Math 572
Winter 2010

EXAM 1 Solutions (Review)

(1a.) $\boxed{\text{Yes}}$: $L(\bar{x}+\bar{y}) = (x_3+y_3+x_2+y_2,\ x_1+y_1+x_2+y_2)^T = (x_3+x_2+y_3+y_2,\ x_1+x_2+y_1+y_2)^T$

$L(\bar{x})+L(\bar{y}) = (x_3+x_2,\ x_1+x_2)^T + (y_3+y_2,\ y_1+y_2)^T = L(\bar{x}+\bar{y})$

$L(\alpha\bar{x}) = (\alpha x_3+\alpha x_2,\ \alpha x_1+\alpha x_2) = \alpha(x_3+x_2,\ x_1+x_2) = \alpha L(\bar{x})$

(1b.) $\boxed{\text{No}}$: $L(\bar{0}) = (0,1)^T \neq (0,0)$

(1c.) $\boxed{\text{Yes}}$

$$L(f(x)+g(x)) = (f+g)'(0)\,x + (f+g)(0) = f'(0)x + f(0) + g'(0)x + g(0) =$$
$$= L(f(x)) + L(g(x))$$

$$L(\alpha f(x)) = (\alpha f)'(0)\,x + (\alpha f)(0) = \alpha\,[f'(0)x + f(0)]$$
$$= \alpha L(f(x))$$

(2.) $\ker(L) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, x_1 - x_2 = 0,\ x_2 - x_3 = 0,\ x_3 - x_1 = 0 \right\}$

$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, x_1 = x_2 = x_3 \right\} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \Rightarrow \boxed{\text{basis for } \ker(L) \text{ is } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}}$

$L(\mathbb{R}^3) = \mathrm{Span}\left(L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right)$

$= \mathrm{Col}(A)$ where $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ $\boxed{\text{a basis for } L(\mathbb{R}^3) \text{ is } \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}}$

<u>Note</u>: A is just the matrix rep. of L wrt the standard basis for $\mathbb{R}^3$

(3a.) $L\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$ $\quad L\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$ $\quad L\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\-4\end{pmatrix}$

$\Rightarrow$ the matrix rep. of L is $\boxed{A=\begin{pmatrix}2&-1&0\\0&1&-4\end{pmatrix}}$

(3b.) $L(x^2) = 2x+1$ $\qquad L(x)=0$ $\qquad L(1)=1$

$= \begin{pmatrix}2\\1\end{pmatrix}$ $\qquad = \begin{pmatrix}0\\0\end{pmatrix}$ $\qquad = \begin{pmatrix}0\\1\end{pmatrix}$

$\Rightarrow$ matrix rep. of L is $\boxed{A=\begin{pmatrix}2&0&0\\1&0&1\end{pmatrix}}$

(4.) $L(\bar{u}_1)=\begin{pmatrix}22\\-13\end{pmatrix}$ $\quad L(\bar{u}_2)=\begin{pmatrix}14\\-4\end{pmatrix}$ $\quad L(\bar{u}_3)=\begin{pmatrix}-23\\17\end{pmatrix}$

$\Rightarrow$ the matrix wrt to $[\bar{u}_1,\bar{u}_2,\bar{u}_3]$ and $[\bar{e}_1,\bar{e}_2]$ is $A=\begin{pmatrix}22&14&-23\\-13&-4&17\end{pmatrix}$

Now the transition matrix from $[\bar{v}_1,\bar{v}_2]$ to $[\bar{e}_1,\bar{e}_2]$ is

$T=\begin{pmatrix}2&-2\\-2&1\end{pmatrix}$ $\Rightarrow$ the transition from $[\bar{e}_1,\bar{e}_2]$ to $[\bar{v}_1,\bar{v}_2]$ is

$T^{-1}=\begin{pmatrix}-1/2&-1\\-1&-1\end{pmatrix}$

Therefore the matrix rep. of L wrt $[\bar{u}_1,\bar{u}_2,\bar{u}_3]$ and $[\bar{v}_1,\bar{v}_2]$

is $\boxed{T^{-1}A=\begin{pmatrix}2&-3&-11/2\\-9&-10&6\end{pmatrix}}$

$$\begin{tikzcd}
\underset{[u_i]}{\mathbb{R}^3} \arrow[r, "A"] \arrow[dr, dotted, "T^{-1}A"'] & \underset{[e_i]}{\mathbb{R}^2} \arrow[d, shift left, "T^{-1}"] \\
 & \underset{[v_i]}{\mathbb{R}^2} \arrow[u, shift left, "T"]
\end{tikzcd}$$

(5.) The transition matrix from $[3x^2+2x+11, x^2+1, -9x^2-8x+7]$

to $[x^2, x, 1]$ is $T = \begin{pmatrix} 3 & 1 & -9 \\ 2 & 0 & -8 \\ 11 & 1 & 7 \end{pmatrix}$

$\Rightarrow$ the matrix rep of L wrt $[3x^2+2x+11, x^2+1, -9x^2-8x+7]$

is given by

$$\boxed{B = T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}}$$

(6a) $\langle 1, x \rangle = \int_{-1}^{1} x \, dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0$

$\langle 1, 3x^2-1 \rangle = \int_{-1}^{1} 3x^2-1 \, dx = x^3 - x\Big|_{-1}^{1} = 0$

$\langle x, 3x^2-1 \rangle = \int_{-1}^{1} 3x^3 - x \, dx = \frac{3x^4}{4} - \frac{x^2}{2}\Big|_{-1}^{1} = 0$

It is an orthogonal basis

(6b.) $\bar{p} = \bar{p}_1 + \bar{p}_2 + \bar{p}_3$

$\bar{p}_1 = \frac{\langle 1, x^{4/3} \rangle}{\langle 1, 1 \rangle} 1 = \frac{\int_{-1}^{1} x^{4/3} \, dx}{\int_{-1}^{1} 1 \, dx} \cdot 1 = \frac{6/7}{2} \cdot 1 = \frac{3}{7} \cdot 1$

$\bar{p}_2 = \frac{\langle x, x^{4/3} \rangle}{\langle x, x \rangle} x = \frac{\int_{-1}^{1} x^{7/3} \, dx}{\int_{-1}^{1} x^2 \, dx} x = 0 \cdot x$

$\bar{p}_3 = \frac{\langle 3x^2 - \cancel{x}, x^{4/3} \rangle}{\langle 3x^2-1, 3x^2-1 \rangle} (3x^2-1) = \frac{\int_{-1}^{1} 3x^{10/3} - x^{4/3} \, dx}{\int_{-1}^{1} 9x^4 - 6x^2 + 1 \, dx} = \frac{48/91}{8/5} (3x^2-1)$

$$\Rightarrow \boxed{x^{4/3} \sim \bar{p} = \frac{3}{7} \cdot 1 + \frac{48/91}{8/5} (3x^2-1)}$$

(7.) $\bar{y}_1 = \begin{pmatrix} -1 \\ -3 \\ 0 \\ -2 \end{pmatrix} \Rightarrow \boxed{\bar{z}_1 = \frac{\bar{y}_1}{\|y_1\|} = \begin{pmatrix} -0.2673 \\ -0.8018 \\ 0 \\ -0.5345 \end{pmatrix}}$

$\bar{y}_2 = \bar{x}_2 - \frac{\langle \bar{x}_2, \bar{y}_1 \rangle}{\langle \bar{y}_1, \bar{y}_1 \rangle} \bar{y}_1 = \begin{pmatrix} -7.5 \\ -1.5 \\ 2 \\ 6 \end{pmatrix} \Rightarrow \boxed{\bar{z}_2 = \frac{\bar{y}_2}{\|y_2\|} = \begin{pmatrix} -0.7557 \\ -0.1511 \\ 0.2015 \\ 0.6046 \end{pmatrix}}$

$\bar{y}_3 = \bar{x}_3 - \frac{\langle \bar{x}_3, y_1 \rangle}{\langle \bar{y}_1, \bar{y}_1 \rangle} \bar{y}_1 - \frac{\langle \bar{x}_3, \bar{y}_2 \rangle}{\langle \bar{y}_2, \bar{y}_2 \rangle} \bar{y}_2 = \begin{pmatrix} -0.2415 \\ 0.3517 \\ 0.5787 \\ -0.4068 \end{pmatrix} \Rightarrow \boxed{\bar{z}_3 = \frac{\bar{y}_3}{\|\bar{y}_3\|} = \begin{pmatrix} -0.2923 \\ 0.4258 \\ 0.7005 \\ -0.4925 \end{pmatrix}}$

$\bar{y}_4 = \bar{x}_4 - \frac{\langle \bar{x}_4, \bar{y}_1 \rangle}{\langle \bar{y}_1, \bar{y}_1 \rangle} \bar{y}_1 - \frac{\langle \bar{x}_4, \bar{y}_2 \rangle}{\langle \bar{y}_2, \bar{y}_2 \rangle} \bar{y}_2 - \frac{\langle \bar{x}_4, \bar{y}_3 \rangle}{\langle \bar{y}_3, \bar{y}_3 \rangle} \bar{y}_3 = \begin{pmatrix} 1.2922 \\ -0.9692 \\ 1.6961 \\ 0.8077 \end{pmatrix} \Rightarrow \boxed{z_4 = \frac{\bar{y}_4}{\|\bar{y}_4\|} = \begin{pmatrix} 0.5216 \\ -0.3912 \\ 0.6846 \\ 0.3260 \end{pmatrix}}$

(8.) $\bar{x}_1 = 1 \quad \bar{x}_2 = x \quad \bar{x}_3 = x^2$

$\bar{y}_1 = \bar{x}_1 \Rightarrow \bar{z}_1 = \frac{\bar{y}_1}{\|\bar{y}_1\|} = \frac{1}{\sqrt{\int_0^1 1\,dx}} = \boxed{1}$

$\bar{y}_2 = \bar{x}_2 - \frac{\langle \bar{x}_2, \bar{y}_1 \rangle}{\langle \bar{y}_1, \bar{y}_1 \rangle} \cdot 1 = x - \frac{\int_0^1 x\,dx}{1} = x - \frac{1/2}{1} = x - \frac{1}{2}$

$\Rightarrow \bar{z}_2 = \frac{\bar{y}_2}{\|\bar{y}_2\|} = \frac{x - 1/2}{\sqrt{\int_0^1 (x-1/2)^2\,dx}} = \frac{x - 1/2}{\sqrt{1/12}} = \boxed{\sqrt{12}\,(x - 1/2)}$

$\bar{y}_3 = \bar{x}_3 - \frac{\langle \bar{x}_3, \bar{y}_1 \rangle}{\langle \bar{y}_1, \bar{y}_1 \rangle} \cdot 1 - \frac{\langle x_3, \bar{y}_2 \rangle}{\langle \bar{y}_2, \bar{y}_2 \rangle} (x - 1/2) = x^2 - \frac{\int_0^1 x^2\,dx}{1} - \frac{\int_0^1 x^3 - 1/2\,x^2\,dx}{\int_0^1 (x-1/2)^2\,dx} (x - 1/2)$

$= x^2 - \frac{1/3}{1} - \frac{1/12}{1/12}(x - 1/2)$

$= x^2 - (x - 1/2) - 1/3 = x^2 - x + 1/6$

$\Rightarrow z_3 = \frac{x^2 - x + 1/6}{\sqrt{\int_0^1 (x^2 - x + 1/6)^2\,dx}} = \frac{x^2 - x + 1/6}{\sqrt{1/180}} = \boxed{\sqrt{180}\,(x^2 - x + 1/6)}$

$\Rightarrow \boxed{\bar{z}_1 = 1} \quad \boxed{\bar{z}_2 = \sqrt{12}\,(x - 1/2)} \quad \boxed{\bar{z}_3 = \sqrt{180}\,(x^2 - x + 1/6)}$