Solutions to HW 4

$\boxed{6.1.1}$

(a.) $A-\lambda I = \begin{pmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{pmatrix} \Rightarrow \det(A-\lambda I) = (3-\lambda)(1-\lambda)-8$

$$= \lambda^2 - 4\lambda + 3 - 8$$
$$= \lambda^2 - 4\lambda - 5$$
$$= (\lambda-5)(\lambda+1)$$

$\Rightarrow$ eigenvalues are $\boxed{\lambda_1 = 5, \quad \lambda_2 = -1}$

$N(A-5I) \Rightarrow$ $\boxed{\text{Basis is } \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}}$

$N(A+I) \Rightarrow$ $\boxed{\text{Basis is } \left\{\begin{pmatrix}-1\\2\end{pmatrix}\right\}}$

(d.) $A-\lambda I = \begin{pmatrix} 3-\lambda & -8 \\ 2 & 3-\lambda \end{pmatrix} \Rightarrow \det(A-\lambda I) = (3-\lambda)(3-\lambda)+16$

$$= \lambda^2 - 6\lambda + 9 + 16$$
$$= \lambda^2 - 6\lambda + 25$$

$\Rightarrow \boxed{\begin{matrix}\lambda_1 = 3+4i \\ \lambda_2 = 3-4i\end{matrix}}$

$N(A-(3+4i)I) \Rightarrow$ Basis is $\boxed{\left\{\begin{pmatrix}2i\\1\end{pmatrix}\right\}}$

$\Rightarrow$ Basis for $N(A+(3-4i)I)$ is $\boxed{\left\{\begin{pmatrix}-2i\\1\end{pmatrix}\right\}}$

(i) $A-\lambda I = \begin{pmatrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{pmatrix} \Rightarrow \det(A-\lambda I) = (4-\lambda)[(-\lambda)(-1-\lambda)+1] - [(-5)(-1-\lambda)-1]$

$$= -\lambda^3 + 3\lambda^2 + 3\lambda + 4 - 4 - 5\lambda$$
$$= -\lambda(\lambda-1)(\lambda-2)$$

$\Rightarrow \boxed{\lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = -2}$

$N(A) \Rightarrow$ Basis is $\boxed{\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}}$

$N(A-I) \Rightarrow$ Basis is $\boxed{\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}\right\}}$

$N(A-2I) \Rightarrow$ Basis is $\boxed{\left\{\begin{pmatrix}7\\3\\1\end{pmatrix}\right\}}$

(6.1.1 cont)

(2.) $A - \lambda I = \begin{pmatrix} 3-\lambda & 0 & 0 & 0 \\ 4 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix} \Rightarrow$

$\det(A - \lambda I) = (3-\lambda)\left[(1-\lambda)(2-\lambda)(2-\lambda)\right] - 4[0]$

$= \lambda^4 - 8\lambda^3 + 23\lambda^2 - 28\lambda + 12$

$\Rightarrow \boxed{\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 3,\ \lambda_4 = 1}$

$\boxed{N(A - 2I) \Rightarrow \text{Basis is } \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}}$

$\boxed{N(A - 3I) \Rightarrow \text{Basis is } \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}}$

$\boxed{N(A - I) \Rightarrow \text{Basis is } \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}}$

6.1.10 or 6.1.13

$A - \lambda I = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} \Rightarrow \det(A - \lambda I) = \left[(\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta\right]$

$= \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta$

$= \lambda^2 - 2\lambda\cos\theta + 1.$

$\boxed{\text{roots are } \cos\theta \pm \sqrt{\cos^2\theta - 1}}$

Since $\cos^2\theta < 1$ if $\theta \neq k\pi \Rightarrow \cos^2\theta - 1 < 0$ if $\theta \neq k\pi$

$\Rightarrow$ roots (eigenvalues) are $\boxed{\text{non-real if } \theta \neq k\pi.}$

Geometrically, a rotation will have a $\boxed{\text{fixed if } \theta = 2k\pi}$ and will send vectors $\boxed{x \text{ to } -x \text{ if } \theta = (2k+1)\pi.}$

6.1.18 or 6.1.21

(a.) let λ be an eigenvalue of Q with eigenvector $\bar{x}$

$\|\bar{x}\| = \|Q\bar{x}\| = \|\lambda\bar{x}\| = |\lambda| \cdot \|\bar{x}\|$

$\Rightarrow |\lambda| = 1.$

(b.) $\det(A^T) = \det(A)$ for all A

$\Rightarrow \det(Q^T) = \det(Q)$ and, since $Q^T = Q^{-1}$, $\det(Q^T) = \det(Q^{-1})$

$\Rightarrow 1 = \det(Q^{-1}Q) = \det(Q) \cdot \det(Q) \Rightarrow \det(Q)^2 = 1$

$\Rightarrow |\det(Q)| = 1$

6.2.1

(a.) $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \Rightarrow$ $\lambda_1 = 2$, $\lambda_2 = 3$; $\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\bar{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\Rightarrow$ $\bar{Y}_1 = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\bar{Y}_2 = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Gen. solution is

$$\boxed{\begin{aligned} \bar{Y} &= c_1\bar{Y}_1 + c_2\bar{Y}_2 \\ &= c_1 e^{2t} + 2c_2 e^{3t} \\ & c_1 e^{2t} + c_2 e^{3t} \end{aligned}}$$

(b.) $A = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \Rightarrow$ $\lambda_1 = 3+2i$, $\lambda_2 = 3-2i$; $\bar{x}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$, $\bar{x}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ $\Rightarrow$ $\bar{Y}_1 = e^{3t}(\cos 2t - \sin 2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\bar{Y}_2 = e^{3t}(\cos 2t + \sin 2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \boxed{\begin{aligned} \bar{Y} &= c_1\bar{Y}_1 + c_2\bar{Y}_2 \\ &= \begin{pmatrix} c_2 e^{3t}\cos 2t + c_2 e^{3t}\sin 2t \\ c_1 e^{3t}\cos 2t - c_1 e^{3t}\sin 2t \end{pmatrix} \end{aligned}}$$

* note: there are other answers by changing $\bar{x}_1$ and $\bar{x}_2$

$\boxed{6.2.5}$

(a.) $y_1' = y_3$ $\qquad$ $\lambda_1 = -\sqrt{2} \sim -1.4142$

$y_2' = y_4 \Rightarrow A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{pmatrix} \Rightarrow$ $\qquad$ $\lambda_2 = -1$

$y_3' = -2y_2$ $\qquad$ $\lambda_3 = \sqrt{2} \sim 1.4142$

$y_4' = y_1 + 3y_2$ $\qquad$ $\lambda_4 = 1$

$$\bar{x}_1 = \begin{pmatrix} +\sqrt{2}/2 \\ -\sqrt{2}/2 \\ +1 \\ -1 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} +2 \\ -1 \\ +2 \\ -1 \end{pmatrix} \quad \bar{x}_3 = \begin{pmatrix} -\sqrt{2}/2 \\ +\sqrt{2}/2 \\ -1 \\ +1 \end{pmatrix} \quad \bar{x}_4 = \begin{pmatrix} -2 \\ +1 \\ -2 \\ +1 \end{pmatrix}$$

$$Y = c_1 \bar{Y}_1 + c_2 \bar{Y}_2 + c_3 \bar{Y}_3 + c_4 \bar{Y}_4$$

$$\bar{Y}_1 = e^{-\sqrt{2}t} \bar{x}_1 \quad \bar{Y}_2 = e^{-t} \bar{x}_2, \quad \bar{Y}_3 = e^{\sqrt{2}t} \bar{x}_3, \quad \bar{Y}_4 = e^{t} \bar{x}_4$$

* note (1.) there is more than one solution.

(2.) only need first two coord of Y_1, Y_2, Y_3, Y_4 (bec. $y_3 = y_1'$ and $y_4 = y_2'$).

(b.) $y_1' = y_3$ $\qquad$ $\lambda_1 = -2$

$y_2' = y_4 \Rightarrow A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \Rightarrow$ $\qquad$ $\lambda_2 = -1$

$y_3' = 2y_1 + y_4$ $\qquad$ $\lambda_3 = 2$

$y_4' = 2y_2 + y_3$ $\qquad$ $\lambda_4 = 1$

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 2 \end{pmatrix} \quad x_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \quad x_4 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\bar{Y}_1 = e^{-2t} \bar{x}_1 \quad \bar{Y}_2 = e^{-t} \bar{x}_2 \quad \bar{Y}_3 = e^{2t} \bar{x}_3 \quad \bar{X}_4 = e^{t} \bar{x}_4$$

$$\bar{Y} = c_1 \bar{Y}_1 + c_2 \bar{Y}_2 + c_3 \bar{Y}_3 + c_4 \bar{Y}_4$$

* see note for (a.).