bar{e}_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ the matrix of L wrt $[\bar{e}_1, \bar{e}_2]$ is

$X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. If T_1 is the transition from $[\bar{u}_1, \bar{u}_2]$ to $[\bar{e}_1, \bar{e}_2]$

then $T_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. $\Rightarrow$ the matrix of L wrt $[\bar{u}_1, \bar{u}_2]$ is

$$B = T_1^{-1} X T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(2a.) If T_2 is the transition from $[\bar{v}_1, \bar{v}_2]$ to $[\bar{e}_1, \bar{e}_2]$ then

$T_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow$ the transition from $[\bar{u}_1, \bar{u}_2]$ to $[\bar{v}_1, \bar{v}_2]$ is given

by $S = T_2^{-1} T_1$

$= \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$

$$\begin{tikzcd} \bar{u}_i \arrow[r, "T_1"] \arrow[dr, dashed, "S"'] & \bar{e}_i \arrow[d, "T_2^{-1}"] \\ & \bar{v}_i \end{tikzcd}$$

(2b.) $A = SBS^{-1} = \begin{pmatrix} 1 & 0 \\ -4 & -1 \end{pmatrix}$

(2c.) $L(\bar{v}_1) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 1 \cdot \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\bar{v}_1} - 4 \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\bar{v}_2} = 1 \cdot \bar{v}_1 - 4 \cdot \bar{v}_2$

$L(\bar{v}_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \cdot \bar{v}_1 - 1 \cdot \bar{v}_2$

$\boxed{4.3}$

(5a.) $L(1) = 0 \qquad L(x) = x \qquad L(x^2) = 2x^2 + 2$

$$\Rightarrow \quad A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(5b.) S from $[1, x, 1+x^2]$ to $[1, x, x^2]$ is given by

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(5c.)

$$B = S^{-1} A S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(5d.) $B \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ a_1 \\ 2a_2 \end{pmatrix} \Rightarrow B^n \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ a_1 \\ 2^n a_2 \end{pmatrix} = a_1 x + 2^n a_2 (1+x^2$