# ANALOGUES OF GOLDSCHMIDT'S THESIS FOR FUSION SYSTEMS

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Abstract. We extend the results of David Goldschmidt's thesis concerning fusion in finite groups to saturated fusion systems and to all primes.

## 1. Introduction

Just recently, David Goldschmidt published his doctoral thesis [6] which had gone unpublished since 1968. In it he shows that if $G$ is a finite simple group and $T$ is a Sylow 2-subgroup of $G$, then the exponent of $Z(T)$ (and hence of $T$) is bounded by a function of the nilpotence class of $T$. He also includes in the write-up a fusion factorization result for an arbitrary finite group involving $\mho^1 Z$ and the Thompson subgroup. In this paper, we generalize these results to saturated fusion systems.

Throughout this paper unless otherwise indicated, $p$ will be a prime number, $n$ a non-negative integer, and $P$ a nontrivial finite $p$-group.

**Theorem 1.** *Suppose $P$ is of nilpotence class at most $n(p-1)+1$, and $\mathcal{F}$ is a saturated fusion system on $P$ with $O_p(\mathcal{F}) = 1$. Then $Z(P)$ has exponent at most $p^n$.*

This bound is sharp for all $n$ and $p$; see Example 1 in Section 3. This also gives a bound on the exponent of $P$ itself, which we certainly do not expect to be sharp.

**Corollary 1.** *Suppose that $P$ is of nilpotence class at most $n(p-1)+1$, and $\mathcal{F}$ is a saturated fusion system on $P$ with $O_p(\mathcal{F}) = 1$. Then $P$ has exponent at most $p^{n^2(p-1)+n}$.*

*Proof.* By Theorem 1, $Z(P)$ has exponent at most $p^n$. We claim that then every upper central quotient also has exponent at most $p^n$, and we shall prove this by induction. Let $k \geqslant 1$, and let $x \in Z^{k+1}(P)$. If $x^{p^n}$ does not lie in $Z^k(P)$, then there exists $t \in P$ such that $[x^{p^n}, t]$ does not lie in $Z^{k-1}(P)$. But by a standard commutator identity, $[x^{p^n}, t] \equiv [x,t]^{p^n} \equiv 1$ modulo $Z^{k-1}(P)$, since by induction $Z^k(P)/Z^{k-1}(P)$ has exponent at most $p^n$. This contradiction establishes the claim. The nilpotence class of $P$ is at most $n(p-1)+1$ by hypothesis, so the exponent of $P$ is at most $p^{n(n(p-1)+1)}$. □

Theorem 1 is obtained from the following.

**Theorem 2.** *Suppose $P$ has nilpotence class at most $n(p-1)+1$ and $\mathcal{F}$ is a saturated fusion system on $P$. Then $\mho^n(Z(P))$ is normal in $\mathcal{F}$.*

In the course of proving this last result in the group case for $p = 2$, Goldschmidt reduces to the situation in which a putative counterexample $G$ has a weakly embedded 2-local

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subgroup. Then his post-thesis classification [5] of such groups gives a contradiction. However, any weakly embedded 2-local $M$ controls 2-fusion, and so the 2-subgroup $O_2(M)$ will show up as a normal subgroup in the fusion system, a shadow of the weakly embedded phenomenon. This allows the corresponding fusion result to hold for an arbitrary prime.

We note that Theorem 2 has the following corollary in the category of groups.

**Theorem 3.** *Let $P$ be a nonabelian Sylow $p$-subgroup of a finite group $G$. Suppose that $P$ has nilpotence class at most $n(p-1)+1$ and that $G$ has no nontrivial strongly closed abelian $p$-subgroup. Then $Z(P)$ has exponent at most $p^n$.*

*Proof.* We can form the saturated fusion system $\mathcal{F}_P(G)$, and Theorem 2 then says that $\mho^n(Z(P))$ is strongly $\mathcal{F}$-closed (see Proposition 1 below), that is, strongly closed in $P$ with respect to $G$. Thus, $\mho^n(Z(P))$ must be trivial. □

Using a recent theorem of Flores and Foote [4], in which they apply the Classification of Finite Simple Groups to describe all finite groups having a strongly closed $p$-subgroup, we get the following direct generalization of Goldschmidt's main theorem.

**Corollary 2.** *Let $P$ be a nonabelian Sylow $p$-subgroup of a finite simple group $G$. If $P$ has nilpotence class at most $n(p-1)+1$, then $Z(P)$ has exponent at most $p^n$.*

*Proof.* Suppose to the contrary that $A := \mho^n(Z(P)) \neq 1$. Then by Theorem 2, $A$ is a nontrivial strongly closed abelian subgroup of $P$. By inspection of the simple groups arising in the conclusion of the main theorem in [4], either $P$ is abelian or $Z(P)$ has exponent $p$. Since $P$ is nonabelian, we must have $n \geqslant 1$ and the corollary follows. □

However, if the hypotheses of Corollary 2 are weakened slightly to assume only that $F^*(G)$ is simple, then the statement is false for all odd primes $p$, as the following example shows. Let $H = \mathrm{PSL}(2,q)$ with $q = r^p$ for some prime power $r$ and with the $p$-part of $q-1$ equal to $p^e$. Let $\sigma$ be a field automorphism of $\mathbf{F}_q$ of order $p$ and $G = H\langle\sigma\rangle$. If $P$ is a Sylow $p$-subgroup of $G$, then $P$ has nilpotence class 2, while $Z(P)$ has exponent $p^{e-1}$, and we may take $e$ as large as we like.

Recall the Thompson subgroup $J(P)$, defined as the group generated by the abelian subgroups of $P$ of maximum order. We also prove the following factorization result.

**Theorem 4.** *Let $\mathcal{F}$ be a saturated fusion system on $P$. Then*

$$\mathcal{F} = \langle\ C_{\mathcal{F}}(\mho^1(Z(P))),\ N_{\mathcal{F}}(J(P))\ \rangle.$$

## 2. Definitions and notation

We collect in this section the necessary information on fusion systems. Since there are by now many good sources of this knowledge [2], in particular in background sections of papers [3, 7] to which this one is similar, we will content ourselves to be brief.

Let $P$ be a finite $p$-group. A *category on $P$* is a category $\mathcal{F}$ with objects the subgroups of $P$ and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ consist of injective group homomorphisms subject to the requirement that every morphism in $\mathcal{F}$ is a composition of an isomorphism in $\mathcal{F}$ and an inclusion.

Let $\mathcal{F}$ be a category on the $p$-group $P$. Let $Q$ and $R$ be subgroups of $P$. We write $\operatorname{Aut}_{\mathcal{F}}(Q)$ for $\operatorname{Hom}_{\mathcal{F}}(Q,Q)$, $\operatorname{Hom}_P(Q,R)$ for the set of group homomorphisms in $\mathcal{F}$ from $Q$ to $R$ induced by conjugation by elements of $P$, and $\operatorname{Out}_{\mathcal{F}}(Q)$ for $\operatorname{Aut}_{\mathcal{F}}(Q)/\operatorname{Aut}_Q(Q)$.

We say $Q$ is

- *fully $\mathcal{F}$-normalized* if $|N_P(Q)| \geqslant |N_P(Q')|$ for all $Q'$ which are $\mathcal{F}$-isomorphic to $Q$,
- *fully $\mathcal{F}$-centralized* if $|C_P(Q)| \geqslant |C_P(Q')|$ for all $Q'$ which are $\mathcal{F}$-isomorphic to $Q$,
- *$\mathcal{F}$-centric* if $C_P(Q') \leqslant Q'$ for all $Q'$ which are $\mathcal{F}$-isomorphic to $Q$, and
- *$\mathcal{F}$-radical* if $O_p(\operatorname{Out}_{\mathcal{F}}(Q)) = 1$.

For a morphism $\varphi : Q \to P$ in $\mathcal{F}$, let

$$N_\varphi = \{\, x \in N_P(Q) \mid \exists y \in N_P(\varphi(Q)),\, \forall z \in Q,\, \varphi(xzx^{-1}) = y\varphi(z)y^{-1} \}$$

Note that we have $QC_P(Q) \leqslant N_\varphi$ for all $\varphi : Q \to P$ in $\mathcal{F}$.

A *saturated fusion system* on $P$ is a category $\mathcal{F}$ on $P$ whose morphism sets contain all group homomorphisms induced by conjugation by elements of $P$, and which satisfies the following two axioms.

- (Sylow axiom) $\operatorname{Aut}_P(P)$ is a Sylow $p$-subgroup of $\operatorname{Aut}_{\mathcal{F}}(P)$, and
- (Extension axiom) for every isomorphism $\varphi : Q \to Q'$ with $Q'$ fully $\mathcal{F}$-normalized, there exists a morphism $\tilde{\varphi} : N_\varphi \to P$ such that $\tilde{\varphi}|_Q = \varphi$.

For the remainder of the paper, $\mathcal{F}$ will denote a saturated fusion system on the finite $p$-group $P$, even though we will often drop the adjective "saturated".

For $Q \leqslant P$, we define the following local subcategories of $\mathcal{F}$. The *normalizer* $N_{\mathcal{F}}(Q)$ of $Q$ in $\mathcal{F}$ is the category on $N_P(Q)$ such that for any $R_1, R_2 \leqslant N_P(Q)$, $\operatorname{Hom}_{N_{\mathcal{F}}(Q)}(R_1, R_2)$ consists of those $\varphi : R_1 \to R_2$ in $\mathcal{F}$ for which there is an extension $\tilde{\varphi} : QR_1 \to QR_2$ of $\varphi$ in $\mathcal{F}$ such that $\tilde{\varphi}(Q) = Q$. The *centralizer* $C_{\mathcal{F}}(Q)$ of $Q$ in $\mathcal{F}$ is the category on $C_P(Q)$ such that for any $R_1, R_2 \leqslant C_P(Q)$, $\operatorname{Hom}_{C_{\mathcal{F}}(Q)}(R_1, R_2)$ consists of those $\varphi : R_1 \to R_2$ in $\mathcal{F}$ for which there is an extension $\tilde{\varphi} : QR_1 \to QR_2$ of $\varphi$ in $\mathcal{F}$ such that $\tilde{\varphi}|_Q = \mathrm{id}_Q$. Lastly, we define $N_P(Q)C_{\mathcal{F}}(Q)$ as we do the normalizer of $Q$, but only allow those $\varphi : R_1 \to R_2$ whose extensions $\tilde{\varphi}$ restrict to automorphisms in $\operatorname{Aut}_P(Q)$.

If $Q$ is fully $\mathcal{F}$-normalized, then $N_{\mathcal{F}}(Q)$ is a saturated fusion system. And if $Q$ is fully $\mathcal{F}$-centralized, then both $C_{\mathcal{F}}(Q)$ and $N_P(Q)C_{\mathcal{F}}(Q)$ are saturated fusion systems.

A *characteristic functor* is a mapping from finite $p$-groups to finite $p$-groups which takes $Q$ to a characteristic subgroup $W(Q)$ of $Q$ such that for any group isomorphism $\varphi : Q \to Q'$, $\varphi(W(Q)) = W(Q')$. We say that a characteristic functor is *positive* provided $W(Q) \neq 1$ whenever $Q \neq 1$. The *center functor*, sending a finite $p$-group $P$ to its center, is a positive characteristic $p$-functor.

A *conjugation family* for $\mathcal{F}$ is a set $\mathcal{C}$ of nonidentity subgroups of $P$ such that $\mathcal{F}$ is generated by compositions and restrictions of morphisms in $\operatorname{Aut}_{\mathcal{F}}(Q)$ as $Q$ ranges over $\mathcal{C}$. Alperin's fusion theorem for saturated fusion systems says that the set of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups is a conjugation family for $\mathcal{F}$, and we call this the *Alperin conjugation family*.

Recall that a subgroup $W$ of $P$ is said to be *weakly $\mathcal{F}$-closed* if for each $\varphi \in \operatorname{Hom}_{\mathcal{F}}(W, P)$, $\varphi(W) = W$. The subgroup $W$ is *strongly $\mathcal{F}$-closed* if for each subgroup $W'$ of $W$ and each

$\varphi \in \mathrm{Hom}_{\mathcal{F}}(W', P)$, $\varphi(W') \leqslant W$. We say $W$ is *normal* in $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(W)$, and denote by $O_p(\mathcal{F})$ the largest such subgroup of $P$.

## 3. Bounding the exponent

The following proposition is slightly misstated in [1, Proposition 1.6], where a normal $W$ is claimed to be contained in every radical subgroup. For this reason, we state a correct version here, but the proof in [1] goes through with little modification.

**Proposition 1.** *Let $\mathcal{F}$ be a fusion system on $P$ and $W \leqslant P$. The following are equivalent.*

(a) *$W$ is normal in $\mathcal{F}$.*
(b) *$W$ is strongly $\mathcal{F}$-closed and is contained in every $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup of $P$.*
(c) *$W$ is weakly $\mathcal{F}$-closed and is contained in every subgroup of some conjugation family for $\mathcal{F}$.*

**Lemma 1.** *Suppose $P$ has nilpotence class at most $n(p-1)+1$. If $Q$ is a subgroup of $P$ with $C_P(\mho^n(Z(Q))) = Q$, then $Q = P$.*

*Proof.* This is Corollary 6 in [6]. □

**Proposition 2.** *Let $W$ be a characteristic subfunctor of the center functor such that $W(P) \leqslant W(Q)$ for all $Q \leqslant P$ with $C_P(Q) \leqslant Q$. Then for any fusion system $\mathcal{F}$ on $P$, either there exists a proper $\mathcal{F}$-centric subgroup $Q$ of $P$ such that $C_P(W(Q)) = Q$, or $W(P)$ is normal in $\mathcal{F}$.*

*Proof.* Suppose there is no proper $\mathcal{F}$-centric subgroup $Q$ of $P$ with $C_P(W(Q)) = Q$. We will show that $W(P)$ is weakly closed in $\mathcal{F}$. In this case, $W(P) \leqslant Z(P)$ is contained in every $\mathcal{F}$-centric subgroup of $P$, hence in every member of an Alperin conjugation family for $\mathcal{F}$. Thus, by Proposition 1, $W(P)$ is in fact normal in $\mathcal{F}$.

Let $Q$ be a fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric subgroup of $P$. Then by hypothesis, $W(P) \leqslant W(Q)$. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$. By Alperin's fusion theorem, it suffices to show that $W(P)$ is invariant under $\alpha$. We do this by induction on $|P : Q|$. If $Q = P$, then $\alpha(W(P)) = W(P)$ since $W(P)$ is a characteristic subgroup of $P$, so suppose that $Q < P$. Then $C_P(W(Q)) > Q$. Let $\beta : W(Q) \to R$ be an isomorphism in $\mathcal{F}$ with $R$ fully $\mathcal{F}$-normalized. Then by the extension axiom, $\beta$ extends to a map $\tilde{\beta} : C_P(W(Q)) \to P$. By induction and Alperin's fusion theorem, we have that $\beta(W(P)) = \tilde{\beta}(W(P)) = W(P)$. But $\beta\alpha|_{W(Q)}$ also extends to $C_P(W(Q))$, and $\beta\alpha(W(P)) = W(P)$ by the same reasoning. Therefore $\alpha(W(P)) = \beta^{-1}\beta\alpha(W(P)) = W(P)$, and this completes the proof. □

We are now ready to prove Theorem 2.

**Theorem 2.** *Suppose $P$ has nilpotence class at most $n(p-1)+1$ and $\mathcal{F}$ is a fusion system on $P$. Then $\mho^n(Z(P))$ is normal in $\mathcal{F}$.*

*Proof.* Let $W = \mho^n Z$. If $C_P(Q) \leqslant Q \leqslant P$, then $Z(P) \leqslant Z(Q)$ and so $W(P) = \mho^n(Z(P)) \leqslant \mho^n(Z(Q)) = W(Q)$. Thus $W$ satisfies the hypotheses of Proposition 2, and Lemma 1 says that there is no proper subgroup of $P$ with $C_P(W(Q)) = Q$. Therefore by Proposition 2, $\mho^n(Z(P))$ is normal in $\mathcal{F}$. □

Theorem 1 now follows immediately from Theorem 2. The following example generalizes a remark of Goldschmidt's in [6], and shows that the bound on the exponent of $Z(P)$ given in Theorem 1 is sharp.

**Example 1.** Let $p$ be an odd prime, let $G = \mathrm{SL}(p+1, q)$ with $|q-1|_p = p^n$, and let $P$ be a Sylow $p$-subgroup of $G$. Then $P$ is a isomorphic to $C_{p^n} \wr C_p$. Let $x$ be the wreathing element, a $p$-cycle permutation matrix. Then $P' = [P, P]$ is isomorphic to a direct product of $p-1$ copies of $C_{p^n}$. Let $P_0 = \langle P', x\rangle$. Let $a_1, \dots, a_{p-1}$ be generators for the $p-1$ cyclic groups of $P'$ of order $p^n$. Then $x$ sends $a_i$ to $a_{i+1}$ for $1 \leqslant i \leqslant p-2$ and $a_{p-1}$ to $a_1^{-1} \cdots a_{p-1}^{-1}$.

Let $\mathcal{F} = \mathcal{F}_P(G)$. We first claim that

$$O_p(\mathcal{F}) = 1. \tag{3.1}$$

Suppose to the contrary and choose $1 \neq N \leqslant P$ normal in $\mathcal{F}$. Then $N$ contains $\Omega_1(Z(P))$. Let $Q$ be the unique maximal abelian subgroup of $P$. Then $\Omega_1(Z(P)) \leqslant Q$, and the alternating group $\mathrm{Alt}(p+1) \leqslant G$ acts irreducibly on $\Omega_1(Q)$, so $N$ contains $\Omega_1(Q)$. As $(p, q) = 1$, the wreathing element $x$ is diagonalizable, hence $x$ is $G$-conjugate to an element in $\Omega_1(Q)$. It follows that $x \in N$. As $N$ is normal in $P$, we have $[P, x] \leqslant N$. Since $[P, x]$ contains an elements of order $p^n$ and $\mathrm{Alt}(p+1)$ also acts irreducibly on the section $Q/\Omega_{n-1}(Q)$, we have that $Q \leqslant N$. Now $P = \langle Q, x\rangle$, so $P$ is normal in $\mathcal{F}$. But $Q$ is a characteristic subgroup of $P$. Since $x \notin Q$, and $x$ is $G$-conjugate to an element of $Q$, this is a contradiction. Thus, (3.1) holds.

Now as $Z(P)$ has exponent $p^n$, the bound in Theorem 1 is sharp for $\mathcal{F}$ provided the class of $P$ is $n(p-1)+1$. For this it suffices to show that $P_0$ has class $n(p-1)$, that is, $P_0$ is of maximal class.

For $n = 1$, $P_0$ is of maximal class $p-1$. We show by induction on $n$ that

$$[P', x; p-1] = \Omega_{n-1}(P') \tag{3.2}$$

and this will complete the proof. Factoring by $\Omega_{n-1}(P')$, we have $[P'/\Omega_{n-1}(P'), x; p-1] = 1$ so that $[P', x; p-1] \leqslant \Omega_{n-1}(P')$ in any case.

Suppose first that $n = 2$. By direct computation,

$$[a_1, x; p-1] = \prod_{k=0}^{p-2} a_{k+1}^{(-1)^k \binom{p-1}{k} - 1}.$$

The sum of the exponents of the $a_i$ in $[a_1, x; p-1]$ is

$$-p + 1 + \sum_{k=0}^{p-2} (-1)^k \binom{p-1}{k} = -p + 1 + (1-1)^{p-1} - \binom{p-1}{p-1} = -p.$$

This means that $[a_1, x; p-1]$ lies outside the sum-zero submodule (which is the unique maximal submodule) for the action of $x$ on $\Omega_1(P')$, and so $[P', x; p-1] = \Omega_1(P')$.

Let $n \geqslant 3$ be arbitrary. Let $N = [P', x; p-1]$. By induction we have that $N$ contains $[\Omega_{n-1}(P'), x; p-1] = \Omega_{n-2}(P')$ and by the $n = 2$ case, we know that $N$ covers $\Omega_1(P'/\Omega_{n-2}(P'))$ modulo $\Omega_{n-2}(P')$. Therefore, $N = \Omega_{n-1}(P')$, proving (3.2).

It now follows that $P$ has class $n(p-1)+1$ while $Z(P)$ has exponent $p^n$, and so the bound of Theorem 1 is sharp.

## 4. A factorization theorem

We now turn to the proof of Theorem 4. We will need a version of the Frattini argument due to Onofrei and Stancu [8, Proposition 3.7].

**Proposition 3.** *Let $\mathcal{F}$ be a fusion system on $P$ and suppose $Q \leqslant P$ is normal in $\mathcal{F}$. Then*

$$\mathcal{F} = \langle\, PC_{\mathcal{F}}(Q),\ N_{\mathcal{F}}(QC_P(Q))\,\rangle.$$

**Lemma 2.** *Suppose $P$ is a $p$-group, $Q \trianglelefteqslant P$, and $C_P(\mho^1(Z(Q))) = Q$. Then $J(P) \leqslant Q$.*

*Proof.* This is Lemma 8 in [6]. □

The *Thompson ordering* on subgroups of $P$ is defined by

$$Q \leqslant_P Q' \quad \text{iff} \quad |N_P(Q)| \leqslant |N_P(Q')| \quad \text{or} \quad |N_P(Q)| = |N_P(Q')| \quad \text{and} \quad |Q| \leqslant |Q'|.$$

We are now ready to prove

**Theorem 4.** *Let $\mathcal{F}$ be a fusion system on $P$. Then*

$$\mathcal{F} = \langle\ C_{\mathcal{F}}(\mho^1(Z(P)),\ N_{\mathcal{F}}(J(P))\ \rangle.$$

*Proof.* Write $\mathcal{F}' = \langle\, C_{\mathcal{F}}(\mho^1(Z(P))),\ N_{\mathcal{F}}(J(P))\,\rangle$. Since each $\mathcal{F}$-centric subgroup of $P$ contains $Z(P)$, it suffices by Alperin's fusion theorem to prove that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ for all $Q \leqslant P$ with $Z(P) \leqslant Q$. We do this by induction on the Thompson ordering. If $Q = P$, then $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$, since $J(P)$ is a characteristic subgroup of $P$, so suppose that $Q <_P P$ with $Z(P) \leqslant Q$ and that $N_{\mathcal{F}}(Q') \subseteq \mathcal{F}'$ for all $Q' >_P Q$ with $Z(P) \leqslant Q'$.

First we reduce to the case where $Q$ is fully $\mathcal{F}$-normalized. Suppose $Q$ is not fully $\mathcal{F}$-normalized. By [7, Lemma 2.2], there exists $\alpha : N_P(Q) \to P$ such that $\alpha(Q)$ is fully $\mathcal{F}$-normalized. Note that $\alpha(Q) >_P Q$, and since $R >_P Q$ for every $R \leqslant P$ with $|N_P(Q)| \leqslant |R|$, we have by induction and Alperin's fusion theorem that $\alpha$ is in $\mathcal{F}'$. Also note that $\alpha(N_P(Q)) \leqslant N_P(\alpha(Q))$; we still denote by $\alpha$ the induced morphism $N_P(Q) \to N_P(\alpha(Q))$. Let $\varphi : R_1 \to R_2$ be a morphism in $N_{\mathcal{F}}(Q)$, and let $\tilde{\varphi}$ be an extension to $QR_1 \leqslant N_P(Q)$. Then $\alpha\tilde{\varphi}\alpha^{-1} : \alpha(Q)\alpha(R_1) \to \alpha(Q)\alpha(R_2)$ restricts to an automorphism of $\alpha(Q)$, whence is contained in $\mathcal{F}'$ by induction. But $\alpha$ is in $\mathcal{F}'$, so $\varphi$ is in $\mathcal{F}'$ too. Thus $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$, so henceforth we assume $Q$ is fully $\mathcal{F}$-normalized.

For brevity, set $W = \mho^1(Z(Q))$, $N = N_P(Q)$, and $C = C_N(W)$. Then $C \trianglelefteqslant N$, so that $N_P(C) \geqslant N$. Suppose first that $C = Q$. Then by Lemma 2, we have $J(N) \leqslant Q$. As $J(N) \trianglelefteqslant N_P(N)$, either $J(N) >_P Q$ or $N = P$. In the first case, since $Z(P) \leqslant J(N)$ and $J(N) = J(Q)$ is a characteristic subgroup of $Q$, we apply induction to get $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(N)) \subseteq \mathcal{F}'$. In the second case we have $J(P) \leqslant Q$, so $J(P) = J(Q)$, and hence $N_{\mathcal{F}}(Q) \subseteq N_{\mathcal{F}}(J(P)) \subseteq \mathcal{F}'$ here as well.

Assume now that $C > Q$. Then $C >_P Q$ because $C \trianglelefteqslant N$. Looking to see that $W \trianglelefteqslant N_{\mathcal{F}}(Q)$, we apply Proposition 3 in this normalizer to get

$$N_{\mathcal{F}}(Q) = \langle\ NC_{N_{\mathcal{F}}(Q)}(W),\ N_{N_{\mathcal{F}}(Q)}(C)\ \rangle.$$

Since $C$ contains $Z(P)$, we have by induction that $N_{N_{\mathcal{F}}(Q)}(C) \subseteq N_{\mathcal{F}}(C) \subseteq \mathcal{F}'$, so to complete the proof, it suffices to show that $NC_{N_{\mathcal{F}}(Q)}(W) \subseteq C_{\mathcal{F}}(\mho^1(Z(P)))$. To see this, let $R_1, R_2 \leqslant N$, and let $\varphi : R_1 \to R_2$ be a morphism in $NC_{N_{\mathcal{F}}(Q)}(W)$. Then there exists $x \in N$

such that $\varphi$ extends to an $\mathcal{F}$-map $\tilde{\varphi} : WR_1 \to WR_2$ with $\tilde{\varphi}\,|_W = c_x$, the conjugation map induced by $x$. But since $Q$ contains $Z(P)$, it follows that $W = \mho^1(Z(Q)) \geqslant \mho^1(Z(P))$, and so $\tilde{\varphi}\,|_{\mho^1(Z(P))} = c_x\,|_{\mho^1(Z(P))} = \mathrm{id}_{\mho^1(Z(P))}$. Therefore, $\varphi \in C_{\mathcal{F}}(\mho^1(Z(P)))$, as was to be shown. We conclude that $N_{\mathcal{F}}(Q) \subseteq \mathcal{F}'$ and the result follows. □

**Remark 1.** In [3, Theorem 4.1], the authors prove in part that for any fusion system $\mathcal{F}$ on $P$, $\mho^1(Z(P)) \cap Z(N_{\mathcal{F}}(J(P))) \leqslant Z(\mathcal{F})$ by reducing to the group case. Theorem 4 gives a reduction-free proof of this fact.

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