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Additivity of the motivic trace and the motivic Euler-characteristic ☆

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ABSTRACT

In this paper, we settle an open conjecture regarding the assertion that the Euler-characteristic of $G/N_G(T)$ for a split reductive group scheme G and the normalizer of a split maximal torus $N_G(T)$ over a field is 1 in the Grothendieck-Witt ring with the characteristic exponent of the field inverted, under the assumption that the base field contains a $\sqrt{-1}$. Numerous applications of this to splittings in the motivic stable homotopy category and to Algebraic K-Theory are worked out in several related papers by Gunnar Carlsson and the authors.

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1. Introduction

This paper is a continuation of the earlier work, [9], where Carlsson and one of the present authors set up motivic and étale variants of the classical Becker-Gottlieb transfer. If one recalls, the power and utility of the classical Becker-Gottlieb transfer stems from the fact it provided a convenient mechanism to obtain splittings to certain maps in the stable homotopy category. In view of the fact that the transfer is a map of spectra, it induces a map of the Atiyah-Hirzebruch spectral sequences associated to generalized cohomology theories, and reduces the question on the existence of stable splittings to the calculation of certain Euler-characteristics in singular cohomology. The most notable example of this is the calculation of the Euler-characteristic of $\mathrm{G}/\mathrm{N_G}(\mathrm{T})$, where G is a compact Lie group and $\mathrm{N_G}(\mathrm{T})$ is the normalizer of a maximal torus in G. Using the transfer, it becomes then possible to show that the generalized cohomology of the Borel construction with respect to G for a space X acted on by G, is a split summand of the generalized cohomology of the corresponding Borel construction for X with respect to $\mathrm{N_G}(\mathrm{T})$. This then provided numerous applications, such as double coset formulae for actions of compact groups, generalizing the well-known double coset formulae for the action of finite groups: see [4], [3], [5], [15], [16].

The motivic analogue of the statement that the Euler characteristic of $\mathrm{G}/\mathrm{N_G}(\mathrm{T})$ is 1 in singular cohomology for compact Lie groups is a conjecture due to Morel (see the next page for more details), that a suitable motivic Euler characteristic in the Grothendieck-Witt group is 1, for $\mathrm{G}/\mathrm{N_G}(\mathrm{T})$, where G is a split connected reductive group and $\mathrm{N_G}(\mathrm{T})$ is the normalizer of a maximal torus in G. We provide an affirmative solution to this conjecture in this paper assuming that the base field k contains a $\sqrt{-1}$, the precise details of which are discussed below.

Let k denote a *perfect field of arbitrary characteristic: we will restrict to the category of smooth quasi-projective schemes over k and adopt the framework of [24].* Throughout, $\mathbf{T}$ will denote $\mathbb{P}^1$ pointed by ∞ and $\mathbf{T}^n$ will denote $\mathbf{T}^{\wedge n}$ for any integer $n \geq 0$. $\mathbb{S}_k$ will denote the corresponding motivic sphere spectrum. Let $\mathbf{Spt}(k_{mot})$ denote the category of motivic spectra over k. The corresponding stable homotopy category will be denoted $\mathcal{SH}(k)$. In positive characteristic p, we consider $\mathbf{Spt}(k_{mot})[\mathrm{p}^{-1}]$: we will identify this with the category of motivic spectra that are module spectra over the localized sphere spectrum $\mathbb{S}_k[\mathrm{p}^{-1}]$. Then assuming $char(k) = 0$, given a smooth scheme X of finite type over k, $\Sigma^\infty_{\mathbf{T}}\mathrm{X}_+$ denotes the $\mathbf{T}$-suspension spectrum of X and $\mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{X}_+) = \mathcal{RH}\mathrm{om}(\Sigma^\infty_{\mathbf{T}}\mathrm{X}_+, \mathbb{S}_k)$, where $\mathcal{RH}om$ denotes the derived internal hom in the category $\mathbf{Spt}(k_{mot})$. When $char(k) = \mathrm{p} > 0$, $\mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{X}_+) = \mathcal{RH}\mathrm{om}(\Sigma^\infty_{\mathbf{T}}\mathrm{X}_+, \mathbb{S}_k[\mathrm{p}^{-1}])$. $\mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{X}_+)$ is the *Spanier-Whitehead dual* of $\Sigma^\infty_{\mathbf{T}}\mathrm{X}_+$. It is known (see [26] and also [25]) that after inverting the characteristic exponent, $\Sigma^{\infty}_{\mathbf{T}}X_+$ is *dualizable* in the sense that the natural map $\Sigma^{\infty}_{\mathbf{T}}X_+ \to D(D(\Sigma^{\infty}_{\mathbf{T}}X_+))$ is an isomorphism in $\mathcal{SH}(k)$.

In this context, we have the *co-evaluation map*

$$\mathbb{S}_k \xrightarrow{c} \Sigma^{\infty}_{\mathbf{T}}X_+ \wedge D(\Sigma^{\infty}_{\mathbf{T}}X_+)$$

and the *evaluation map*

$$D(\Sigma^{\infty}_{\mathbf{T}}X_+) \wedge \Sigma^{\infty}_{\mathbf{T}}X_+ \to \mathbb{S}_k$$

in characteristic 0. In positive characteristic p, we also have the co-evaluation map

$$\mathbb{S}_k[p^{-1}] \xrightarrow{c} \Sigma^{\infty}_{\mathbf{T}}X_+[p^{-1}] \wedge D(\Sigma^{\infty}_{\mathbf{T}}X_+[p^{-1}])$$

and the *evaluation map*

$$D(\Sigma^{\infty}_{\mathbf{T}}X_+[p^{-1}]) \wedge \Sigma^{\infty}_{\mathbf{T}}X_+[p^{-1}] \to \mathbb{S}_k[p^{-1}].$$

See [11, p. 87]. Let $f : X \to X$ denote a self-map.

Definition 1.1. Assume the above setting. Then, in characteristic 0, the following composition in $\mathcal{SH}(k)$ defines the *trace* $\tau_X(f_+)$:

$$\mathbb{S}_k \xrightarrow{c} \Sigma^{\infty}_{\mathbf{T}}X_+ \wedge D(\Sigma^{\infty}_{\mathbf{T}}X_+) \xrightarrow{\tau} D(\Sigma^{\infty}_{\mathbf{T}}X_+) \wedge \Sigma^{\infty}_{\mathbf{T}}X_+ \xrightarrow{id\wedge f} D(\Sigma^{\infty}_{\mathbf{T}}X_+) \wedge \Sigma^{\infty}_{\mathbf{T}}X_+ \xrightarrow{e} \mathbb{S}_k. \quad (1.0.1)$$

In positive characteristic p, the following composition in $\mathcal{SH}(k)[p^{-1}]$ defines the corresponding trace, which will be denoted $\tau_{X,\mathbb{S}_k[p^{-1}]}(f_+)$:

$$\begin{aligned}\mathbb{S}_k[p^{-1}] &\xrightarrow{c} \Sigma^{\infty}_{\mathbf{T}}[p^{-1}]X_+ \wedge D(\Sigma^{\infty}_{\mathbf{T}}[p^{-1}]X_+) \xrightarrow{\tau} D(\Sigma^{\infty}_{\mathbf{T}}[p^{-1}]X_+) \wedge \Sigma^{\infty}_{\mathbf{T}}[p^{-1}]X_+ \\ &\xrightarrow{id\wedge f} D(\Sigma^{\infty}_{\mathbf{T}}[p^{-1}]X_+) \wedge \Sigma^{\infty}_{\mathbf{T}}[p^{-1}]X_+ \xrightarrow{e} \mathbb{S}_k[p^{-1}].\end{aligned} \quad (1.0.2)$$

Here τ is the map interchanging the two factors. When $f = id_X$, the corresponding trace $\tau_{X_+} = \tau_X(id_{X+})$ $(\tau_{X_+,\mathbb{S}_k[p^{-1}]} = \tau_{X,\mathbb{S}_k[p^{-1}]}(id_{X_+}))$ will be denoted $\chi_{mot}(X)$ and *called the motivic Euler-characteristic* of X.

By [22] (see also [23]), $\pi_{0,0}(\mathbb{S}_k)$ identifies with the Grothendieck-Witt ring of the field k, $GW(k)$, and therefore $\chi_{mot}(X)$ is a class in $GW(k)$ in characteristic 0 and in $GW(k)[p^{-1}]$ in positive characteristic. (In [22] the isomorphism of $\pi_{0,0}(\mathbb{S}_k)$ with the Grothendieck-Witt ring of the field k was proven under the assumption $char(k) \neq 2$: the above restriction on the characteristic of the field k is removed in [6, Theorem 10.12].)

Then an open conjecture in this setting (due to Morel (see [19])) was the following: let G denote a split reductive group over k, with T a split maximal torus and $N_G(T)$ its normalizer in G. Then the conjecture states that $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T})) = 1$ in $\mathrm{GW}(k)$ with the characteristic exponent of the field inverted. In fact, this is the *strong form* of the conjecture. The *weak form* of the conjecture is simply the statement that $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ is a *unit* in $\mathrm{GW}(k)$ with the characteristic exponent of the field inverted.

The main result of the current paper is an affirmative solution of the above conjecture as stated in the following theorem.

Theorem 1.2. *Let* G *denote a split linear algebraic group over the perfect field k, with* T *a split maximal torus and* N(T) *its normalizer in* G. *Then the following are true:*

(i) $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T})) = 1$ *in* $\mathrm{GW}(k)$ *if* $char(k) = 0$ *and k contains a* $\sqrt{-1}$.
(ii) $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T})) = 1$ *in* $\mathrm{GW}(k)[\mathrm{p}^{-1}]$ *if* $char(k) = \mathrm{p} > 0$ *and k contains a* $\sqrt{-1}$.

The statements (i) and (ii) already were used in the preprint [17] in the context of proving the additivity also for the transfer and proving various applications of these in the motivic stable homotopy category. Then, Ananyevskiy proved independently the *weak form* of the conjecture in [1]. In this paper, we also show how to simplify the proof discussed in [1], by making use of our proof of the strong form of the conjecture for fields that contain $\sqrt{-1}$. This is discussed in our proof of the following Corollary.

Corollary 1.3. *Assume as in Theorem 1.2 that* G *denotes a split linear algebraic group over k, with* T *a split maximal torus and* N(T) *its normalizer in* G. *Then the following hold:*

(i) $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ *in* $\mathrm{GW}(k)$ *is a unit if* $char(k) = 0$, *and*
(ii) $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ *in* $\mathrm{GW}(k)[\mathrm{p}^{-1}]$ *is a unit if* $char(k) = p > 0$.

In view of the interest in these results, and because a proof of additivity for the trace is relatively straight-forward,[1] we have decided to write this short paper entirely devoted to a self-contained proof of these results. On feeding this result into the motivic variant of the transfer constructed in [9] and [10], we obtain a number of splitting results. The following result should serve as a proto-typical example of such applications.

Let $\mathrm{E} \to \mathrm{B}$ denote a G-torsor for the action of *any* linear algebraic group G with both E and B smooth quasi-projective schemes over k, with B *connected.* Let Y denote a G-scheme or an unpointed simplicial presheaf provided with a G-action. Let $q : \mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N_G(T)}}{\times}\mathrm{Y}) \to \mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{Y}$ denote the map induced by the map $\mathrm{G}\underset{\mathrm{N_G(T)}}{\times}\mathrm{Y} \to \mathrm{Y}$ sending $(g, y) \mapsto gy$. (In case the group G is *special* in the sense of Grothendieck, the quotient construction above can be carried out on the Nisnevich site, but in general this needs to

[1] I.e., unlike additivity for the transfer which is much more involved, and needs the notion of rigidity in an essential manner.

be carried out on the etale site and then pushed forward to the Nisnevich site, by means of a derived push-forward: the details are in [10, 3.4.2].)

Then the induced map

$$\mathrm{q}^* : \mathrm{h}^{\bullet,*}(\mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{Y}, \mathrm{M}) \to \mathrm{h}^{\bullet,*}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N_G(T)}}{\times}\mathrm{Y}), \mathrm{M})$$

is a split injection, where $\mathrm{h}^{*,\bullet}$ denotes any generalized motivic cohomology theory with respect to the motivic spectrum M.

In order to show that the map q^* is a split monomorphism, one needs the *the transfer*:

$$tr(\mathrm{Y}) : \Sigma^{\infty}_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \to \Sigma^{\infty}_{\mathbf{T}}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N_G(T)}}{\times}\mathrm{Y})_+,$$

which is a map in $\mathcal{SH}(k)$ ($\mathcal{SH}(k)[\mathrm{p}^{-1}]$, respectively) so that the composition $tr(\mathrm{Y})^* \circ q^*$ is multiplication by $\chi_{mot}(\mathrm{G/N(T)})$. Therefore, knowing $\chi_{mot}(\mathrm{G/N(T)})$ is a unit in $\mathrm{GW}(k)$ shows q^* is a split injection.

See also [18], where such splittings obtained from the motivic transfer proves a variant of the classical Segal-Becker theorem for Algebraic K-Theory.

Remark 1.4. Certain special cases of the above splitting results, for groups that are *special* seem to be also worked out in [19], *under the assumption the above conjecture is true*. Observe that a linear algebraic group G is *special* in the sense of Grothendieck (see [8]), if any torsor for G is locally trivial on the Zariski site. *Special* groups include $\{\mathrm{GL}_n, \mathrm{SL}_n | n\}$, but *exclude all orthogonal groups as well as finite groups*. For groups G that are *not special*, G-torsors are locally trivial only on the étale site. The construction of the transfer, worked out in [9] and [10, Chapter 3] apply for all such groups.

Here is an overview of the paper. One of the key techniques that is used in the proof of Theorem 1.2 is to show that the trace and the motivic Euler-characteristic are *additive up to multiplication by a sign* in general, and additive when the base field k has a $\sqrt{-1}$. We devote section 2 of the paper to establishing this additivity. Section 3 then completes the proof of the above theorem closely following the ideas for a proof of the corresponding result as in [7, Lemma 3.5] in the étale setting.

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In addition, the authors are also grateful to the referee for providing us very valuable feedback, which surely have helped us sharpen some results and improve the overall organization.

2. Additivity of the motivic trace

The main goal of this section is to establish additivity properties for the pre-transfer and trace. But we begin by establishing certain properties of a general nature for the pre-transfer and the trace.

2.1. Basic properties of the pre-transfer and trace

It is convenient to reformulate the trace in terms of the pre-transfer, which we proceed to discuss next. At the same time, we extend the framework as follows. The following discussion is a variant of what appears in [20, Chapter III]. See also [21], [13] and [14] for related discussions.

Definition 2.1. (Co-module structures) Assume that C is an unpointed simplicial presheaf, i.e., C is a contravariant functor from a given site to the category of unpointed simplicial sets. Let C_+ denote the corresponding pointed simplicial presheaf. Then the diagonal map $\Delta : C_+ \to C_+ \wedge C_+$ together with the augmentation $\epsilon : C_+ \to S^0$ defines the structure of an associative co-algebra of simplicial presheaves on C_+. A pointed simplicial presheaf P will be called a right C_+-co-module, if it comes equipped with maps $\Delta : P \to P \wedge C_+$ so that the diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & P \wedge C_+ \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle id\wedge\Delta} \\ P \wedge C_+ & \xrightarrow{\Delta\wedge id} & P \wedge C_+ \wedge C_+ \end{array} \quad \text{and} \quad \begin{array}{ccc} P & & \\ \downarrow{\scriptstyle \Delta} & \searrow^{id} & \\ P \wedge C_+ & \xrightarrow{id\wedge\epsilon} & P \wedge S^0 \end{array} \tag{2.1.1}$$

commute. *The most common choice of* P *is with* $P = C_+$ and with the obvious diagonal map $\Delta : C_+ \to C_+ \wedge C_+$ as providing the co-module structure. However, the reason we are constructing the pre-transfer in this generality (see the definition below) is so that we are able to obtain strong additivity results as in Theorem 2.5.

Definition 2.2. (*The pre-transfer*) Assume that the pointed simplicial presheaf P is such that $\Sigma^{\infty}_{\mathbf{T}} P$ is dualizable in $\mathbf{Spt}(k_{mot})$ and is provided with a map $f : P \to P$. Assume further that C is an unpointed simplicial presheaf so that P is a right C_+-co-module. Then the *pre-transfer with respect to* C_+ is defined to be a map $tr'(f) : \mathbb{S}_k \to \Sigma^{\infty}_{\mathbf{T}} C_+$, which is the composition of the following maps. Let $e : D(\Sigma^{\infty}_{\mathbf{T}} P) \wedge \Sigma^{\infty}_{\mathbf{T}} P \to \mathbb{S}_k$ denote the evaluation map. We take the dual of this map to obtain:

$$c = \mathrm{D(e)} : \mathbb{S}_k \simeq \mathrm{D}(\mathbb{S}_k) \to \mathrm{D}(\mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \wedge (\Sigma^\infty_{\mathbf{T}}\mathrm{P})) \stackrel{\simeq}{\leftarrow} \mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \wedge (\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \stackrel{\tau}{\to} (\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \wedge \mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{P}). \tag{2.1.1}$$

Here τ denotes the obvious flip map interchanging the two factors and c denotes the co-evaluation. The reason that taking the double dual yields the same object up to weak-equivalence is because we are in fact taking the dual in the setting discussed above. Observe that all the maps that *go in the left-direction are weak-equivalences*. All the maps involved in the definition of the co-evaluation map are *natural maps*.

To complete the definition of the pre-transfer, one simply composes the co-evaluation map with the following composite map:

$$\begin{aligned} &(\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \wedge \mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \stackrel{\tau}{\to} \mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \wedge (\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \stackrel{\mathrm{id}\wedge \mathrm{f}}{\to} \mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \wedge (\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \\ &\stackrel{id\wedge\Delta}{\to} \mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \wedge (\Sigma^\infty_{\mathbf{T}}\mathrm{P}) \wedge (\Sigma^\infty_{\mathbf{T}}\mathrm{C}_+) \stackrel{\mathrm{e}\wedge\mathrm{id}}{\to} \mathbb{S}_k \wedge (\Sigma^\infty_{\mathbf{T}}\mathrm{C}_+) \simeq \Sigma^\infty_{\mathbf{T}}\mathrm{C}_+. \end{aligned} \tag{2.1.1}$$

The corresponding *trace* $\tau(\mathrm{f})$, is defined as the composition of the above pre-transfer $tr'(\mathrm{f})$ with the projection π sending C_+ to S^0_+.

When $\mathrm{f} = \mathrm{id}_{\mathrm{P}}$, the pre-transfer (trace) will be denoted tr'_{P} (τ_{P}, respectively), and when $\mathrm{P} = \mathrm{C}_+$ and $\mathrm{f} = \mathrm{id}_{\mathrm{P}}$, the pre-transfer (trace) will be denoted tr'_{C_+} (τ_{C_+}, respectively).

Remark 2.3. Observe that now the trace map τ_{C_+} identifies with the following composite map:

$$\tau_{\mathrm{C}_+} : \mathbb{S}_k \stackrel{c}{\to} \Sigma^\infty_{\mathbf{T}}\mathrm{C}_+ \wedge \mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{C}_+) \stackrel{\tau}{\to} \mathrm{D}(\Sigma^\infty_{\mathbf{T}}\mathrm{C}_+) \wedge \Sigma^\infty_{\mathbf{T}}\mathrm{C}_+ \stackrel{\mathrm{e}}{\to} \mathbb{S}_k.$$

Definition 2.4. If $\mathcal{E}$ denotes *any commutative ring spectrum* in $\mathbf{Spt}(k_{mot})$, for example, $\mathbb{S}_k[\mathrm{p}^{-1}]$, we will let $\mathbf{Spt}(k_{mot}, \mathcal{E})$ denote the category of $\mathcal{E}$-module spectra over $\mathcal{E}$. Then one may replace the sphere spectrum $\mathbb{S}_k$ everywhere by $\mathcal{E}$ and define the pre-transfer and trace similarly, provided the unpointed simplicial presheaf C is such that $\mathcal{E} \wedge \mathrm{C}_+$ is dualizable in $\mathbf{Spt}(k_{mot}, \mathcal{E})$ and is provided with a map $\mathrm{f} : \mathrm{C} \to \mathrm{C}$. When $\mathrm{P} = \mathrm{C}_+$, these will be denoted $tr(\mathrm{f}_+)'_{\mathcal{E}}$, $tr'_{\mathrm{C}_+,\mathcal{E}}$, $\tau_{\mathrm{C}_+,\mathcal{E}}$, etc.

Let

$$\mathrm{U}_+ \stackrel{\mathrm{j}_+}{\to} \mathrm{X}_+ \stackrel{\mathrm{k}_+}{\to} \mathrm{X}/\mathrm{U} = \mathrm{Cone}(\mathrm{j}_+) \to \mathrm{S}^1 \wedge \mathrm{U}_+ \tag{2.1.2}$$

denote a cofiber sequence where both U and X are unpointed simplicial presheaves, with j_+ a cofibration. Now a key point to observe is that all of U, X and X/U have the structure of right X_+-co-modules. The right X_+-co-module structure on X_+ is given by the diagonal map $\Delta : \mathrm{X}_+ \to \mathrm{X}_+ \wedge \mathrm{X}_+$, while the right X_+-co-module structure on U_+ is given by the map $\Delta : \mathrm{U}_+ \stackrel{\Delta}{\to} \mathrm{U}_+ \wedge \mathrm{U}_+ \stackrel{\mathrm{id}\wedge \mathrm{j}_+}{\to} \mathrm{U}_+ \wedge \mathrm{X}_+$, where $j : \mathrm{U} \to \mathrm{X}$ is the given map. The right X_+-co-module structure on X/U is obtained in view of the commutative square

$$\begin{array}{ccc} U & \xrightarrow{(id\times j)\circ\Delta} & U \times X \\ \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle j\times id} \\ X & \xrightarrow{\Delta} & X \times X \end{array} \tag{2.1.3}$$

which provides the map

$$X/U \rightarrow (X{\times}X)/(U{\times}X) \cong (X/U){\wedge}X_+. \tag{2.1.4}$$

We begin with the following results, which are variants of [20, Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV] adapted to our contexts.

Theorem 2.5. *Let* $U_+ \overset{j_+}{\rightarrow} X_+ \overset{k_+}{\rightarrow} X/U = \mathrm{Cone}(j) \rightarrow S^1{\wedge}U_+$ *denote a cofiber sequence as in* (2.1.2)*. Let* $f: U_+ \rightarrow U_+$*,* $g: X_+ \rightarrow X_+$ *denote two pointed maps so that the diagram*

$$\begin{array}{ccc} U_+ & \xrightarrow{j_+} & X_+ \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} \\ U_+ & \xrightarrow{j_+} & X_+ \end{array}$$

commutes. Let $h: X/U \rightarrow X/U$ *denote the corresponding induced map. Then, with the right* X_+*-co-module structures discussed above, one obtains the following commutative diagram:*

$$\begin{array}{ccccccc} U_+ & \xrightarrow{j_+} & X_+ & \xrightarrow{k_+} & X/U & \xrightarrow{l} & S^1 \wedge U_+ \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle S^1\Delta} \\ U_+ \wedge X_+ & \xrightarrow{j_+\wedge id} & X_+ \wedge X_+ & \xrightarrow{k_+\wedge id} & (X/U) \wedge X_+ & \xrightarrow{l\wedge id} & S^1 \wedge U_+ \wedge X_+. \end{array} \tag{2.1.5}$$

Assume further that the $\mathbf{T}$*-suspension spectra of all the above simplicial presheaves are dualizable in* $\mathbf{Spt}(k_{mot})$*. Then, one obtains in* $\mathcal{SH}(k)$*:*

$$tr'(g) = tr'(f) + tr'(h), \quad \text{and} \quad \tau(g) = \tau(f) + \tau(h).$$

Let $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}(k_{mot})$*. Then the corresponding results also hold if the smash products of the above simplicial presheaves with the ring spectrum* $\mathcal{E}$ *are dualizable in* $\mathbf{Spt}(k_{mot}, \mathcal{E})$*.*

Theorem 2.6. *Let* $F = F_1 \sqcup_{F_3} F_2$ *denote a pushout of unpointed simplicial presheaves on the big Nisnevich site of the base scheme, with the corresponding maps* $F_3 \rightarrow F_2$*,*

$\mathrm{F_3} \to \mathrm{F_1}$ *and* $\mathrm{F_j} \to \mathrm{F}$*, for* $j = 1, 2, 3$*, assumed to be cofibrations (that is, injective maps of presheaves). Assume further the following: the* $\mathbf{T}$*-suspension spectra of all the above simplicial presheaves are dualizable in* $\mathbf{Spt}(k_{mot})$*. Let* $i_j : \mathrm{F_j} \to \mathrm{F}$ *denote the inclusion* $\mathrm{F_j} \to \mathrm{F}$*,* $j = 1, 2, 3$*. Then, one obtains in* $\mathcal{SH}(k)$*:*

- (i) $tr'_{\mathrm{F}_+} = i_1 \circ tr'_{\mathrm{F}_{1+}} + i_2 \circ tr'_{\mathrm{F}_{2+}} - i_3 \circ tr'_{\mathrm{F}_{3+}}$ *and* $\tau_{\mathrm{F}_+} = \tau_{\mathrm{F}_{1+}} + \tau_{\mathrm{F}_{2+}} - \tau_{\mathrm{F}_{3+}}$*, where* tr'_{F_+} *and* $tr'_{\mathrm{F}_{j+}}$*,* $j = 1, 2, 3$ *(*τ_{F_+}*,* $\tau_{\mathrm{F}_{j+}}$*,* $j = 1, 2, 3$*) denote the pre-transfer maps (trace maps, respectively).*
- (ii) *In particular, taking* $\mathrm{F_2} = *$*, and* $\mathrm{F} = \mathrm{Cone}(\mathrm{F_3} \to \mathrm{F_1})$*, we obtain in* $\mathcal{SH}(k)$*:* $tr'_{\mathrm{F}} = i_1 \circ tr'_{\mathrm{F}_{1+}} - i_3 \circ tr'_{\mathrm{F}_{3+}}$ *and* $\tau_{\mathrm{F}} = \tau_{\mathrm{F}_{1+}} - \tau_{\mathrm{F}_{3+}}$*.*

Let $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}(k_{\mathrm{mot}})$*. Then the corresponding results also hold if the smash products of the above simplicial presheaves with the ring spectrum* $\mathcal{E}$ *are dualizable in* $\mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$*.*

Our next goal is to provide proofs of these two theorems. We will discuss the proofs explicitly only for the case of spectra in $\mathbf{Spt}(k_{mot})$, as the corresponding results readily extend to spectra in $\mathbf{Spt}(k_{mot}, \mathcal{E})$ for a commutative ring spectrum $\mathcal{E}$ in $\mathbf{Spt}(k_{mot}, \mathcal{E})$. The additivity of the trace follows readily from the additivity of the pre-transfer, as it is obtained by composing with the projection $\Sigma^{\infty}_{\mathbf{T}} \mathrm{X}_+ \to \mathbb{S}_{\mathrm{k}}$.

Since this is discussed in the topological framework in [20, Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV], our proof amounts to verifying carefully and in a detailed manner that the same arguments there carry over to our framework. This is possible, largely because the arguments in the proof of [20, Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV] depend only on a theory of Spanier-Whitehead duality in a symmetric monoidal triangulated category framework and [11] shows that the entire theory of Spanier-Whitehead duality works in such general frameworks. Nevertheless, it seems prudent to show explicitly that at least the key arguments in [20, Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV] carry over to our framework. It may be important to point out that the discussion in [20, Chapters III and IV] is carried out in the equivariant framework: as all our discussion is taking place with no group actions, one may take the group to be trivial in the discussion in [20].

The very first observation is that the hypotheses of Theorem 2.5 readily imply the commutativity of the diagram:

$$\begin{array}{ccccccc}
\mathrm{U}_+ & \xrightarrow{j_+} & \mathrm{X}_+ & \xrightarrow{k_+} & \mathrm{X/U} & \xrightarrow{l} & \mathrm{S}^1 \wedge \mathrm{U}_+ \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{S}^1 f} \\
\mathrm{U}_+ & \xrightarrow{j_+} & \mathrm{X}_+ & \xrightarrow{k_+} & \mathrm{X/U} & \xrightarrow{l} & \mathrm{S}^1 \wedge \mathrm{U}_+ .
\end{array}$$

Next we proceed to verify the commutativity of the diagram (2.1.5). Since the first square clearly commutes, it suffices to verify the commutativity of the second square. This follows readily in view of the following commutative square of pairs:

$$\begin{array}{ccc} (\mathrm{X}, \phi) & \longrightarrow & (\mathrm{X}, \mathrm{U}) \\ \downarrow \Delta & & \downarrow \Delta \\ (\mathrm{X} \times \mathrm{X}, \phi) & \longrightarrow & (\mathrm{X} \times \mathrm{X}, \mathrm{U} \times \mathrm{X}). \end{array}$$

Observe, as a consequence that we have verified that the hypotheses of [20, Theorem 7.10, Chapter III] are satisfied by the $\Sigma^{\infty}_{\mathbf{T}}$-suspension spectra of all the simplicial presheaves appearing in (2.1.5).

The next step is to observe that the $\mathrm{F_i}$, $i = 1, 2, 3$ (F) in our Theorem 2.6, correspond to the $\mathrm{F_i}$ (F, respectively) in [20, Theorem 2.9, Chapter IV]. Now observe that

$$\mathrm{F}_{3+} \to (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ \to \mathrm{F}_+ \to \mathrm{S}^1 \wedge \mathrm{F}_{3,+} \tag{2.1.6}$$

is a distinguished triangle. Moreover as $\mathrm{F}_1 \sqcup \mathrm{F}_2$ has a natural map (which we will call k) into F, there is a commutative diagram:

$$\begin{array}{ccc} (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ & \xrightarrow{k} & \mathrm{F}_+ \\ \downarrow \Delta & & \downarrow \Delta \\ (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ \wedge \mathrm{F}_+ & \xrightarrow{k \wedge id} & \mathrm{F}_+ \wedge \mathrm{F}_+. \end{array}$$

Then the distinguished triangle (2.1.6) provides the commutative diagram:

$$\begin{array}{ccccccc} (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ & \xrightarrow{k} & \mathrm{F}_+ & \longrightarrow & \mathrm{S}^1 \wedge \mathrm{F}_{3,+} & \longrightarrow & \mathrm{S}^1 \wedge (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ \\ \downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta \\ (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+ \wedge \mathrm{F}_+ & \xrightarrow{k \wedge id} & \mathrm{F}_+ \wedge \mathrm{F}_+ & \longrightarrow & (\mathrm{S}^1 \wedge \mathrm{F}_{3,+}) \wedge \mathrm{F}_+ & \longrightarrow & (\mathrm{S}^1 \wedge (\mathrm{F}_1 \sqcup \mathrm{F}_2)_+) \wedge \mathrm{F}_+ \end{array}$$

so that the hypotheses of [20, Theorem 7.10, Chapter III] are satisfied with X, Y and Z there equal to the $\Sigma^{\infty}_{\mathbf{T}}$-suspension spectra of $(\mathrm{F}_1 \sqcup \mathrm{F}_2)_+$, F_+ and $\mathrm{S}^1 \wedge \mathrm{F}_{3,+}$. These arguments, therefore reduce the proof of Theorem 2.6 to that of Theorem 2.5.

Therefore, what we proceed to verify is that, then the proof of [20, Theorem 7.10, Chapter III] carries over to our framework. This will then complete the proof of Theorem 2.5. A key step of this amounts to verifying that the big commutative diagram given on [20, p. 166] carries over to our framework. One may observe that this big diagram is broken up into various sub-diagrams, labeled (I) through (VII) and that it suffices to verify that each of these sub-diagrams commutes up to homotopy. This will prove that additivity holds for the trace.

For this, it seems best to follow the terminology adopted in [20, Theorem 7.10, Chapter III]: therefore we will let U_+ (X_+ and X/U) in Theorem 2.5 be denoted X (Y and Z, respectively) for the remaining part of the proof of Theorem 2.5. Let $k : X \to Y$ ($i : Y \to Z$ and $\pi : Z \to S^1 \wedge X$) denote the corresponding maps $j_+ : U_+ \to X_+$ ($k_+ : X_+ \to X/U$, and the map $l : X/U \to S^1 \wedge U_+$) as in Theorem 2.5. Then the very first step in this direction is to verify that the three squares

$$\begin{array}{ccc} DY \wedge X & \xrightarrow{id\wedge k} & DY \wedge Y \\ \downarrow{\scriptstyle Dk\wedge id} & & \downarrow{\scriptstyle e} \\ DX \wedge X & \xrightarrow{e} & \mathbb{S}_k \end{array}, \quad \begin{array}{ccc} DZ \wedge Y & \xrightarrow{id\wedge i} & DZ \wedge Z \\ \downarrow{\scriptstyle Di\wedge id} & & \downarrow{\scriptstyle e} \\ DY \wedge Y & \xrightarrow{e} & \mathbb{S}_k \end{array}, \text{ and}$$

$$\begin{array}{ccc} D(S^1 \wedge X) \wedge Z & \xrightarrow{id\wedge\pi} & D(S^1 \wedge X) \wedge (S^1 \wedge X) \\ \downarrow{\scriptstyle D\pi\wedge id} & & \downarrow{\scriptstyle e} \\ DZ \wedge Z & \xrightarrow{e} & \mathbb{S}_k \end{array} \tag{2.1.7}$$

commute up to homotopy. (The homotopy commutativity of these squares is a formal consequence of Spanier-Whitehead duality: see [27, pp. 324-325] for proofs in the classical setting.) As argued on [20, page 167, Chapter III], the composite $e \circ (D\pi \wedge i) : D(S^1 \wedge X) \wedge Y \to \mathbb{S}_k$ is equal to $e \circ ((id \wedge \pi) \circ (id \wedge i)$ and is therefore the trivial map. Therefore, if j denotes the inclusion of DZ∧Z in the cofiber of $D\pi \wedge i$, one obtains the induced map $\bar{e} : (DZ \wedge Z)/(D(S^1 \wedge X) \wedge Y) \to \mathbb{S}_k$ so that the triangle

$$\begin{array}{ccc} DZ \wedge Z & \xrightarrow{e} & \mathbb{S}_k \\ \downarrow{\scriptstyle j} & \nearrow{\scriptstyle \bar{e}} & \\ (DZ \wedge Z)/(D(S^1 \wedge X) \wedge Y) & & \end{array} \tag{2.1.8}$$

homotopy commutes. This provides the commutative triangle denoted (I) in [20, p. 166] there and the commutative triangle denoted (II) there commutes by the second and third commutative squares in (2.1.7). The duals of (I) and (II) are the triangles denoted (I*) and (II*) (on [20, p. 166]) and therefore, they also commute.

Next we briefly consider the homotopy commutativity of the remaining diagram beginning with the squares labeled (III), (IV) and (V) in [20, p. 166]. Since the maps denoted δ are weak-equivalences, it suffices to show that these squares homotopy commute when the maps denoted δ^{-1} are replaced by the corresponding maps δ going in the opposite direction. Such maps δ appearing there are all special instances of the following natural map: $\delta : \mathrm{DB}\wedge\mathrm{A} \to \mathrm{D}(\mathrm{DA}\wedge\mathrm{B})$, for two spectra A and B in $\mathbf{Spt}(k_{mot})$. The homotopy commutativity of the squares (III), (IV) and (V) are reduced therefore to the naturality of the above map in the arguments A and B: see the discussion in [20, pp. 167-168]. The commutativity of the triangle labeled (VI) follows essentially from the definition of the maps there. Finally the homotopy commutativity of the square (VII) is reduced to the following lemma, which is simply a restatement of [20, Lemma 7.11, Chapter III]. These will complete the proof for the additivity property for the trace and hence the proofs of Theorems 2.5 and 2.6.

Lemma 2.7. *Let* $\mathrm{f} : \mathrm{A} \to \mathrm{X}$ *and* $\mathrm{g} : \mathrm{B} \to \mathrm{Y}$ *be maps in* $\mathbf{Spt}(k_{mot})$ *and let* $i : \mathrm{X} \to \mathrm{Cone(f)}$ *and* $j : \mathrm{Y} \to \mathrm{Cone(g)}$ *be the inclusions into their cofibers. Then the boundary map* $\delta : \Sigma_{\mathrm{S}^1}^{-1} Cone(i \wedge j) \to Cone(\mathrm{f}\wedge\mathrm{g})$ *in the cofiber sequence* $Cone(\mathrm{f} \wedge \mathrm{g}) \to \mathrm{Cone}((\mathrm{i}\circ\mathrm{f}) \wedge (\mathrm{j} \circ \mathrm{g}))$ $\to \mathrm{Cone}(\mathrm{i} \wedge \mathrm{j})$ *is the sum of the two composites:*

$$\Sigma_{\mathrm{S}^1}^{-1} Cone(i \wedge j) \overset{\Sigma_{\mathrm{S}^1}^{-1} Cone(i\wedge id)}{\to} Cone(id_{Cone(\mathrm{f})} \wedge j) = Cone(\mathrm{f}) \wedge \mathrm{B} \cong \mathrm{Cone}(\mathrm{f} \wedge \mathrm{id_B})$$
$$\overset{\mathrm{Cone(id\wedge g)}}{\to} \mathrm{Cone}(\mathrm{f} \wedge \mathrm{g}),$$

$$\Sigma_{\mathrm{S}^1}^{-1} Cone(i \wedge j) \overset{\Sigma_{\mathrm{S}^1}^{-1} Cone(id\wedge j)}{\to} \Sigma_{\mathrm{S}^1}^{-1} Cone(i \wedge id_{Cone(\mathrm{g})}) = \mathrm{A} \wedge \mathrm{Cone(g)} \cong \mathrm{Cone}(\mathrm{id_A} \wedge \mathrm{g})$$
$$\overset{\mathrm{Cone(f\wedge id)}}{\to} \mathrm{Cone}(\mathrm{f} \wedge \mathrm{g}).$$

Proposition 2.8. *(Multiplicative property of the pre-transfer and trace) Assume* $\mathrm{F_i}$, $i = 1, 2$ *are simplicial presheaves, and let* $\mathrm{f_i} : \mathrm{F_i} \to \mathrm{F_i}$, $i = 1, 2$ *denote a given map. Let* $\mathrm{F} = \mathrm{F}_{1+}\wedge\mathrm{F}_{2+}$ *and let* $\mathrm{f} = \mathrm{f}_{1+}\wedge\mathrm{f}_{2+}$. *Then*

$$tr'_{\mathrm{F}}(\mathrm{f}) = \mathrm{tr}'_{\mathrm{F}_{1+}}(\mathrm{f}_{1+}) \wedge \mathrm{tr}'_{\mathrm{F}_2}(\mathrm{f}_{2+}), \ \ and$$
$$\tau_{\mathrm{F}}(\mathrm{f}) = \tau_{\mathrm{F}_{1+}}(\mathrm{f}_{1+}) \wedge \tau_{\mathrm{F}_{2+}}(\mathrm{f}_{2+}).$$

A corresponding result holds if F_2 *is a pointed simplicial presheaf with* $\mathrm{F} = \mathrm{F}_{1+}\wedge\mathrm{F}_2$.

Proof. A key point to observe is that the evaluation $e_{\mathrm{F}} : \mathrm{D(F)}\wedge\mathrm{F} \to \mathbb{S}_k$ is given by starting with $e_{\mathrm{F}_{1+}} \wedge e_{\mathrm{F}_{2+}} : \mathrm{D}(\mathrm{F}_{1+})\wedge\mathrm{F}_{1+}\wedge\mathrm{D}(\mathrm{F}_{2+})\wedge\mathrm{F}_{2+} \to \mathbb{S}_k \wedge \mathbb{S}_k \simeq \mathbb{S}_k$ and by precomposing it with the map $\mathrm{D(F)}\wedge\mathrm{F} = \mathrm{D}(\mathrm{F}_{1+}\wedge\mathrm{F}_{2+})\wedge\mathrm{F}_{1+}\wedge\mathrm{F}_{2+} \overset{\tau}{\to} \mathrm{D}(\mathrm{F}_{1+}) \wedge \mathrm{F}_{1+} \wedge \mathrm{D}(\mathrm{F}_{2+})\wedge\mathrm{F}_{2+}$, where τ is the obvious map that interchanges the factors. Similarly the co-evaluation map $c : \mathbb{S}_k \simeq \mathbb{S}_k \wedge \mathbb{S}_k \overset{c_{\mathrm{F}_{1+}} \wedge c_{\mathrm{F}_{2+}}}{\to} \mathrm{F}_{1+}\wedge\mathrm{D}(\mathrm{F}_{1+})\wedge\mathrm{F}_{2+} \wedge \mathrm{D}(\mathrm{F}_{2+})$ provides the co-evaluation map for F. The multiplicative property of the pre-transfer follows readily from the above two observations as well as from the definition of the pre-transfer as in Definition 2.2. In view of the definition of the trace as in Definition 2.2, the multiplicative property of the trace follows from the multiplicative property of the pre-transfer. These prove the statements when $F_+ = F_{1+} \wedge F_{2+}$. The corresponding statements when F_2 is already a pointed simplicial presheaf may be proven along entirely similar lines. □

2.2. Additivity of the motivic trace

The goal of this section is to prove the following theorem.

Theorem 2.9. *(Mayer-Vietoris and Additivity for the Trace)*

(i) *Let* X *denote a smooth quasi-projective scheme and let* $i_j : X_j \to X$, $j = 1, 2$ *denote the open immersion of two Zariski open subschemes of* X, *with* $X = X_1 \cup X_2$. *Let* $U \to X$ *denote the open immersion of a Zariski open subscheme of* X, *with* $U_i = U \cap X_i$. *Then adopting the terminology above, (that is, where* τ_P *denotes the trace associated to the pointed simplicial presheaf* P*), and when char*$(k) = 0$,

$$\tau_{X/U} = \tau_{X_1/U_1} + \tau_{X_2/U_2} - \tau_{(X_1 \cap X_2)/(U_1 \cap U_2)} \text{ in } \mathcal{SH}(k). \tag{2.2.1}$$

In case char$(k) = p > 0$,

$$\tau_{X/U, \mathbb{S}_k[p^{-1}]} = \tau_{X_1/U_1, \mathbb{S}_k[p^{-1}]} + \tau_{X_2/U_2, \mathbb{S}_k[p^{-1}]} - \tau_{(X_1 \cap X_2)/(U_1 \cap U_2), \mathbb{S}_k[p^{-1}]}, \text{ in } \mathcal{SH}(k)[p^{-1}]. \tag{2.2.2}$$

Throughout the following discussion, let $< -1 >$ *denote the class in the Grothendieck-Witt ring associated to* $-1 \in k$ *as in [22, p. 252].*

(ii) *Let* $i : Z \to X$ *denote a closed immersion of smooth schemes with* $j : U \to X$ *denoting the corresponding open complement. Let* $\mathcal{N}$ *denote the normal bundle associated to the closed immersion* i *and let* $\mathrm{Th}(\mathcal{N})$ *denotes its Thom-space. Let* c *denote the codimension of* Z *in* X. *Then adopting the terminology above, we obtain in* $\mathcal{SH}(k)$ *when char*$(k) = 0$:

$$\tau_{X_+} = \tau_{U_+} + \tau_{X/U}, \text{ and } \tau_{X/U} = \tau_{\mathrm{Th}(\mathcal{N})} =< -1 >^c \tau_{Z_+}. \tag{2.2.3}$$

In case $\sqrt{-1} \in k$, *it follows that*

$$\tau_{X/U} = \tau_{\mathrm{Th}(\mathcal{N})} = \tau_{Z_+}.$$

In case char$(k) = p > 0$, *we obtain in* $\mathcal{SH}(k)[p^{-1}]$:

$$\tau_{X_+, \mathbb{S}_k[p^{-1}]} = \tau_{U_+, \mathbb{S}_k[p^{-1}]} + \tau_{X/U, \mathbb{S}_k[p^{-1}]}, \tau_{X/U, \mathbb{S}_k[p^{-1}]} = \tau_{\mathrm{Th}(\mathcal{N}), \mathbb{S}_k[p^{-1}]} =< -1 >^c \tau_{Z_+, \mathbb{S}_k[p^{-1}]}, \tag{2.2.4}$$

and assuming $\sqrt{-1} \in k$

$$\tau_{X/U, \mathbb{S}_k[p^{-1}]} = \tau_{\mathrm{Th}(\mathcal{N}), \mathbb{S}_k[p^{-1}]} = \tau_{Z_+, \mathbb{S}_k[p^{-1}]}.$$

(iii) *Let* $\{S_\alpha|\alpha\}$ *denote a stratification of the smooth scheme* X *into finitely many locally closed and smooth subschemes* S_α. *Let* c_α *denote the codimension of* S_α *in* X. *Then we obtain in* $\mathcal{SH}(k)$ *when* $char(k) = 0$*:*

$$\tau_{X_+} = \Sigma_\alpha < -1 >^{c_\alpha} \tau_{S_{\alpha+}} \text{ and assuming } \sqrt{-1} \in k, \qquad (2.2.5)$$
$$\tau_{X_+} = \Sigma_\alpha \tau_{S_{\alpha+}}.$$

In case $char(k) = p > 0$, *we obtain in* $\mathcal{SH}(k)[p^{-1}]$*:*

$$\tau_{X_+, \mathbb{S}_k[p^{-1}]} = \Sigma_\alpha < -1 >^{c_\alpha} \tau_{S_{\alpha+}, \mathbb{S}_k[p^{-1}]}, \text{ and again assuming } \sqrt{-1} \in k, \qquad (2.2.6)$$
$$\tau_{X_+, \mathbb{S}_k[p^{-1}]} = \Sigma_\alpha \tau_{S_{\alpha+}, \mathbb{S}_k[p^{-1}]}. \quad \square$$

Proof. We will explicitly discuss only the case in characteristic 0, as proofs in positive characteristics will follow along the same lines.

First one recalls the stable homotopy cofiber sequence (see [24, p. 115, Theorem 2.23])

$$\Sigma^\infty_{\mathbf{T}} U_+ \to \Sigma^\infty_{\mathbf{T}} X_+ \to \Sigma^\infty_{\mathbf{T}} (X/U) \simeq \Sigma^\infty_{\mathbf{T}} \wedge Th(\mathcal{N}) \qquad (2.2.7)$$

in the stable motivic homotopy category over the base scheme. The first statement in (2.2.3) follows by applying Theorem 2.5 to the stable homotopy cofiber sequence in (2.2.7).

Next we will consider (i), namely the Mayer-Vietoris sequence. For this, one begins with the stable cofiber sequences

$$\Sigma^\infty_{\mathbf{T}} (U_1 \cap U_2)_+ \to \Sigma^\infty_{\mathbf{T}} (U_1 \sqcup U_2)_+ \to \Sigma^\infty_{\mathbf{T}} (U)_+,$$
$$\Sigma^\infty_{\mathbf{T}} (X_1 \cap X_2)_+ \to \Sigma^\infty_{\mathbf{T}} (X_1 \sqcup X_2)_+ \to \Sigma^\infty_{\mathbf{T}} (X)_+.$$

Then one applies Theorem 2.6(i) to both of them, which will prove:

$$\tau_{U_+} = \tau_{(U_1 \cup U_2)_+} = \tau_{U_{1+}} + \tau_{U_{2+}} - \tau_{(U_1 \cap U_2)_+} \text{ and} \qquad (2.2.8)$$
$$\tau_{X_+} = \tau_{(X_1 \cup X_2)_+} = \tau_{X_{1+}} + \tau_{X_{2+}} - \tau_{(X_1 \cap X_2)_+}.$$

On applying the first statement in (ii) to $U_i \subseteq X_i$, $i = 1, 2$ and $U_1 \cap U_2 \subseteq X_1 \cap X_2$ we obtain:

$$\tau_{X_i/U_i} = \tau_{X_{i+}} - \tau_{U_{i+}}, i = 1, 2 \text{ and}$$
$$\tau_{(X/U)_+} = \tau_{(X_1 \cup X_2)/(U_1 \cup U_2)} = \tau_{(X_1 \cup X_2)_+} - \tau_{(U_1 \cup U_2)_+}.$$

The required statement in (2.2.1) now follows on substituting from (2.2.8). This completes the proof of (i).

We proceed to establish the remaining statement in (2.2.3). First we will consider the case where the normal bundle $\mathcal{N}$ is trivial, mainly because this is an important special case to consider. When the normal bundle is trivial, we observe that

$X/U \simeq Th(\mathcal{N}) \simeq \mathbf{T}^c \wedge Z_+$. Next, the multiplicative property of the trace as in Lemma 2.8 shows that

$$\tau_{\Sigma^\infty_{\mathbf{T}}(\mathbf{T}^c \wedge Z_+)} = (\tau_{\Sigma^\infty_{\mathbf{T}}\mathbf{T}})^{\wedge^c} \wedge \tau_{\Sigma^\infty_{\mathbf{T}} Z_+} \tag{2.2.9}$$

as classes in $\pi_{0,0}(\mathbb{S}_k)$. In general, it is known that the class of $\tau_{\Sigma^\infty_{\mathbf{T}}\mathbf{T}} =< -1 >$ in the Grothendieck-Witt group $GW(k)$: recall that $GW(k)$ identifies with $\pi_{0,0}(\mathbb{S}_k)$, in view of [22, Theorem 6.2.2]. (Here it may be important to recall that $\mathbf{T}$ is the pointed simplicial presheaf $\mathbb{P}^1$ pointed by ∞.) This implies that $\tau_{\Sigma^\infty_{\mathbf{T}}\mathbf{T}} = -1$ in $\pi_{0,0}(\mathbb{S}_k)$ and proves the second statement in (2.2.3) when $\mathcal{N}$ is trivial.

Next we assume that $\sqrt{-1} \in k$. Then the quadratic form $< -1 >$ gets identified with $< 1 >$ in the Grothendieck-Witt group $GW(k)$: see, for example, [28, p. 44]. Therefore, $\tau_{\Sigma^\infty_{\mathbf{T}}\mathbf{T}} =< 1 >$, hence $\tau_{\Sigma^\infty_{\mathbf{T}}\mathbf{T}^c \wedge Z_+} = \tau_{\Sigma^\infty_{\mathbf{T}} Z_+}$. This completes the proof of (ii), when the normal bundle to Z in X is trivial.

To consider the general case when the normal bundle $\mathcal{N}$ is not necessarily trivial, one takes a finite Zariski open cover $\{U_i | i = 1, \cdots, n\}$ so that $\mathcal{N}_{|U_i}$ is trivial for each i. Then the Mayer-Vietoris property considered in (i) and ascending induction on n, together with the case where the normal bundle is trivial considered above, completes the proof in this case. (Observe that any scheme Z over k of finite type is always quasi-compact, so that such a finite open cover always exists.) These complete the proof of all the statements in (ii).

Next we consider the statement in (iii). This will follow from the second statement in (ii) using ascending induction on the number of strata. However, as this induction needs to be handled carefully, we proceed to provide an outline of the relevant argument. We will assume that the stratification of X defines the following increasing filtrations:

(a) $\phi = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = X$, where each X_i is closed and the strata $X_i - X_{i-1}$, $i = 0, \cdots, n$ are smooth.

(b) $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n = X$, where each U_i is open in X (and therefore smooth), with $U_i - U_{i-1} = X_{n-i} - X_{n-i-1}$, for all $i = 0, \cdots n$. Now observe that each $U_k \to X$ is an open immersion, while each $X_k - X_{k-1} \to X - X_{k-1}$ is a closed immersion. Let c_k denote the corresponding codimension.

We now apply Theorem 2.9(ii) with $U = U_{n-1}$, and $Z = U_n - U_{n-1} = X_0 - X_{-1} = X_0$, the closed stratum. Since X is now smooth and so is Z, the hypotheses of Theorem 2.9(ii) are satisfied. This provides us

$$\tau_{X_+} = \tau_{U_{n-1+}} + \tau_{X/U_{n-1}} \text{ and } \tau_{X/U_{n-1}} =< -1 >^{c_0} \tau_{X_{0+}} \tag{2.2.10}$$

Next we replace X by U_{n-1}, U by U_{n-2} and Z by $X_1 - X_0$. Since $X_1 - X_0$ is smooth, Theorem 2.9(ii) now provides us

$$\tau_{U_{n-1+}} = \tau_{U_{n-2+}} + < -1 >^{c_1} \tau_{(X_1 - X_0)_+}. \tag{2.2.11}$$

Substituting these in (2.2.10), we obtain

$$\tau_{\mathrm{X}_+} = \tau_{\mathrm{U_{n-2+}}} + < -1 >^{c_1} \tau_{(\mathrm{X}_1 - \mathrm{X}_0)_+} + < -1 >^{c_0} \tau_{\mathrm{X}_{0+}}.$$

Clearly this may be continued inductively to deduce statement (iii) in Theorem 2.9 from Theorem 2.9(ii). □

3. Proofs of the main theorems

We begin by discussing the following Proposition, which seems to be rather well-known. (See for example, [29, Proposition 4.10] or [7, (3.6)].)

Proposition 3.1. *Let* T *denote a split torus acting on a separated scheme* X *all defined over the given perfect base field* k*.*

Then the following hold.

X *admits a decomposition into a disjoint union of finitely many locally closed,* T*-stable subschemes* $\mathrm{X_j}$ *so that*

$$\mathrm{X_j} \cong (\mathrm{T}/\Gamma_\mathrm{j}) \times \mathrm{Y_j}. \tag{3.0.1}$$

Here each Γ_j *is a sub-group-scheme of* T*, each* $\mathrm{Y_j}$ *is a scheme of finite type over* k *which is also regular and on which* T *acts trivially with the isomorphism in* (3.0.1) *being* T*-equivariant.*

Proof. One may derive this from the generic torus slice theorem proved in [29, Proposition 4.10], which says that if a split torus acts on a reduced separated scheme of finite type over a perfect field, then the following are satisfied:

(1) there is an open subscheme U which is regular and stable under the T-action
(2) a geometric quotient U/T exists, which is a regular scheme of finite type over k
(3) U is isomorphic as a T-scheme to $\mathrm{T}/\Gamma \times \mathrm{U}/\mathrm{T}$ where Γ is a diagonalizable subgroup scheme of T and T acts trivially on U/T.

(See also [7, (3.6)] for a similar decomposition.) □

Next we consider the following theorem.

Theorem 3.2. *Under the assumption that the base field* k *is of characteristic* 0*, the following hold, where* τ_{X_+} *denotes the trace associated to the pointed scheme* X_+*:*

(i) $\tau_{\mathbb{G}_{m+}} = 1 - < -1 >$ *in* $\mathrm{GW}(k)$*, and if* T *is a split torus of rank* n*,* $\tau_{\mathrm{T}_+} = (1 - < -1 >)^n$ *in* $\mathrm{GW}(k)$*. Therefore, it follows that when* k *contains a* $\sqrt{-1}$*,* $\tau_{\mathbb{G}_{m+}} = 0$ *and* $\tau_{\mathrm{T}_+} = 0$ *in* $\mathrm{GW}(k)$*.*

(ii) *Let* T *denote a split torus acting on a smooth scheme* X*. Then* X^T *is also smooth, and* $\tau_{X_+} - \tau_{X^T_+}$ *belongs to the ideal generated by* $(1- < -1 >)$ *in* GW(k)*. In particular, when* k *contains a* $\sqrt{-1}$*,* $\tau_{X_+} = \tau_{X^T_+}$ *in* GW(k)*.*

If the base field is of positive characteristic p*, the corresponding assertions hold with the trace of a pointed smooth scheme* Y_+ *replaced by* $\tau_{Y_+, \mathbb{S}_k[p^{-1}]}$ *and the Grothendieck-Witt ring replaced by the Grothendieck-Witt ring with the prime* p *inverted.*

Proof. We will only consider the proofs when the base field is of characteristic 0, since the proofs in the positive characteristic case are entirely similar. However, it is important to point out that in positive characteristics p, it is important to invert p: for otherwise, one no longer has a theory of Spanier-Whitehead duality. Next observe from Definition 2.2, that the trace τ_{X_+} associated to any smooth scheme X is a map $\mathbb{S}_k \to \mathbb{S}_k$: as such, we will identify τ_{X_+} with the corresponding class $\tau^*_{X_+}(1)$ in the Grothendieck Witt-ring of the base field.

Next we consider (i). We observe that the scheme $\mathbb{A}^1$ is the disjoint union of the closed point $\{0\}$ and $\mathbb{G}_m$. If $i_1 : \{0\} \to \mathbb{A}^1$ and $j_1 : \mathbb{G}_m \to \mathbb{A}^1$ are the corresponding immersions, Theorems 2.9(ii) and (iii) show that

$$\tau_{\mathbb{A}^1_+} = \tau_{\mathbb{G}_{m+}} + \tau_{\mathbb{A}^1/\mathbb{G}_m} = \tau_{\mathbb{G}_{m+}} + \tau_{\mathbf{T}} = \tau_{\mathbb{G}_{m+}} + < -1 > . \tag{3.0.2}$$

Therefore, it follows that

$$\tau_{\mathbb{G}_{m+}} = \tau_{\mathbb{A}^1_+} - < -1 >= 1- < -1 >, \tag{3.0.3}$$

where $\tau_{\mathbb{A}^1_+} = \tau_{\{0\}_+} = 1$ by $\mathbb{A}^1$-contractibility. One may readily see this from the definition of the pre-transfer as in Definition 2.2, which shows that both the pre-transfer $tr'_{C_+} = tr'_{C_+}(id)$ and hence the corresponding trace, $\tau_{C_+} = \pi \circ tr'_{C_+}$ depends on C_+ only up to its class in the motivic stable homotopy category. Since T is a split torus, we may assume $T = \mathbb{G}^n_m$ for some positive integer n. Then the multiplicative property of the trace and pre-transfer (see Proposition 2.8) prove that $\tau_{T_+} = (1- < -1 >)^n$. In particular, when k contains a $\sqrt{-1}$, it follows that $\tau_{\mathbb{G}_{m+}} = 0$ and $\tau_{\mathbf{T}+} = 0$ in GW(k). These complete the proof of statement (i).

Therefore, we proceed to prove the statement in (ii). First, we invoke Proposition 3.1 to conclude that X^T is the disjoint union of the subschemes X_j for which $\Gamma_j = T$.

Let $i_j : X_j \cong (T/\Gamma_j) \times Y_j \to X$ denote the locally closed immersion. Next observe that the additivity of the trace proven in Theorem 2.9, and the multiplicativity of the pre-transfer and trace proven in Proposition 2.8 along with the decomposition in (3.0.1) show that

$$\tau_{X_+} = \Sigma_j \tau_{X_{j+}} = \Sigma_j(\tau_{T/\Gamma_j+}) \wedge \tau_{Y_{j+}}. \tag{3.0.4}$$

Now statement (i) in the theorem shows that the term $\tau_{T/\Gamma_j+} = (1- < -1 >)^{n_j}$, if T/Γ_j is a split torus of rank n_j. Since X^T is the disjoint union of the subschemes $X_j = T/\Gamma_j \times Y_j$ with $\Gamma_j = T$, the additivity of the trace proven in Theorem 2.9 and applied to X^T proves the sum of such terms on the right-hand-side of (3.0.4) is $\tau_{X^T_+}$. Therefore, it follows that $\tau_{X_+} - \tau_{X^T_+}$ belongs to the ideal in $GW(k)$ generated by $1- < -1 >$. In particular, when k contains a $\sqrt{-1}$, it follows that $\tau_{X+} = \tau_{X^T_+}$. These complete the proof of the statements in (ii). □

Proof of Theorem 1.2. We point out that it is important to assume the base field k is perfect in the following arguments: this will ensure that all the schemes considered here are defined over the same base field. First we will show that we can reduce to the case G is *connected*. Let G^o denote the connected component of G containing the identity element and let T denote a split maximal torus in G. Then, one first obtains the isomorphisms $G/N_G(T) \cong \{gTg^{-1} | g \, \varepsilon \, G\}$ and $G^o/N_{G^o}(T) \cong \{g_oTg_o^{-1} | g_o \, \varepsilon \, G^o\}$. Next observe that gTg^{-1}, being a maximal torus and hence a connected subgroup of G, is in fact a maximal torus in G^o for each $g \, \varepsilon \, G$. These show that

$$G/N_G(T) = \{gTg^{-1} | g \, \varepsilon \, G\} \cong \{g_oTg_o^{-1} | g_o \, \varepsilon \, G^o\} = G^o/N_{G^o}(T).$$

Therefore, we may assume the group G is connected.

Moreover, we may take the quotient by the unpointed radical $R_u(G)$, which is a normal subgroup (and is isomorphic to an affine space), with the quotient $G_{red} = G/R_u(G)$ reductive. Now $G/N_G(T) \cong G_{red}/N_{G_{red}}(T)$ (since the intersection of a maximal torus in G with the unpointed radical $R_u(G)$ is trivial), so that we may assume G is a connected split reductive group.

Then we observe that since $G/N_G(T)$ is the variety of all split maximal tori in G, T has an action on $G/N_G(T)$ (induced by the left translation action of T on G) so that there is exactly a single fixed point, namely the coset $eN_G(T)$, that is, $(G/N_G(T))^T = \{eN_G(T)\} = \{\mathrm{Spec}\, k\}$. (To prove this assertion, one may reduce to the case where the base field is algebraically closed, since the formation of fixed point schemes respects change of base fields as shown in [12, p. 33, Remark (3)]. See also [7, Lemma 3.5]. In fact, one may see this directly as follows. Making use of the identification of $G/N_G(T)$ with $\{gTg^{-1} | g \, \varepsilon \, G\}$, one sees that if $g_0Tg_0^{-1}$ is fixed by the conjugation action of T, then $g_0^{-1}Tg_0 \subseteq N_G(T)^o = T$, so that $g_0 \, \varepsilon \, N_G(T)$. Thus the coset $g_0N_G(T) = eN_G(T)$.)

Next we will first consider the case where the base field is of characteristic 0. Therefore, by Theorem 3.2(ii),

$$\tau_{G/N_G(T)_+} = \tau_{(G/N_G(T))^T_+} = \tau_{\mathrm{Spec}\, k_+} = id_{\mathbb{S}_k},$$

which is the identity map of the motivic sphere spectrum. Therefore,

$$\chi_{mot}(\mathrm{G}/\mathrm{N_G}(T)) = \tau^*_{\mathrm{G}/\mathrm{N_G(T)}_+}(1) = 1.$$

The motivic stable homotopy group $\pi_{0,0}(\mathbb{S}_k)$ identifies with the Grothendieck-Witt ring by [22]. This completes the proof of the statement on $\tau_{\mathrm{G}/\mathrm{N_G(T)}_+}$ in Theorem 1.2 in this case. In case the base field is of positive characteristic p, one observes that $\Sigma^\infty_{\mathbf{T}}\mathrm{G}/\mathrm{N_G}(\mathrm{T})_+$ will be dualizable only in $\mathbf{Spt}(k_{mot})[\mathrm{p}^{-1}]$. But once the prime p is inverted the same arguments as before carry over proving the corresponding statement. These complete the proof of the theorem. □

Proof of Corollary 1.3. Observe, first that if $\bar{k}$ is the algebraic closure of the given field, then it contains a $\sqrt{-1}$, and therefore the conclusions of the theorem hold in this case. In positive characteristic p, we proceed to show that this already implies that $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ is a *unit* in the group $\mathrm{GW}(k)[\mathrm{p}^{-1}]$, without the assumption on the existence of a square root of -1 in k. For this, one may first observe the commutative diagram, where $\bar{k}$ is an algebraic closure of k:

$$\begin{array}{ccc} \mathrm{GW}(\bar{k})[\mathrm{p}^{-1}] & \xrightarrow[\cong]{rk} & \mathbb{Z}[\mathrm{p}^{-1}] \\ \uparrow & & \uparrow{\scriptstyle id} \\ \mathrm{GW}(k)[\mathrm{p}^{-1}] & \xrightarrow{rk} & \mathbb{Z}[\mathrm{p}^{-1}]. \end{array} \tag{3.0.5}$$

Here the left vertical map is induced by the change of base fields from k to $\bar{k}$, and rk denotes the *rank* map. Since the motivic Euler-characteristic of $\mathrm{G}/\mathrm{N_G}(\mathrm{T})$ over $Spec\, k$ maps to the motivic Euler-characteristic of the corresponding $\mathrm{G}/\mathrm{N_G}(\mathrm{T})$ over $Spec\, \bar{k}$, it follows that the rank of $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ over $Spec\, k$ is in fact 1. By [1, Lemma 2.9(2)], this shows that the $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ over $Spec\, k$ is in fact a unit in $\mathrm{GW}(k)[\mathrm{p}^{-1}]$, that is, when k has positive characteristic. (For the convenience of the reader, we will summarize a few key facts discussed in [1, Proof of Lemma 2.9(2)]. It is observed there that when the base field k is *not* formally real, then $\mathrm{I}(k) = kernel(\mathrm{GW}(k) \stackrel{rk}{\to} \mathbb{Z})$ is the nil radical of $\mathrm{GW}(k)$: see [2, Theorem V.8.9, Lemma V.7.7 and Theorem V. 7.8]. Therefore, if $char(k) = \mathrm{p} > 0$, and the rank of $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ is 1 in $\mathbb{Z}[\mathrm{p}^{-1}]$, then $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ is $1 + q$ for some nilpotent element q in $\mathrm{I}(k)[\mathrm{p}^{-1}]$ and the conclusion follows.)

An alternative shorter proof is the following: observe that $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T})) - \chi_{\mathrm{mot}}((\mathrm{G}/\mathrm{N_G}(\mathrm{T}))^{\mathrm{T}}) = \chi_{\mathrm{mot}}(\mathrm{G}/\mathrm{N_G}(\mathrm{T})) - 1$ belongs to the ideal generated by $1- < -1 >$. $1- < -1 >$ clearly belongs to $\mathrm{I}(k) = kernel(\mathrm{GW}(k) \stackrel{rk}{\to} \mathbb{Z})$. When k is not formally real, the above ideal is nilpotent as observed above, and therefore, $\chi_{mot}(\mathrm{G}/\mathrm{N_G}(\mathrm{T}))$ is a unit when k is not formally real.

In characteristic 0, the commutative diagram

$$\begin{array}{ccc} \mathrm{GW}(\bar{k}) & \xrightarrow[\cong]{rk} & \mathbb{Z} \\ \uparrow & & \uparrow{\scriptstyle id} \\ \mathrm{GW}(k) & \xrightarrow{rk} & \mathbb{Z}, \end{array} \tag{3.0.6}$$

shows that once again the rank of $\chi_{mot}(\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T}))$ is 1. Therefore, to show that the class $\chi_{mot}(\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T}))$ is a unit in $\mathrm{GW}(k)$, it suffices to show its signature is a unit: this is proven in [1, Theorem 5.1(1)]. (Again, for the convenience of the reader, we summarize some details from the proof of [1, Theorem 5.1(1)]. When the field k is not formally real, the discussion in the last paragraph applies, so that by [1, Lemma 2.12] one reduces to considering only the case when k is a real closed field. In this case, one lets $\mathbb{R}^{alg}$ denote the real closure of $\mathbb{Q}$ in $\mathbb{R}$. Then, one knows the given real closed field k contains a copy of $\mathbb{R}^{alg}$ and that there exists a reductive group scheme $\widetilde{\mathrm{G}}$ over $\mathrm{Spec}\,\mathbb{R}^{alg}$ so that $\mathrm{G} = \widetilde{\mathrm{G}} \times_{\mathrm{Spec}\,\mathbb{R}^{\mathrm{alg}}} \mathrm{Spec}\,\mathrm{k}$. Let $\mathrm{G}_{\mathbb{R}} = \widetilde{\mathrm{G}} \times_{\mathrm{Spec}\,\mathbb{R}^{\mathrm{alg}}} \mathrm{Spec}\,\mathbb{R}$. Then one also observes that the Grothendieck-Witt groups of the three fields k, $\mathbb{R}^{alg}$ and $\mathbb{R}$ are isomorphic, and the motivic Euler-characteristics $\chi_{mot}(\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T}))$, $\chi_{mot}(\widetilde{\mathrm{G}}/\widetilde{\mathrm{N}_{\mathrm{G}}(\mathrm{T})})$ and $\chi_{mot}(\mathrm{G}_{\mathrm{Spec}\,\mathbb{R}}/\mathrm{N}(\mathrm{T})_{\mathrm{Spec}\,\mathbb{R}})$ over the above three fields identify under the above isomorphisms, so that one may assume the base field k is $\mathbb{R}$. Then it is shown in [1, Proof of Theorem 5.1(1)] that, in this case, knowing the rank and signature of the motivic Euler characteristic $\chi_{mot}(\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T}))$ are 1 suffices to prove it is a unit in the Grothendieck-Witt group.) These complete the proof of the corollary. □

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