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Equivariant algebraic K-theory, G-theory and derived completions [☆]

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A B S T R A C T

In the mid 1980s, while working on establishing completion theorems for equivariant Algebraic K-Theory similar to the well-known Atiyah-Segal completion theorem for equivariant topological K-theory, the late Robert Thomason found the strong finiteness conditions that are required in such theorems to be too restrictive. Then he made a conjecture on the existence of a completion theorem in the sense of Atiyah and Segal for equivariant algebraic G-theory, for actions of linear algebraic groups on schemes that holds without any of the strong finiteness conditions that are required in such theorems proven by him, and also appearing in the original Atiyah-Segal theorem. *The main goal of the present paper is to provide a proof of this conjecture in as broad a context as possible, making use of the technique of derived completion, and to consider several of the applications.*

Our solution is broad enough to allow actions by all linear algebraic groups, irrespective of whether they are connected or not, and acting on any quasi-projective scheme of finite type over a field, irrespective of whether they are regular or projective. This allows us therefore to consider the equivariant algebraic G-Theory of large classes of varieties like all toric varieties (for the action of a torus) and all spherical varieties (for the action of a reductive group). Restricting to actions by split tori, we are also able to consider actions on algebraic

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spaces. Moreover, the restriction that the base scheme be a field is also not required often, but is put in mainly to simplify some of our exposition. These enable us to obtain a wide range of applications, some of which are briefly sketched and which we plan to explore in detail in the future. In fact, we discuss an extension of our results to equivariant homotopy K-theory along with various Riemann-Roch theorems in a sequel. A comparison of our results with previously known results, none of which made use of derived completions, shows that without the use of derived completions one can only obtain results which are indeed very restrictive.

Contents

1. Introduction

An important problem in algebraic K-theory is to understand the behavior of equivariant K-theory. This is important not only for manifestly equivariant problems, but also for descent problems arising in non-equivariant problems over a non-algebraically closed base field, or more generally a scheme. The problem we are considering in this paper can be summarized as follows. Let X denote a scheme provided with the action of a linear algebraic group G. Then let $\mathbf{K}(\mathrm{X}, \mathrm{G})$ denote the spectrum associated to the symmetric monoidal category of G-equivariant vector bundles on X. Next let $\mathrm{EG} \to \mathrm{BG}$ denote a principal G-bundle with BG the classifying space for G in a certain sense as made clear later on. Then the pull-back along the projection $p_2 : \mathrm{EG} \times \mathrm{X} \to \mathrm{X}$ induces a map of spectra $p_2^* : \mathbf{K}(\mathrm{X}, \mathrm{G}) \to \mathbf{K}(\mathrm{EG} \times \mathrm{X}, \mathrm{G}) \simeq \mathbf{K}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})$, where $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ is the Borel construction. The *main goal* of the present paper is to prove that the map p_2^* becomes a weak-equivalence after a certain derived completion has been performed at the spectrum level on $\mathbf{K}(\mathrm{X}, \mathrm{G})$ provided one replaces the category of G-equivariant vector bundles by the category of G-equivariant coherent sheaves.

Observe that the statement above is at the level of spectra in the familiar sense and do not involve any equivariant spectra or equivariant homotopy theory. However, our approach is in fact suggested by the powerful results in equivariant homotopy theory such as [4], [6] and the following lines should put our work in perspective. For both of the above referenced results, one finds that the equivariant theory (equivariant topological K-theory in the case of [4], equivariant stable homotopy theory in the case of [6]) satisfies an approximation result that asserts that the equivariant theory, after application of a suitable notion of completion is equivalent to a homotopy fixed point set of the equivariant theory, and the latter is equivalent to the corresponding non-equivariant theory applied to a suitable form of the Borel construction. In the setting of topological K-theory, the above statement reduces to the one in the last paragraph, though completion meant one on the homotopy groups till now. Homotopy fixed point sets can be considered homotopy theoretic information, and there are standard spectral sequences for computing them. Informally we will say that equivariant topological K-theory and equivariant stable homotopy theory are *homotopy computable*. The approach to analyzing equivariant algebraic K-theory is now to ask to what extent it too is homotopy computable. The answer we provide is that equivariant algebraic G-theory is also homotopy computable, but with a somewhat different notion of homotopy fixed point set more suited to motivic frameworks, such as those in [59] or [36], thereby also solving affirmatively the conjecture posed in [56].

Apart from [8], derived completions have never been used in the context of the Atiyah-Segal framework for equivariant algebraic K-theory. The focus in [8] is on actions of profinite groups, such as the absolute Galois groups of fields, and therefore, is essentially disjoint from the context of actions by linear algebraic groups considered in this paper. However, work in progress by the authors suggest that the results of the present paper may also find application even when the primary focus is actions by profinite groups, and also may have a bearing on the Norm-Residue Theorem: see [19], [60]. We expect our results to have a number of applications, several of which are sketched later on (see section 6), such as equivariant forms of higher Riemann-Roch theorems, computation of the homotopy groups of the derived completion using the motivic Atiyah-Hirzebruch spectral sequences, as well as to equivariant forms of homotopy K-theory in the sense Weibel: see [63].

In order to put our work in context, it seems essential to provide a quick review of related results in the literature, with a more detailed comparison left to section 6. The first of these are the results of Atiyah and Segal in their landmark paper in the 1960s: see [4], which we will presently recall. If X is a compact topological space provided with the action of a compact Lie group, we let $\mathrm{K}_0^{\mathrm{G}}(\mathrm{X})$ ($\mathrm{K}_0(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})$) denote the Grothendieck group of isomorphism classes of G-equivariant topological vector bundles (the corresponding topological K-theory in degree 0 of the Borel construction $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$, respectively). The relationship between $\mathrm{K}_0^{\mathrm{G}}(\mathrm{X})$ and $\mathrm{K}_0(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})$ was studied in this setting in [4]. Their main result was that, under strong assumptions on X which would imply that $\mathrm{K}_0^{\mathrm{G}}(\mathrm{X})$ is finite as a module over the representation ring $\mathrm{R}(\mathrm{G})$, the pull-back

p_2^* considered in the first paragraph induces an isomorphism between $\mathrm{K}_0(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})$ and the completion of $\mathrm{K}_0^{\mathrm{G}}(\mathrm{X})$ with respect to the augmentation ideal in $\mathrm{R}(\mathrm{G})$.[1]

The need for such strong assumptions may be understood by recalling completions for commutative rings. Let A denote a commutative ring with a multiplicative unit and let I denote an ideal in A. If M is an A-module, recall that the I-adic completion of M along I is $\mathrm{M}\hat{} = \lim_{\infty \leftarrow n} \mathrm{M}/\mathrm{I}^n\mathrm{M}$. Unfortunately, the functor $\mathrm{M} \mapsto \mathrm{M}\hat{}$ is neither right-exact nor left-exact, in general: see [3, Chapter 10, Proposition 10.12] for basic results on completion in the Noetherian case. Therefore, one needs the higher derived functors of the above completion functor, which are nontrivial in general, that is, when A is no longer required to be Noetherian and/or M is no longer required to be finitely generated over A.

For the case considered by Atiyah and Segal in [4], the ring in question is R(G) which is Noetherian (see [45, Corollary 3.3]), so that strong assumptions on X ensure that the R(G)-module, $\mathrm{K}_0^{\mathrm{G}}(\mathrm{X})$ is finite, which then ensure that the completion is well-behaved.

The next results in this direction are those of Thomason, where a corresponding relationship for the action of algebraic groups on schemes was studied by him in a series of papers in the 1980s: see [54], [56] and [57]. In his seminal paper [56], Thomason proves that with certain strong restrictions on an algebraic group G and scheme X with G-action, a Bott periodic form of algebraic G-theory is indeed homotopy computable. The result is clearly very powerful, but it suffers from two deficiencies.

(1) It requires that we deal with the Bott-element inverted algebraic G-theory with finite coefficients prime to the characteristic. While it is true that there is a strong relationship between algebraic G-theory and its Bott-element inverted version, this relationship is not strong enough to permit exact calculations. In fact, the Bott-element inverted version of algebraic G-theory with finite coefficients prime to the characteristic is essentially topological (or étale) G-theory, so that Thomason's results are only for this form of G-theory and not for algebraic G-theory itself.
We will in fact present an example in 6.1.1 that shows that the map from equivariant algebraic G-theory with finite coefficients to the corresponding equivariant algebraic G-theory with the Bott-element inverted is *not* an isomorphism for many varieties, so that Thomason's results do *not* provide an Atiyah-Segal type completion theorem for equivariant algebraic G-theory, but only for its topological variant, namely the Bott-element inverted form of equivariant algebraic G-theory. Therefore, one needs a completion theorem for equivariant algebraic G-theory itself.
(2) There are certain awkward restrictions to the result, in particular the requirement that the scheme or algebraic space be over a separably closed base field or possibly a

[1] The need for completion in this setting seems to be because of the failure of compactness of $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ in general. But this observation is of secondary importance for us, since the main goal of the paper is to be able to relax the strong finiteness assumptions in the Atiyah-Segal theorem by making use of derived completions.

scheme with finitely generated K-groups, such as rings of integers in number fields. Such restrictions are mainly to ensure that the forms of equivariant algebraic K-groups that are considered are finite over R(G).

The homotopy fixed point scheme that comes up in Thomason's results is based on a simplicial model of the classifying space for an algebraic group, and this seems to be enough primarily because Thomason is considering algebraic G-theory with the Bott-element inverted. To be able to consider algebraic G-theory itself, one needs a model of classifying spaces that is more suited to motivic contexts, such as those in [59] or [36], which is what we consider in this paper. More details on Thomason's theorem and his strategy are discussed in section 6

A third and recent attempt to prove an Atiyah-Segal type theorem for equivariant algebraic K-theory, (that is, without using finite coefficients or inverting the Bott element) is [29, Theorem 1.2]. (See also [30].) This makes use of the usual completion of the homotopy groups of the K-theory spectrum at the augmentation ideal of the representation ring. Though [29, Theorem 1.2] avoids Thomason's strong hypotheses, and could very well be the strongest result that could be obtained by the traditional completion, it requires a number of very strong and restrictive assumptions, such as the schemes have to be projective and smooth over a field, the groups have to be connected and split reductive etc., which very seriously restrict its utility. (See section 6 for further details.)

In fact, Thomason clearly understood the source of the difficulties and foresaw the key ingredients necessary to resolve them. We have the following quote from one of Thomason's papers (see [Th86], pp. 795-796) (slightly edited to remove references to the context of his paper):

"This is because the isomorphisms (involving the comparison between the two types of equivariant theories) are expressed at the superficial level of the corresponding homotopy groups, rather than the deeper level of spectra, and so depend on the $\mathrm{I_G}$-adic completion process being exact on the level of the homotopy groups. This necessitates strong finiteness assumptions which can be met only in special cases. If the proper homotopy theoretic construction on the spectrum of equivariant K-theory could be found, that would induce $\mathrm{I_G}$-adic completion on the homotopy groups in nice cases, and do something more complicated involving the derived functors of the inverse limit functor in general, then it could be used to formulate an extension of the Atiyah-Segal completion theorem to more general settings. It would also extend the Atiyah-Segal theorem without their finiteness hypotheses."

The first author has constructed exactly such a homotopy theoretic construction (called *derived completion*) in [7] and the results of the present paper make use of this derived completion, following the approach conjectured by Thomason. A comparison of our main theorems, Theorems 1.2, 1.6 below show that, they do not suffer from any of the above restrictions that showed up in [56] or [29].

Though we state our main results only when the base scheme S is the spectrum of a field, as pointed out above, several of our results extend to a more general setting.

Therefore, in order to extend the validity of our results to as general a context as possible, we will assume the following framework for the paper. The base scheme S will be a separated Noetherian scheme which is regular. For the most part, this means S will be the spectrum of a finitely generated algebra over a field k, which is regular. Our approach is general enough that some of our results also extend to actions of group schemes (especially tori) on algebraic spaces. The algebraic spaces and schemes we consider will always be assumed to be *Noetherian and separated over the base scheme* S. The group schemes we consider will be affine group schemes which are finitely presented, separated and faithfully flat over S. When we say a group scheme G is affine, we mean that it admits a closed immersion into some $\mathrm{GL}_{n,\mathrm{S}}$, for some integer $n > 0$. Observe that any finite group G may be viewed as an affine group-scheme over S in the obvious manner, for example, by imbedding it as a subgroup of the monomial matrices in some GL_n.

Our main results will be stated only under the following basic assumptions.

1.1. Standing hypotheses

(i) We will assume the base scheme S is the spectrum of a perfect field k of arbitrary characteristic p. Let H denote a not-necessarily-connected linear algebraic group defined over k. For considering actions of such linear algebraic groups other than split tori, we will always restrict to H-schemes X of finite type over k, for which there exists a closed H-equivariant immersion into a regular H-scheme $\tilde{\mathrm{X}}$ of finite type over k. When H is a split torus (or a diagonalizable subgroup of a split torus), we will allow actions on all separated algebraic spaces of finite type over S.

(ii) We will fix an *ambient bigger group*, denoted G throughout, and which contains the given linear algebraic group H as a closed sub-group-scheme and satisfies the following strong conditions: G will denote a connected split reductive group over the field k so that if T denotes a maximal torus in G, then R(T) is free of finite rank over R(G) and R(G) is Noetherian. (Here R(T) and R(G) denote the corresponding representation rings.)

(iii) The above hypothesis is satisfied by $\mathrm{G} = \mathrm{GL_n}$ or SL_n, for any positive integer n, or any finite product of these groups. It is also trivially satisfied by all split tori. (A basic hypothesis that guarantees this condition is that the algebraic fundamental group $\pi_1(\mathrm{G})$ is torsion-free: see section 2.3.)

Next, it seems important to clarify our basic strategy as discussed in the following key reductions 1.2 as well as Proposition 1.1 below.

1.2. Key reductions and the basic strategy

(i) Observe that any linear algebraic group H can be imbedded into G (as a closed sub-group-scheme), where G is a general linear group (that is, a GL_n) or a finite product of such groups. Our basic strategy is to show by the following arguments that we may reduce to considering the action of the ambient group G, which will be a finite product of GL_ns. Then we reduce to considering actions by a maximal split torus, and eventually to the case of a 1-dimensional torus.

The reduction to the case where the split reductive group is a finite product of GL_n*s is handled by Proposition 1.1 (which makes use of Theorem 1.6 in its proof), while the reduction from a finite product of* GL_n*s to a split torus is handled by Theorems 3.6 and 4.7, and the case when the group is a split torus is handled by Theorem 5.6.*

Let X denote a scheme as in the Standing hypotheses 1.1 and provided with an action by the not-necessarily connected linear algebraic group H. For the purposes of this introduction, we will let $\mathbf{K}(\mathrm{X},\mathrm{H})$ ($\mathbf{G}(\mathrm{X},\mathrm{H})$) denote the spectrum obtained from the category of H-equivariant vector bundles (coherent sheaves) on X, though a more precise definition is given in section 2.

(ii) Assume the above situation. Then we let $\mathrm{G}\times\mathrm{H}$ act on $\mathrm{G}\times\mathrm{X}$ by $(g_1,h_1)\circ(g,x) = (g_1gh_1^{-1},h_1x)$, $g_1,g\,\epsilon\,\mathrm{G}$, $h_1\,\epsilon\,\mathrm{H}$ and $x\,\epsilon\,\mathrm{X}$. Now one may observe that $\mathrm{G}\times\mathrm{H}$ has an induced action on $\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}$ (defined the same way), and that $\mathrm{G}\times\mathrm{H}$ acts on X through the given action of H on X. (Here $\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}$ denotes the quotient of $\mathrm{G}\times\mathrm{X}$ by the action of H given by $h(g,x)=(gh^{-1},hx)$.) The maps $s:\mathrm{G}\times\mathrm{X}\to\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}$ and $r:\mathrm{G}\times\mathrm{X}\to\mathrm{G}\underset{\mathrm{G}}{\times}\mathrm{X}=\mathrm{X}$ are $\mathrm{G}\times\mathrm{H}$ equivariant maps and the pull-backs

$$s^*:\mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X},\mathrm{G})\to\mathbf{G}(\mathrm{G}\times\mathrm{X},\mathrm{G}\times\mathrm{H})\text{ and }\mathrm{r}^*:\mathbf{G}(\mathrm{X},\mathrm{H})\to\mathbf{G}(\mathrm{G}\times\mathrm{X},\mathrm{G}\times\mathrm{H}) \tag{1.2.1}$$

are weak-equivalences of module-spectra over $\mathbf{K}(\mathrm{S},\mathrm{G}\times\mathrm{H})$. (See Lemma 2.7 below for more details.) Moreover, if X is a quasi-projective scheme, then so is $\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}$. These observations, along with Proposition 1.1 below, enable us to just consider the equivariant **G**-theory and **K**-theory with respect to the action of the ambient group G.

(iii) One may also observe that the induced action by the closed subgroup $\mathrm{G}\times\{1\}$ of $\mathrm{G}\times\mathrm{H}$ on $\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}$ identifies with the left-action by G on $\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}$. Similarly the induced action by the closed subgroup $\mathrm{H}\cong\{1\}\times\mathrm{H}$ on X identifies with the given action of H on X.

(iv) If H is a connected split reductive group, then $\mathrm{R}(\mathrm{H})\cong\mathrm{R}(\mathrm{T})^{\mathrm{W}}$, where W denotes the Weyl group of H. Therefore R(H) is Noetherian. Therefore, one may find a closed imbedding $\mathrm{H}\to\mathrm{GL}_{\mathrm{n}_1}\times\cdots\mathrm{GL}_{\mathrm{n}_\mathrm{m}}$ for some $n_1,\cdots,n_m$ such that the restriction $\mathrm{R}(\mathrm{GL}_{\mathrm{n}_1}\times\cdots\mathrm{GL}_{\mathrm{n}_\mathrm{m}})\to\mathrm{R}(\mathrm{H})$ is surjective, so that Theorem 1.6 below applies. Moreover the same conclusions apply to any linear algebraic group H for which R(H) is Noetherian.

(v) In view of these observations, assuming the ambient group G is a finite product of GL_ns is not a serious restriction at all.

Recall from [52, Theorem 2.5] that if H is a connected linear algebraic group and X is any H-scheme which is normal and quasi-projective over the base scheme $\mathrm{S} = \mathrm{Spec}\, k$, one may find an H-equivariant locally closed immersion of X into a projective space $\mathbb{P}^n$ on which H acts linearly. Therefore, one may find an open H-stable subscheme $\tilde{\mathrm{X}}$ of this $\mathbb{P}^n$ into which X admits an H-equivariant closed immersion. (To see this, observe first that one may find an open subscheme U of the above projective space into which X admits a closed immersion. If we let V denote the image of $\mathrm{H} \times \mathrm{U}$ under the action map $\mathrm{H}\times\mathbb{P}^{\mathrm{n}} \to \mathbb{P}^{\mathrm{n}}$, then V is open and H-stable and contains X as a locally closed subscheme. It suffices to show that X is in fact closed in V, which should follow from the fact that any point of V is in the orbit of a point of U and because X is closed in U.) Clearly such an $\tilde{\mathrm{X}}$ is regular. (It is shown in [57, p. 629] how to remove the restriction that H be connected in the above discussion. See also 6.3.)

We will let $\mathrm{EG}^{\mathrm{gm}}$ denote the geometric classifying space for G which is constructed in 2.1 as an ind-object of schemes. We let $\rho_{\mathrm{H}} : \mathbf{K}(\mathrm{S}, \mathrm{H}) \to \mathbf{K}(\mathrm{S})$ denote the map of commutative ring spectra defined by restriction to the trivial subgroup-scheme. (See section 3, especially Definition 3.1 for further details.) For a prime $\ell \neq p$, let $\rho_\ell : \mathbb{S} \to \mathbb{H}(\mathbb{Z}/\ell)$ denote the mod$-\ell$ reduction map, where $\mathbb{S}$ denotes the sphere spectrum and $\mathbb{H}(\mathbb{Z}/\ell)$ denotes the (usual) $\mathbb{Z}/\ell$-Eilenberg-Maclane spectrum. Let $\rho_\ell \circ \rho_{\mathrm{H}} : \mathbf{K}(\mathrm{S}, \mathrm{H}) \to \mathbf{K}(\mathrm{S})\underset{\mathbb{S}}{\wedge}\mathbb{H}(\mathbb{Z}/\ell)$ denote the composition of ρ_{H} and the mod$-\ell$ reduction map $id_{\mathbf{K}(\mathrm{S})} \wedge \rho_\ell : \mathbf{K}(\mathrm{S}) \to \mathbf{K}(\mathrm{S})\underset{\mathbb{S}}{\wedge}\mathbb{H}(\mathbb{Z}/\ell)$.

We then state the following Proposition that completes the reduction to considering actions by the ambient group G.

Proposition 1.1. *(i) Making use of the weak-equivalences in (1.2.1) as module-spectra over* $\mathbf{K}(\mathrm{S}, \mathrm{G} \times \mathrm{H})$*, one obtains the weak-equivalences:*

$$s^*\widehat{\ }_{\rho_{\mathrm{G}\times\mathrm{H}}} : \mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G})\widehat{\ }_{\rho_{\mathrm{G}\times\mathrm{H}}} \overset{\simeq}{\to} \mathbf{G}(\mathrm{G} \times \mathrm{X}, \mathrm{G} \times \mathrm{H})\widehat{\ }_{\rho_{\mathrm{G}\times\mathrm{H}}} \text{ and}$$

$$r^*\widehat{\ }_{\rho_{\mathrm{G}\times\mathrm{H}}} : \mathbf{G}(\mathrm{X}, \mathrm{H})\widehat{\ }_{\rho_{\mathrm{G}\times\mathrm{H}}} \overset{\simeq}{\to} \mathbf{G}(\mathrm{G} \times \mathrm{X}, \mathrm{G} \times \mathrm{H})\widehat{\ }_{\rho_{\mathrm{G}\times\mathrm{H}}}.$$

(ii) Since the restriction maps $\mathrm{R}(\mathrm{G} \times \mathrm{H}) \to \mathrm{R}(\mathrm{G}) = \mathrm{R}(\mathrm{G} \times \{1\})$ *and* $\mathrm{R}(\mathrm{G} \times \mathrm{H}) \to \mathrm{R}(\mathrm{H}) = \mathrm{R}(\{1\} \times \mathrm{H})$ *are split-surjective, Theorem 1.6 and the observation (iii) above, provide the weak-equivalences:*

$$\mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G})\widehat{\ }_{\rho_{\mathrm{G}}} \simeq \mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G})\widehat{\ }_{\rho_{\mathrm{G}\times\mathrm{H}}} \text{ and } \mathbf{G}(\mathrm{X}, \mathrm{H})\widehat{\ }_{\rho_{\mathrm{H}}} \simeq \mathbf{G}(\mathrm{X}, \mathrm{H})\widehat{\ }_{\rho_{\mathrm{G}\times\mathrm{H}}}.$$

(iii) Thus combining (i) and (ii), we obtain:

$$\mathbf{G}(\mathrm{X}, \mathrm{H})\widehat{\ }_{\rho_{\mathrm{H}}} \simeq \mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G})\widehat{\ }_{\rho_{\mathrm{G}}}.$$

With the above reductions in place, we obtain the following main result.

Theorem 1.2. *Assume that the base scheme* $\mathrm{S} = \mathrm{Spec}\, k$ *for a perfect infinite field* k *and that* X *denotes any scheme of finite type over* S *satisfying the Standing hypotheses in* 1.1 *and provided with an action by the not-necessarily-connected linear algebraic group* H*. Let* G *denote a fixed ambient linear algebraic group which is a finite product of* $\mathrm{GL}_n s$ *containing* H *as a closed sub-group-scheme. Let* $\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X}$ *denote the ind-scheme defined by the Borel construction as in section* 2.1*. (The K-theory and G-theory of these objects are defined in Definition* 2.5*.)*

(i) Then the map $\mathbf{G}(\mathrm{X},\mathrm{H}) \simeq \mathbf{K}(\mathrm{S},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{X},\mathrm{H}) \rightarrow \mathbf{K}(\mathrm{EG}^{\mathrm{gm}},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{X},\mathrm{H}) \rightarrow \mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \times \mathrm{X},\mathrm{H}) \simeq \mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})$ *factors through the derived completion of* $\mathbf{G}(\mathrm{X},\mathrm{H})$ *at* ρ_{H} *and induces a weak-equivalence*

$$\mathbf{G}(\mathrm{X},\mathrm{H})\hat{}_{\rho_{\mathrm{H}}} \overset{\simeq}{\rightarrow} \mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X}).$$

The spectrum on the left-hand-side is the derived completion of $\mathbf{G}(\mathrm{X},\mathrm{H})$ *along the map* ρ_{H}*. The above map is contravariantly functorial for flat* H*-equivariant maps.*

(ii) Let ℓ *denote a prime different from the characteristic of* k*. Then one also obtains a weak-equivalence*

$$\mathbf{G}(\mathrm{X},\mathrm{H})_{\ell}\hat{}_{\rho_{\mathrm{H}}} \overset{\simeq}{\rightarrow} \mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})_{\ell}$$

where the subscript ℓ *denotes mod*$-\ell$*-variants of the appropriate spectra. (See* (3.0.11) *for their precise definitions.) Therefore, one also obtains the weak-equivalence:*

$$\mathbf{G}(\mathrm{X},\mathrm{H})\hat{}_{\rho_{\ell} \circ \rho_{\mathrm{H}}} \overset{\simeq}{\rightarrow} \mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})\hat{}_{\rho_{\ell}}.$$

The spectrum on the left-side (right-side) denotes the derived completion of $\mathbf{G}(\mathrm{X},\mathrm{H})$ *(*$\mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})$*) with respect to the composite map* $\rho_{\ell} \circ \rho_{\mathrm{H}}$ *(the map* ρ_{ℓ}*, respectively). The above map is also contravariantly functorial for flat* H*-equivariant maps.*

(iii) One obtains a weak-equivalence: $\mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X}) \simeq \mathbf{G}(\mathrm{EH}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})$*, where* $\mathrm{EH}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X}$ *denotes the ind-scheme associated to* H *and defined by the Borel construction as in section* 2.1*.*

(iv) If the group G *is replaced by a split torus, and* H *denotes the same torus, then all of the above results extend to the case where* X *is a separated algebraic space of finite type over any base field* k*.*

Remark 1.3. Similar to the mod-ℓ variant considered in Theorem 1.2 (ii), the other key results in this paper, such as Corollary 1.4, Lemma 5.1, Proposition 5.2, as well as Theorems 3.6, 5.6 all have mod-ℓ-variants. As these may be deduced in the same manner from the corresponding results stated integrally, we choose not to discuss these explicitly.

Before we proceed further, a few brief comments on the subtle nature of derived completion seem to be also in order.

(i) First, observe that *a key problem* with completion with respect to an ideal in a commutative ring with unit is that, in general, it is *neither left exact nor right exact*. Therefore, a key role of derived completion is to rectify this problem: that is, one has to rectify the failure of both left-exactness and right exactness. This means that a simple-minded approach by taking either a projective resolution (that is, a cofibrant replacement) or an injective resolution (that is, a fibrant replacement) will *not suffice*.

(ii) Instead, one has to combine suitable cofibrant and fibrant replacements, and also at the same time, the resulting derived completion for spectra has to be computable in terms of the derived completion of their homotopy groups.

(iii) As Thomason states in his conjecture, the derived completion also has to have the derived functor of the inverse limit functor built into it. Finally, the technology to resolve the above issues needed important improvements in the theory of spectra and stable homotopy theory (see for example, [50], [49]) that became generally available only during the last 15 years.

These are among the reasons, why the existence of a derived completion with desirable properties remained a conjecture for many years and the issues were all successfully resolved only in the 2008 paper [7] by the first author. Moreover, in order to apply those results to the context of this paper, we have to amplify some of those results: these are discussed in section 3 as well as in the appendices A and C. The fact that we are able to resolve successfully all the problems discussed above with the Atiyah-Segal completion theorem, shows the effectiveness of the derived completion.

We provide the following remarks and corollaries to clarify the range of applications of the above Theorem.

First we would like to point out that though the statements (ii) and (iii) in Theorem 1.2 seem natural and easily provable in the case of topological K-theory, they are by no means obvious in our setting and take some effort to be proven. See the discussion in sections 3 and 8 for details on their proofs.

Observe that any finite group H may be imbedded in a symmetric group Σ_n, for some $n > 0$, as a subgroup, and the symmetric group Σ_n imbeds into $\mathrm{G} = \mathrm{GL_n}$ as the closed subgroup of all permutation matrices. Moreover, if X is any quasi-projective scheme of finite type over k, provided with an action by H, one may imbed X as a closed H-stable subscheme of $\tilde{\mathrm{X}}$, which is a regular scheme of finite type over k and provided with an action by H. (For example, one may imbed X non-equivariantly into a smooth scheme $\tilde{\mathrm{X}}$, making use of the quasi-projectivity assumption on X and then take the diagonal imbedding of X into the product of copies of $\tilde{\mathrm{X}}$ indexed by H.) Then we obtain:

Corollary 1.4. *(Finite group actions) Assume* H *is a finite group imbedded as a closed subgroup of some* GL_n. *Let* $\mathrm{G} = \mathrm{GL_n}$. *Then the map* $\mathbf{G}(\mathrm{X},\mathrm{H}) \simeq \mathbf{K}(\mathrm{S},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{X},\mathrm{H}) \rightarrow$

$\mathbf{G}(\mathrm{EG^{gm}},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{X},\mathrm{H}) \to \mathbf{G}(\mathrm{EG^{gm}} \times \mathrm{X},\mathrm{H}) \simeq \mathbf{G}(\mathrm{EG^{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})$ *factors through the derived completion of* $\mathbf{G}(\mathrm{X},\mathrm{H})$ *at* ρ_{H} *and induces a weak-equivalence*

$$\mathbf{G}(\mathrm{X},\mathrm{H})\hat{}_{\rho_{\mathrm{H}}} \xrightarrow{\simeq} \mathbf{G}(\mathrm{EG^{gm}} \underset{\mathrm{H}}{\times} \mathrm{X}) \simeq \mathbf{G}(\mathrm{EH^{gm}} \underset{\mathrm{H}}{\times} \mathrm{X}).$$

The above mentioned results of Sumihiro (see [52, Theorem 2.5]) enable us to see that Theorem 1.2 applies to all toric varieties and spherical varieties, irrespective of whether they are regular or projective. Recall a *normal* variety X provided with the action of a split torus T (of a connected split reductive group H) is called *a toric variety* (*an* H-*spherical variety*) if T has only finitely many orbits on X (H and a Borel subgroup of H both have only finitely many orbits on X, respectively). (See [14], [53].)

Corollary 1.5. *(Toric varieties and Spherical varieties) Theorem 1.2 applies to all toric varieties, with the group* H *denoting the given torus and to all* H-*spherical varieties, where* H *is the given connected reductive group.*

Here are some *remaining features* of our main Theorem.

(i) Algebraic spaces are allowed as long as the groups acting on them are split tori (or smooth diagonalizable subgroups of them: see the discussion following Theorem 1.6 for this extension).
(ii) Our Theorem 1.7 shows that when applied to projective smooth varieties over a field, the homotopy groups of the derived completion identify with the usual completion of the homotopy groups, so that we recover all the results of [29] as a corollary to our results.
(iii) Making use of our results, and combining with the usual spectral sequence relating motivic cohomology and algebraic K-theory, we obtain a computation of the equivariant K-groups of smooth schemes after applying derived completions, in terms of their equivariant motivic cohomology groups. We also obtain as corollaries, strong forms of equivariant Riemann-Roch theorems valid for all normal quasi-projective G-schemes for the action of any linear algebraic group G, and involving higher equivariant G-theory and equivariant motivic (and other forms of) cohomology.
(iv) More details on all of these are discussed in section 6.

In fact, if one restricts to actions of split tori, one may also consider more general base schemes than a field: but we choose not to discuss this extension in detail, mainly for keeping the discussion simpler. The *strategy we adopt* to proving the above theorem has *several similarities* as well as *some key differences* with the proof by Atiyah and Segal (see [4]) of their theorem. Both proofs proceed by reducing to actions by groups that are easier to understand. In our case, we first reduce to the case where the given not-necessarily-connected linear algebraic group H is replaced by the ambient connected split reductive group G, then this group G is replaced by a finite product of GL_ns. Finally we reduce to the case where this product of GL_ns is replaced by its split maximal torus.

In this case, an equivariant form of the Kunneth-formula as in [2] and [28] together with some properties of the derived completion provides a proof. The key difference between our proof and the proof of the classical Atiyah-Segal theorem is in the use of the derived completion, which is essential for our proof. Moreover, unlike in the classical case, for a closed sub-group-scheme H in G, the derived completion of a $\mathbf{K}(\mathrm{S},\mathrm{H})$-module spectrum with respect to the maps $\rho_H : \mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S})$ and $\rho_{\mathrm{G}} : \mathbf{K}(\mathrm{S},\mathrm{G}) \to \mathbf{K}(\mathrm{S})$ will be different in general. The main exception to this is discussed in the following theorem.

Theorem 1.6. *Let* $\mathrm{H} \subseteq \mathrm{G}$ *denote a closed algebraic subgroup of the linear algebraic group* G *so that the restriction map* $\mathrm{R}(\mathrm{G}) \to \mathrm{R}(\mathrm{H})$ *of representation rings is surjective. Let* X *denote a scheme of finite type over* k*, provided with an action by* H *and satisfying the Standing hypotheses 1.1. Then, for any module spectrum* M *over* $\mathbf{K}(\mathrm{X},\mathrm{H})$*, which is* s*-connected for some integer* s*, the derived completions of* M *along the augmentations* $\rho_{\mathrm{H}} : \mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S})$ *and* $\rho_{\mathrm{G}} : \mathbf{K}(\mathrm{S},\mathrm{G}) \to \mathbf{K}(\mathrm{S})$ *are weakly-equivalent, where* $\mathrm{S} = \mathrm{Spec}\, k$ *denotes the base scheme.*

Since the proof of this theorem is quite short, we will state it here itself.

Proof of Theorem 1.6. One observes that the hypotheses of [7, Corollary 7.11] are satisfied with the ring spectrum $A = \mathbf{K}(\mathrm{S},\mathrm{G})$, the ring spectrum $B = \mathbf{K}(\mathrm{S},\mathrm{H})$ and the ring spectrum $C = \mathbf{K}(\mathrm{S})$. □

Here are *several examples* where the last theorem applies.

(i) A notable example where this Theorem finds application is Proposition 1.1, which enables us to reduce to considering derived completions with respect to the ambient group G.
(ii) Another example for the above situation is when H is a diagonalizable subgroup of a split torus G. For example, the above conclusions hold if H is a finite abelian group imbedded as a closed subgroup of a split torus.
(iii) Another is when G denotes a Borel subgroup of a split reductive group, and H is a diagonalizable subgroup of the maximal torus contained in G.
(iv) Next assume that H is a finite group, or more generally that H is any linear algebraic group for which R(H) is Noetherian. Since the representation ring of H (of representations over k) is Noetherian, one may imbed H into a finite product of general linear groups, $\mathrm{GL}_{n_1}, \cdots, \mathrm{GL}_{n_m}$ so that the induced map $\mathrm{R}(\mathrm{GL}_{\mathrm{n}_1} \times \cdots \times \mathrm{GL}_{\mathrm{n}_\mathrm{m}}) \to \mathrm{R}(\mathrm{H})$ is surjective. Therefore, the above theorem applies in this case as well, where $\mathrm{G} = \mathrm{GL}_{\mathrm{n}_1} \times \cdots \times \mathrm{GL}_{\mathrm{n}_\mathrm{m}}$.

[57, Proposition 4.1] shows the analysis in [45, Corollary 3.3] carries over to show that the representation ring of any linear algebraic group over any algebraically closed field of characteristic 0 is Noetherian. [46] shows that for any connected split reductive group

G defined over a field, the representation ring R(G) is Noetherian. One may observe as a direct corollary to the above remarks and the last statement in Theorem 1.2, that our results extend to actions of all smooth split diagonalizable group schemes on schemes and algebraic spaces of finite type over k.

We also provide the following theorem, which shows when the derived completions reduce to the usual completions at the augmentation ideal.

Theorem 1.7. *(i) Let* G *denote a connected split reductive group satisfying our Standing hypotheses 1.1 acting on a projective smooth scheme* X *over the field k. Then one obtains the isomorphism:* $\pi_*(\mathbf{K}(\mathrm{X},\mathrm{G})\widehat{\ }_{\rho_{\mathrm{G}}}) \cong \pi_*(\mathbf{K}(\mathrm{X},\mathrm{G}))\widehat{\ }_{\mathrm{I_G}}$*, where the term on the right denotes the completion of* $\pi_*((\mathbf{K}(\mathrm{X},\mathrm{G}))$ *at the augmentation ideal* $I_{\mathrm{G}} \subseteq R(\mathrm{G})$*.*

(ii) If G *is any connected split reductive group acting on the projective smooth scheme* X *over the field k and* GL_n *denotes a general linear group containing* G *as a closed subgroup-scheme, then* $\pi_*(\mathbf{K}(\mathrm{X},\mathrm{G})\widehat{\ }_{\rho_{\mathrm{GL_n}}}) \cong \pi_*(\mathbf{K}(\mathrm{X},\mathrm{G}))\widehat{\ }_{\mathrm{I_G}}$*.*

This paper originated in our efforts to provide a proof of the descent conjecture for the K-theory of fields due to the first author: see [8]. It is also motivated by a similar derived completion theorem proven there for actions of profinite groups, making use of another model for the classifying spaces. In that framework, one needs to consider representation rings of profinite groups, which are in general, non-Noetherian. In the present framework, since the representation rings of linear algebraic groups are Noetherian, the need for derived completion is only so as not to put any strong restrictions on the schemes whose equivariant K-theory and G-theory we consider.

Here is an *outline* of the paper. We review the basic properties of equivariant K-theory, G-theory and the geometric classifying spaces of linear algebraic groups in section 2. Section 3 is devoted to establishing several basic results on derived completion that we use in later sections of the paper. This supplements the results of [7] on derived completion, with the main result being Theorem 3.6. Sections 4 and 5 are devoted to a detailed proof of Theorems 1.2 and 1.6, with section 4 discussing the reduction to the case where the group is a split torus, making strong use of Theorem 3.6. In this section, we also re-interpret Theorem 1.2 in terms of pro-spectra. Section 5 discusses the proof of Theorem 1.2 for the action of a split torus. We point out that several results in this section hold more generally for actions on algebraic spaces, though still over a base field. Section 6 discusses several examples and compares our results with earlier results in the literature. We also provide a proof of Theorem 1.7. It also discusses various applications such as computing the completed equivariant G-theory in terms of equivariant motivic cohomology as well as equivariant Riemann-Roch theorems. These should show the power and utility of our techniques. The appendices A though C discuss various results of a technical nature: Appendix A discusses the passage between Eilenberg-Maclane spectra and chain complexes, Appendix B discusses the role of motivic slices in establishing key properties of Borel-style equivariant K-theories and Appendix C contains supplementary technical results.

1.3. Basic conventions and terminology

(i) The main framework we need to work in can be readily understood by observing that, if H is a linear algebraic group acting on the algebraic space X, the spectra $\mathbf{K}(\mathrm{X}, \mathrm{H})$ and $\mathbf{G}(\mathrm{X}, \mathrm{H})$ considered in (2.0.1) are S^1-spectra in the *usual sense*: in particular they are *not* equivariant spectra in the sense of [32] for example. This is because they are obtained by applying the (well-known) construction of Waldhausen K-theory to the category of perfect complexes and pseudo-coherent complexes of H-equivariant $\mathcal{O}_{\mathrm{X}}$-modules. Observe that here S^1 denotes the simplicial 1-sphere, which clearly has no action by H.[2]

We will let $\mathbf{Spt}_{\mathrm{S}^1}$ denote the category of S^1-symmetric spectra throughout the rest of the paper. The reason for this notation will become clear from the discussion in (v) and (vi) below. One may observe that the sphere spectrum considered in [49] (see also [24]) is the sequence of spaces $\{\mathrm{S}^{\mathrm{n}}|\mathrm{n} \geq 0\}$, where S^{n} denotes the simplicial n-sphere provided with the action of the symmetric group Σ_n: we will denote this by $\mathbb{S}$. This spectrum has the structure of a commutative ring spectrum, any object $\mathcal{X} \in \mathbf{Spt}_{\mathrm{S}^1}$ is then a module spectrum over $\mathbb{S}$ and vice-versa. In fact the category $\mathbf{Spt}_{\mathrm{S}^1}$ of S^1-symmetric spectra is symmetric monoidal under smash product, and the unit of this category is the symmetric sphere spectrum $\mathbb{S}$. Thus the category of module spectra over $\mathbb{S}$ is synonymous with the category $\mathbf{Spt}_{\mathrm{S}^1}$ and therefore, all the results of [49] and [47] carry over to $\mathbf{Spt}_{\mathrm{S}^1}$.

In particular, it comes equipped with a stable model structure that is cofibrantly generated and compatible with the closed symmetric monoidal structure given by the smash product, making it a monoidal model category satisfying the monoidal axiom as in [47, Definitions 3.1, 3.3]. It is also well-known (see, for example [16, Appendix]) that $\mathbf{Spt}_{S^1}$ contains the K-theory spectra associated to any category with cofibrations and weak-equivalences in the sense of [62]. Therefore, all the constructions involved in the derived completion can be carried out in $\mathbf{Spt}_{S^1}$.

(ii) We will make extensive use of the model structures defined in [47, Theorem 4.1] to produce cofibrant replacements for commutative algebra spectra over a given commutative ring spectrum. The commutative ring spectra that show up in the paper are largely the K-theory spectra associated to the actions of a linear algebraic group.

(iii) In addition, we will also need to consider the usual Eilenberg-Maclane spectra associated to commutative rings. (This is discussed in detail in Appendix A.) Let R denote such a commutative ring and let $Alg(R)$ denote the category of commutative algebras over R. In this case we will consider the free-commutative algebra functor to find a simplicial resolution of a commutative R-algebra $\mathcal{S}$, and use it defines a

[2] This is also entirely similar to the version of equivariant topological K-theory defined by Atiyah and Segal in [4]: that is the topological K-theory of a category of equivariant topological vector bundles each of which is provided with an action by the given fixed group.

commutative algebra spectrum $\mathbb{H}(\tilde{\mathcal{S}})$ which will be a cofibrant replacement of $\mathbb{H}(\mathcal{S})$ in the model category of module spectra over $\mathbb{H}(R)$. See the proof of Theorem 3.6 and A.0.3 for more details on this technique.

(iv) Let $\mathbf{Spt}_{S^1}$ denote the category of S^1-symmetric spectra as in (i), and let I denote a small category. Then we provide the category of diagrams of spectra of type I, $\mathbf{Spt}_{S^1}^{\mathrm{I}}$, with the projective model structure as in [5, Chapter XI]. Here the fibrations (weak-equivalences) are maps $\{f^i : K^i \to L^i | i \epsilon \mathrm{I}\}$ so that each f^i is a fibration (weak-equivalence, respectively) with the cofibrations defined by the lifting property with respect to trivial fibrations. Since the homotopy inverse limit functor is not well-behaved unless one restricts to fibrant objects in this model structure, we will always implicitly replace a given diagram of spectra functorially by a fibrant one before applying the homotopy inverse limit. If $\{K^i | i \epsilon \mathrm{I}\}$ is a diagram of spectra, $\mathrm{holim}\{K^i | i \epsilon \mathrm{I}\}$ actually denotes $\mathrm{holim}\{R(K^i) | i \epsilon \mathrm{I}\}$ where $\{R(K^i) | i \epsilon \mathrm{I}\}$ is a fibrant replacement of $\{K^i | i \epsilon \mathrm{I}\}$ in $\mathbf{Spt}_{S^1}^{\mathrm{I}}$.

(v) Let Sm_{S} denote the category of smooth schemes of finite type over the given base scheme S, which one may recall is assumed to be separated, Noetherian and regular. Now $\mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}})$ will denote the category of presheaves of S^1-symmetric spectra on the category Sm_{S} provided with the Nisnevich topology and where the affine line $\mathbb{A}^1_{\mathrm{S}}$ is inverted. Model structures on the category $\mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}})$ are discussed in [11]. Observe that if $\mathcal{X}$ denotes a fibrant object in $\mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}})$, and X is a smooth scheme of finite type over S, sending $\mathcal{X} \mapsto \Gamma(\mathrm{X}, \mathcal{X})$ defines a functor $\mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}}) \to \mathbf{Spt}_{\mathrm{S}^1}$. Since the spectrum representing algebraic K-theory $\mathrm{X} \mapsto \mathbf{K}(\mathrm{X})$ as in (i) is $\mathbb{A}^1$-invariant and satisfies Nisnevich excision when restricted to smooth schemes X, it defines a fibrant object in $\mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}})$. As a result we are able to apply motivic arguments to this functor.

As pointed out in Remark 2.6, the only place we need to invoke motivic spectra is in order to prove that the Borel-style equivariant K-theory is independent of the choice of the geometric classifying space as in (2.1.4). Here we will use the fact that the non-equivariant algebraic K-theory spectrum is a fibrant motivic S^1-spectrum (that is, it is a presheaf of S^1-spectra which is $\mathbb{A}^1$-invariant and satisfies Nisnevich excision when restricted to smooth schemes over the base scheme S) and is -1-connected when restricted to smooth schemes over the base scheme S, so that we can apply the slice tower to it. See Appendix B.

(vi) The discussion in the last paragraph should also explain why we use the notation $\mathbf{Spt}_{\mathrm{S}^1}$ in (i): this is mainly to avoid any confusion with motivic $\mathbb{P}^1$-spectra, which are also commonly used in the motivic contexts.

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2. Equivariant K-theory and G-theory: basic terminology and properties

Throughout the paper we will let G denote a split reductive group satisfying the Standing hypotheses in 1.1, quite often this being a GL_n or a finite product of GL_ns. Let H denote a closed linear algebraic subgroup of G. (In particular it is a smooth group-scheme over the base field.) Let X denote a separated algebraic space or scheme over S provided with an action by H.

Let $\mathrm{Pscoh}(\mathrm{X}, \mathrm{H})$ $(\mathrm{Perf}(\mathrm{X}, \mathrm{H}))$ denote the category of pseudo-coherent complexes of H-equivariant $\mathcal{O}_X$-modules with bounded coherent cohomology sheaves (the category of perfect complexes of H-equivariant $\mathcal{O}_X$-modules, respectively). Recall that a complex of H-equivariant $\mathcal{O}_X$-modules is *pseudo-coherent (perfect)* if it is quasi-isomorphic locally on the appropriate topology on X (that is, if X is a scheme, we use the Zariski topology, and if X is an algebraic space, and not a scheme, we use the étale topology) to a bounded above complex (a bounded complex, respectively) of locally free $\mathcal{O}_{\mathrm{X}}$-modules with bounded coherent cohomology sheaves. We provide these categories with the structure of bi-Waldhausen categories with cofibrations, fibrations and weak-equivalences by letting the cofibrations be the maps of complexes that are degree-wise split monomorphisms (fibrations be the maps of complexes that are degree-wise split epimorphisms, weak-equivalences be the maps that are quasi-isomorphisms, respectively). $\mathbf{G}(\mathrm{X}, \mathrm{H})$ $(\mathbf{K}(\mathrm{X}, \mathrm{H}))$ will denote the K-theory spectrum obtained from the above bi-Waldhausen category structure on $\mathrm{Pscoh}(\mathrm{X}, \mathrm{H})$ $(\mathrm{Perf}(\mathrm{X}, \mathrm{H})$, respectively).

One may also consider the category $\mathrm{Vect}(\mathrm{X}, \mathrm{H})$ of H-equivariant vector bundles on X. This is an exact category and one may apply Quillen's construction to it to produce another variant of the equivariant K-theory spectrum of X. If we assume that every H-equivariant coherent sheaf on X is the H-equivariant quotient of an H-equivariant vector bundle on X, then one may observe that this produces a spectrum weakly-equivalent to $\mathbf{K}(\mathrm{X}, \mathrm{H})$: see [58, 2.3.1 Proposition] or [27, Proposition 2.8]. It follows from [54, Theorem 5.7 and Corollary 5.8] that this holds in many well-known examples. It is shown in [26, section 2] that, in general, the map from the K-theory spectrum to the G-theory spectrum (sending a perfect complex to itself, but viewed as a pseudo-coherent complex) is a weak-equivalence

$$\mathbf{K}(\mathrm{X}, \mathrm{H}) \simeq \mathbf{G}(\mathrm{X}, \mathrm{H}) \tag{2.0.1}$$

provided X is regular. In general, such a result fails to be true for the Quillen K-theory of G-equivariant vector bundles, which is the reason for our preference to the Waldhausen style K-theory and G-theory considered above.

We will nevertheless make the assumption that every H-equivariant coherent sheaf on the base scheme S is the H-equivariant quotient of an H-equivariant vector bundle.

(Recall that, for some parts of the paper, we do not require the base scheme to be the spectrum of a field and that, nevertheless, we require that it be always a Noetherian regular scheme.) This hypothesis is clearly satisfied if the base scheme is the spectrum of a field and more generally if the base-scheme is the spectrum of a Noetherian regular ring and the algebraic group H is split reductive: see [54, Corollary 5.2]. It now follows from the hypotheses that $\mathbf{K}(\mathrm{S},\mathrm{H})$ is weakly-equivalent to the Quillen K-theory of H-equivariant vector bundles on S and that one obtains the weak-equivalence $\mathbf{G}(\mathrm{S},\mathrm{H}) \simeq \mathbf{K}(\mathrm{S},\mathrm{H})$.

If ℓ is a prime different from the residue characteristics, $\rho_\ell : \mathbb{S} \to \mathbb{H}(Z/\ell)$ will denote the obvious map from the sphere spectrum to the mod$-\ell$ Eilenberg-Maclane spectrum.

2.1. The geometric classifying space

We begin by recalling briefly the construction of the *geometric classifying space of a linear algebraic group*: see for example, [59, section 1], [36, section 4]. Let H denote a linear algebraic group over $\mathrm{S} = \mathrm{Spec}\, k$, that is, a closed subgroup-scheme in GL_n over S for some n. For a (closed) embedding $i : \mathrm{H} \to \mathrm{GL_n}$ *the geometric classifying space* $\mathrm{B_{gm}(H;i)}$ of H with respect to i is defined as follows. For $m \geq 1$, let $\mathrm{EH^{gm,m}} = \mathrm{U_m(H)} = \mathrm{U}(\mathbb{A}^{\mathrm{nm}})$ be the open sub-scheme of $\mathbb{A}^{nm}$ where the diagonal action of H determined by i is free. By choosing m large enough, one can always ensure that $\mathrm{U}(\mathbb{A}^{\mathrm{nm}})$ is non-empty and the quotient $\mathrm{U}(\mathbb{A}^{\mathrm{nm}})/\mathrm{H}$ is a quasi-projective scheme: see the discussion following Definition 2.1, where a more detailed discussion of the geometric classifying spaces appears.

Let $\mathrm{BH^{gm,m}} = \mathrm{V_m(H)} = \mathrm{U_m(H)/H}$ denote the quotient S-scheme (which will be a quasi-projective scheme) for the action of H on $\mathrm{U_m(H)}$ induced by this (diagonal) action of H on $\mathbb{A}^{nm}$; the projection $\mathrm{U_m(H)} \to \mathrm{V_m(H)}$ defines $\mathrm{V_m(H)}$ as the quotient of $\mathrm{U_m(H)}$ by the free action of H and $\mathrm{V_m(H)}$ is thus smooth. We have closed embeddings $\mathrm{U_m(H)} \to \mathrm{U_{m+1}(H)}$ and $\mathrm{V_m(H)} \to \mathrm{V_{m+1}(H)}$ corresponding to the embeddings $Id \times \{0\} \colon \mathbb{A}^{\mathrm{nm}} \to \mathbb{A}^{\mathrm{nm}} \times \mathbb{A}^{\mathrm{n}}$. We set $\mathrm{EH^{gm}} = \{\mathrm{U_m(H)}|\mathrm{m}\} = \{\mathrm{EH^{gm,m}}|\mathrm{m}\}$ and $\mathrm{BH^{gm}} = \{\mathrm{V_m(H)}|\mathrm{m}\}$ which are ind-objects in the category of schemes. (If one prefers, one may view each $\mathrm{EH^{gm,m}}$ ($\mathrm{BH^{gm,m}}$) as a sheaf on the big Nisnevich (étale) site of smooth schemes over k and then view $\mathrm{EH^{gm}}$ ($\mathrm{BH^{gm}}$) as the corresponding colimit taken in the category of sheaves on $(\mathrm{Sm/S})_{Nis}$ or on $(\mathrm{Sm/S})_{et}$.)

Given a scheme X of finite type over S with an H-action satisfying the Standing hypotheses 1.1, we let $\mathrm{U_m(H)}\underset{\mathrm{H}}{\times}\mathrm{X}$ denote the *balanced product*, where (u, x) and (ug^{-1}, gx) are identified for all $(u, x)\ \epsilon\ \mathrm{U_m} \times \mathrm{X}$ and $g\ \epsilon$ H. Since the H-action on $\mathrm{U_m(H)}$ is free, $\mathrm{U_m(H)}\underset{\mathrm{H}}{\times}\mathrm{X}$ exists as a geometric quotient which is also a quasi-projective scheme in this setting, in case X is assumed to be quasi-projective: see [35, Proposition 7.1]. (In case X is an algebraic space of finite over S, the above quotient also exists, but as an algebraic space of finite type over S.)

It needs to be pointed out that the construction of the geometric classifying space is not unique in general. When X is an algebraic space or a scheme in general, we will let $\{\mathrm{U_m(H)}\underset{\mathrm{H}}{\times}\mathrm{X}|\mathrm{m}\}$ denote the ind-object constructed as above for a chosen ind-scheme

$\{U_m(H)|m \geq 0\}$. When X is restricted to the category of smooth schemes over Spec k, one can apply the result below in (2.1.4) to show that the choice of the ind-scheme $\{U_m(H)|m \geq 0\}$ is irrelevant, as long as it satisfies the hypotheses in Definition 2.1.

Next we recall a particularly nice way to construct geometric classifying spaces through what are called *admissible gadgets*.

Definition 2.1. (Admissible gadgets) (See [36, 4.2] and also [29, 3.1].) The first step in constructing an admissible gadget is to start with a good pair (W, U) for G: this is a pair (W, U) of smooth schemes over k where W is a k-rational representation of G and $U \subsetneq W$ is a G-invariant non-empty open subset which is a smooth scheme with a free action by G, so that the quotient U/G is a scheme. It is known (*cf.* [59, Remark 1.4]) that a good pair for G always exists. The following choice of a good pair is often convenient. Choose a faithful k-rational representation R of G of dimension n, that is, G admits a closed immersion into GL(R). Then G acts freely on an open subset U of $W = \mathrm{End}_k(R)$. (Observe that if $R = k^{\oplus n}$, $W = R^{\oplus n}$. Now one may take $U = GL(R)$.) Let $Z = W \setminus U$. A sequence of pairs $\{(W_i, U_i)|i \geq 1\}$ of smooth schemes over k is called an *admissible gadget* for G, if there exists a good pair (W, U) for G such that $W_i = W^{\times^i}$ and $U_i \subsetneq W_i$ is a G-invariant open subset such that the following conditions hold for each $i \geq 1$.

(1) $(U_i \times W) \cup (W \times U_i) \subseteq U_{i+1}$ as G-invariant open subsets.
(2) $\{\mathrm{codim}_{U_{i+1}}(U_{i+1} \setminus (U_i \times W))|i\}$ is a strictly increasing sequence, that is, $\mathrm{codim}_{U_{i+2}}(U_{i+2} \setminus (U_{i+1} \times W)) > \mathrm{codim}_{U_{i+1}}(U_{i+1} \setminus (U_i \times W))$.
(3) $\{\mathrm{codim}_{W_i}(W_i \setminus U_i)|i\}$ is a strictly increasing sequence, that is, $\mathrm{codim}_{W_{i+1}}(W_{i+1} \setminus U_{i+1}) > \mathrm{codim}_{W_i}(W_i \setminus U_i)$.
(4) U_i is a smooth quasi-projective scheme over k with a free G-action, so that the quotient U_i/G is also a smooth quasi-projective scheme over k.

A *particularly nice example* of an admissible gadget for G can be constructed as follows. Given a good pair (W, U), we now let

$$W_i = W^{\times^i}, U_1 = U \text{ and } EG^{gm,i} = U_{i+1} = (U_i \times W) \cup (W \times U_i) \text{ for } i \geq 1, \qquad (2.1.1)$$

where U_{i+1} is viewed as a subscheme of $W^{\times^{i+1}}$. Observe that

$$EG^{gm,i} = U_{i+1} = U \times W^{\times i} \cup W \times U \times W^{\times i-1} \cup \cdots \cup W^{\times i} \times U. \qquad (2.1.2)$$

Setting $Z_1 = Z$ and $Z_{i+1} = U_{i+1} \setminus (U_i \times W)$ for $i \geq 1$, one checks that $W_i \setminus U_i = Z^{\times^i}$ and $Z_{i+1} = Z^{\times^i} \times U$. In particular, $\mathrm{codim}_{W_i}(W_i \setminus U_i) = i(\mathrm{codim}_W(Z))$ and $\mathrm{codim}_{U_{i+1}}(Z_{i+1}) = (i+1)d - i(\dim(Z)) - d = i(\mathrm{codim}_W(Z))$, where $d = \dim(W)$. Moreover, $U_i \to U_i/G$ is a principal G-bundle and that the quotient $V_i = U_i/G$ exists as a smooth quasi-projective scheme (since the G-action on U is free and U/G is a quasi-projective scheme). We will often use $EG^{gm,i}$ to denote the $i+1$-th term of an admissible gadget $\{U_i|i\}$.

In particular if $\mathrm{G} = \mathrm{GL_n}$, one starts with a faithful representation on the affine space $\mathrm{V} = \mathbb{A}^{\mathrm{n}}$. Let $\mathrm{W} = \mathrm{End(V)}$ and $\mathrm{U} = \mathrm{GL(V)} = \mathrm{GL_n}$.

In case $\mathrm{G} = \mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$, $\mathrm{ET^{gm,i}}$ will be defined as follows. Assume first that $\mathrm{T} = \mathbb{G}_{\mathrm{m}}$. Then one may let $\mathrm{ET^{gm,i}} = \mathbb{A}^{\mathrm{i+1}} - 0$ with the diagonal action of $\mathrm{T} = \mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^{i+1}$. Now $\mathrm{BT^{gm,i}} = \mathbb{P}^{\mathrm{i}}$, which clearly has a Zariski open covering by $i+1$ affine spaces, each isomorphic to $\mathbb{A}^i$. If $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$, then

$$\mathrm{ET^{gm,i}} = (\mathbb{A}^{\mathrm{i+1}} - 0)^{\times \mathrm{n}} \tag{2.1.3}$$

with the j-th copy of $\mathbb{G}_m$ acting diagonally on the j-th factor $\mathbb{A}^{i+1} - 0$. Now $\mathrm{BT^{gm,i}} = (\mathbb{P}^{\mathrm{i}})^{\times \mathrm{n}}$.

Next we consider the following preliminary result.

Proposition 2.2. *If* $\mathrm{G} = \mathrm{GL_n}$*, a finite product of* GL_n*s or a split torus, then there exists a finite Zariski open covering,* $\{\mathrm{V}_\alpha | \alpha\}$ *of each* $\mathrm{BG^{gm,i}}$ *with each* V_α *being* $\mathbb{A}^1$*-acyclic, that is, each* V_α *has a* k*-rational point* p_α *so that* V_α *and* p_α *are equivalent in the* $\mathbb{A}^1$*-homotopy category. In fact, one may choose a common* k*-rational point* p *in the intersection of all the open* V_α*. Moreover, in all three cases, one may choose a Zariski open covering of* $\mathrm{BG^{gm,i}}$ *of the above form, to consist of* $i+1$ *or more open sets.*

Proof. We will recall the construction of the geometric classifying spaces $\mathrm{BG^{gm,i}}$ given above. Observe that when $\mathrm{G} = \mathrm{GL_n}$, $\mathrm{U} = \mathrm{GL_n}$ so that

$$\mathrm{EG^{gm,i}} = \mathrm{U_{i+1}} = \mathrm{GL_n} \times \mathrm{W}^{\times \mathrm{i}} \cup \mathrm{W} \times \mathrm{GL_n} \times \mathrm{W}^{\times \mathrm{i-1}} \cup \cdots \cup \mathrm{W}^{\times \mathrm{i}} \times \mathrm{GL_n}$$

Therefore $\mathrm{BG^{gm,i}}$ is covered by the Zariski open cover $\mathrm{V}_1 = \mathrm{GL_n} \underset{\mathrm{GL_n}}{\times} \mathrm{W}^{\times \mathrm{i}}, \mathrm{V}_2 = \mathrm{W} \underset{\mathrm{GL_n}}{\times} \mathrm{GL_n} \times \mathrm{W}^{\times \mathrm{i-1}}, \cdots, \mathrm{V_{i+1}} = \mathrm{W}^{\times \mathrm{i}} \underset{\mathrm{GL_n}}{\times} \mathrm{GL_n}$ each of which is isomorphic to $\mathrm{W}^{\times \mathrm{i}}$ and hence $\mathbb{A}^1$-acyclic. Observe that there are exactly $i+1$-open sets in this cover. Moreover, one may choose a k-rational point $q \,\epsilon\, \mathrm{GL}_n = \mathrm{U}$ and let $p' = (q, \cdots, q) \,\epsilon\, \mathrm{W}^{\times \mathrm{i+1}}$ denote a k-rational point. Clearly this belongs to $\mathrm{EG^{gm,i}}$ and provides a common k-rational point in the intersection of the above open cover of $\mathrm{BG^{gm,i}}$. When $\mathrm{G} = \mathrm{GL_{n_1}} \times \cdots \times \mathrm{GL_{n_m}}$, one may take for $\mathrm{BG^{g,m}} = \mathrm{B\,GL_{n_1}^{gm,m}} \times \cdots \times \mathrm{B\,GL_{n_m}^{gm,m}}$, so that the conclusion is also clear in this case.

If $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$, then recall $\mathrm{ET^{gm,i}} = (\mathbb{A}^{\mathrm{i+1}} - 0)^{\times \mathrm{n}}$ with the j-th copy of $\mathbb{G}_m$ acting on the j-th factor $\mathbb{A}^{i+1} - 0$. Now $\mathrm{BT^{gm,i}} = (\mathbb{P}^{\mathrm{i}})^{\times \mathrm{n}}$. The conclusions are clear in this case as well. $\square$

Remark 2.3. Let $\mathrm{G} = \mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$. Now we have already two constructions for $\mathrm{ET^{gm,i}}$. The first, which we denote by $\mathrm{ET^{gm,i}(1)}$ starts with $\mathrm{ET^{gm,1}(1)} = \mathrm{GL_n}$ with the successive $\mathrm{ET^{gm,i}(1)}$ obtained by applying the construction in (2.1.1) with $\mathrm{ET^{gm,i}(1)} = \mathrm{E\,GL_n^{gm,i}}$. Let $\mathrm{ET^{gm,i}(2)}$ denote the second construction with $\mathrm{ET^{gm,i}} = (\mathbb{A}^{\mathrm{i+1}} - 0)^{\mathrm{n}}$. The explicit relationship between two such constructions is often delicate. But as shown in Appendix B, the equivariant Borel-style generalized cohomology is independent of the choice of the ind-scheme $\{\mathrm{EG}^{\mathrm{gm,m}}|\mathrm{m}\}$ used in defining a Borel-style generalized cohomology, at least on considering smooth schemes of finite type over k.

Example 2.4. Assume H is a subgroup of Σ_n, which is the symmetric group on n-letters. Then one may choose $\mathrm{U_m}(\Sigma_\mathrm{n}) = \{(\mathrm{x_i}, \cdots, \mathrm{x_n}) \ \epsilon \ (\mathbb{A}^\mathrm{m})^\mathrm{n}|\mathrm{x_i} \neq \mathrm{x_j}, \mathrm{i} \neq \mathrm{j}\}$, since the action on this $\mathrm{U_m}(\Sigma_\mathrm{n})$ by the symmetric group Σ_n is free.

Definition 2.5. (Borel style equivariant K-theory and G-theory) Assume first that H is a linear algebraic group or a finite group viewed as an algebraic group by imbedding it in some GL_n. We define the Borel style equivariant K-theory of X to be $\mathbf{K}(\mathrm{EH^{gm}}\underset{\mathrm{H}}{\times}\mathrm{X}) = \underset{\infty\leftarrow\mathrm{m}}{\mathrm{holim}}\mathbf{K}(\mathrm{U_m(H)}\underset{\mathrm{H}}{\times}\mathrm{X})$. $\mathbf{G}(\mathrm{EH^{gm}}\underset{\mathrm{H}}{\times}\mathrm{X})$ is defined similarly by observing that the closed imbeddings $\mathrm{U_m(H)}\underset{\mathrm{H}}{\times}\mathrm{X} \rightarrow \mathrm{U_{m+1}(H)}\underset{\mathrm{H}}{\times}\mathrm{X}$ are all regular closed imbeddings, and that, therefore the corresponding pull-backs are defined at the level of G-theory. If ℓ is a prime different from the characteristic of the field k, then $\mathbf{K}(\mathrm{EH^{gm}}\underset{\mathrm{H}}{\times}\mathrm{X})\hat{}_{\rho_\ell} = \underset{\infty\leftarrow\mathrm{m}}{\mathrm{holim}}\mathbf{K}(\mathrm{U_m(H)}\underset{\mathrm{H}}{\times}\mathrm{X})\hat{}_{\rho_\ell}$. $\mathbf{G}(\mathrm{EH^{gm}}\underset{\mathrm{H}}{\times}\mathrm{X})\hat{}_{\rho_\ell}$ is defined similarly.

Since the H-action on $\mathrm{EH^{gm}}$ is free, one obtains the weak-equivalences:
$\mathbf{K}(\mathrm{EH^{gm}} \times \mathrm{X}, \mathrm{H}) \simeq \mathbf{K}(\mathrm{EH^{gm}}\underset{\mathrm{H}}{\times}\mathrm{X})$, $\mathbf{G}(\mathrm{EH^{gm}} \times \mathrm{X}, \mathrm{H}) \simeq \mathbf{G}(\mathrm{EH^{gm}}\underset{\mathrm{H}}{\times}\mathrm{X})$
and similarly for the ρ_ℓ-completed version.

Next we make the following observations.

- Let $\{\mathrm{EH^{gm,m}}|\mathrm{m}\}$ denote the ind-scheme defined above associated to the algebraic group H. Then if X is any scheme or algebraic space over k, viewing everything as simplicial presheaves on the Nisnevich site, we obtain $\lim_{m\rightarrow\infty} \mathrm{EH^{gm,m}}\underset{\mathrm{H}}{\times}\mathrm{X} \cong (\lim_{\mathrm{m}\rightarrow\infty} \mathrm{EH^{gm,m}})\underset{\mathrm{H}}{\times}\mathrm{X} = \mathrm{EH^{gm}}\underset{\mathrm{H}}{\times}\mathrm{X}$. (This follows readily from the observation that the H action on $\mathrm{EH^{gm,m}}$ is free and that filtered colimits commute with the balanced product construction above.)
- It follows therefore, that if E is any $\mathbb{A}^1$-local spectrum and X is a smooth scheme of finite type over k, then one obtains a weak-equivalence:

$$\mathrm{Map}(\Sigma^\infty_{\mathrm{S^1}}(\mathrm{EH^{gm}}\underset{\mathrm{H}}{\times}\mathrm{X})_+, \mathrm{E}) = \underset{\infty\leftarrow\mathrm{m}}{\mathrm{holim}}\{\mathrm{Map}(\Sigma^\infty_{\mathrm{S^1}}(\mathrm{EH^{gm,m}}\underset{\mathrm{H}}{\times}\mathrm{X})_+, \mathrm{E})|\mathrm{m}\}.$$

where $\mathrm{Map}(\quad, E)$ denotes the simplicial mapping spectrum. (This is the space of global sections of the internal hom $\mathcal{M}ap(\quad,\quad)$ in $\mathbf{Spt}_{\mathrm{S^1}}(\mathrm{S})$, considered in (8.0.1).) It is shown in Appendix B, making use of the properties of motivic slices that for any two different models of geometric classifying spaces given by $\{\mathrm{EH^{gm,m}}|\mathrm{m}\}$ and $\{\widetilde{\mathrm{EH}}^{gm,m}|m\}$, one obtains a weak-equivalence for any $\mathbb{A}^1$-local spectrum E and any smooth scheme X:

$$\underset{\infty\leftarrow m}{\mathrm{holim}}\{\mathrm{Map}(\Sigma^\infty_{\mathrm{S^1}}(\widetilde{\mathrm{EH}}^{gm,m}\underset{\mathrm{H}}{\times}\mathrm{X})_+, E)|m\} = \underset{\infty\leftarrow m}{\mathrm{holim}}\{\mathrm{Map}(\Sigma^\infty_{\mathrm{S^1}}(\mathrm{EH}^{gm,m}\underset{\mathrm{H}}{\times}\mathrm{X})_+, E)|m\}. \tag{2.1.4}$$

Remark 2.6. It may be worth pointing out that the only need to restrict to schemes satisfying the Standing hypotheses as in 1.1 is to be able to reduce G-theory to K-theory for regular schemes, where one may make use of the above arguments to show the Borel style equivariant K-theory (and hence Borel-style equivariant G-theory) is independent of the choice of the geometric classifying spaces. This is essential, since in the proof of Theorem 1.2 one needs to use two different models of geometric classifying spaces for the action of a maximal torus T in a split reductive group G, one of these being $\{\mathrm{EH}^{\mathrm{gm,m}}/\mathrm{T}|\mathrm{m}\}$ and the other being $\{\mathrm{ET}^{\mathrm{gm,m}}/\mathrm{T}|\mathrm{m}\}$. It is only in this step that we need to invoke motivic arguments. If one restricts to actions of split tori, then one may use a single model of the geometric classifying space, namely $\{\mathrm{ET}^{\mathrm{gm,m}}/\mathrm{T}|\mathrm{m}\}$, and therefore, Theorem 1.2 holds for actions of split tori on all separated algebraic spaces of finite type over any base field.

2.2. Key properties of equivariant K- and G-theories

All of the properties below extend to the ρ_ℓ-completed versions though we do not state them explicitly. Throughout X will denote an algebraic space of finite type over S, except in statements that involve comparison of two distinct models of geometric classifying spaces, where it is required to be a scheme satisfying the Standing hypotheses in 1.1. One may also consider the Quillen style K-theory of the abelian category of H-equivariant coherent sheaves on X. This will always produce a spectrum weakly-equivalent to $\mathbf{G}(\mathrm{X}, \mathrm{H})$: see, for example, [27, Proposition 2.7].

Lemma 2.7. *Assume as in (1.2.1) that X is provided with an action by the linear algebraic group H and that G is a bigger linear algebraic group containing H as a closed subgroup scheme. Then one obtains the commutative diagram*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{S}, \mathrm{G}\times\mathrm{H}) \wedge \mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G}) \arrow[r] \arrow[d, "id\wedge s^*"'] & \mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G}) \arrow[d, "s^*"'] \\
\mathbf{K}(\mathrm{S}, \mathrm{G}\times\mathrm{H}) \wedge \mathbf{G}(\mathrm{G}\times\mathrm{X}, \mathrm{G}\times\mathrm{H}) \arrow[r] & \mathbf{G}(\mathrm{G}\times\mathrm{X}, \mathrm{G}\times\mathrm{H}) \\
\mathbf{K}(\mathrm{S}, \mathrm{G}\times\mathrm{H}) \wedge \mathbf{G}(\mathrm{X}, \mathrm{H}) \arrow[r] \arrow[u, "id\wedge r^*"] & \mathbf{G}(\mathrm{X}, \mathrm{H}), \arrow[u, "r^*"]
\end{tikzcd}$$

where the maps r and s are as in (1.2.1). Moreover both the maps s^* and r^* are weak-equivalences.

Proof. One may consider the corresponding diagram of tensor-product pairings of categories of equivariant vector bundles (or perfect complexes of equivariant $\mathcal{O}$-modules) and equivariant coherent sheaves: then one may readily see that such a diagram commutes. The commutativity of the diagram in Lemma 2.7 follows. The fact that s and r are weak-equivalences follows from descent theory applied to the corresponding categories of equivariant coherent sheaves. □

(i) Let H denote a closed subgroup scheme of G. Then, as observed above, one obtains the weak-equivalences: $\mathbf{G}(X, H) \simeq \mathbf{G}(G \underset{H}{\times} X, G)$ and $\mathbf{K}(X, H) \simeq \mathbf{K}(G \underset{H}{\times} X, G)$. Under the same hypotheses, one also obtains the weak-equivalences of inverse systems (that is, a weak-equivalence on taking their homotopy inverse limits): $\{\mathbf{K}(EH^{gm,m} \underset{H}{\times} X)|m\} \simeq \{\mathbf{K}(EG^{gm,m} \underset{G}{\times} G \underset{H}{\times} X)|m\}$ and $\{\mathbf{G}(EH^{gm,m} \underset{H}{\times} X)|m\} \simeq \{\mathbf{G}(EG^{gm,m} \underset{G}{\times} G \underset{H}{\times} X)|m\}$. (These follow readily in view of the results in Appendix B, and the property (vii) below, which shows how to reduce to the case X is a regular scheme.)

For the remaining items (ii) through (vi), we will assume that the action of the subgroup H on X extends to an action by the ambient group G. Moreover, where these properties are used is in section 4, where G will denote a split reductive group, and H will denote either a Borel subgroup or a maximal split subtorus.

(ii) The G-equivariant flat map $\pi : G \underset{H}{\times} X \to X$, $(gh^{-1}, hx) \mapsto gh^{-1}hx = gx$ induces maps $\pi^* : \mathbf{G}(X, G) \to \mathbf{G}(G \underset{H}{\times} X, G) \simeq \mathbf{G}(X, H)$, $\pi^* : \mathbf{G}(EG^{gm,m} \underset{G}{\times} X) \to \mathbf{G}(EG^{gm,m} \underset{G}{\times} (G \underset{H}{\times} X))$ which identify with the corresponding maps obtained by restricting the group action from G to H.

(iii) In view of the above properties, given any linear algebraic group G, we will fix a closed imbedding $G \to GL_n$ for some n and identify $\mathbf{G}(X, G)$ with $\mathbf{G}(GL_n \underset{G}{\times} X, GL_n)$, $\mathbf{K}(X, G)$ with $\mathbf{K}(GL_n \underset{G}{\times} X, GL_n)$,

$$\mathbf{G}(EG^{gm,m} \underset{G}{\times} X) \text{ with } \mathbf{G}(EGL_n{}^{gm,m} \underset{GL_n}{\times} (GL_n \underset{G}{\times} X)) \text{ and}$$
$$\mathbf{K}(EG^{gm,m} \underset{G}{\times} X) \text{ with } \mathbf{K}(EGL_n{}^{gm,m} \underset{GL_n}{\times} (GL_n \underset{G}{\times} X)).$$

(iv) Next assume that G is a split reductive group over S and H = B, that is, H is a Borel subgroup of G. Then using the observation that G/B is proper over S and $R^n\pi_* = 0$ for n large enough, one sees that the map π also induces push-forwards $\pi_* : \mathbf{G}(G \underset{B}{\times} X, G) \to \mathbf{G}(X, G)$ and $\pi_* : \mathbf{G}(EG^{gm,m} \underset{G}{\times} G \underset{B}{\times} X) \to \mathbf{G}(EG^{gm,m} \underset{G}{\times} X)$, induced by the derived direct image functor $R\pi_*$. (Such a derived direct image functor may be made functorial at the level of complexes by considering pseudo-coherent complexes which are injective $\mathcal{O}_X$-modules in each degree, or by using the functorial Godement resolution.)

(v) Assume the above situation. Then the projection formula applied to $R\pi_*$ shows that the composition $R\pi_*\pi^*(F) = F \otimes R\pi_*(\mathcal{O}_{G \underset{B}{\times} X}) \simeq F$, since

$$R^n\pi_*(\mathcal{O}_{G \underset{B}{\times} X}) = \mathcal{O}_X, \text{ if n=0 and} \tag{2.2.1}$$

$$= 0, \text{ if } n > 0.$$

It follows that π^* is a split monomorphism in this case, with the splitting provided by π_*. (See [57, Theorem 1.13]. See (4.0.10) where this is applied to reduce equivariant **G**-theory with respect to the action of a split reductive group G to that of a Borel subgroup B and hence to that of a maximal torus T.)

(vi) Assume the situation as above with T = the maximal torus in G. Then $\mathrm{B}/\mathrm{T} = \mathrm{R_u}(\mathrm{B})$= an affine space. Now one obtains the weak-equivalences:

$$\begin{aligned}\mathbf{G}(\mathrm{X},\mathrm{B}) \simeq \mathbf{G}(\mathrm{B}\underset{\mathrm{T}}{\times}\mathrm{X},\mathrm{B}) \simeq \mathbf{G}(\mathrm{X},\mathrm{T}) \text{ and } \mathbf{G}(\mathrm{EB}^{\mathrm{gm,m}}\underset{\mathrm{B}}{\times}\mathrm{X}) \simeq \mathbf{G}(\mathrm{EB}^{\mathrm{gm,m}}\underset{\mathrm{B}}{\times}\mathrm{B}\underset{\mathrm{T}}{\times}\mathrm{X})\\ \simeq \mathbf{G}(\mathrm{EB}^{\mathrm{gm,m}}\underset{\mathrm{T}}{\times}\mathrm{X})\end{aligned}$$

where the first weak-equivalences are from the homotopy property of **G**-theory and the observation that $\mathrm{R_u}(\mathrm{B})$ is an affine space over S.

(vii) Next assume that X is a scheme satisfying the Standing hypotheses as in 1.1. Let $i : \mathrm{X} \to \tilde{\mathrm{X}}$ denote an H-equivariant closed immersion into a regular H-scheme. Then one obtains the weak-equivalence: $\mathbf{G}(\mathrm{X},\mathrm{H}) \simeq (\mathrm{hofib}(\mathrm{i}^* : \mathbf{K}(\tilde{\mathrm{X}},\mathrm{H}) \to \mathbf{K}(\tilde{\mathrm{X}}-\mathrm{X},\mathrm{H})))$ where hofib denotes the canonical homotopy fiber. Similarly, for each non-negative integer m, $\mathbf{G}(\mathrm{EH}^{\mathrm{gm,m}}\underset{\mathrm{H}}{\times}\mathrm{X})$ is weakly-equivalent to the canonical homotopy fiber of the restriction $i^* : \mathbf{K}(\mathrm{EH}^{\mathrm{gm,m}}\underset{\mathrm{H}}{\times}\tilde{\mathrm{X}}) \to \mathbf{K}(\mathrm{EH}^{\mathrm{gm,m}}\underset{\mathrm{H}}{\times}(\tilde{\mathrm{X}}-\mathrm{X}))$. These identifications enable us to readily extend the main results of this paper to schemes that are not regular, but satisfy the Standing hypotheses 1.1.

2.3. Properties of the representation ring: the algebraic fundamental group of a split reductive group

First recall that the algebraic fundamental group associated to a split reductive group G over the base field k with maximal torus T may be defined as $\Lambda/\mathrm{X}(\mathrm{T})$, where Λ ($\mathrm{X}(\mathrm{T})$) denotes the weight lattice (the lattice of characters of T, respectively): see for example, [34, 1.1]. Then it is observed in [34, Proposition 1.22], making use of [51, Theorem 1.3], that if this fundamental group is torsion-free, then R(T) is a free module over R(G). Here R(G) (R(T)) denotes the representation ring of G (T, respectively). Making use of the observation that SL_n is simply-connected (that is, the above fundamental group is trivial), for any n, one may conclude that $\pi_1(\mathrm{GL}_n) \cong \pi_1(\mathbb{G}_m) \cong \mathbb{Z}$ where $\mathbb{G}_m$ denotes the central torus in GL_n. Therefore R(T) is free over $\mathrm{R}(\mathrm{GL_n})$, where T denotes a maximal torus in GL_n.

3. Derived completions

In this section, we begin by reviewing some of the basic results from [7] and then go onto establish several supplemental results on the derived completion which will play a key role in later sections of the paper. The essential use of derived smash products in Proposition 5.2 and Theorem 5.6, which play a key-role in our results, make the framework of the derived completions as worked out in [7] the appropriate framework for this paper. (See also [15] for related results.)

Let $f : B \to C$ denote a map of commutative ring spectra in $\mathbf{Spt}_{S^1}$ which are both -1-connected and let $\tilde{C}$ denote a cofibrant replacement of C in the category of commutative algebra spectra over B. Let $Mod(B)$ ($Mod(\tilde{C})$) denote the category of module spectra over B ($\tilde{C}$, respectively). Then one defines a triple $Mod(B) \to Mod(B)$ by sending a B-module spectrum M to $M \underset{B}{\wedge} \tilde{C}$, which belongs to $Mod(\tilde{C})$, and then by sending it to the corresponding B-module spectrum using restriction of scalars from $\tilde{C}$ to B. Iterating this defines a cosimplicial object $\mathcal{T}_B^\bullet(M, \tilde{C})$ in $Mod(B)$ whose n-th degree term is the iterated smash product:

$$\overbrace{M \underset{B}{\wedge} \tilde{C} \cdots \underset{B}{\wedge} \tilde{C}}^{n+1}.$$

We will often denote this object as $\overbrace{M \overset{L}{\underset{B}{\wedge}} C \cdots \overset{L}{\underset{B}{\wedge}} C}^{n+1}$, meaning that an implicit choice of a cofibrant replacement of C has been made. Similarly $\mathcal{T}_B^\bullet(M, \tilde{C})$ will often be denoted $\mathcal{T}_B^\bullet(M, C)$, meaning an implicit choice of a cofibrant replacement of C has been made.

Definition 3.1. In this set-up, *the derived completion of M along f* is defined to be the homotopy inverse limit: $\underset{\Delta}{\text{holim}}\, \mathcal{T}_B^\bullet(M, C)$ and often denoted $M\hat{}_f$.

The derived completion is therefore a completion with respect to the above triple in the sense of [40].

Remark 3.2. Let $Alg(B)$ denote the category of commutative algebra spectra over B. Observe that in the model structure on $Alg(B)$ defined in [49], the weak-equivalences are maps in $Alg(B)$ which are stable equivalences of symmetric spectra and that any cofibrant object in $Alg(B)$ is also cofibrant in $Mod(B)$. Observe also that the weak-equivalences in $Mod(B)$ are maps in $Mod(B)$ which are stable equivalences of symmetric spectra. Therefore first observe that if $\tilde{C}$ is chosen as above, then it is also a cofibrant replacement of C in $Mod(B)$. Therefore, if $\hat{C} \epsilon Alg(B)$ is such that it is a cofibrant replacement of C in $Mod(B)$, then for any $M \epsilon Mod(B)$, one obtains a weak-equivalence between $M \underset{B}{\wedge} \tilde{C}$ and $M \underset{B}{\wedge} \hat{C}$ as well as between the corresponding iterated smash products with $\tilde{C}$ and $\hat{C}$ over B. Therefore, one may also define the cosimplicial object $\mathcal{T}_B^\bullet(M, \tilde{C})$ using $\hat{C}$ in the place of $\tilde{C}$. This observation will be used in the proof of Theorem 3.6.

3.0.1. Partial derived completions

The partial derived completion of M along f up to order m will be defined to be the $\underset{\Delta_{\leq m}}{\text{holim}}\, \tau_{\leq m}\mathcal{T}^{\bullet}_B(M, C)$, where $\tau_{\leq m}\mathcal{T}^{\bullet}_B(M, C) = \{\mathcal{T}^i_B(M, C)|i \leq m\}$. This will be denoted $M\widehat{\ }_{f,m}$.

Let I_B denote the homotopy fiber of f. This is a B-module spectrum and it is observed in [7, Corollary 6.7] that one obtains the stable cofiber sequence:

$$(\overset{L}{\underset{B}{\wedge}}{}_{i=1}^{m+1} I_B)\overset{L}{\underset{B}{\wedge}} M \to M \to \underset{\Delta_{\leq m}}{\text{holim}}\, \tau_{\leq m}\mathcal{T}^{\bullet}_B(M, C) = M\widehat{\ }_{f,m} \tag{3.0.2}$$

Given a commutative ring R (a module M over R) $\mathbb{H}(R)$ ($\mathbb{H}(M)$, respectively) will denote the Eilenberg-Maclane spectrum associated to R (M, respectively) as in Appendix A. (Recall that $\mathbb{H}(R)$ ($\mathbb{H}(M)$) has a single non-trivial stable homotopy group in degree 0, where it is R (M, respectively).) Then one observes readily that $\mathbb{H}(R)$ is a commutative connected ring spectrum with $\mathbb{H}(M)$ a module-spectrum over $\mathbb{H}(R)$.

Lemma 3.3. *Suppose R is a commutative Noetherian ring with I an ideal in R so that I is flat as an R-module. Let M be a flat R-module. Then one obtains the following weak-equivalences, where $f : \mathbb{H}(R) \to \mathbb{H}(R/I)$ denotes the map induced by the map $R \to R/I$:*

$$\begin{aligned}(\overset{L}{\underset{\mathbb{H}(R)}{\wedge}}{}_{i=1}^{m+1} \mathbb{H}(I))\overset{L}{\underset{\mathbb{H}(R)}{\wedge}} \mathbb{H}(M) &\simeq \mathbb{H}((\overset{L}{\underset{R}{\otimes}}{}_{i=1}^{m+1} I)\overset{L}{\underset{R}{\otimes}} M) \textit{ and} \\ H(M)\widehat{\ }_{f,m} &= \underset{\Delta_{\leq m}}{\mathit{holim}}\, \tau_{\leq m}\mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(M), \mathbb{H}(R/I)) \\ &\simeq \mathbb{H}(\underset{\infty\leftarrow m}{\mathit{lim}}\, \{M/I^k|k \leq m + 1\})\end{aligned} \tag{3.0.3}$$

where one makes use of the free resolutions by the free commutative algebra functor discussed in A.0.3 to define the truncated cosimplicial object $\tau_{\leq m}\mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(M), \mathbb{H}(R/I))$.

Proof. In view of the flatness assumptions, the spectral sequence computing the derived smash products of the left-hand-side in (3.0.3) degenerates. This proves the first weak-equivalence: observe that the flatness assumptions imply the left-derived functors of the tensor product there may be replaced by the tensor product. Next observe that the iterated derived tensor product $M\overbrace{\overset{L}{\underset{R}{\otimes}} I \cdots \overset{L}{\underset{R}{\otimes}} I}^{m+1}$ identifies with $M\overbrace{\underset{R}{\otimes} I \cdots \underset{R}{\otimes} I}^{m+1} = M \otimes I^{m+1}$ for similar reasons, and that therefore, these maps injectively to $M\overbrace{\underset{R}{\otimes} R \cdots \underset{R}{\otimes} R}^{m+1} \cong M$. Then the resulting inverse system $\{M \otimes R/I^k = M/(I^k M)|k\}$ has surjective transition maps so that there are no lim^1-terms. This provides the second weak-equivalence. □

Remark 3.4. A corresponding result for the completion $M \mapsto M\widehat{\ }_f$, for M a finitely generated R-module, is proven in [7, Theorem 4.4] which holds more generally without any of the flatness assumptions above. However, that proof does not apply to partial derived completions: hence the need for the above result.

3.0.4. Basic spectral sequences and reduction to derived completions of modules over rings

If M is an s-connected module spectrum over B, (for some integer s) there exist convergent second quadrant spectral sequences:

$$E_1^{p,q} = \pi_{q+2p}(\mathbb{H}(\pi_{-p}(M))\widehat{\ }_{\mathbb{H}(\pi_0(f))}) \Rightarrow \pi_{q+p}(M\widehat{\ }_f), \tag{3.0.5}$$
$$E_1^{p,q} = \pi_{q+2p}(\mathbb{H}(\pi_{-p}(M))\widehat{\ }_{f,m}) \Rightarrow \pi_{q+p}(M\widehat{\ }_{f,m}).$$

The first is discussed explicitly in [7, Theorem 7.1] and the second is obtained in an entirely similar manner using [7, Proposition 7.7 and Corollary 7.8]. One may want to observe that one cannot, in general, identify the $E_1^{p,q}$-terms for the partial derived completion with $\pi_{q+2p}(\mathbb{H}(\pi_{-p}(M))\widehat{\ }_{\mathbb{H}(\pi_0(f)),m})$.

However, since the convergence of the above spectral sequences may not be strong, what seems more useful are the arguments in [7, Corollary 7.8] that enable one to reduce questions on derived completions to derived completions of modules over rings. Accordingly let $M \epsilon\ Mod(B)$ for some commutative ring spectrum B, let $f : B \rightarrow C$ denote a map of commutative ring spectra which are both -1-connected. Let $\{\cdots \rightarrow M < t > \rightarrow M < t-1 > \cdots |t\}$ denote the canonical Postnikov tower for M. We will assume that M is s-connected, for some integer s. Then the fiber of the fibration $M < t > \rightarrow M < t-1 >$ is an Eilenberg Maclane spectrum $K(\pi_t(M), t)$. We will denote this by $\mathbb{H}(\pi_t(M))[t]$. Observe that each $M < t >$ is a module spectrum over $\mathbb{H}(\pi_0(B), 0)$: using the map $B \rightarrow \mathbb{H}(\pi_0(B), 0)$ of ring spectra, one may view each $M < t >$ and the fiber of the map $M < t > \rightarrow M < t-1 >$ as module spectra over B. It is shown in [7, Corollary 7.8] that, then

(i) $\{\cdots \rightarrow (M < t >)\widehat{\ }_f \rightarrow (M < t-1 >)\widehat{\ }_f \cdots |t\}$ is also a tower of fibrations (up to homotopy),
(ii) that the homotopy fiber of the map $(M < t >)\widehat{\ }_f \rightarrow (M < t-1 >)\widehat{\ }_f$ is $\mathbb{H}(\pi_t(M))\widehat{\ }_{\mathbb{H}(\pi_0(f))}[t]$, where $\mathbb{H}(\pi_t(M))\widehat{\ }_{\mathbb{H}(\pi_0(f))}$ denotes the derived completion of $\mathbb{H}(\pi_t(M))$ along the map $\mathbb{H}(\pi_0(f))$
(with $\mathbb{H}(\pi_t(M))\widehat{\ }_{\mathbb{H}(\pi_0(f))}[t]$ denoting a suitable shift of $\mathbb{H}(\pi_t(M))\widehat{\ }_{\mathbb{H}(\pi_0(f))}$), and that
(iii) $M\widehat{\ }_f \simeq \underset{\infty \leftarrow t}{\mathrm{holim}}(M < t > \widehat{\ }_f)$.

Since M is assumed to be s-connected, $M < s >= K(\pi_s(M), s)$, so that we may start an inductive argument at s and step along the terms in the tower in (i) and finally use (iii) to deduce properties of the derived completion on M.

3.0.6. Mod-ℓ completions

Given a commutative ring R with a unit, let $\mathbb{H}(\mathrm{R})$ denote the Eilenberg-Maclane spectrum. Let ℓ denote a fixed prime. Then the mod-ℓ completion for us will denote the completion with respect to the mod-ℓ reduction map $\rho_\ell : \mathbb{S} \to \mathbb{H}(\mathbb{Z}/\ell)$, where $\mathbb{S}$ denotes the symmetric (S^1)-sphere spectrum. *We will assume throughout that* $\mathbb{H}(\mathbb{Z}/\ell)$ *is a cofibrant (commutative) algebra-spectrum over the sphere spectrum* $\mathbb{S}$.

3.0.7. Derived completions for equivariant K-theory

For the rest of this discussion we will assume that S is the spectrum of a field and that H is a linear algebraic group defined over S. We denote the restriction map

$$\begin{aligned} &\mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S}) \text{ by } \rho_{\mathrm{H}} \text{ and the Postnikov-truncation map} \\ &\mathbf{K}(\mathrm{S}) \to \mathbf{K}(\mathrm{S}) <0> \simeq \mathbb{H}(\pi_0(\mathbf{K}(\mathrm{S}))) \text{ by } <0>. \end{aligned} \tag{3.0.8}$$

The composite map $\mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S}) <0> \simeq \mathbb{H}(\pi_0(\mathbf{K}(\mathrm{S})))$ will be denoted $\tilde{\rho}_{\mathrm{H}}$. Observe that $\pi_0(\mathbf{K}(\mathrm{S},\mathrm{H})) = \mathrm{R}(\mathrm{H})=$ the representation ring of H. Clearly the restriction ρ_{H} induces a surjection on taking π_0: this is clear, since if V is any finite dimensional representation of H of dimension d, the d-th exterior power of V will be a 1-dimensional representation of H. The Postnikov-truncation map $<0>$: $\mathbf{K}(\mathrm{S}) \to \mathbf{K}(\mathrm{S}) <0> \simeq \mathbb{H}(\pi_0(\mathbf{K}(\mathrm{S})))$ clearly induces an isomorphism on π_0. Therefore, the hypotheses of [7, Theorem 6.1] are satisfied and it shows that the derived completion functors with respect to the two maps ρ_{H} and $\tilde{\rho}_{\mathrm{H}}$ identify up to weak-equivalence. Therefore, while we will always make use of the completion with respect to the map ρ_{H}, we will often find it convenient to identify this completion with the completion with respect to $\tilde{\rho}_{\mathrm{H}}$.

Observe that the above completion ρ_{H} has the following explicit description. Let $Mod(\mathbf{K}(\mathrm{S},\mathrm{H}))$ ($Mod(\mathbf{K}(\mathrm{S}))$) denote the category of module-spectra over $\mathbf{K}(\mathrm{S},\mathrm{H})$ ($\mathbf{K}(\mathrm{S})$, respectively). Then sending a module spectrum $M \,\epsilon\, Mod(\mathbf{K}(\mathrm{S},\mathrm{H}))$ to $M \overset{L}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{K}(\mathrm{S})$ and then viewing it as a $\mathbf{K}(\mathrm{S},\mathrm{H})$-module spectrum using the ring map $\mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathrm{K}(\mathrm{S})$ defines a triple. The corresponding cosimplicial object of spectra is given by

$$\mathcal{T}^{\bullet}_{\mathbf{K}(\mathrm{S},\mathrm{H})}(M,\mathbf{K}(\mathrm{S})) : \mathrm{M} \overset{\rightrightarrows}{\leftarrow} \mathrm{M} \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{K}(\mathrm{S}) \overset{\rightrightarrows}{\leftarrow} \cdots \mathrm{M} \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{K}(\mathrm{S}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \cdots \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{K}(\mathrm{S}) \cdots$$

with the obvious structure maps. Then the derived completion of M with respect to ρ_{H} is

$$M\,\widehat{}_{\rho_{\mathrm{H}}} = \underset{\Delta}{\mathrm{holim}}\, \mathcal{T}^{\bullet}_{\mathbf{K}(\mathrm{S},\mathrm{H})}(M,\mathbf{K}(\mathrm{S})) \tag{3.0.9}$$

and the corresponding partial derived completion to degree m will be denoted $\rho_{\mathrm{H},\mathrm{m}}$.

We may also consider the following variant. For each prime ℓ, we let

$$\rho_\ell \circ \rho_{\mathrm{H}} : \mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S})/\ell = \mathbf{K}(\mathrm{S}) \underset{\mathbb{S}}{\wedge} \mathbb{H}(\mathbb{Z}/\ell) \tag{3.0.10}$$

denote the composition of ρ_{H} and the mod-ℓ reduction map $id_{\mathbf{K}(\mathrm{S})} \wedge \rho_\ell$. The derived completion with respect to $\rho_\ell \circ \rho_{\mathrm{H}}$ clearly has a description similar to the one in (3.0.9).

If X is a scheme or algebraic space provided with an action by the group H, we let

$$\begin{aligned} \mathbf{K}(\mathrm{X},\mathrm{H})_\ell &= \overbrace{\mathbf{K}(\mathrm{X},\mathrm{H}) \underset{\mathbb{S}}{\wedge} \mathbb{H}(\mathbb{Z}/\ell) \cdots \underset{\mathbb{S}}{\wedge} \mathbb{H}(\mathbb{Z}/\ell)}^{\mathrm{n}} \\ \mathbf{K}(\mathrm{EH}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})_\ell &= \overbrace{\mathbf{K}(\mathrm{EH}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X}) \underset{\mathbb{S}}{\wedge} \mathbb{H}(\mathbb{Z}/\ell) \cdots \underset{\mathbb{S}}{\wedge} \mathbb{H}(\mathbb{Z}/\ell)}^{\mathrm{n}} \end{aligned} \tag{3.0.11}$$

for some positive integer n. If it becomes important to indicate the n above, we will denote $\mathbf{K}(\mathrm{X},\mathrm{H})_\ell$ ($\mathbf{K}(\mathrm{EH}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})_\ell$) by $\mathbf{K}(\mathrm{X},\mathrm{H})_{\ell,\mathrm{n}}$ ($\mathbf{K}(\mathrm{EH}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})_{\ell,\mathrm{n}}$, respectively). One defines $\mathbf{G}(\mathrm{X},\mathrm{H})_\ell$ and $\mathbf{G}(\mathrm{EH}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X})_\ell$ similarly. (Recall from 3.0.6, the smash products above are all derived smash products.)

Next we proceed to compare the derived completions with respect to the map $\rho_{\mathrm{G}}: \mathbf{K}(\mathrm{S},\mathrm{G}) \to \mathbf{K}(\mathrm{S})$ for a split reductive group G and the derived completion with respect to the map $\rho_{\mathrm{T}}: \mathbf{K}(\mathrm{S},\mathrm{T}) \to \mathbf{K}(\mathrm{S})$, where T is a maximal torus in G. Given a module spectrum M over $\mathbf{K}(\mathrm{S},\mathrm{T})$, $res_*(M)$ will denote M with the induced $\mathbf{K}(\mathrm{S},\mathrm{G})$-module structure, where $res: \mathbf{K}(\mathrm{S},\mathrm{G}) \to \mathbf{K}(\mathrm{S},\mathrm{T})$ denotes the restriction map. Clearly there is a map $res_*(M) \to M$ (which is the identity map on the underlying spectra) compatible with the restriction map $\mathbf{K}(\mathrm{S},\mathrm{G}) \to \mathbf{K}(\mathrm{S},\mathrm{T})$. Therefore one obtains induced maps

$$res_*(M) \overset{L}{\underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge}} \mathbf{K}(\mathrm{S}) \cdots \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge}} \mathbf{K}(\mathrm{S}) \to \mathrm{M} \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{T})}{\wedge}} \mathbf{K}(\mathrm{S}) \cdots \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{T})}{\wedge}} \mathbf{K}(\mathrm{S}), \tag{3.0.12}$$

which induce a map from the derived completion of $res_*(M)$ with respect to ρ_{G} to the derived completion of M with respect to ρ_{T}. (Similar conclusions hold for the maps $\rho_\ell \circ \rho_{\mathrm{G}}$ and $\rho_\ell \circ \rho_{\mathrm{T}}$.) These may be seen more readily from the following lemma.

Lemma 3.5. *Let $A \to B \to C$ denote maps of commutative ring spectra that are -1-connected. Let $f: B \to C$ and $res: A \to B$ denote the given maps. Assume that C is a cofibrant object in the category of commutative algebra spectra over B and that $f: B \to C$ is the resulting cofibration. Then the following hold.*
(i) One may find a cofibrant replacement $\tilde{B} \to B$ of B in the model category of commutative algebra spectra over A and $\tilde{C} \to C$ of C in the model category of commutative algebra spectra over $\tilde{B}$ so that the induced maps $A \to \tilde{B}$ and $\tilde{B} \to \tilde{C}$ are cofibrations in the model category of commutative algebra spectra over A.
(ii) Let $\widetilde{res}: A \to \tilde{B}$ and $\tilde{f}: \tilde{B} \to \tilde{C}$ denote the induced maps of ring spectra. For any B-module spectrum M, one obtains an induced map of inverse systems of truncated cosimplicial) objects of spectra:

$$\{\tau_{\leq m}\mathcal{T}^\bullet_A(res_*(M), \tilde{C}) \to \tau_{\leq m}\mathcal{T}^\bullet_{\tilde{B}}(M, \tilde{C}) \xrightarrow{\simeq} \tau_{\leq m}\mathcal{T}^\bullet_B(M, C) | m\}$$

where $res_*(M)$ denote M viewed as an A-module spectrum by restriction of scalars.
(iii) Assume that $A \to B$ and $B \to C$ are both cofibrations in the category of commutative A-algebra spectra. Let I_B (I_A) denote the canonical homotopy fiber of the map $f : B \to C$ (the composite map $A \to C$, respectively). Then there exist cofibrant replacements $\tilde{I}_B \to I_B$ in the category of module spectra over B ($\tilde{I}_A \to I_A$ in the category of module spectra over A) together with a map $\tilde{I}_A \to \tilde{I}_B$ compatible with the maps $A \to B$. It follows that for a B-module spectrum as in (ii), one obtains an inverse system of compatible maps of spectra:

$$\{\tilde{I}_A^{m+1} = \overbrace{\tilde{I}_A \underset{A}{\wedge} \tilde{I}_A \cdots \underset{A}{\wedge} \tilde{I}_A}^{m+1} \underset{A}{\wedge} (res_*(M)) \to \tilde{I}_B^{m+1} = \overbrace{\tilde{I}_B \underset{B}{\wedge} \tilde{I}_B \cdots \underset{B}{\wedge} \tilde{I}_B}^{m+1} \underset{B}{\wedge} (M) | m\}.$$

Proof. (i) This follows readily from [12, (3.10) Remark], since the category of commutative $\tilde{B}$-algebra spectra may be viewed as the *under category* $\tilde{B} \downarrow$(commutative A-algebra spectra) with the model structure induced from the model structure on the category of commutative A-algebra spectra, so that the cofibration $\tilde{B} \to \tilde{C}$ will be a cofibration in the category of commutative A-algebra spectra. This proves (i). Then the existence of the map $\{\tau_{\le m}\mathcal{T}_A^\bullet(res_*(M), \tilde{C}) \to \tau_{\le m}\mathcal{T}_{\tilde{B}}^\bullet(M, \tilde{C})$ in (ii) is immediate, making use of the map $A \to \tilde{B}$ and observing that the composite map $A \to \tilde{B} \overset{f}{\to} \tilde{C}$ is a cofibration in the category of commutative A-algebra spectra. The existence of the map $\tau_{\le m}\mathcal{T}_{\tilde{B}}^\bullet(M, \tilde{C}) \overset{\simeq}{\to} \tau_{\le m}\mathcal{T}_B^\bullet(M, C)|m\}$ is clear. That it is a weak-equivalence follows from the fact that the maps $\tilde{B} \to B$ and $\tilde{C} \to C$ are weak-equivalences as well as the assumption that C is a cofibrant algebra over B. This proves (ii).

(iii) Let $\tilde{I}_B \to I_B$ denote a cofibrant replacement in the category of B-module spectra. Observe that this is a trivial fibration in the same model category and hence a trivial fibration of the underlying spectra. Therefore, let $\hat{I}_A = I_A \underset{I_B}{\times} \tilde{I}_B$. Then $\hat{I}_A$ is a module spectrum over A and the induced map $\hat{I}_A \to I_A$ is a map of A-module spectra which is a trivial fibration of A-module spectra. Therefore a cofibrant replacement $\tilde{I}_A$ of $\hat{I}_A$ in the category of A-module spectra is a cofibrant replacement of I_A in the category of A-module spectra and provided with a map $\tilde{I}_A \to \tilde{I}_B$ compatible with the map $A \to B$. The last statement is clear. □

We will invoke the above lemma in the following theorem with $A = \mathbf{K}(\mathrm{S}, \mathrm{G})$, $B = \mathbf{K}(\mathrm{S}, \mathrm{T})$ and $C = \mathbf{K}(\mathrm{S})$ with $res : A \to B$ and $f : B \to C$ denoting the obvious maps induced by restriction of groups.

Theorem 3.6. *Assume that* $\mathrm{S} = \mathrm{Spec}\, k$ *for a field* k *and that the split reductive group* G *has* $\pi_1(\mathrm{G})$ *torsion-free. Let* M *be an* s*-connected module spectrum over* $\mathbf{K}(\mathrm{S}, \mathrm{T})$ *for some integer* s*. Then the inverse system of maps defined in* (3.0.12),

$$\{res_*(M)\widehat{\ }_{\rho_{\mathrm{G}}, m} \to M\widehat{\ }_{\rho_{\mathrm{T}}, m} | m\}$$

induces a weak-equivalence on taking the homotopy inverse limit as $m \to \infty$.

Proof. In view of the arguments in the second paragraph of 3.0.4, one reduces immediately to proving the corresponding statement when M is an $R(\mathrm{T})$-module and the completions are with respect to the rank-maps $R(\mathrm{T}) \to \mathbb{Z}$ and $R(\mathrm{G}) \to \mathbb{Z}$. Throughout this proof, we will denote the rank-map $R(\mathrm{T}) \to \mathbb{Z}$ by ρ_{T} and the rank-map $R(\mathrm{G}) \to \mathbb{Z}$ by ρ_{G}. With this notation, we now reduce to proving

$$\mathbb{H}(res_*(M))\widehat{\ }_{\mathbb{H}(\rho_{\mathrm{G}})} \simeq \mathbb{H}(M)\widehat{\ }_{\mathbb{H}(\rho_{\mathrm{T}})}$$

where $\mathbb{H}$ denotes the Eilenberg-Maclane spectrum functor discussed in detail in Appendix A. Next one observes the isomorphisms:

$$R(\mathrm{T}) \underset{R(\mathrm{G})}{\otimes} R(\mathrm{G})/\ker(\rho_{\mathrm{G}}) \cong R(\mathrm{T}) \underset{R(\mathrm{T})}{\otimes} R(\mathrm{T})/(\mathrm{R(T)}.\ker(\rho_{\mathrm{G}})) \cong R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}})).$$

Next we make use of Remark 3.2 to find a more convenient definition of the derived completions. Let

$$R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})_\bullet \to R(\mathrm{G})/\ker(\rho_{\mathrm{G}})$$

denote a simplicial resolution by the free commutative algebra functor over $R(\mathrm{G})$ as discussed in A.0.3. It follows that if N denotes the functor sending a simplicial abelian group to the corresponding normalized chain complex,

$$N(R(\mathrm{T}) \underset{R(\mathrm{G})}{\otimes} R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})_\bullet) \to R(\mathrm{T}) \underset{R(\mathrm{G})}{\otimes} R(\mathrm{G})/\ker(\rho_{\mathrm{G}}) \cong R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}})) \tag{3.0.13}$$

is a flat resolution by a commutative dg-algebra over $R(\mathrm{T})$. One may observe that this step strongly uses the assumption that $\pi_1(\mathrm{G})$ is torsion-free, so that $R(\mathrm{T})$ is free over $R(\mathrm{G})$. In view of the discussion in Appendix A (see especially (A.0.4)), this implies the weak-equivalence:

$$\mathbb{H}(R(\mathrm{T})) \underset{\mathbb{H}(R(\mathrm{G}))}{\wedge} \mathbb{H}(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})) \simeq \mathbb{H}(R(\mathrm{T})/R(\mathrm{T})\ker(\rho_{\mathrm{G}})). \tag{3.0.14}$$

Recall from the lines following (A.0.4), $\mathbb{H}(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})) = \underset{\Delta}{\mathrm{hocolim}}\{\mathbb{H}(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})_n)| n\}$ is a commutative algebra spectrum over $\mathbb{H}(R(\mathrm{G}))$ and is cofibrant as an object in $Mod(\mathbb{H}(R(\mathrm{G})))$. Now (3.0.13) implies the identifications:

$$M \overset{L}{\underset{R(\mathrm{T})}{\otimes}} R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}})) \cong M \underset{R(\mathrm{T})}{\otimes} R(\mathrm{T}) \underset{R(\mathrm{G})}{\otimes} \mathrm{N}(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})_\bullet) \tag{3.0.15}$$

$$res_*(M) \overset{L}{\underset{R(\mathrm{G})}{\otimes}} R(\mathrm{G})/(\ker(\rho_{\mathrm{G}})) \cong res_*(M) \underset{R(\mathrm{G})}{\otimes} N(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})_\bullet).$$

Here $res_*(M)$ denotes M viewed as an $R(\mathrm{G})$-module using restriction of scalars. These, together with (A.0.2), then also provide the identifications:

$$\begin{aligned}&\mathbb{H}(M)\underset{\mathbb{H}(R(\mathrm{T}))}{\overset{L}{\wedge}}\mathbb{H}(R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}})))\\ &\quad\cong \mathbb{H}(M)\underset{\mathbb{H}(R(\mathrm{T}))}{\wedge}\mathbb{H}(R(\mathrm{T}))\underset{\mathbb{H}(R(\mathrm{G}))}{\wedge}\mathbb{H}(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})) \qquad (3.0.16)\\ &\mathbb{H}(res_*(M))\underset{\mathbb{H}(R(\mathrm{G}))}{\overset{L}{\wedge}}\mathbb{H}(R(\mathrm{G})/(\ker(\rho_{\mathrm{G}}))) \cong \mathbb{H}(res_*(M))\underset{\mathbb{H}(R(\mathrm{G}))}{\wedge}\mathbb{H}(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})).\end{aligned}$$

Recall again that $\mathbb{H}(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})) = \underset{\Delta}{\mathrm{hocolim}}\{\mathbb{H}(R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})_n)|n\}$ is a commutative algebra spectrum over $\mathbb{H}(R(\mathrm{G}))$ and is cofibrant as an object in $Mod(\mathbb{H}(R(\mathrm{G})))$. Therefore, one may identify the two right-hand-sides of (3.0.16) and therefore the corresponding left-hand-sides.

Next one repeats the same argument with M replaced by the terms in each degree of $M\underset{R(\mathrm{T})}{\otimes}R(\mathrm{T})\underset{R(\mathrm{G})}{\otimes}R(\widetilde{\mathrm{G})/\ker}(\rho_{\mathrm{G}})_\bullet$ so that one obtains an identification of the cosimplicial objects of spectra:

$$\begin{aligned}&\mathcal{T}^\bullet_{\mathbb{H}(R(\mathrm{G}))}(\mathbb{H}(res_*(M)),\mathbb{H}(R(\mathrm{G})/(\ker(\rho_{\mathrm{G}}))))\\ &\quad\cong \mathcal{T}^\bullet_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(M),\mathbb{H}(R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}})))) \qquad (3.0.17)\end{aligned}$$

Observe that the proof will be complete, if we show that there is a weak-equivalence of the homotopy inverse limit of the term on the right with the homotopy inverse limit of the cosimplicial object
$\mathcal{T}^\bullet_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(M),\mathbb{H}(R(\mathrm{T})/(\ker(\rho_{\mathrm{T}})))).$

Therefore we will let I denote either one of the ideals $\ker(\rho_{\mathrm{T}})$ or $R(\mathrm{T}).\ker(\rho_{\mathrm{G}})$. Now Proposition 3.8 below provides the weak-equivalence:

$$\mathrm{holim}\,\mathcal{T}^\bullet_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(M),\mathbb{H}(R(\mathrm{T})/I)) \simeq \underset{\infty\leftarrow \mathrm{k}}{\mathrm{holim}}\{\mathbb{H}(\mathrm{M})\underset{\mathbb{H}(R(\mathrm{T}))}{\overset{\mathrm{L}}{\wedge}}(\mathbb{H}(R(\mathrm{T})/I^{\mathrm{k}}))|\mathrm{k}\} \qquad (3.0.18)$$

where the first homotopy inverse limit is of the cosimplicial object there and the second homotopy inverse limit is that of the inverse system. (Again we make use of the simplicial resolution by the free commutative algebra functor over $R(\mathrm{T})$ as discussed in A.0.3 to define $\mathcal{T}^\bullet_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(M),\mathbb{H}(R(\mathrm{T})/I))$.)

Next one observes that $R(\mathrm{T})$ is a finite $R(\mathrm{G})$-module and hence that there exists some positive integer n_0 so that $\ker(\rho_{\mathrm{T}})^{n_0}\subseteq R(\mathrm{T}).\ker(\rho_{\mathrm{G}})\subseteq\ker(\rho_{\mathrm{T}})$. Therefore, we proceed to show that the inverse systems

$$\{\mathbb{H}(M)\underset{\mathbb{H}(R(\mathrm{T}))}{\overset{L}{\wedge}}\mathbb{H}(R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}}))^{\mathrm{k}})|\mathrm{k}\geq 1\} \text{ and}$$

$$\{\mathbb{H}(M)\underset{\mathbb{H}(R(\mathrm{T}))}{\overset{L}{\wedge}}\mathbb{H}(R(\mathrm{T})/(\ker(\rho_{\mathrm{T}}))^{\mathrm{k}})|\mathrm{k}\geq 1\}$$

define weakly-equivalent homotopy inverse limits. By taking a simplicial resolution $\widetilde{M}^{\bullet} \to M$ of M by free $R(\mathrm{T})$-modules, one may replace $\mathbb{H}(M)$ by $\underset{\Delta}{\mathrm{hocolim}}\mathbb{H}(\widetilde{M}^{\bullet})$. Let $\widetilde{\mathbb{H}(M)} = \underset{\Delta}{\mathrm{hocolim}}\mathbb{H}(\widetilde{M}^{\bullet})$. Now it suffices to show that,

$$\{\widetilde{\mathbb{H}(M)} \underset{\mathbb{H}(R(\mathrm{T}))}{\wedge} \mathbb{H}(R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}}))^{\mathrm{k}})|\mathrm{k} \geq 1\} \text{ and}$$
$$\{\widetilde{\mathbb{H}(\mathrm{M})} \underset{\mathbb{H}(R(\mathrm{T}))}{\wedge} \mathbb{H}(R(\mathrm{T})/(\ker(\rho_{\mathrm{T}}))^{\mathrm{k}})|\mathrm{k} \geq 1\} \tag{3.0.19}$$

are isomorphic as pro-objects of $\mathbb{H}(R(\mathrm{T}))$-module spectra. Recall that if $P = \{P_\alpha|\alpha\}$ and $Q = \{Q_\beta|\beta\}$ are pro-objects of $\mathbb{H}(R(\mathrm{T}))$-modules spectra, then,

$$Hom(P, Q) = \underset{\beta}{\lim}\underset{\alpha}{\mathrm{colim}} Hom_{\mathbb{H}(R(\mathrm{T}))}(P_\alpha, Q_\beta).$$

Therefore, it suffices to show that for any $\mathbb{H}(R(\mathrm{T}))$-module spectrum K, one obtains an isomorphism:

$$\lim_{n\to\infty} Hom_{\mathbb{H}(R(\mathrm{T}))}(\widetilde{\mathbb{H}(M)} \underset{\mathbb{H}(R(\mathrm{T}))}{\wedge} \mathbb{H}(R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}}))^{\mathrm{n}}), \mathrm{K})$$
$$\cong \lim_{n\to\infty} Hom_{\mathbb{H}(R(\mathrm{T}))}(\widetilde{\mathbb{H}(M)} \underset{\mathbb{H}(R(\mathrm{T}))}{\wedge} \mathbb{H}(R(\mathrm{T})/(\ker(\rho_{\mathrm{T}}))^{\mathrm{n}}), \mathrm{K})$$

But the left-hand-side identifies with $\lim_{n\to\infty} Hom_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}}))^{\mathrm{n}}), \mathcal{H}\mathrm{om}_{\mathbb{H}(R(\mathrm{T}))}(\widetilde{\mathbb{H}(\mathrm{M})}, \mathrm{K}))$ and the right-hand-side identifies with $\lim_{n\to\infty} Hom_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(R(\mathrm{T})/(\ker(\rho_{\mathrm{T}}))^{\mathrm{n}}), \mathcal{H}\mathrm{om}_{\mathbb{H}(R(\mathrm{T}))}(\widetilde{\mathbb{H}(\mathrm{M})}, \mathrm{K}))$. Here $\mathcal{H}om_{\mathbb{H}(R(\mathrm{T}))}$ denotes the internal hom in the category of $\mathbb{H}(R(\mathrm{T}))$-spectra. Observe also that since $\mathbb{H}(R(\mathrm{T}))$ is a commutative ring spectrum, the internal Hom in the category of modules over $\mathbb{H}(R(\mathrm{T}))$ belongs to the same category. Since the pro-objects $\{\mathbb{H}(R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}}))^{\mathrm{n}})|\mathrm{n}\}$ and $\{\mathbb{H}(R(\mathrm{T})/\ker(\rho_{\mathrm{T}})^{\mathrm{n}})|\mathrm{n}\}$ are isomorphic, it follows that one obtains an isomorphism of the pro-objects in (3.0.19).

In view of (3.0.18), we have shown that the two cosimplicial objects

$$\mathcal{T}^{\bullet}_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(M), \mathbb{H}(R(\mathrm{T})/(R(\mathrm{T}).\ker(\rho_{\mathrm{G}})))) \text{ and } \mathcal{T}^{\bullet}_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(\mathrm{M}), \mathbb{H}(R(\mathrm{T})/(\ker(\rho_{\mathrm{T}}))))$$

define weakly-equivalent objects on taking the homotopy inverse limits. In view of the identification in (3.0.17), it therefore also follows that the two cosimplicial objects

$$\mathcal{T}^{\bullet}_{\mathbb{H}(R(\mathrm{G}))}(\mathbb{H}(res_*(M)), \mathbb{H}(R(\mathrm{G})/(\ker(\rho_{\mathrm{G}})))) \text{ and } \mathcal{T}^{\bullet}_{\mathbb{H}(R(\mathrm{T}))}(\mathbb{H}(\mathrm{M}), \mathbb{H}(R(\mathrm{T})/(\ker(\rho_{\mathrm{T}}))))$$

define weakly-equivalent homotopy inverse limits, that is, $\mathbb{H}(res_*(M))\widehat{\ }_{\mathbb{H}(\rho_{\mathrm{G}})} \simeq \mathbb{H}(M)\widehat{\ }_{\mathbb{H}(\rho_{\mathrm{T}})}$. The proof of the corresponding statement for the derived completions with respect to $\rho_\ell \circ \rho_{\mathrm{G}}$ and $\rho_\ell \circ \rho_{\mathrm{T}}$ is similar. $\square$

Remark 3.7. 1. For the applications in the rest of the paper, we will restrict to the case where the group $\mathrm{G} = \mathrm{GL_n}$ for some n or a finite product $\Pi_{i=1}^{m} \mathrm{GL}_{n_i}$, since the hypotheses of the Theorem are satisfied in this case.

2. It may be worth pointing out that for the usual completions, the I_{T}-adic and the $I_{\mathrm{G}}\mathrm{R}(\mathrm{T})$-adic completions are the same on any $\mathrm{R}(\mathrm{T})$-module (as proven in [4, p. 8]). Thus Theorem 3.6 shows this extends to derived completions, provided the hypotheses of Theorem 3.6 hold.

Proposition 3.8. *Let R denote a Noetherian ring and let I denote an ideal in R. For each $k \geq 1$, let $S_k = R/I^k$ and let $\tilde{S}_k$ denote a free resolution of S_k as a commutative dg-algebra over R as in A.0.3. Let $\tilde{S} = \tilde{S}_1$.*

Let $T_{\mathbb{H}(R)}(\mathcal{M}, \mathbb{H}(\tilde{S}))$ denote the triple sending an $\mathbb{H}(R)$-module spectrum $\mathcal{M}$ to the $\mathbb{H}(R)$-module spectrum $\mathcal{M} \underset{\mathbb{H}(R)}{\otimes} \mathbb{H}(\tilde{S})$ and let $\mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathcal{M})$ denote the corresponding cosimplicial resolution of the $\mathbb{H}(R)$-module $\mathcal{M}$. If M is any R-module, not necessarily finitely generated, then one obtains the identification (up to weak-equivalence):

$$holim\, \mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(M), \mathbb{H}(\tilde{S})) \simeq \underset{\infty \leftarrow k}{holim} \{\mathbb{H}(M) \underset{\mathbb{H}(R)}{\wedge} \mathbb{H}(\tilde{S}_k)|k\} \simeq \underset{\infty \leftarrow k}{holim} \{\mathbb{H}(M \underset{R}{\otimes} \tilde{S}_k)|k\} \simeq \underset{\infty \leftarrow k}{holim} \{\mathbb{H}(M \overset{L}{\underset{R}{\otimes}} R/I^k)|k\}.$$

(Again we make use of the simplicial resolution by the free commutative algebra functor over R as discussed in A.0.3 to define $\mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(M), \mathbb{H}(\tilde{S}))$.)

Proof. See [7, Corollary 5.5] for a somewhat similar result. First, the definition of the cosimplicial object $\mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(M), \mathbb{H}(\tilde{S}))$ provides the identification:

$$\mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(M), \mathbb{H}(\tilde{S})) \cong \mathbb{H}(M) \overset{L}{\underset{\mathbb{H}(R)}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(R), \mathbb{H}(R/I)).$$

At this point, observe that one obtains a distinguished triangle of pro-objects of cosimplicial objects (that is, a diagram of pro-cosimplicial spectra, which is a stable cofiber sequence for each k and each cosimplicial degree):

$$\begin{aligned}\{\mathbb{H}(M) \overset{L}{\underset{\mathbb{H}(R)}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(I^k), H(R/I))|k\} &\to \{\mathbb{H}(M) \overset{L}{\underset{\mathbb{H}(R)}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(R), \mathbb{H}(R/I))|k\} \\ &\to \{\mathbb{H}(M) \overset{L}{\underset{\mathbb{H}(R)}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(R/I^k), \mathbb{H}(R/I))|k\}\end{aligned} \tag{3.0.20}$$

The middle term $\{\mathbb{H}(M) \overset{L}{\underset{\mathbb{H}(R)}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(R), \mathbb{H}(R/I))|k\}$ is the constant pro-object consisting of the same object $\mathbb{H}(M) \overset{L}{\underset{\mathbb{H}(R)}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(R)}(\mathbb{H}(R), \mathbb{H}(R/I))$ for all k. For each fixed k, and each fixed cosimplicial degree m, the homotopy groups of $\mathcal{T}^{m}_{\mathbb{H}(R)}(\mathbb{H}(I^k), \mathbb{H}(R/I))$ are computed by a strongly convergent spectral sequence whose E_2-terms are the cohomology of the iterated derived tensor product $I^k \overset{L}{\underset{R}{\otimes}} R/I \cdots \overset{L}{\underset{R}{\otimes}} R/I$.

For a finitely generated module K over R, we will denote the cohomology groups of the iterated derived tensor product $K\overset{L}{\underset{R}{\otimes}}R/I\cdots\overset{L}{\underset{R}{\otimes}}R/I$ as $MultiTor^m_{R,*}(K, R/I, \cdots, R/I)$. One considers the functor $K \mapsto \{I^kK|k\} = \underline{I}K$ sending a finitely generated module K over the Noetherian ring R to the pro-object $\{I^kK|k\}$ in the category of R-modules. As observed in the proof of [7, Theorem 4.4], the pro-object

$\{MultiTor^m_{R,*}(I^k, R/I, \cdots, R/I)|k\}$

identifies with the pro-object

$\underline{I}MultiTor^m_{R,*}(R, R/I, \cdots, R/I) = \{I^kMultiTor^m_{R,*}(R, R/I, \cdots, R/I)|k\}$.

Since the cohomology groups of the iterated derived tensor product $R\overset{L}{\underset{R}{\otimes}}R/I\cdots\overset{L}{\underset{R}{\otimes}}R/I$ are R/I-modules, it follows that I^k acts trivially on the above cohomology groups. These observations prove that the pro-object $\{\mathcal{T}^\bullet_R(I^k, R/I)|k\}$, and therefore the pro-objects

$$\{M\overset{L}{\underset{R}{\otimes}}\mathcal{T}^\bullet_R(I^k, R/I)|k\} \text{ and } \{\pi_n(H(M)\overset{L}{\underset{\mathbb{H}(R)}{\wedge}}\mathcal{T}^\bullet_{\mathbb{H}(R)}(\mathbb{H}(I^k), \mathbb{H}(R/I)))|k\}$$

for any fixed integer n and each fixed cosimplicial degree are trivial. This shows that the last map in (3.0.20) induces a weak-equivalence on taking the homotopy inverse limit of the pro-objects (for each fixed cosimplicial degree).

Next we fix an integer k and consider the homotopy inverse limit of the cosimplicial object

$\mathbb{H}(M)\overset{L}{\underset{\mathbb{H}(R)}{\wedge}}\mathcal{T}^\bullet_{\mathbb{H}(R)}(\mathbb{H}(R/I^k), \mathbb{H}(R/I))$. The discussion in [7] immediately following Proposition 4.3, as well as Lemma 3.9 below now show that for any fixed k, the homotopy inverse limit of the last cosimplicial object identifies with $\mathbb{H}(M)\overset{L}{\underset{\mathbb{H}(R)}{\wedge}}(\mathbb{H}(R/I^k))$. Finally we take the homotopy inverse limits over $k \to \infty$ followed by the homotopy inverse limits of the resulting cosimplicial objects for each of the three terms in (3.0.20). The above discussion shows that this induces a weak-equivalence of the resulting last two terms. However, one may clearly interchange the two homotopy inverse limits. Since the middle term in (3.0.20) is a constant pro-object, the above observations now provide the weak-equivalences:

$$\begin{aligned}
\operatorname{holim}\mathcal{T}^\bullet_{\mathbb{H}(R)}(\mathbb{H}(M), \mathbb{H}(R/I)) &\simeq \operatorname{holim}(\mathbb{H}(M)\overset{L}{\underset{\mathbb{H}(R)}{\wedge}}\mathcal{T}^\bullet_{\mathbb{H}(R)}(\mathbb{H}(R), \mathbb{H}(R/I)))\\
&\simeq \underset{\infty\leftarrow k}{\operatorname{holim}}\{\mathbb{H}(M)\overset{L}{\underset{\mathbb{H}(R)}{\wedge}}(\mathbb{H}(R/I^k))|k\}\\
&= \underset{\infty\leftarrow k}{\operatorname{holim}}\{\mathbb{H}(M)\underset{\mathbb{H}(R)}{\wedge}\mathbb{H}(\tilde{S}_k)|k\} \simeq \underset{\infty\leftarrow k}{\operatorname{holim}}\{\mathbb{H}(M\underset{R}{\otimes}\tilde{S}_k)|k\}
\end{aligned} \tag{3.0.21}$$

where the first two homotopy inverse limits are of the cosimplicial objects there and the remaining homotopy inverse limits are that of the inverse system. □

Lemma 3.9. *Let $\rho : A \to B$ denote a surjective map of commutative Noetherian rings with $I = \ker(\rho)$. For any chain complexes of A-modules M and N that are trivial in negative degrees, the augmentation*

$$\mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} \mathbb{H}(M/I^n) \to \underset{\Delta}{holim}\, \mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(A)}(\mathbb{H}(M/I^n), \mathbb{H}(B))$$

is a weak-equivalence for any fixed positive integer n. (Again we make use of the simplicial resolution by the free commutative algebra functor over A as discussed in A.0.3 to define $\mathcal{T}^{\bullet}_{\mathbb{H}(A)}(\mathbb{H}(M/I^n), \mathbb{H}(B))$.)

Proof. The proof will be by ascending induction on n. The assertion for $n = 1$, is essentially [7, Proposition 3.2, 6]. In this case, it suffices to observe that the cosimplicial object $\mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(A)}(\mathbb{H}(M/I^n), \mathbb{H}(B))$ has an extra co-degeneracy which provides a contracting homotopy. Next we consider the case of a general n. By taking a free resolution, we may assume M and N consist of free A-modules in each degree. In view of the above flatness assumptions on M, one has the short-exact sequence (of complexes):

$0 \to M \underset{A}{\otimes} I^{k-1}/I^k \to M/I^k \to M/I^{k-1} \to 0.$

This provides a commutative diagram of cosimplicial objects:

$$\begin{tikzcd}
\mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} (\mathbb{H}(M \underset{A}{\otimes} I^{k-1}/I^k)) \arrow[r] \arrow[d] & \mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} (\mathbb{H}(M/I^k)) \arrow[r] \arrow[d] & \mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} (\mathbb{H}(M/I^{k-1})) \arrow[d] \\
\mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(A)}(\mathbb{H}(M \underset{A}{\otimes} I^{k-1}/I^k), \mathbb{H}(B)) \arrow[r] & \mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(A)}(\mathbb{H}(M/I^k), \mathbb{H}(B)) \arrow[r] & \mathbb{H}(N) \underset{\mathbb{H}(A)}{\overset{L}{\wedge}} \mathcal{T}^{\bullet}_{\mathbb{H}(A)}(\mathbb{H}(M/I^{k-1}), \mathbb{H}(B))
\end{tikzcd}$$

The two rows are clearly stable cofiber sequences and remain so on taking the homotopy inverse limit of the cosimplicial objects in the last row. Observe that $M \underset{A}{\otimes} I^{k-1}/I^k$ is an $A/I = B$-module. Therefore the case $n = 1$ applies to show the first vertical map is a weak-equivalence on taking the homotopy inverse limit of the cosimplicial objects. By the inductive assumption, the last vertical map induces a weak-equivalence on taking the homotopy inverse limit. Therefore, the middle map also induces a weak-equivalence on taking the homotopy inverse limit. □

4. Reduction to the case of a torus in G and an outline of the proof of the main theorem

Recall the following hypotheses of Theorem 1.2: let X denote a scheme of finite type over the base scheme S = Spec k, satisfying the Standing hypotheses 1.1 and provided with an action by the not-necessarily connected linear algebraic group H. G denotes a connected split reductive group containing H as a closed subgroup scheme. We begin with the following Proposition which shows we may reduce to just considering actions by the ambient group G.

Proposition 4.1. *If the derived completion with respect to the map ρ_{G} induces a weak-equivalence:*

$$\mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G})\widehat{\ }_{\rho_{\mathrm{G}}} \overset{\simeq}{\to} \mathbf{G}(\mathrm{EG}^{\mathrm{gm}}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}))$$

then one obtains a weak-equivalence:

$$\mathbf{G}(\mathrm{X}, \mathrm{H})\widehat{\ }_{\rho_{\mathrm{H}}} \overset{\simeq}{\to} \mathbf{G}(\mathrm{EG}^{\mathrm{gm}}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X})) \simeq \mathbf{G}(\mathrm{EH}^{\mathrm{gm}}\underset{\mathrm{H}}{\times}\mathrm{X}).$$

Proof. We first make use of the weak-equivalence $\mathbf{G}(\mathrm{X}, \mathrm{H}) \simeq \mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G})$ (see Lemma 2.7), and Proposition 1.1 to obtain the weak-equivalence: $\mathbf{G}(\mathrm{X}, \mathrm{H})\widehat{\ }_{\rho_{\mathrm{H}}} \simeq \mathbf{G}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}, \mathrm{G})\widehat{\ }_{\rho_{\mathrm{G}}}$. The weak-equivalence $\mathbf{G}(\mathrm{EH}^{\mathrm{gm}}\underset{\mathrm{H}}{\times}\mathrm{X}) \simeq \mathbf{G}(\mathrm{EG}^{\mathrm{gm}}\underset{\mathrm{H}}{\times}(\mathrm{G}\underset{\mathrm{H}}{\times}\mathrm{X}))$ follows from Appendix B. □

Therefore, we may assume without loss of generality, in the rest of this section that $\mathrm{G} = \mathrm{GL_n}$ for some $n \geq 1$ or a finite product of such GL_ns.

Definition 4.2. Let the base scheme $\mathrm{S} = \mathrm{Spec}\, k$. Then we make the following definitions.

(i) Let B denote a smooth scheme of finite type over S and let A denote a locally closed smooth subscheme of B. Then we let $\mathbf{K}(\mathrm{B}, \mathrm{A})$ = the canonical homotopy fiber of the restriction $\mathbf{K}(\mathrm{B}) \to \mathbf{K}(\mathrm{A})$. In case B has a k-rational point p, we will let $\mathbf{K}(\mathrm{B}, \mathrm{p})$ be denoted by $\tilde{\mathbf{K}}(\mathrm{B})$.

 (Observe that $\tilde{\mathbf{K}}(\mathrm{B})$ identifies up to weak-equivalence with the homotopy cofiber of the map $\pi_{\mathrm{B}}^* : \mathbf{K}(p) \to \mathbf{K}(\mathrm{B})$, where $\pi_{\mathrm{B}} : \mathrm{B} \to \mathrm{Spec}\, k = \mathrm{p}$ is the structure map of X. Therefore, $\tilde{\mathbf{K}}(\mathrm{B})$ is independent of the choice of the k-rational point p in B.)

(ii) Let E denote a smooth scheme of finite type over S provided with an action by the linear algebraic group G and let C denote a locally closed G-stable smooth subscheme of E. Then we let $\mathbf{K}(\mathrm{E}, \mathrm{C}, \mathrm{G})$ = the canonical homotopy fiber of the restriction $\mathbf{K}(\mathrm{E}, \mathrm{G}) \to \mathbf{K}(\mathrm{C}, \mathrm{G})$.

(iii) Next assume that G acts freely on the smooth scheme E and B is the corresponding geometric quotient, with $\pi : \mathrm{E} \to \mathrm{B}$ denoting the corresponding quotient map. Let p denote a k-rational point of B. Then we will denote $\mathbf{K}(\mathrm{E}, \pi^{-1}(\mathrm{p}), \mathrm{G})$ by $\tilde{\mathbf{K}}(\mathrm{E}, \mathrm{G})$.

Now we obtain the following result whose proof is straightforward and is therefore skipped.

Lemma 4.3. *Let* X *and* Y *be smooth schemes provided with actions by the linear algebraic group* G*. Let* A *(*B*) denote a locally closed smooth* G*-stable subscheme of* X *(*Y*, respectively). Then one obtains a pairing:*

$$\mathbf{K}(\mathrm{X}, \mathrm{A}, \mathrm{G}) \underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\overset{\mathrm{L}}{\wedge}} \mathbf{K}(\mathrm{Y}, \mathrm{B}, \mathrm{G}) \to \mathbf{K}(\mathrm{X} \times \mathrm{Y}, \mathrm{A} \times \mathrm{Y} \cup \mathrm{X} \times \mathrm{B}, \mathrm{G})$$

that is contravariantly functorial in all the arguments.

Proposition 4.4. *Assume that the linear algebraic group* G *acts freely on the smooth scheme* E *and* B *is the corresponding geometric quotient, with* $\pi : \mathrm{E} \to \mathrm{B}$ *denoting the corresponding quotient map. Assume that* B *has a Zariski open cover by* $\{\mathrm{V_i}|\mathrm{i} = 1, \cdots, \mathrm{n}\}$ *so that (i) there exists a positive integer* m*, so that each* $\mathrm{V_i} \cong \mathbb{A}^{\mathrm{m}}$ *and (ii) there exists a* k*-rational point* $\mathrm{p} \,\epsilon\, \cap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{V_i}$*.*

Then the composite map $\overbrace{\tilde{\mathbf{K}}(\mathrm{E},\mathrm{G}) \wedge^{\mathrm{L}} \tilde{\mathbf{K}}(\mathrm{E},\mathrm{G}) \wedge^{\mathrm{L}} \cdots \wedge^{\mathrm{L}} \tilde{\mathbf{K}}(\mathrm{E},\mathrm{G})}^{n} \to \tilde{\mathbf{K}}(\mathrm{E}^{\times^{\mathrm{n}}},\mathrm{G}) \overset{\Delta^*}{\to} \tilde{\mathbf{K}}(\mathrm{E},\mathrm{G})$ *is null-homotopic.*

Proof. Observe first that $p \simeq \mathrm{V_i}$, for each $i = 1, \cdots, n$, in the $\mathbb{A}^1$-homotopy category. Next, the cofiber-sequence $(\mathrm{V_i}, \mathrm{p}) \to (\mathrm{B}, \mathrm{p}) \to (\mathrm{B}, \mathrm{V_i})$ shows that one obtains the stable fiber sequence: $\mathbf{K}(\mathrm{B}, \mathrm{V_i}) \to \mathbf{K}(\mathrm{B}, \mathrm{p}) \to \mathbf{K}(\mathrm{V_i}, \mathrm{p})$. Since $\mathbf{K}(\mathrm{V_i}, \mathrm{p})$ is weakly-contractible, it follows that the map $\mathbf{K}(\mathrm{B}, \mathrm{V_i}) \to \mathbf{K}(\mathrm{B}, \mathrm{p})$ is a weak-equivalence for each i.

Next observe the weak-equivalences (since the action of G is assumed to be free on E):

$$\begin{aligned}\mathbf{K}(\mathrm{B}) \simeq \mathbf{K}(\mathrm{E},\mathrm{G}),\ \mathbf{K}(\mathrm{B},\mathrm{V_i}) \simeq \mathbf{K}(\mathrm{E},\pi^{-1}(\mathrm{V_i}),\mathrm{G}), \mathbf{K}(\mathrm{B},\mathrm{p}) \simeq \mathbf{K}(\mathrm{E},\pi^{-1}(\mathrm{p}),\mathrm{G}) \text{ and}\\ \mathbf{K}(\mathrm{V_i},\mathrm{p}) \simeq \mathbf{K}(\pi^{-1}(\mathrm{V_i}),\pi^{-1}(\mathrm{p}),\mathrm{G}).\end{aligned} \tag{4.0.1}$$

In particular, it follows that all terms on the right sides above are module spectra over $\mathbf{K}(\mathrm{S},\mathrm{G})$. Next recall that the cardinality of the open cover $\{\mathrm{V_i}|\mathrm{i}\}$ is n.

Let $\mathcal{V} = \mathrm{V_1} \times \mathrm{B^{n-1}} \bigcup \mathrm{B} \times \mathrm{V_2} \times \mathrm{B^{n-2}} \bigcup \mathrm{B} \times \mathrm{B} \times \mathrm{V_3} \times \mathrm{B^{n-3}} \bigcup \cdots \bigcup \mathrm{B^{n-1}} \times \mathrm{V_n}$. We let $\tilde{\mathrm{V}}_\mathrm{i} = \pi^{-1}(\mathrm{V_i})$, $\tilde{\mathcal{V}} = \tilde{\mathrm{V}}_1 \times \mathrm{E}^{n-1} \bigcup \mathrm{E} \times \tilde{\mathrm{V}}_2 \times \mathrm{E}^{\mathrm{n}-2} \bigcup \mathrm{E} \times \mathrm{E} \times \tilde{\mathrm{V}}_3 \times \mathrm{E}^{\mathrm{n}-3} \bigcup \cdots \bigcup \mathrm{E}^{\mathrm{n}-1} \times \tilde{\mathrm{V}}_\mathrm{n}$.

Observe that $\tilde{\mathbf{K}}(\mathrm{E},\mathrm{G})$= the homotopy fiber of the map $\mathbf{K}(\mathrm{E},\mathrm{G}) \to \mathbf{K}(\pi^{-1}(\mathrm{p}),\mathrm{G})$, and $\mathbf{K}(\mathrm{E},\tilde{\mathrm{V}}_\mathrm{i},\mathrm{G})$ = the homotopy fiber of the map $\tilde{\mathbf{K}}(\mathrm{E},\mathrm{G}) = \mathbf{K}(\mathrm{E},\pi^{-1}(\mathrm{p}),\mathrm{G}) \to \mathbf{K}(\pi^{-1}(\mathrm{V_i}),\pi^{-1}(\mathrm{p}),\mathrm{G})$. (Since the last term is weakly contractible, it follows that the map $\mathbf{K}(\mathrm{E},\tilde{\mathrm{V}}_\mathrm{i},\mathrm{G}) \to \tilde{\mathbf{K}}(\mathrm{E},\mathrm{G})$ is a weak-equivalence.) $\tilde{\mathbf{K}}(\mathrm{E}^{\times^n},\mathrm{G})$ is defined as $\mathbf{K}(\mathrm{E}^{\times \mathrm{n}},\pi^{-1}(\mathrm{p})^{\times \mathrm{n}},\mathrm{G})$. Now the following diagram

$$\begin{tikzcd}
\mathbf{K}(\mathrm{E},\tilde{\mathrm{V}}_1,\mathrm{G}) \wedge^{\mathrm{L}} \mathbf{K}(\mathrm{E},\tilde{\mathrm{V}}_2,\mathrm{G}) \wedge^{\mathrm{L}} \cdots \wedge^{\mathrm{L}} \mathbf{K}(\mathrm{E},\tilde{\mathrm{V}}_\mathrm{n},\mathrm{G}) \arrow[r, "\simeq"] \arrow[d, "\times"] & \tilde{\mathbf{K}}(\mathrm{E},\mathrm{G}) \wedge^{\mathrm{L}} \tilde{\mathbf{K}}(\mathrm{E},\mathrm{G}) \wedge^{\mathrm{L}} \cdots \tilde{\mathbf{K}}(\mathrm{E},\mathrm{G}) \arrow[d] \\
\mathbf{K}(\mathrm{E}^{\times \mathrm{n}},\tilde{\mathcal{V}},\mathrm{G}) \arrow[r] \arrow[d, "\Delta^*"] & \tilde{\mathbf{K}}(\mathrm{E}^{\times^{\mathrm{n}}},\mathrm{G}) \arrow[d, "\Delta^*"] \\
\mathbf{K}(\mathrm{E},\mathrm{E},\mathrm{G}) \arrow[r] & \tilde{\mathbf{K}}(\mathrm{E},\mathrm{G})
\end{tikzcd} \tag{4.0.2}$$

commutes up to homotopy. The pairings forming the top vertical maps in the left and right column are provided by Lemma 4.3. The contravariant functoriality of the pairings there show that the top square homotopy commutes. Since all maps are maps of module spectra over $\mathbf{K}(\mathrm{S},\mathrm{G})$ (as observed in (4.0.1)), the $\wedge^{\mathrm{L}}$ above is over $\mathbf{K}(\mathrm{S},\mathrm{G})$. Since the top horizontal map is a weak-equivalence, and since $\mathbf{K}(\mathrm{E},\mathrm{E},\mathrm{G})$ is weakly contractible, it follows that the composition of the maps in the right column is null-homotopic. This completes the proof of the Proposition. □

Proposition 4.5. *Let* G *denote* GL_n *for some* n *or a split torus. Let* X *denote a scheme or algebraic space of finite type over* $\mathrm{S} = \mathrm{Spec}\, k$*. Then, with the choice of the classifying spaces as in* (2.1.2) *and* (2.1.3)*, the following hold.*

(i) If I_G *denotes the homotopy fiber of the restriction map* $\mathbf{K}(\mathrm{S},\mathrm{G}) \to \mathbf{K}(\mathrm{S})$*, then the (obvious) map*

$$\mathbf{G}(\mathrm{X},\mathrm{G}) \to \mathbf{G}(\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X},\mathrm{G}) \simeq \mathbf{G}(\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \mathrm{X})$$

factors through $\mathbf{G}(\mathrm{X},\mathrm{G})/\mathrm{I}_{\mathrm{G}}^{\mathrm{Lm}'+1}\mathbf{G}(\mathrm{X},\mathrm{G})$ *for some positive integer* $m' \geq m$*. Here*

$$I_{\mathrm{G}}^{Lm'+1}\mathbf{G}(\mathrm{X},\mathrm{G}) = \overbrace{I_{\mathrm{G}} \overset{L}{\underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge}} I_{\mathrm{G}} \cdots \overset{L}{\underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge}} I_{\mathrm{G}}}^{\mathrm{m}'+1} \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge}} \mathbf{G}(\mathrm{X},\mathrm{G}),$$

and $\mathbf{G}(\mathrm{X},\mathrm{G})/\mathrm{I}_{\mathrm{G}}^{\mathrm{Lm}'+1}\mathbf{G}(\mathrm{X},\mathrm{G})$ *is the homotopy cofiber of the map* $I_{\mathrm{G}}^{Lm'+1}\mathbf{G}(\mathrm{X},\mathrm{G}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge}} \mathbf{G}(\mathrm{X},\mathrm{G}) \to \mathbf{G}(\mathrm{X},\mathrm{G})$.

(ii) It follows that the map $\mathbf{G}(\mathrm{X},\mathrm{G}) \to \mathbf{G}(\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X},\mathrm{G})$ *factors through the partial derived completion* $\mathbf{G}(\mathrm{X},\mathrm{G})\hat{}_{\rho_{\mathrm{G}},\mathrm{m}'}$ *for some positive integer* $m' \geq m+1$*. If* $\pi : \mathrm{X} \to \mathrm{Y}$ *is a* G*-equivariant map that is also flat (proper), then the above factorization is compatible with the induced map* $\pi^* : \mathbf{G}(\mathrm{Y},\mathrm{G}) \to \mathbf{G}(\mathrm{X},\mathrm{G})$ *(the induced map* $\pi_* : \mathbf{G}(\mathrm{X},\mathrm{G}) \to \mathbf{G}(\mathrm{Y},\mathrm{G})$*, respectively).*

(iii) More generally, if $\mathrm{G_i}$*,* $i = 1, \cdots, q$ *are either general linear groups or split tori, the map*

$$\mathbf{G}(\mathrm{X},\mathrm{G}_1 \times \cdots \times \mathrm{G_q}) \to \mathbf{G}(\mathrm{EG}_1^{\mathrm{gm,m}} \times \cdots \times \mathrm{EG}_{\mathrm{q}}^{\mathrm{gm,m}} \times \mathrm{X},\mathrm{G}_1 \times \cdots \times \mathrm{G_q})$$

factors through

$$\mathbf{G}(\mathrm{X},\mathrm{G}_1 \times \cdots \times \mathrm{G_q})\hat{}_{\rho_{\mathrm{G}_1 \times \cdots \mathrm{G_q}},\mathrm{m}'}$$

for some positive integer $m' \geq m+1$*. Let* $\pi : \mathrm{X} \to \mathrm{Y}$ *denote an equivariant map with respect to the action of* $\mathrm{G} = \mathrm{G}_1 \times \cdots \times \mathrm{G_q}$*. If* $\pi : \mathrm{X} \to \mathrm{Y}$ *is also a flat map (also a proper map), then the above factorization is compatible with the induced map* $\pi^* : \mathbf{G}(\mathrm{Y},\mathrm{G}_1 \times \cdots \times \mathrm{G_q}) \to \mathbf{G}(\mathrm{X},\mathrm{G}_1 \times \cdots \times \mathrm{G_q})$ *(the induced map* $\pi_* : \mathbf{G}(\mathrm{X},\mathrm{G}_1 \times \cdots \times \mathrm{G_q}) \to \mathbf{G}(\mathrm{Y},\mathrm{G}_1 \times \cdots \times \mathrm{G_q})$*, respectively).*

Proof. This parallels the original proof in [4] for the usual completion for equivariant topological K-theory. We will invoke Proposition 4.4 with $\mathrm{B} = \mathrm{BG}^{\mathrm{gm,m}}$, $\mathrm{E} = \mathrm{EG}^{\mathrm{gm,m}}$ for some fixed m and let $\{\mathrm{V_i} | \mathrm{i} = 1, \cdots, \mathrm{n}\}$ denote a finite Zariski open cover of B with chosen k-rational point $p \,\epsilon\, \cap_i \mathrm{V_i}$, so that $p \simeq \mathrm{V_i}$ in the $\mathbb{A}^1$-homotopy category. Recall Proposition 2.2 that shows the existence of such Zariski open coverings of $\mathrm{B} = \mathrm{BG}^{\mathrm{gm,m}}$. Then Proposition 4.4 shows that the composite map

$$\begin{aligned}&\overbrace{\tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S,G})}{\wedge}} \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}) \cdots \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S,G})}{\wedge}} \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G})}^{n} \\ &\to \tilde{\mathbf{K}}((\mathrm{EG}^{\mathrm{gm,m}})^{\times \mathrm{n}}, \mathrm{G}) \overset{\Delta^*}{\to} \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}) \end{aligned} \tag{4.0.3}$$

is null-homotopic.

Next consider the homotopy commutative diagram

$$\begin{tikzcd} I_{\mathrm{G}} \arrow[r] \arrow[d] & \mathbf{K}(\mathrm{S}, \mathrm{G}) \arrow[r] \arrow[d] & \mathbf{K}(\mathrm{S}) \arrow[d, "id"] \\ \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}) \arrow[r] & \mathbf{K}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}) \arrow[r] & \mathbf{K}(\mathrm{G}, \mathrm{G}) \simeq \mathbf{K}(\mathrm{S}). \end{tikzcd} \tag{4.0.4}$$

Taking $\mathrm{B} = \mathrm{BG}^{\mathrm{gm,m}}$, $\mathrm{E} = \mathrm{EG}^{\mathrm{gm,m}}$ in (4.0.2), we see that the map

$$I_{\mathrm{G}}^{L\wedge^n} = \overbrace{I_{\mathrm{G}} \overset{L}{\underset{\mathbf{K}(\mathrm{S,G})}{\wedge}} I_{\mathrm{G}} \cdots \overset{L}{\underset{\mathbf{K}(\mathrm{S,G})}{\wedge}} I_{\mathrm{G}}}^{n} \to \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}) \simeq \tilde{\mathbf{K}}(\mathrm{BG}^{\mathrm{gm,m}})$$

is null-homotopic, since it factorizes as

$$\begin{aligned} I_{\mathrm{G}}^{L\wedge^n} &\to \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G})^{\mathrm{L}\wedge^{\mathrm{n}}} \\ &= \overbrace{\tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S,G})}{\wedge}} \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}) \cdots \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S,G})}{\wedge}} \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G})}^{\mathrm{n}} \\ &\to \tilde{\mathbf{K}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G}). \end{aligned}$$

Therefore, the middle vertical map in (4.0.4), namely $\mathbf{K}(\mathrm{S}, \mathrm{G}) \to \mathbf{K}(\mathrm{EG}^{\mathrm{gm,m}}, \mathrm{G})$, factors through $\mathbf{K}(\mathrm{S}, \mathrm{G})/\mathrm{I}_{\mathrm{G}}^{\mathrm{L}\wedge^{\mathrm{n}}}$ which is the homotopy cofiber of the map $I_{\mathrm{G}}^{L\wedge^n} \to \mathbf{K}(\mathrm{S}, \mathrm{G})$. Finally one makes use of the pairing: $\mathbf{K}(\mathrm{S}, \mathrm{G}) \overset{\mathrm{L}}{\wedge} \mathbf{G}(\mathrm{X}, \mathrm{G}) \to \mathbf{G}(\mathrm{X}, \mathrm{G})$ and the homotopy commutative square (where the vertical maps are the obvious ones)

$$\begin{tikzcd}
I_{\mathrm{G}}^{L\wedge^n} \overset{L}{\wedge} \mathbf{G}(\mathrm{X},\mathrm{G}) \arrow[r] \arrow[d] & \mathbf{K}(\mathrm{S},\mathrm{G}) \overset{\mathrm{L}}{\wedge} \mathbf{G}(\mathrm{X},\mathrm{G}) \arrow[r] \arrow[d] & \mathbf{G}(\mathrm{X},\mathrm{G}) \arrow[d] \\
\mathbf{K}(\mathrm{EG}^{\mathrm{gm,m}},\mathrm{G}) \overset{\mathrm{L}}{\wedge} \mathbf{G}(\mathrm{X},\mathrm{G}) \arrow[r] & \mathbf{K}(\mathrm{EG}^{\mathrm{gm,m}},\mathrm{G}) \overset{\mathrm{L}}{\wedge} \mathbf{G}(\mathrm{X},\mathrm{G}) \arrow[r] & \mathbf{G}(\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X},\mathrm{G}).
\end{tikzcd} \tag{4.0.5}$$

These show the composite map from the top left-corner to the bottom-right corner is null-homotopic. This proves the first statement by letting $m' = n \geq m+1$.

Next we consider the first statement in (ii) as well as the functoriality of the factorizations in (i) and (ii). Recall (see [7, Corollary 6.7]) that the partial derived completion $\mathbf{G}(\mathrm{X},\mathrm{G})\hat{}_{\rho_{\mathrm{G}},\mathrm{m}'}$ = the homotopy cofiber of the map

$$\overbrace{\tilde{I}_{\mathrm{G}} \underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge} \tilde{I}_{\mathrm{G}} \cdots \underset{\mathbf{K}(\mathrm{Spec}\, k,\mathrm{G})}{\wedge} \tilde{I}_G}^{m'+1} \underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge} \widetilde{\mathbf{G}(\mathrm{X},\mathrm{G})} \to I_G^{\wedge^{m'+1}} \underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge} \widetilde{\mathbf{G}(\mathrm{X},\mathrm{G})}$$
$$\to \mathbf{K}(\mathrm{S},\mathrm{G}) \underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge} \widetilde{\mathbf{G}(\mathrm{X},\mathrm{G})} = \widetilde{\mathbf{G}(\mathrm{X},\mathrm{G})}$$

which maps into $(\mathbf{K}(\mathrm{S},\mathrm{G})/\mathrm{I}_{\mathrm{G}}^{\mathrm{L}\wedge^{\mathrm{m}'+1}}) \underset{\mathbf{K}(\mathrm{S},\mathrm{G})}{\wedge} \widetilde{\mathbf{G}(\mathrm{X},\mathrm{G})}$. Here $\tilde{I}_{\mathrm{G}} \to I_{\mathrm{G}}$ ($\widetilde{\mathbf{G}(\mathrm{X},\mathrm{G})} \to \mathbf{G}(\mathrm{X},\mathrm{G})$) is a cofibrant replacement in the category of $\mathbf{K}(\mathrm{S},\mathrm{G})$-module spectra. This proves the first statement in (ii). The functoriality of the factorization in (i) and (ii) follows readily by observing that both π^* (when π is flat) and π_* (when π is proper) are module maps over $\mathbf{K}(\mathrm{S},\mathrm{G})$. Therefore, these maps will induce maps between the diagrams in (4.0.5) corresponding to X and Y.

To prove the last statement, one first observes that $\mathbf{G}(\mathrm{X},\mathrm{G}_1 \times \cdots \times \mathrm{G}_{\mathrm{q}})$ is a module spectrum over $\mathbf{K}(\mathrm{S},\mathrm{G}_1 \times \cdots \times \mathrm{G}_{\mathrm{q}})$. Now the homotopy commutative square (where the vertical maps are the obvious ones)

$$\begin{tikzcd}
\mathbf{K}(\mathrm{S},\Pi_{\mathrm{i}=1}^{\mathrm{q}}\mathrm{G}_{\mathrm{i}}) \wedge \mathbf{G}(\mathrm{X},\Pi_{\mathrm{i}=1}^{\mathrm{q}}\mathrm{G}_{\mathrm{i}}) \arrow[r] \arrow[d] & \mathbf{G}(\mathrm{X},\Pi_{\mathrm{i}=1}^{\mathrm{q}}\mathrm{G}_{\mathrm{i}}) \arrow[d] \\
\mathbf{K}(\mathrm{EG}_1^{\mathrm{gm,m}} \times \cdots \mathrm{EG}_{\mathrm{n}}^{\mathrm{gm,m}},\Pi_{\mathrm{i}=1}^{\mathrm{q}}\mathrm{G}_{\mathrm{i}}) \wedge \mathbf{G}(\mathrm{X},\Pi_{\mathrm{i}=1}^{\mathrm{q}}\mathrm{G}_{\mathrm{i}}) \arrow[r] & \mathbf{G}(\mathrm{EG}_1^{\mathrm{gm,m}} \times \cdots \mathrm{EG}_{\mathrm{n}}^{\mathrm{gm,m}} \times \mathrm{X},\Pi_{\mathrm{i}=1}^{\mathrm{q}}\mathrm{G}_{\mathrm{i}})
\end{tikzcd}$$

and an argument exactly as in the case of a single group, proves the factorization in the last statement. The functoriality of this factorization may be proven as in the case of a single group. $\square$

Definition 4.6. For each linear algebraic group G which is either a finite product of general linear groups or split tori acting on a scheme X satisfying the Standing hypotheses as in 1.1 and each choice of admissible gadgets as in Proposition 2.2, we will define a function $\alpha : \mathbb{N} \to \mathbb{N}$, by $\alpha(m) = m'$ where $m' \geq m+1$ is some choice of m' as in the last Proposition.

Let $\pi_{n,\mathrm{X}} : \mathrm{EG}^{\mathrm{gm,n}} \times \mathrm{X} \to \mathrm{X}$ denote the obvious projection. Then, $\pi_{n,\mathrm{X}}$ is G-equivariant for the diagonal action of G on $\mathrm{EG}^{\mathrm{gm,n}} \times \mathrm{X}$ and the given action of G on X. Now we consider the map:

$$\pi_{\mathrm{X}}^* : \mathbf{G}(\mathrm{X}, \mathrm{G}) \to \mathbf{G}(\mathrm{EG} \times \mathrm{X}, \mathrm{G}) = \underset{\infty \leftarrow \mathrm{n}}{\mathrm{holim}} \{\mathbf{G}(\mathrm{EG}^{\mathrm{gm,n}} \times \mathrm{X}, \mathrm{G})\} \tag{4.0.6}$$

As we observed earlier, in view of the fact that the G-action on $\mathrm{EG}^{\mathrm{gm,n}}$ is free, we may identify $\mathbf{G}(\mathrm{EG} \times \mathrm{X}, \mathrm{G})$ with $\mathbf{G}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})$. The main result of this section will be the following Theorem.

Theorem 4.7. *Let* G *denote a finite product of general linear groups acting on a scheme* X *of finite type over a field k satisfying the Standing hypotheses 1.1. Let* T *denotes the maximal split torus in* G. *Then the maps in the diagram*

$$\pi_{\mathrm{X},\, \widehat{\rho_{\mathrm{G}}}}^* : \mathbf{G}(\mathrm{X}, \mathrm{G})\widehat{\ }_{\rho_{\mathrm{G}}} \to \mathbf{G}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})\widehat{\ }_{\rho_{\mathrm{G}}} \leftarrow \mathbf{G}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})$$

are both weak-equivalences of spectra provided the corresponding maps

$$\pi_{\mathrm{X},\, \widehat{\rho_{\mathrm{T}}}}^* : \mathbf{G}(\mathrm{X}, \mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}}} \to \mathbf{G}(\mathrm{ET} \times_{\mathrm{T}} \mathrm{X})\widehat{\ }_{\rho_{\mathrm{T}}} \leftarrow \mathbf{G}(\mathrm{ET} \times_{\mathrm{T}} \mathrm{X})$$

are both weak-equivalence of spectra. Moreover, the map

$$\pi_{\mathrm{X},\, \widehat{\rho_{\mathrm{T}}}}^* : \mathbf{G}(\mathrm{X}, \mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}}} \to \mathbf{G}(\mathrm{ET} \times_{\mathrm{T}} \mathrm{X})\widehat{\ }_{\rho_{\mathrm{T}}} \leftarrow \mathbf{G}(\mathrm{ET} \times_{\mathrm{T}} \mathrm{X})$$

is a weak-equivalence for any G*-scheme* X *satisfying the Standing Hypothesis 1.1(i).*

The inverse system of map $\{\mathbf{G}(\mathrm{X}, \mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m}} \to \mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}} \times \mathrm{X}, \mathrm{T})|\mathrm{m}\}$ *forms a homotopy coherent map of inverse systems and induces a weak-equivalence on taking the homotopy inverse limit as $m \to \infty$: this holds for any separated algebraic space* X *of finite type over k provided with an action by* T*, when the* $\{\mathrm{ET}^{\mathrm{gm,m}}|\mathrm{m}\}$ *is chosen as in* (2.1.3).

Proof. The rest of this section will be devoted to a proof of the above theorem. We start with the following diagram

$$\begin{tikzcd} \mathbf{G}(\mathrm{Y}, \mathrm{G}) \arrow[r, "\{\pi_{m,\mathrm{Y}}^*|m\}"] \arrow[d, "h"] & \{\mathbf{G}(\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{Y}, \mathrm{G})|\mathrm{m}\} \arrow[d, "\{h_m|m\}"'] \\ \mathbf{G}(\mathrm{X}, \mathrm{G}) \arrow[r, "\{\pi_{m,\mathrm{X}}^*|m\}"] & \{\mathbf{G}(\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X}, \mathrm{G})|\mathrm{m}\}, \end{tikzcd} \tag{4.0.7}$$

where X and Y are schemes provided with actions by G, and either $h = f^*$, for a G-equivariant flat map f or $h = Rf_*$ for a proper G-equivariant map. Then the above diagram strictly commutes when $h = f^*$ and also for $h = Rf_*$, with a suitable choice of the right derived functor Rf_* as defined below. Observe that the maps $\pi_{m,\mathrm{Y}}$ and $\pi_{m,\mathrm{X}}$ are all flat maps. Therefore, the strict commutativity of the diagram (4.0.7) for $h = f^*$ follows from the observation that G-theory is strictly contravariantly functorial for pull-back by flat maps.

Therefore, we will next consider the case where $h = Rf_*$. In this case, by applying the functorial Godement resolution, we may replace every pseudo-coherent complex of $\mathcal{O}$-modules, up to quasi-isomorphism by a complex of flabby $\mathcal{O}$-modules up to quasi-isomorphism. Let G denote this functor. Then we claim the above square commutes strictly with $h = Rf_*$, and this may be seen as follows. The composition of the top horizontal map and the right vertical map sends a pseudo-coherent complex of G-equivariant $\mathcal{O}$-modules Q to

$$(id \times f)_* \pi^*_{m,Y}(G(Q)) = (id \times f)_*(\mathcal{O}_{\mathrm{EG^{gm,m}}} \boxtimes G(Q)) = \mathcal{O}_{\mathrm{EG^{gm,m}}} \boxtimes (f_* G(Q)).$$

The composition of the left vertical map and the bottom horizontal map sends the same complex Q to

$$\pi^*_{m,X}(f_*(G(Q))) = \mathcal{O}_{\mathrm{EG^{gm,m}}} \boxtimes (f_*(G(Q))).$$

It follows that the diagram in (4.0.7) which now becomes

$$\begin{array}{ccc} \mathbf{G}(\mathrm{Y},\mathrm{G}) & \xrightarrow{\{\pi^*_{m,\mathrm{Y}} \circ G|m\}} & \{\mathbf{G}(\mathrm{EG^{gm,m}} \times \mathrm{Y}, \mathrm{G})|\mathrm{m}\} \\ \downarrow {\scriptstyle f_* \circ G} & & \downarrow {\scriptstyle \{(id_m \times f)_*|m\}} \\ \mathbf{G}(\mathrm{X},\mathrm{G}) & \xrightarrow{\{\pi^*_{m,\mathrm{X}}|m\}} & \{\mathbf{G}(\mathrm{EG^{gm,m}} \times \mathrm{X}, \mathrm{G})|\mathrm{m}\}, \end{array} \tag{4.0.8}$$

also commutes strictly in this case.

Therefore, on taking the homotopy inverse limit of the diagram in (4.0.7), we obtain the commutative diagram:

$$\begin{array}{ccc} \mathbf{G}(\mathrm{Y},\mathrm{G}) & \xrightarrow{\pi^*_{\mathrm{Y}}} & \mathbf{G}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{Y}) \\ \downarrow {\scriptstyle f_*} & & \downarrow {\scriptstyle \hat{f}_* = \mathrm{holim}\{f_{m*}|m\}} \\ \mathbf{G}(\mathrm{X},\mathrm{G}) & \xrightarrow{\pi^*_{\mathrm{X}}} & \mathbf{G}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}). \end{array} \tag{4.0.9}$$

Let B denote a fixed Borel subgroup of G and let $\pi : \mathrm{G} \underset{\mathrm{B}}{\times} \mathrm{X} \to \mathrm{X}$ denote the map considered in section 2.2(ii). This map is both flat and proper. Therefore, invoking the above result, we see that the maps π^* and π_* associated to $\pi : \mathrm{G} \underset{\mathrm{B}}{\times} \mathrm{X} \to \mathrm{X}$ induce maps that make the following diagram commute:

$$\begin{tikzcd}
\mathbf{G}(\mathrm{X},\mathrm{G}) \arrow[r] \arrow[d, "\pi^*"] & \mathbf{G}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}) \arrow[d, "\pi^*"] \\
\mathbf{G}(\mathrm{X},\mathrm{B}) \simeq \mathbf{G}(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X},\mathrm{G}) \arrow[r] \arrow[d, "\pi_*"] & \mathbf{G}(\mathrm{EG}\underset{\mathrm{B}}{\times}\mathrm{X}) \simeq \mathbf{G}(\mathrm{EG}\times(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X}),\mathrm{G}) \arrow[d, "\pi_*"] \\
\mathbf{G}(\mathrm{X},\mathrm{G}) \arrow[r] & \mathbf{G}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}).
\end{tikzcd} \tag{4.0.10}$$

Observe that the composition of the maps π^* and π_* is a weak-equivalence in both the right and left columns: see 2.2(v).

On applying the derived completion with respect to ρ_{G} to the above diagram, we obtain the commutative diagram, where the composition of the vertical maps π^* and π_* are both weak-equivalences:

$$\begin{tikzcd}
\mathbf{G}(\mathrm{X},\mathrm{G})\,\widehat{}_{\rho_{\mathrm{G}}} \arrow[r] \arrow[d, "\pi^*"] & \mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{G})\,\widehat{}_{\rho_{\mathrm{G}}} \arrow[d, "\pi^*"] & \mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{G}) \arrow[l] \arrow[d, "\pi^*"] \\
\mathbf{G}(\mathrm{X},\mathrm{B})\,\widehat{}_{\rho_{\mathrm{G}}} \simeq \mathbf{G}(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X},\mathrm{G})\,\widehat{}_{\rho_{\mathrm{G}}} \arrow[r] \arrow[d, "\pi_*"] & \mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{B})\,\widehat{}_{\rho_{\mathrm{G}}} \simeq \mathbf{G}(\mathrm{EG}\times(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X}),\mathrm{G})\,\widehat{}_{\rho_{\mathrm{G}}} \arrow[d, "\pi_*"] & \mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{B}) \arrow[l] \arrow[d, "\pi_*"] \\
\mathbf{G}(\mathrm{X},\mathrm{G})\,\widehat{}_{\rho_{\mathrm{G}}} \arrow[r] & \mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{G})\,\widehat{}_{\rho_{\mathrm{G}}} & \mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{G}) \arrow[l]
\end{tikzcd} \tag{4.0.11}$$

Next observe that the equivariant K-theory spectrum with respect to the Borel subgroup B canonically identifies with the equivariant K-theory spectrum with respect to fixed maximal torus T. Since all the terms in the middle row are equivariant G-theory spectra with respect to the action of the Borel subgroup B, invoking Theorem 3.6, it suffices to prove that both the maps in

$$\mathbf{G}(\mathrm{X},\mathrm{T})\,\widehat{}_{\rho_{\mathrm{T}}} \rightarrow \mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{T})\,\widehat{}_{\rho_{\mathrm{T}}} \leftarrow \mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{T}) \tag{4.0.12}$$

are weak-equivalences. This completes the proof of the first statement in Theorem 4.7.

Next we consider the proofs of the remaining statements in the theorem. One may also observe from (B.0.4) and the *Standing Hypothesis* 1.1(i) that there is a weak-equivalence $\mathbf{G}(\mathrm{EG}\times\mathrm{X},\mathrm{T}) \simeq \mathbf{G}(\mathrm{ET}\times\mathrm{X},\mathrm{T})$.

Next we observe from Theorem 5.6 that each of the maps $\mathbf{G}(\mathrm{X},\mathrm{T}) \rightarrow \mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}}\times \mathrm{X},\mathrm{T})$ factors through $\mathbf{G}(\mathrm{X},\mathrm{T})\,\widehat{}_{\rho_{\mathrm{T}},\mathrm{m}}$ and that the corresponding induced map $\mathbf{G}(\mathrm{X},\mathrm{T})\,\widehat{}_{\rho_{\mathrm{T}},\mathrm{m}} \rightarrow \mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{X},\mathrm{T})$ is a weak-equivalence, that is natural in m.

Now we consider the commutative diagram of towers:

$$\begin{tikzcd}
\{\mathbf{G}(\mathrm{X},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m}}|\mathrm{m}\} \arrow[r] \arrow[d, "id"] & \{\mathbf{G}(\mathrm{ET}\times\mathrm{X},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m}}|\mathrm{m}\} \arrow[d, "\beta=\{\beta_m|m\}"] \\
\{\mathbf{G}(\mathrm{X},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m}}|\mathrm{m}\} \arrow[r, "\simeq"'] \arrow[dr] & \{\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{X},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m}}|\mathrm{m}\} \\
 & \{\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{X},\mathrm{T})|\mathrm{m}\} \arrow[u, "\alpha=\{\alpha_m|m\}"']
\end{tikzcd} \tag{4.0.13}$$

Theorem 5.6 provides the identification $\{\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{X},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m}}|\mathrm{m}\}\simeq\{\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{X},\mathrm{T})|\mathrm{m}\}$, where T acts diagonally on $\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{X}$. Therefore, (B.0.4) and the *Standing Hypothesis* 1.1(i) shows that the map $\alpha=\{\alpha_m|m\}$ induces a weak-equivalence on taking the homotopy inverse limit over $m\to\infty$.

Next consider the double tower

$$\{\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{X},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{n}}|\mathrm{n},\mathrm{m}\}. \tag{4.0.14}$$

Taking the homotopy inverse limit along m first provides the term $\{\mathbf{G}(\mathrm{ET}\times\mathrm{X},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{n}}|\mathrm{n}\}$: see [7, Corollary 7.8]. On the other hand, taking the diagonal of the above double tower provides the tower $\{\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}}\times\mathrm{X},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m}}|\mathrm{m}\}$. Now the diagonal map $\Delta:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ is cofinal, which shows the comma category $\Delta/(n,m)$ for any pair of integers $n,m\,\epsilon\,\mathbb{N}$ is contractible. Therefore, [5, Chapter XI, 9.2 Cofinality Theorem] shows that the map β induces a weak-equivalence on taking the homotopy inverse limit as $m\to\infty$. Therefore, it follows that both maps in (4.0.12) are weak-equivalences, thereby completing the proof of all but the last statement in Theorem 4.7. The last statement in Theorem 4.7 is proven in Theorem 5.6. □

5. Proof in the case of a split torus

In view of the above reductions, it suffices to assume that the group scheme H is a split torus $\mathbb{G}_m^n$. In case the group scheme is a smooth diagonalizable group-scheme, we may imbed that as a closed subgroup scheme of a split torus. Then, making use of the remarks following Theorem 1.6, one may extend the results in this section to actions of smooth diagonalizable group schemes also. However, we will not discuss this case explicitly. Thus assuming $\mathrm{H}=\mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$= a split torus, a key step is provided by the following result.

First assume that the base scheme S is the spectrum if a field k and that X and Y are schemes or separated algebraic spaces of finite type over S. Assume X and Y are acted on by an affine group-scheme H over S. In this case the external tensor product of coherent $\mathcal{O}$-modules induces a pairing $\mathbf{G}(\mathrm{X},\mathrm{H})\wedge\mathbf{G}(\mathrm{Y},\mathrm{H})\to\mathbf{G}(\mathrm{X}\underset{\mathrm{S}}{\times}\mathrm{Y},\mathrm{H})$ where G-theory denotes the Quillen K-theory spectra of the exact category of coherent $\mathcal{O}$-modules. The action of H on $\mathrm{X}\underset{\mathrm{S}}{\times}\mathrm{Y}$ is the diagonal action. This pairing is compatible with the structure of the above spectra as module-spectra over the ring spectrum $\mathbf{K}(\mathrm{S},\mathrm{H})$ so that one obtains the induced pairing:

$$p_1^* \wedge p_2^* : \mathbf{G}(\mathrm{X},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{Y},\mathrm{H}) \to \mathbf{G}(\mathrm{X}\underset{\mathrm{S}}{\times}\mathrm{Y},\mathrm{H}).$$

One obtains a similar pairing where the Quillen-style G-theory and K-theory above are replaced by the corresponding Waldhausen-style theories. Once again, the flatness hypothesis on X and Y over S ensures that the external product preserves cofibrations and weak-equivalences after fixing one of the arguments. In addition one may check that if $K' \to K$ is a cofibration of $\mathcal{O}_{\mathrm{X}}$-modules and $M' \to M$ is a cofibration of $\mathcal{O}_{\mathrm{Y}}$-modules, then the induced map $K \underset{\mathcal{O}_{\mathrm{S}}}{\otimes} M' \underset{K' \underset{\mathcal{O}_{\mathrm{S}}}{\otimes} M'}{\oplus} K' \underset{\mathcal{O}_{\mathrm{S}}}{\otimes} M \to K \underset{\mathcal{O}_{\mathrm{S}}}{\otimes} M$ is a cofibration which is a quasi-isomorphism if either of the maps $K' \to K$ or $M' \to M$ are quasi-isomorphisms.

Lemma 5.1. *(See [28, Proposition 3.1].) Assume that the base scheme* S *is the spectrum of a field k. Let* X *denote a cellular scheme, that is, a scheme stratified by locally closed smooth subschemes all of which are affine spaces over the base scheme Assume that* H *is an affine group scheme over* S *acting on* X *so that the strata of* X *are* H*-stable. Let* Y *denote any scheme or algebraic space of finite type over* S *also provided with an* H*-action. Then the map (induced by the above external product)*

$$p_1^* \wedge p_2^* : \mathbf{G}(\mathrm{X},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{Y},\mathrm{H}) \to \mathbf{G}(\mathrm{X}\underset{\mathrm{S}}{\times}\mathrm{Y},\mathrm{H}) \tag{5.0.1}$$

is a weak-equivalence of spectra. Here $p_1 : \mathrm{X}\underset{\mathrm{S}}{\times}\mathrm{Y} \to \mathrm{X}$, $p_2 : \mathrm{X}\underset{\mathrm{S}}{\times}\mathrm{Y} \to \mathrm{Y}$ is the projection to the i-th factor. When X *and* Y *are regular, we may replace the G-theory spectrum by the K-theory spectrum everywhere.*

Proof. This is essentially [28, Proposition 3.1] where it is stated in terms of the K-theory and G-theory of quotient stacks. One considers the localization sequences in H-equivariant G-theory defined using the given strata on X and the induced strata on $\mathrm{X} \times \mathrm{Y}$. The map $p_1^* \wedge p_2^*$ is compatible with these localization sequences. Therefore, one reduces to the case where X is an affine space over S, in which case the conclusion follows by the homotopy property of H-equivariant G-theory. When X and Y are regular, the equivariant G-theory spectrum is weakly equivalent to the corresponding equivariant K-theory spectrum as observed earlier, thereby proving the last statement. □

Proposition 5.2. *Assume that the base scheme* S *is the spectrum of a field k. Assume that $\mathrm{H} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$ = a split torus over* S *and let* Y *denote any scheme or separated algebraic space of finite type over* S *provided with an* H*-action. In this case the map (defined as in* (5.0.1)*)*

$$\mathbf{K}(\mathrm{EH}^{\mathrm{gm,m}},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{Y},\mathrm{H}) \to \mathbf{G}(\mathrm{EH}^{\mathrm{gm,m}} \times \mathrm{Y},\mathrm{H}) \simeq \mathbf{G}(\mathrm{EH}^{\mathrm{gm,m}}\underset{\mathrm{H}}{\times}\mathrm{Y})$$

is a weak-equivalence for all positive integers m, with the product $\mathrm{EH}^{\mathrm{gm,m}} \times \mathrm{Y}$ denoting the fibered product over the base-scheme S*. Therefore, the induced map*

$$\underset{\infty \leftarrow m}{holim}\{\mathbf{K}(\mathrm{EH}^{\mathrm{gm,m}}, \mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S,H})}{\wedge}} \mathbf{G}(\mathrm{Y,H})|\mathrm{m}\} \to \underset{\infty \leftarrow \mathrm{m}}{holim}\{\mathbf{G}(\mathrm{EH}^{\mathrm{gm,m}} \times \mathrm{Y,H})|\mathrm{m}\}$$
$$\simeq \underset{\infty \leftarrow \mathrm{m}}{holim}\{\mathbf{G}(\mathrm{EH}^{\mathrm{gm,m}} \underset{\mathrm{H}}{\times} \mathrm{Y})|\mathrm{m}\}$$

is also a weak-equivalence. The corresponding assertions also hold with the G-theory spectrum replaced by the K-theory spectrum when Y *is regular.*

Proof. A point to observe is that $\mathrm{EH}^{gm,m}$ is not a cellular scheme, so that the last lemma does not apply directly. We first recall that if $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$ is a split torus, then $\mathrm{ET}^{\mathrm{gm,m}} = (\mathbb{A}^{\mathrm{m}+1} - 0)^{\mathrm{n}}$ and $\mathrm{BT}^{\mathrm{gm,m}} = \mathrm{ET}^{\mathrm{gm,m}}/\mathrm{T} = ((\mathbb{A}^{\mathrm{m}+1} - 0)/\mathbb{G}_{\mathrm{m}})^{\mathrm{n}} = (\mathbb{P}^{\mathrm{m}})^{\mathrm{n}}$. Therefore the first statement in the proposition says the obvious map

$$\mathbf{G}((\mathbb{A}^{m+1} - 0)^n, \mathbb{G}_m^n) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})}{\wedge}} \mathbf{G}(\mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}) \to \mathbf{G}((\mathbb{A}^{\mathrm{m}+1} - 0)^{\mathrm{n}} \times \mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})$$
$$\simeq \mathbf{G}((\mathbb{A}^{\mathrm{m}+1} - 0)^{\mathrm{n}} \underset{\mathbb{G}_{\mathrm{m}}^{\mathrm{n}}}{\times} \mathrm{Y})$$

is a weak-equivalence. The last weak-equivalence follows from the observation that $\mathbb{G}_m^n$ acts freely on the product $(\mathbb{A}^{m+1} - 0)^n$, with each factor of $\mathbb{G}_m$ acting on the corresponding factor of $\mathbb{A}^{m+1} - 0$. Now it remains to prove the first map is a weak-equivalence.

For the proof, it will be convenient to denote the rank of the split torus as n and the n in $(\mathbb{A}^{m+1} - 0)^n$ as u. For $u < n$, the scheme $(\mathbb{A}^{m+1} - 0)^u$ will be identified as the closed subscheme $(\mathbb{A}^{m+1} - 0)^u \times 0^{n-u}$ in $(\mathbb{A}^{m+1} - 0)^u \times (\mathbb{A}^{m+1})^{n-u}$ with the last $n - u$ factors being the origin of $\mathbb{A}^{m+1}$ and the remaining factors being $(\mathbb{A}^{m+1} - 0)$. In this case, the $\mathbb{G}_m$ forming the last $n - u$ factors will clearly act trivially on the corresponding factors which are identified with the origin in $\mathbb{A}^{m+1}$. Therefore, making use of the localization sequences in G-equivariant G-theory, one may prove this by ascending induction on u. We will consider the map

$$\mathbf{G}((\mathbb{A}^{m+1} - 0)^u \times 0^{n-u}, \mathbb{G}_m^n) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})}{\wedge}} \mathbf{G}(\mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})$$
$$\to \mathbf{G}((\mathbb{A}^{m+1} - 0)^u \times \mathbb{A}^{m+1} \times 0^{n-u-1}, \mathbb{G}_m^n). \tag{5.0.2}$$

Assume that the above map is a weak-equivalence for a fixed $u < n$, $0 \leq u$; we will then show that it is also a weak-equivalence for u replaced by $u + 1$ as follows. Since both the source and the target of the above map are compatible with localization sequences, we obtain the following commutative diagram with each row a stable cofiber sequence:

$$\begin{tikzcd}
\mathbf{G}((\mathbb{A}^{m+1} - 0)^u \times 0^{n-u}, \mathbb{G}_m^n) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})}{\wedge}} \mathbf{G}(\mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}) \arrow[r] \arrow[d] & \mathbf{G}((\mathbb{A}^{m+1} - 0)^u \times \mathbb{A}^{m+1} \times 0^{n-u-1}, \mathbb{G}_m^n) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})}{\wedge}} \mathbf{G}(\mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}) \arrow[d] \\
\mathbf{G}((\mathbb{A}^{m+1} - 0)^u \times 0^{n-u} \times \mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}) \arrow[r] & \mathbf{G}((\mathbb{A}^{m+1} - 0)^u \times \mathbb{A}^{m+1} \times 0^{n-u-1} \times \mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})
\end{tikzcd}$$

$$\begin{array}{c} \longrightarrow \mathbf{G}((\mathbb{A}^{m+1}-0)^{u+1}\times 0^{n-u-1}, \mathbb{G}_m^n) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathbb{G}_{\mathrm{m}}^{\mathrm{n}})}{\wedge}} \mathbf{G}(\mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}) \\ \downarrow \\ \longrightarrow \mathbf{G}((\mathbb{A}^{m+1}-0)^{u+1}\times 0^{n-u-1}\times \mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}) \end{array}$$

The first vertical map is a weak-equivalence by the inductive assumption and the second vertical map is a weak-equivalence making use of this inductive assumption and the homotopy property for equivariant G-theory. Therefore, so is the last vertical maps which completes the induction. This reduces everything to the case where $u = 0$, which is clear. When Y is regular, the weak-equivalences $\mathbf{K}(\mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}) \simeq \mathbf{G}(\mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})$, $\mathbf{K}((\mathbb{A}^{m+1} - 0)^n, \mathbb{G}_m^n) \simeq \mathbf{G}(\mathbb{A}^{m+1} - 0)^n, \mathbb{G}_m^n)$, $\mathbf{K}((\mathbb{A}^{m+1} - 0)^n \times \mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}) \simeq \mathbf{G}((\mathbb{A}^{\mathrm{m}+1} - 0)^{\mathrm{n}} \times \mathrm{Y}, \mathbb{G}_{\mathrm{m}}^{\mathrm{n}})$ and $\mathbf{K}((\mathbb{A}^{m+1} - 0)^n \underset{\mathbb{G}_m^n}{\times} \mathrm{Y}) \simeq \mathbf{G}((\mathbb{A}^{\mathrm{m}+1} - 0)^{\mathrm{n}} \underset{\mathbb{G}_{\mathrm{m}}^{\mathrm{n}}}{\times} \mathrm{Y})$ complete the proof. □

5.0.3. Multiple derived completions

We remind the reader that the discussion in this section also takes place in $\mathbf{Spt}_{\mathrm{S}^1}$, which is the category of all symmetric S^1-spectra. Let R denote a commutative ring spectrum and let $Alg(R)$ denote the category of commutative R-algebra spectra. We will provide $Alg(R)$ with the cofibrantly generated model category structure discussed in [49]. Let n denote a fixed positive integer and let $B_i, C_i \,\epsilon\, Alg(R)$, $i = 1, \cdots, n$ be provided with maps of commutative algebra spectra $f_i : B_i \to C_i$ in $Alg(R)$. We will also assume that one is provided with augmentations $C_i \to R$ that are maps in $Alg(R)$. (In the examples that we consider below, each $C_i = R$, so this last condition is automatic.) We will assume the B_i, C_i $i = 1, \cdots, n$ are cofibrant in $Alg(R)$, so that each map $B_i \to C_i$ is a cofibration in $Alg(R)$. We will let

$$B = B_1 \wedge_R B_2 \wedge_R \cdots \wedge_R B_n \text{ and } C = C_1 \wedge_R C_2 \wedge_R \cdots \wedge_R C_n.$$

Since each $B_i \to C_i$ is a cofibration, it follows that the induced map $B \to C$ is a cofibration in $Alg(R)$. Then one first observes that these are commutative algebra spectra over R and that $f = f_1 \wedge_R f_2 \wedge_R \wedge_R \cdots \wedge_R f_n$ is a map of such algebra spectra. We proceed to consider derived completions of a module spectrum $M \,\epsilon\, Mod(B)$ with respect to the map f and explore how it relates to the derived completions with respect to the maps f_i, $i = 1, \cdots, n$.

It needs to be pointed out that there is an entirely parallel set-up where R denotes a commutative ring and B_i, C_i, $i = 1, \cdots, n$ are commutative algebras over R (or commutative dg-algebras over R) which will all be assumed to be flat over R. In this set-up M will be a module (or dg-module) over B.

The *main example* to keep in mind is what appears in Proposition 5.2, namely $R = \mathbf{K}(\mathrm{S})$, $B_i = \mathbf{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}})$ with the $\mathbb{G}_m$ forming the i-th factor in $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$ and C_i being suitable cofibrant replacements of $\mathbf{K}(\mathrm{S})$ for all $i = 1, \cdots, n$. Then $\mathbf{K}(\mathrm{S}, \mathrm{T})$ identifies with $\mathbf{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}}) \wedge_{\mathbf{K}(\mathrm{S})} \mathrm{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}}) \cdots \wedge_{\mathbf{K}(\mathrm{S})} \mathbf{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}})$ so that we will let $B = \mathbf{K}(\mathrm{S}, \mathrm{T})$.

For each $i = 1, \cdots, n$, let $T_{B_i} : Mod(B) \to Mod(B)$ denote the triple defined by $M \mapsto M \wedge_{B_i} C_i$ which is $M \wedge_{B_i} C_i$ viewed as a B-module through the obvious B-bi-module structure on M. Clearly $T_{B_i}(M)$ has a (right) C_i-module structure. Now one obtains an induced B_i-module structure on $T_{B_i}(M)$ obtained by restriction of scalars along the map $B_i \to C_i$. Since all the ring spectra we consider are commutative ring spectra, one may observe that this B_i-module structure is compatible with the B-module structure on $T_{B_i}(M)$ considered earlier (and used in the definition of the functor T_{B_i}). We let $\mathcal{T}^{\bullet}_{B_i}$ denote the cosimplicial object obtained by iterating T_{B_i} and denoted $\mathcal{T}^{\bullet}_{B_i}(M, C_i)$ elsewhere. Observe that it is also possible to compose T_{B_i} and T_{B_j} and thereby define a triple $T_{B_1,\cdots,B_n} : Mod(B) \to Mod(B)$ by $M \mapsto M \wedge_{B_1} C_1 \wedge_{B_2} C_2 \cdots \wedge_{B_n} C_n$, that is,

$$T_{B_1,\cdots,B_n} = T_{B_n} \circ \cdots \circ T_{B_1}$$

Observe $T_{B_1,\cdots,B_n}(M)$ has the obvious structure of a (right) $C = C_1 \wedge_R \cdots \wedge_R C_n$-module and that the B-module structure on it obtained by restriction of scalars along the map $B_1 \wedge_R \cdots \wedge_R B_n \to C_1 \wedge_R \cdots \wedge_R C_n$ is compatible with the B-module structure on $T_{B_1,\cdots,B_n}(M)$ used in its definition: that is, if $m \,\epsilon\, M, b_i \,\epsilon\, B_i, c_i \,\epsilon\, C_i$ and $r_i \,\epsilon\, R$, $i = 1, \cdots, n$, then $m \wedge r_1 b_1 c_1 \wedge r_2 b_2 c_2 \wedge \cdots \wedge r_n b_n c_n$ is identified with $m r_1 \cdots r_n b_1 \cdots b_n \wedge c_1 \wedge c_2 \wedge \cdots \wedge c_n$. Therefore, one obtains the identifications for any $M \,\epsilon\, Mod(B)$:

$T_{B_n}(T_{B_{n-1}} \cdots (T_{B_1}(M))) = T_{B_1,\cdots,B_n}(M) = T_{B_1 \wedge_R \cdots \wedge_R B_n}(M) = M \underset{B_1 \wedge_R \cdots \wedge_R B_n}{\wedge} C_1 \wedge_R \cdots \wedge_R C_n$,

where the B_i-module structure on $T_{B_{i-1}} \cdots T_{B_1}(M)$ is from the original B-module structure on M. Moreover if $M = M_1 \wedge_R \cdots \wedge_R M_n$, then $T_{B_1,\cdots,B_n}(M) = T_{B_1}(M_1) \wedge_R T_{B_2}(M_2) \wedge_R \cdots \wedge_R T_{B_n}(M_n)$.

One may iterate the above constructions to obtain an n-fold multi-cosimplicial object: $\mathcal{T}^{\bullet,\cdots,\bullet}_{B_1,\cdots,B_n}(M)$ defined by

$\mathcal{T}^{m_1,\cdots,m_n}_{B_1,\cdots,B_n}(M) = \mathcal{T}^{m_n}_{B_n}(\cdots \mathcal{T}^{m_1}_{B_1}(M))$

where an $M \,\epsilon\, Mod(B)$ is viewed as an object in $Mod(B_1)$ for forming $\mathcal{T}^{m_1}_{B_1}(M)$ and each $\mathcal{T}^{m_{i-1}}_{B_{i-1}}(\cdots \mathcal{T}^{m_1}_{B_1}(M))$ is given the B_i-module structure induced from the original B-module structure on M before applying $\mathcal{T}^{m_i}_{B_i}$. For a sequence of non-negative integers $m_1, \cdots, m_n$, we define $\sigma_{\le m_1,\cdots,m_n} \mathcal{T}^{\bullet,\cdots,\bullet}_{B_1,\cdots,B_n}(M) = \{\mathcal{T}^{i_1,\cdots,i_n}_{B_1,\cdots,B_n}(M) | i_1 \le m_1, \cdots .i_n \le m_n\}$, which is a truncated multi-cosimplicial object. Then we let

$$M\hat{\ }_{f,m} = \underset{\Delta}{\text{holim}}(\sigma_{\le m}(\Delta \mathcal{T}^{\bullet,\cdots,\bullet}_{B_1,\cdots,B_n}(M))) \text{ where } \Delta \text{ denotes the diagonal and} \tag{5.0.4}$$

$$M\hat{\ }_{f,m_1,\cdots,m_n} = \underset{\Delta_{\le m_1}}{\text{holim}} \cdots \underset{\Delta_{\le m_n}}{\text{holim}}(\sigma_{\le m_1,\cdots,\le m_n}(\mathcal{T}^{\bullet,\cdots,\bullet}_{B_1,\cdots,B_n}(M))). \tag{5.0.5}$$

This denotes the partial derived completion up to degrees $m_1, \cdots, m_n$.

Proposition 5.3. *Assume the above situation. Then the following hold for an $M \,\epsilon\, Mod(B)$.*

(i) For any sequence of positive integers $m_1, \cdots, m_n$,

$M\hat{\ }_{f,m_1,\cdots,m_n} = \underset{\Delta_{\le m_1}}{holim} \cdots \underset{\Delta_{\le m_n}}{holim}(\sigma_{\le m_1,\cdots,\le m_n}(\mathcal{T}^{\bullet,\cdots,\bullet}_{B_1,\cdots,B_n}(M)))$

$\simeq \underset{\Delta_{\leq m_n}}{holim}\,\sigma_{\leq m_n}\mathcal{T}^{\bullet}_{B_n}(\underset{\Delta_{\leq m_2}}{holim}\,\sigma_{\leq m_{n-1}}\mathcal{T}^{\bullet}_{B_{n-1}}(\cdots(\underset{\Delta_{\leq m_1}}{holim}\,\sigma_{\leq m_1}\mathcal{T}^{\bullet}_{B_1}(M))))$

(ii) $M\widehat{\ }_f = holim\, M\widehat{\ }_{f,m} \simeq \underset{\infty,\cdots,\infty\leftarrow(m_1,\cdots,m_n)}{holim} M\widehat{\ }_{f,m_1,\cdots,m_n}$ *which denotes the derived completion along* f.

(iii) In case $M = M_1 \wedge_R M_2 \wedge_R \cdots \wedge_R M_n$, *with* $M_i \,\epsilon\, Mod(B_i)$, *then* $M\widehat{\ }_{f,m_1,\cdots,m_n} \simeq \underset{\Delta_{\leq m_1}}{holim}\,\sigma_{\leq m_1}(\mathcal{T}^{\bullet}_{B_1}(M_1, C_1) \wedge_R \cdots \wedge_R \underset{\Delta_{\leq m_n}}{holim}(\sigma_{\leq m_n}\mathcal{T}_{B_n}(M, C_n)) \simeq M_1\widehat{\ }_{f_1,m_1} \wedge_R M_2\widehat{\ }_{f_2,m_2} \wedge_R \cdots \wedge_R M_n\widehat{\ }_{f_n,m_n}$.

Proof. The weak-equivalence in the second line in (i) follows from [7, Proposition 2.9]. (The main point here is that the partial derived completions involve only finite homotopy pull-backs.) The statement in (ii) now follows because the iterated homotopy inverse limit of a multi-cosimplicial object may be replaced by the homotopy inverse limit of the diagonal cosimplicial object. (See, for example, [55, Lemma 5.33] for a proof.) The last statement then follows from the observation that the triples T_{B_i} commute with each other and by an application of [7, Proposition 2.9]. □

Remark 5.4. For $i = 1, \cdots, n$, let $\rho_{\mathrm{T_i}} = \mathrm{f_i} : \mathbf{K}(\mathrm{S}, \mathbb{G}_{\mathrm{m}}) \to \mathbf{K}(\mathrm{S})$ denote the map induced by the restriction to the trivial subgroup, where $\mathbb{G}_m$ is the i-th factor in the split torus $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\times \mathrm{n}}$. Let $f = f_1 \wedge \cdots \wedge f_n$. Then using the terminology above, we proceed to establish the weak-equivalences (in the following theorem):

$$\mathbf{K}(\mathrm{S},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m_1},\cdots,\mathrm{m_n}} \simeq \mathbf{K}(\mathrm{BT}^{\mathrm{gm},\mathrm{m_1},\cdots,\mathrm{m_n}}) = \mathbf{K}(\mathbb{P}^{\mathrm{m_1}} \times \cdots \mathbb{P}^{\mathrm{m_n}}), \text{ and} \tag{5.0.6}$$
$$\mathbf{K}(\mathrm{S},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}}} \simeq \mathbf{K}(\mathrm{BT}^{\mathrm{gm}}) = \underset{\infty\leftarrow \mathrm{m}}{\mathrm{holim}}\mathbf{K}(\mathbb{P}^{\mathrm{m}} \times \cdots \times \mathbb{P}^{\mathrm{m}})$$

Corresponding statements hold for the completion with respect to the map $\rho_\ell \circ \rho_T : \mathbf{K}(\mathrm{S},\mathrm{T}) \to \mathbf{K}(\mathrm{S}) \to \mathbf{K}(\mathrm{S})\underset{\mathbb{S}}{\wedge}\mathbb{H}(\mathbb{Z}/\ell)$ for a fixed prime ℓ different from $char(k) = p$.

The following results will play a major role in the proof of the main theorem. First observe that if $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$ is an n-dimensional split torus, then $\mathrm{ET}^{\mathrm{gm},\mathrm{m}} = (\mathbb{A}^{\mathrm{m}+1} - 0)^{\mathrm{n}}$ and $\mathrm{BT}^{\mathrm{gm},\mathrm{m}} = (\mathbb{P}^{\mathrm{m}})^{\mathrm{n}}$. The obvious map $\mathrm{ET}^{\mathrm{gm},\mathrm{m}} \to \mathrm{S}$ induces a compatible collection of maps $\{\mathbf{K}(\mathrm{S},\mathrm{T}) \to \mathbf{K}(\mathrm{ET}^{\mathrm{gm},\mathrm{m}},\mathrm{T}) = \mathbf{K}(\mathrm{BT}^{\mathrm{gm},\mathrm{m}}) | \mathrm{m} \geq 0\}$.

Theorem 5.5. *Assume the base scheme* S *is the spectrum of a field* k *and that* $\mathrm{T} = \mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$ *is a split torus over* S. *For each* $i = 1, \cdots, n$, *let* T_i *denote the* $\mathbb{G}_m$ *forming the* i*-th factor of* T. *Let* $\rho_{\mathrm{T}} : \mathbf{K}(\mathrm{S},\mathrm{T}) \to \mathbf{K}(\mathrm{S})$ *denote the restriction map as in 3.0.7. Then the map* $\mathbf{K}(\mathrm{S},\mathrm{T}) \to \mathbf{K}(\mathrm{ET}^{\mathrm{gm},\mathrm{m}},\mathrm{T}) = \mathbf{K}(\mathrm{BT}^{\mathrm{gm},\mathrm{m}})$ *factors through the multiple partial derived completion*

$$\mathbf{K}(\mathrm{S},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T}},\mathrm{m},\cdots,\mathrm{m}} = \underset{\Delta_{\leq \mathrm{m}}}{holim}\,\sigma_{\leq \mathrm{m}} \cdots \underset{\Delta_{\leq \mathrm{m}}}{holim}\,\sigma_{\leq \mathrm{m}}\mathcal{T}^{\bullet,\cdots,\bullet}_{\mathbf{K}(\mathrm{S},\mathrm{T_1}),\cdots,\mathbf{K}(\mathrm{S},\mathrm{T_n})}(\mathbf{K}(\mathrm{S},\mathrm{T}),\mathbf{K}(\mathrm{S}))$$

and induces a weak-equivalence, that is natural in m:

$$\mathbf{K}(\mathrm{S},\mathrm{T})\widehat{\ }_{\rho_{\mathrm{T},\mathrm{m},\cdots,\mathrm{m}}} \simeq \mathbf{K}(\mathrm{BT}^{\mathrm{gm},\mathrm{m}} \times \cdots \times \mathrm{BT}^{\mathrm{gm},\mathrm{m}}).$$

Proof. First observe that

$$\pi_*(\mathbf{K}(\mathrm{S},\mathrm{T})) = R(\mathrm{T}) \underset{\mathbb{Z}}{\otimes} \pi_*(\mathbf{K}(\mathrm{S})) \tag{5.0.7}$$

Step 1. We will next restrict to the case where the torus T is one dimensional. Let λ denote the 1-dimensional representation of T corresponding to a generator of $R(\mathrm{T})$. For each $m \geq 0$, $(\lambda - 1)^{m+1}$ defines a class in $\pi_0(\mathbf{K}(\mathrm{S},\mathrm{T})) = R(\mathrm{T}) \otimes \pi_0(\mathbf{K}(\mathrm{S}))$. Viewing this class as a map $\mathbb{S} \to \mathbf{K}(\mathrm{S},\mathrm{T})$ (recall $\mathbb{S}$ denotes the S^1-sphere spectrum), the composition $\mathbb{S} \wedge \mathbf{K}(\mathrm{S},\mathrm{T}) \to \mathbf{K}(\mathrm{S},\mathrm{T}) \wedge \mathbf{K}(\mathrm{S},\mathrm{T}) \xrightarrow{\mu} \mathbf{K}(\mathrm{S},\mathrm{T})$ defines a map of spectra

$$\mathbf{K}(\mathrm{S},\mathrm{T}) \to \mathbf{K}(\mathrm{S},\mathrm{T}),$$

which will be still denoted $(\lambda - 1)^{m+1}$. (Here μ is the ring structure on the spectrum $\mathbf{K}(\mathrm{S},\mathrm{T})$.) *The heart of the proof of this theorem involves showing that the homotopy cofiber of the map* $(\lambda - 1)^{m+1}$ *identifies with the partial derived completion to the order* m*, along the restriction map* $\rho_{\mathrm{T}} : \mathbf{K}(\mathrm{S},\mathrm{T}) \to \mathbf{K}(\mathrm{S})$.

Step 2. Let $\mathrm{S} \times \mathbb{A}^{\mathrm{m}+1}$ denote the vector bundle of rank $m+1$ over S on which the torus T acts diagonally. The *Koszul-Thom class* of this bundle is $\pi^*(\lambda - 1)^{m+1}$, where $\pi : \mathrm{S} \times \mathbb{A}^{\mathrm{m}+1} \to \mathrm{S}$ is the projection. This defines a class in $\pi_0(\mathbf{K}(\mathrm{S} \times \mathbb{A}^{\mathrm{m}+1}, \mathrm{S} \times (\mathbb{A}^{\mathrm{m}+1} - 0), \mathrm{T}))$. Next consider the diagram:

$$\begin{tikzcd}
\mathbf{K}(\mathrm{S},\mathrm{T}) \arrow[r, "(\lambda-1)^{m+1}"] \arrow[d, "\simeq"] & \mathbf{K}(\mathrm{S},\mathrm{T}) \arrow[r] \arrow[d, "\simeq"] & cofiber((\lambda-1)^{m+1}) \arrow[d] \\
\mathbf{K}(\mathrm{S} \times \mathbb{A}^{\mathrm{m}+1}, \mathrm{S} \times (\mathbb{A}^{\mathrm{m}+1} - 0), \mathrm{T}) \arrow[r] & \mathbf{K}(\mathrm{S} \times \mathbb{A}^{\mathrm{m}+1}, \mathrm{T}) \arrow[r] & \mathbf{K}(\mathrm{S} \times (\mathbb{A}^{\mathrm{m}+1} - 0), \mathrm{T})
\end{tikzcd} \tag{5.0.8}$$

where the bottom row is the stable homotopy fiber sequence associated to the homotopy cofiber sequence: $\mathrm{S} \times (\mathbb{A}^{\mathrm{m}+1} - 0) \to \mathrm{S} \times \mathbb{A}^{\mathrm{m}+1} \to (\mathrm{S} \times \mathbb{A}^{\mathrm{m}+1})/(\mathrm{S} \times (\mathbb{A}^{\mathrm{m}+1} - 0))$ in the $\mathbb{A}^1$-stable homotopy category. The left-most vertical map is provided by *Thom-isomorphism*, that is, by cup-product with the Koszul-Thom-class $\pi^*(\lambda - 1)^{m+1}$. Therefore, this map is a weak-equivalence. The second vertical map is π^* and is a weak-equivalence by the homotopy property. In view of the fact that the unit of the ring spectrum $\mathbf{K}(\mathrm{S},\mathrm{T})$ is sent to the Koszul-Thom class $\pi^*(\lambda - 1)^{m+1}$ by the left-most vertical map and the middle map is the pull-back π^*, the left-most square commutes. Therefore, the right square also commutes and the right most vertical map is also a weak-equivalence.

Step 3. Next we take $m = 0$. Then it follows from what we just showed that the map $cofiber(\lambda - 1) \to \mathbf{K}(\mathrm{S} \times \mathbb{A}^1 - 0, \mathrm{T}) = \mathbf{K}(\mathrm{S})$ (forming the last vertical map in (5.0.8)) is a weak-equivalence. Now the map $\mathbf{K}(\mathrm{S}\times\mathbb{A}^{\mathrm{m}+1},\mathbb{G}_{\mathrm{m}})\to\mathbf{K}(\mathrm{S}\times\mathbb{G}_{\mathrm{m}},\mathbb{G}_{\mathrm{m}})=\mathbf{K}(\mathrm{S})$ forming the bottom row in the right-most square of (5.0.8) identifies with restriction map $\mathbf{K}(\mathrm{S},\mathbb{G}_{\mathrm{m}})\to\mathbf{K}(\mathrm{S})$. Therefore, by the homotopy commutativity of the right-most square of (5.0.8), it follows that the homotopy fiber of the restriction map ρ_{T} : $\mathbf{K}(\mathrm{S},\mathrm{T})\to\mathbf{K}(\mathrm{S})$ identifies up to weak-equivalence with the homotopy fiber of the map $\mathbf{K}(\mathrm{S},\mathrm{T})\to\mathrm{cofiber}(\lambda-1)$, that is, with the map $(\lambda-1):\mathbf{K}(\mathrm{S},\mathrm{T})\to\mathbf{K}(\mathrm{S},\mathrm{T})$.

Step 4. Next recall (using [7, Corollary 6.7]) that the partial derived completion $\mathbf{K}(\mathrm{S},\mathrm{T})\hat{}_{\rho_{\mathrm{T}},\mathrm{m}}$ may be identified as follows. Let $\tilde{I}_{\mathrm{T}}\to I_{\mathrm{T}}$ denote a cofibrant replacement in the category of module spectra over $\mathbf{K}(\mathrm{S},\mathrm{T})$, with I_{T} denoting the homotopy fiber of the restriction $\mathbf{K}(\mathrm{S},\mathrm{T})\to\mathbf{K}(\mathrm{S})$. Then

$$\mathbf{K}(\mathrm{S},\mathrm{T})\hat{}_{\rho_{\mathrm{T}},\mathrm{m}}=\mathrm{Cofib}(\overbrace{\tilde{I}_{\mathrm{T}}\underset{\mathbf{K}(\mathrm{S},\mathrm{T})}{\wedge}\tilde{I}_{\mathrm{T}}\underset{\mathbf{K}(\mathrm{S},\mathrm{T})}{\wedge}\cdots\underset{\mathbf{K}(\mathrm{S},\mathrm{T})}{\wedge}\tilde{I}_{\mathrm{T}}}^{\mathrm{m}+1}\to\mathbf{K}(\mathrm{S},\mathrm{T})).$$

What we have just shown in Step 3 is that the homotopy fiber I_{T} of the restriction map $\mathbf{K}(\mathrm{S},\mathrm{T})\to\mathbf{K}(\mathrm{S})$ identifies with the map $\mathbf{K}(\mathrm{S},\mathrm{T})\overset{(\lambda-1)}{\to}\mathbf{K}(\mathrm{S},\mathrm{T})$ and hence is clearly cofibrant over $\mathbf{K}(\mathrm{S},\mathrm{T})$: that is, $\tilde{I}_{\mathrm{T}}=I_{\mathrm{T}}$ is simply $\mathbf{K}(\mathrm{S},\mathrm{T})$ mapping into $\mathbf{K}(\mathrm{S},\mathrm{T})$ by the map $(\lambda-1)$. Therefore the homotopy cofiber

$$Cofib(\overbrace{\tilde{I}_{\mathrm{T}}\underset{\mathbf{K}(\mathrm{S},\mathrm{T})}{\wedge}\tilde{I}_{\mathrm{T}}\underset{\mathbf{K}(\mathrm{S},\mathrm{T})}{\wedge}\cdots\underset{\mathbf{K}(\mathrm{S},\mathrm{T})}{\wedge}\tilde{I}_{\mathrm{T}}}^{m+1}\to\mathbf{K}(\mathrm{S},\mathrm{T}))=\mathrm{Cofiber}(\mathbf{K}(\mathrm{S},\mathrm{T})\overset{(\lambda-1)^{\mathrm{m}+1}}{\to}\mathbf{K}(\mathrm{S},\mathrm{T})).$$

Making use of the diagram (5.0.8), these observations prove that

$$\mathbf{K}(\mathrm{S},\mathrm{T})\hat{}_{\rho_{\mathrm{T}},\mathrm{m}}\simeq\mathrm{cofiber}(\lambda-1)^{\mathrm{m}+1}\simeq\mathbf{K}(\mathrm{S}\times(\mathbb{A}^{\mathrm{m}+1}-0),\mathrm{T})=\mathbf{K}(\mathrm{S}\times\mathbb{P}^{\mathrm{m}})=\mathbf{K}(\mathrm{BT}^{\mathrm{gm},\mathrm{m}}).$$

Observe that as m varies, the diagram in (5.0.8) forms an inverse system of commutative diagrams, which provides the compatibility of the above weak-equivalence as m varies. (If one prefers, one may also apply Lemma C.5 to the inverse system of squares forming the last vertical map in (5.0.8), as m varies.) This completes the proof for the case of the 1-dimensional torus. (It may be worthwhile pointing out that classical argument due to Atiyah and Segal for the usual completion for torus actions in equivariant topological K-theory is very similar: see [4, section 3, Step 1], the main difference being that in [4] the corresponding argument is applied at the level of homotopy groups.)

Step 5. Next suppose that $\mathrm{T}=\mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$. Letting $\mathrm{T}_{\mathrm{n}-1}=$ the product of the first $(n-1)$ copies of $\mathbb{G}_m$, one observes that $\mathbf{K}(\mathrm{S},\mathrm{T})\simeq\mathbf{K}(\mathrm{S},\mathrm{T}_{\mathrm{n}-1})\wedge_{\mathbf{K}(\mathrm{S})}\mathbf{K}(\mathrm{S},\mathbb{b}\mathrm{G}_{\mathrm{m}})$. Therefore one may use ascending induction on n and the discussion on multiple derived completions in Proposition 5.3 to complete the proof of the theorem. □

Theorem 5.6. *Assume the base scheme* S *is the spectrum of a field* k *and that* X *is either a scheme or a separated algebraic space of finite type over* S. *Let* $\mathrm{H}=\mathbb{G}_{\mathrm{m}}^{\mathrm{n}}$ *denote a split torus over* S *acting on* X.

(i) *Then, for each fixed positive integer* m, *the map*

$$\mathbf{G}(\mathrm{X},\mathrm{H}) \simeq \mathbf{K}(\mathrm{S},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{X},\mathrm{H}) \to \mathbf{K}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}},\mathrm{H}) \overset{\mathrm{L}}{\underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge}} \mathbf{G}(\mathrm{X},\mathrm{H})$$
$$\to \mathbf{G}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}} \underset{\mathrm{H}}{\times} \mathrm{X})$$

factors through the partial derived completion $\mathbf{G}(\mathrm{X},\mathrm{H})\,\hat{}_{\rho_{\mathrm{H}},\mathrm{m}}$, *the induced map* $\mathbf{G}(\mathrm{X},\mathrm{H})\,\hat{}_{\rho_{\mathrm{H}},\mathrm{m}} \to \mathbf{G}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}} \underset{\mathrm{H}}{\times} \mathrm{X})$ *is functorial in* m *and is a weak-equivalence. Moreover, this weak-equivalence is natural in both* X *and* H.

(ii) *Therefore, we obtain an induced weak-equivalence* $\mathbf{G}(\mathrm{X},\mathrm{H})\,\hat{}_{\rho_{\mathrm{H}}} \to \underset{\infty \leftarrow m}{holim}\, \mathbf{G}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}} \underset{\mathrm{H}}{\times} \mathrm{X})$ *that is functorial in* X *and* H.

(iii) *The corresponding assertions hold with the G-theory spectrum replaced by the K-theory spectrum when* X *is regular.*[3]

Proof. We will first consider the case when H is a one dimensional torus, and then use ascending induction on the rank of the split torus H to prove the theorem in general.

Therefore we will assume presently that H is a one dimensional torus. Let $\mathcal{I}_{\mathrm{H}}$ denote the homotopy fiber of the map $\mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S})$ induced by the restriction map. One may readily see that it is module spectrum over $\mathbf{K}(\mathrm{S},\mathrm{H})$. Then, we will replace this by a cofibrant object in $Mod(\mathbf{K}(\mathrm{S},\mathrm{H}))$ and denote the resulting object by the same symbol. Therefore, one may now form the derived smash-product

$$\mathcal{I}_{\mathrm{H}}^{m+1} = \overbrace{\mathcal{I}_{\mathrm{H}} \underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge} \mathcal{I}_{\mathrm{H}} \cdots \underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge} \mathcal{I}_{\mathrm{H}}}^{m+1}. \tag{5.0.9}$$

[7, Corollary 6.7] and Theorem 5.5 then show that

$$\mathcal{I}_{\mathrm{H}}^{m+1} \to \mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S},\mathrm{H})\,\hat{}_{\rho_{\mathrm{H}},\mathrm{m}} \simeq \mathbf{K}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}},\mathrm{H}) \tag{5.0.10}$$

is a stable fibration sequence of spectra, and therefore also a stable cofibration sequence of spectra. Let $\widetilde{\mathbf{G}(\mathrm{X},\mathrm{H})}$ denote a cofibrant replacement of $\mathbf{G}(\mathrm{X},\mathrm{H})$ in the model category $Mod(\mathbf{K}(\mathrm{S},\mathrm{H}))$. Now taking smash-product over $\mathbf{K}(\mathrm{S},\mathrm{H})$ with the spectrum $\widetilde{\mathbf{G}(\mathrm{X},\mathrm{H})}$ provides the stable cofibration (or equivalently stable fibration) sequence:

$$\mathcal{I}_{\mathrm{H}}^{m+1} \underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge} \widetilde{\mathbf{G}(\mathrm{X},\mathrm{H})} \to \widetilde{\mathbf{G}(\mathrm{X},\mathrm{H})} \to \mathbf{K}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}},\mathrm{H}) \underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge} \widetilde{\mathbf{G}(\mathrm{X},\mathrm{H})} \simeq \mathbf{G}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}} \underset{\mathrm{H}}{\times} \mathrm{X}). \tag{5.0.11}$$

[3] Work in progress shows that it is also possible to prove a variant of this theorem where equivariant G-theory is replaced by homotopy equivariant K-theory, which would be an equivariant version of the homotopy K-theory considered in [63].

The last weak-equivalence is provided by Proposition 5.2. One also obtains the stable cofibration (or equivalently stable fibration) sequence (by smashing the stable cofiber sequence $\mathcal{I}_{\mathrm{H}}^{m+1} \to \mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S},\mathrm{H})/\mathcal{I}_{\mathrm{H}}^{\mathrm{m}+1}$ with $\mathbf{G}\widetilde{(\mathrm{X},\mathrm{H})}$ over $\mathbf{K}(\mathrm{S},\mathrm{H})$):

$$\mathcal{I}_{\mathrm{H}}^{m+1} \underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge} \mathbf{G}\widetilde{(\mathrm{X},\mathrm{H})} \to \mathbf{G}\widetilde{(\mathrm{X},\mathrm{H})} \to \mathbf{K}(\mathrm{S},\mathrm{H})/\mathcal{I}_{\mathrm{H}}^{\mathrm{m}+1} \underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge} \mathbf{G}\widetilde{(\mathrm{X},\mathrm{H})}$$
$$\simeq \mathbf{G}\widetilde{(\mathrm{X},\mathrm{H})}/(\mathcal{I}_{\mathrm{H}}^{\mathrm{m}+1} \underset{\mathbf{K}(\mathrm{S},\mathrm{H})}{\wedge} \mathbf{G}\widetilde{(\mathrm{X},\mathrm{H})}). \tag{5.0.12}$$

By [7, Corollary 6.7] again, one sees that last term above identifies with $\mathbf{G}(\mathrm{X},\mathrm{H})\,\widehat{}_{\rho_{\mathrm{H}},\mathrm{m}}$. Therefore, one obtains the weak-equivalence

$$\mathbf{G}(\mathrm{X},\mathrm{H})\,\widehat{}_{\rho_{\mathrm{H}},\mathrm{m}} \simeq \mathbf{G}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}} \underset{\mathrm{H}}{\times} \mathrm{X}). \tag{5.0.13}$$

These complete the proof of statement (i) in this case, and statement (ii) is clear from the above discussion. Taking the homotopy inverse limit as $m \to \infty$, one obtains statement (iii). This completes the proof when H is a one dimensional torus.

Next let H denote a split torus of rank $n > 1$. We will assume using ascending induction on n that the theorem is true for all split tori of rank $< n$. Let $\mathrm{H} = \mathrm{H}_1 \times \mathrm{H}_2$, where H_1 is a one dimensional torus and H_2 is a split torus of rank $n - 1$.

Observe that $\mathbf{K}(\mathrm{S},\mathrm{H}) = \mathbf{K}(\mathrm{S},\mathrm{H}_1 \times \mathrm{H}_2) \simeq \mathbf{K}(\mathrm{S},\mathrm{H}_1) \wedge_{\mathbf{K}(\mathrm{S})} \mathbf{K}(\mathrm{S},\mathrm{H}_2)$, so that Proposition 5.3(iii), Proposition 5.2 along with the inductive assumption shows that

$$\begin{aligned}\mathbf{K}(\mathrm{S},\mathrm{H}_1 \times \mathrm{H}_2)\,\widehat{}_{\rho_{\mathrm{H}},\mathrm{m},\mathrm{m}} &\simeq \mathbf{K}(\mathrm{S},\mathrm{H}_1)\,\widehat{}_{\rho_{\mathrm{H}_1},\mathrm{m}} \wedge_{\mathbf{K}(\mathrm{S})} \mathbf{K}(\mathrm{S},\mathrm{H}_2)\,\widehat{}_{\rho_{\mathrm{H}_2},\mathrm{m}} \\ &\simeq \mathbf{K}(\mathrm{EH}_1^{\mathrm{gm},\mathrm{m}},\mathrm{H}_1) \wedge_{\mathbf{K}(\mathrm{S})} \mathbf{K}(\mathrm{EH}_2^{\mathrm{gm},\mathrm{m}},\mathrm{H}_2) \\ &\simeq \mathbf{K}(\mathrm{EH}_1^{\mathrm{gm},\mathrm{m}} \times_{\mathrm{S}} \mathrm{EH}_2^{\mathrm{gm},\mathrm{m}},\mathrm{H}) = \mathbf{K}(\mathrm{EH}^{\mathrm{gm},\mathrm{m}},\mathrm{H}).\end{aligned} \tag{5.0.14}$$

(One may observe that Proposition 5.2 provides the third weak-equivalence.) Therefore, one sees that the stable cofibration sequence in (5.0.10) still holds with $\mathrm{H} = \mathrm{H}_1 \times \mathrm{H}_2$, and where $\mathcal{I}_{\mathrm{H}}$ denotes the homotopy fiber of the map $\mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S})$ induced by the restriction map. Now taking smash product over $\mathbf{K}(\mathrm{S},\mathrm{H})$ of the terms of in the stable cofiber sequence in (5.0.10) with $\mathbf{G}\widetilde{(\mathrm{X},\mathrm{H})}$ once again produces the stable cofibration sequence in (5.0.11).

By smashing the stable cofiber sequence $\mathcal{I}_{\mathrm{H}}^{m+1} \to \mathbf{K}(\mathrm{S},\mathrm{H}) \to \mathbf{K}(\mathrm{S},\mathrm{H})/\mathcal{I}_{\mathrm{H}}^{\mathrm{m}+1}$ with $\mathbf{G}\widetilde{(\mathrm{X},\mathrm{H})}$ over $\mathbf{K}(\mathrm{S},\mathrm{H})$) provides the stable cofibration (or equivalently stable fibration) sequence in (5.0.12). The above arguments show that one obtains the weak-equivalence in (5.0.13) for the split torus H. These observations complete the proof of statements (i) and (ii) in this case. Taking the homotopy inverse limit as $m \to \infty$, one obtains statement (iii). This concludes the proof of the theorem for the split torus H. □

We conclude this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. Let G denote a split reductive group over k, which is a finite product of GL_ns and containing the given linear algebraic group H as a closed subgroup. Then Proposition 1.1 provides the weak-equivalence:

$$\mathbf{G}(\mathrm{X}, \mathrm{H})\widehat{}_{\rho_\mathrm{H}} \simeq \mathbf{G}(\mathrm{G} \times_\mathrm{H} \mathrm{X}, \mathrm{G})\widehat{}_{\rho_\mathrm{G}}.$$

At this point Theorem 4.7 (aided by Theorem 3.6) provides the weak-equivalence:

$$\mathbf{G}(\mathrm{G} \times_\mathrm{H} \mathrm{X}, \mathrm{G})\widehat{}_{\rho_\mathrm{G}} \simeq \mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{G}}{\times} (\mathrm{G} \times_\mathrm{H} \mathrm{X})) \simeq \mathbf{G}(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{H}}{\times} \mathrm{X}),$$

by reducing this statement to the corresponding statement when G is replaced by a maximal torus T in G. Theorem 5.6 proves this statement for the case when the group G is a split torus. These complete the proof of the statement in (i).

The statement in (ii) is a variant of (i) and may be proven similarly. The statement in (iii) follows now from (B.0.4). Observe that Theorem 5.6 holds for all separated algebraic spaces of finite type over the given base scheme S. This observation proves the last statement in Theorem 1.2. □

6. Comparison with existing results, examples and applications

6.1. Comparison with Thomason's theorem

In this section we will first compare our results with prior results on Atiyah-Segal type completion theorems in equivariant G-theory, the earliest of which are those of Thomason. In [56, Theorem 3.2], Thomason proves the following theorem:

Theorem 6.1. *Let* G *denote a linear algebraic group scheme over a separably closed field* k *so that it is the product of a group scheme smooth over* k *and an infinitesimal group scheme. Let* X *denote a separated algebraic space on which* G *acts. Let* ℓ *denote a prime that is invertible in* k*, let* β *denote the Bott element and let* I_G *denote the kernel of rank map* $R(\mathrm{G}) \to \mathbb{Z}$*. Then the map*

$$\pi_*(\mathbf{G}/\ell^\nu(\mathrm{X}, \mathrm{G})[\beta^{-1}])\widehat{}_{\mathrm{I_G}} \xrightarrow{\simeq} \pi_* \mathbf{G}/\ell^\nu(\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{G}}{\times} \mathrm{X})[\beta^{-1}])$$

is an isomorphism, where the completion on the left denotes the completion of the homotopy groups
$\pi_*(\mathbf{G}/\ell^\nu(\mathrm{X}, \mathrm{G})[\beta^{-1}])$ *at the ideal* I_G*.*

In the above theorem of Thomason, the strategy is to show that a spectral sequence whose E_2-terms are the cohomology groups computed on a site called the *isovariant étale site* associated to the G-scheme X with $\mathbb{Z}/\ell^\nu$-coefficients converges strongly to the above homotopy groups. (The above spectral sequence is an equivariant analogue of the spectral sequence relating étale cohomology to mod$-\ell^\nu$-algebraic K-theory with the Bott-element inverted as in [55].) *All the stringent hypotheses stated there seem necessary to obtain strong convergence of this spectral sequence and thereby to conclude that the homotopy groups* $\pi_*(\mathbf{G}/\ell^\nu(\mathrm{X},\mathrm{G})[\beta^{-1}])$ *are finite modules over the representation ring* $R(\mathrm{G})$*.* Nevertheless, the following simple example shows that for lots of varieties with group actions, the Bott-element inverted form of equivariant algebraic K-theory with finite coefficients is distinct from the corresponding equivariant algebraic K-theory with finite coefficients. Thus Thomason's theorem does *not provide* an Atiyah-Segal type completion theorem for equivariant algebraic K-theory, but only for its topological variant, that is, with the Bott element inverted.

6.1.1. Equivariant algebraic K-theory and the equivariant algebraic K-theory with the Bott element inverted are not isomorphic in general

Let k denote an algebraically closed field and let ℓ denote a prime different from $char(k)$. Let $s \geq 1$ be an integer and let $\mathrm{X} = \mathbb{G}_m^s$. Then we make the following observation:

Proposition 6.2. *The map* $\pi_i(\mathbf{K}(\mathbb{G}_m^s)/\ell^\nu) \to \pi_i(\mathbf{K}(\mathbb{G}_m^s)/\ell^\nu[\beta^{-1}])$ *is an isomorphism for all* $i \geq 0$ *if* $s = 1$ *and for* $s \geq 2$*, it is not surjective for* $i < s-1$*. In particular, for all* $i < s-1$*, the above map is only an injection.*

Proof. For $s = 1$, this follows readily from Quillen's calculation of the K-groups of $\mathbb{G}_m$: that is, $\pi_i(\mathbf{K}(\mathbb{G}_m)) = \pi_i(\mathbf{K}(\mathrm{Spec}\, k)) \oplus \pi_{i-1}(\mathbf{K}(\mathrm{Spec}\, k))$, for $i \geq 0$. (See [39, Corollary to Theorem 8].) In fact the above result follows more generally from the localization sequence for any smooth scheme X:

$$\mathbf{K}(\mathrm{X} \times \{0\}) \to \mathbf{K}(\mathrm{X} \times \mathbb{A}^1) \xrightarrow{\mathrm{j}^*} \mathbf{K}(\mathrm{X} \times \mathbb{G}_\mathrm{m}),$$

along with the observation that the composite map $j^* \circ \pi^*$ is a split monomorphism, where $\pi : \mathrm{X} \times \mathbb{A}^1 \to \mathrm{X}$ is the projection and $j : \mathrm{X} \times \mathbb{G}_\mathrm{m}^1 \to \mathrm{X} \times \mathbb{A}^1$ is the open immersion. Therefore, the same conclusions hold for $\mathbf{K}(\quad)/\ell^n$ and $\mathbf{K}(\quad)/\ell^n[\beta^{-1}]$. Hence, one may compute

$$\begin{aligned}\pi_0(\mathbf{K}(\mathbb{G}_m^2)/\ell^n[\beta^{-1}]) &\cong \pi_0(\mathbf{K}(\mathrm{Spec}\, k)/\ell^n[\beta^{-1}]) \oplus \pi_{-2}(\mathbf{K}(\mathrm{Spec}\, k)/\ell^n[\beta^{-1}]) \\ &\cong \mathbb{Z}/\ell^n \oplus \mathbb{Z}/\ell^n, \text{ while} \\ \pi_0(\mathbf{K}(\mathbb{G}_m^2)/\ell^n) &\cong \pi_0(\mathbf{K}(\mathrm{Spec}\, k)/\ell^n) \cong \mathbb{Z}/\ell^n.\end{aligned}$$

Therefore, for $s = 2$, the map $\pi_i(\mathbf{K}(\mathbb{G}_m^s)/\ell^\nu) \to \pi_i(\mathbf{K}(\mathbb{G}_m^s)/\ell^\nu[\beta^{-1}])$ is *not* surjective. One may similarly obtain the same conclusion for all $s \geq 2$.

Then, observing that $\mathbb{G}_m$ is a linear scheme, one invokes the derived Kunneth formula in [25, Theorem 4.2]. Then one observes that the corresponding spectral sequences degenerate (in view of the computation of the K-groups of $\mathbb{G}_m$ above), thereby providing the isomorphisms

$$\pi_*(\mathbf{K}(\mathbb{G}_m^s)/\ell^\nu) \cong \pi_*(\mathbf{K}(\mathbb{G}_m^{s-1})/\ell^\nu) \underset{\pi_*(\mathbf{K}(\mathrm{Spec}\, k)/\ell^n)}{\otimes} \pi_*(\mathbf{K}(\mathbb{G}_m)/\ell^\nu), \tag{6.1.1}$$

$$\begin{aligned}\pi_*(\mathbf{K}(\mathbb{G}_m^s)/\ell^\nu[\beta^{-1}]) &\cong \pi_*(\mathbf{K}(\mathbb{G}_m^{s-1})/\ell^\nu[\beta^{-1}]) \underset{\pi_*(\mathbf{K}(\mathrm{Spec}\, k)/\ell^\nu[\beta^{-1}])}{\otimes} \pi_*(\mathbf{K}(\mathbb{G}_m)/\ell^\nu[\beta^{-1}]) \\ &\cong \pi_*(\mathbf{K}(\mathbb{G}_m^{s-1})/\ell^\nu[\beta^{-1}) \underset{\pi_*(\mathbf{K}(\mathrm{Spec}\, k)/\ell^\nu)}{\otimes} \pi_*(\mathbf{K}(\mathbb{G}_m)/\ell^\nu).\end{aligned}$$

Now an induction on s, will show that the induced map of the right-hand-sides is an injection on homotopy groups for all $i \geq 0$. $\square$

Now let $s \geq 3$, $\mathrm{G} = \mathbb{G}_\mathrm{m}$ acting by translation on the first factor in $\mathrm{X} = \mathbb{G}_\mathrm{m}^\mathrm{s}$ and trivially on the other factors. Observing that the classifying space for $\mathbb{G}_m$ is $\mathbb{P}^\infty$, one can readily see that $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X} \simeq \mathrm{EG}\times\mathbb{G}_\mathrm{m}^{\mathrm{s}-1} \simeq \mathbb{G}_\mathrm{m}^{\mathrm{s}-1}$. Therefore, it follows that the homotopy groups of the mod-ℓ^ν algebraic K-theory spectrum of $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ and the homotopy groups of the corresponding K-theory spectrum with the Bott-element inverted are not isomorphic in degrees $< s-2$. The case where $\mathbb{G}_m$ acts diagonally on $\mathbb{G}_m^s$ may be reduced to this case by viewing $\mathbb{G}_m^s \cong \Delta(\mathbb{G}_m) \times (\mathbb{G}_m^s/\Delta(\mathbb{G}_m))$. Moreover, the above example shows that for varieties provided with an action by a torus $\mathbb{G}_m$, and provided with strata that are products of affine spaces and split tori (as is the case with all linear varieties), the $\mathbb{G}_m$-equivariant algebraic K-theory is likely to be distinct from the corresponding Bott-element inverted variant.

Our main result, Theorem 1.2, is a huge improvement over the above theorem of Thomason, since we do not need to work modulo a prime nor with the Bott element inverted, nor with any other serious restriction such as the base field being separably closed. This is made possible by the use of derived completions.

6.2. Comparison with the completion theorem in [29]

Next we proceed to compare our results in detail with the most recent attempt in [29], at proving Atiyah-Segal completion theorems, for equivariant algebraic K-theory integrally and without inverting the Bott element, making use of the classical completion of the homotopy groups of the equivariant K-theory spectra at the augmentation ideal of the representation ring.

6.2.1. The range of allowed group actions in [29]

In [29, Theorem 1.2] one has to restrict to connected groups, thereby disallowing even actions by finite abelian groups. In contrast, in this paper the following are the assumptions.

(i) We may assume the base scheme S is a regular Noetherian affine scheme of finite type over a field, in general, though, we may restrict to the case where S is the spectrum of a field to keep the discussion simple enough. Then we allow any smooth linear algebraic group scheme G acting on any quasi-projective variety (scheme) that admits a G-equivariant closed immersion into a regular quasi-projective variety (scheme) provided with a G-action.

(ii) This puts very little restriction on the groups that are allowed: clearly all finite groups may be imbedded as closed subgroups of a suitable GL_n and therefore are allowed in Theorem 1.2.

6.2.2. The range of allowed objects with group action in [29]

(i) The objects considered have to be *schemes of finite type over a field.* In particular, this means natural objects like *algebraic spaces cannot* be considered.

(ii) The schemes have to be *both projective as well as smooth.* In particular, varieties that are non-smooth, like *normal varieties* are not allowed in general. Recall that *toric varieties and spherical varieties* are defined to be *normal varieties* satisfying certain further hypotheses. This shows that large classes of commonly occurring varieties like toric varieties that are singular or non-projective (when the group is a torus) are not allowed. Similarly, the large class of varieties called *spherical varieties (when the group is a reductive group, even as simple as* GL_n *or* SL_n*) that are singular or non-projective cannot be considered.* A variant of this is the counter-example discussed in [29, Theorem 1.5] where the scheme is the homogeneous space $\mathbb{G}_m/\mu_2$ for the action of $\mathbb{G}_m$, all over $\mathbb{C}$. This shows that [29, Theorem 1.2], *fails even for smooth varieties such as homogeneous spaces for a linear algebraic group, which are not projective.* The source of the counter-example is that the first map in [29, (10.3)] is not surjective.

On using derived completions, the corresponding map from the partial derived completion to the corresponding term of the Borel-style equivariant G-theory spectrum is a weak-equivalence as shown in (5.0.13). Therefore, this counter-example does not arise on using derived completions.

(iii) The assumption of projective smoothness over a field and the restriction to actions by connected split reductive groups in [29] is to be able to reduce to the case of schemes stratified by strata that are affine spaces over a smooth projective scheme with trivial action, by making strong use of Bialynicki-Birula decomposition. All toric and spherical varieties have only finitely many fixed points for the action of a maximal torus. This means the only toric and spherical varieties that are allowed by [29, Theorem 1.2] are those schemes stratified by strata that are affine spaces, which forms an extremely narrow class of varieties. Moreover for such varieties over separably closed fields, the map from equivariant algebraic K-theory to equivariant algebraic K-theory with the Bott element inverted, and both with coefficients prime to the characteristics, is an isomorphism in non-negative degrees, so that [29, Theorem 1.2] in this case is implied by Thomason's Theorem (that is, [56, Theorem 3.2]).

Remark 6.3. The kind of dimension shifting that occurs in [29, Theorem 1.5] is quite natural from our point of view. In our case, the derived completion when applied to rings and modules, creates higher dimensional homotopy. For example, the derived completion at the ring homomorphism $\mathbb{Z} \to \mathbb{F}_\ell$ of the module $\mathbb{Q}/\mathbb{Z}$ has $\pi_0 \cong 0$ and $\pi_1 \cong \mathbb{Z}_\ell$. In the counter example in [29, Theorem 1.5], this is accounted for by the fact that the derived completion of the module $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ at the augmentation homomorphism $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \to \mathbb{Z}$ has $\pi_0 \cong \mathbb{Z}$ and $\pi_1 \cong \mathbb{Z}_2$.

The rest of this section will focus on various applications and good features of the derived completion developed in the earlier sections.

6.3. Equivariant G-theory and K-theory of G*-quasi-projective varieties and schemes: toric varieties, spherical varieties and linear varieties*

We will again restrict to the case the base scheme is a field. For a given linear algebraic group G, a convenient class of schemes we consider are what are called G-*quasi-projective schemes*: a quasi-projective scheme X with a given G-action is G-quasi-projective, if it admits a G-equivariant locally closed immersion into a projective space, Proj(V), for a representation V of G. Then a well-known theorem of Sumihiro, (see [52, Theorem 2.5]) shows that any normal quasi-projective variety provided with the action of a connected linear algebraic group G is G-quasi-projective. This may be extended to the case where G is not necessarily connected by invoking the above result for the action of G^o and then considering the closed G-equivariant closed immersion of the given G-scheme into $\Pi_{g\ \epsilon\ G/G^o}\mathrm{Proj}(V)$ followed by a G-equivariant closed immersion of the latter into $\mathrm{Proj}(\otimes_{g\ \epsilon\ G/G^o}V)$. (See for example. [57, p. 629].)

Now the important observation is that *quasi-projective toric varieties* are defined to be *normal* varieties on which a split torus T acts with finitely many orbits. Therefore, such toric varieties are T-quasi-projective. This already includes many familiar varieties: see [37], for example.

Given a connected split reductive group G, a *quasi-projective* G-*spherical variety* is defined to be a normal G-variety on which G, as well as a Borel subgroup of G, have finitely many orbits. (See [53].) This class of varieties clearly includes all toric varieties (when the group is a split torus), but also includes many other varieties. The discussion in the first paragraph above, now shows these are also G-quasi-projective varieties.

Finally we recall that linear varieties are those varieties that have a stratification where each stratum is a product of an affine space and a split torus. We will restrict to quasi-projective linear varieties with an action by the linear algebraic group G, where each stratum is also G-stable. It is clear the G-equivariant **G**-theory of such varieties can be readily understood in terms of the corresponding G-equivariant **G**-theory of the strata.

Examples of such varieties are normal quasi-projective varieties on which a connected split solvable group acts with finitely many orbits. (See [43, p. 119].)

It is important to observe that all the main theorems in this paper, Theorems 1.2 and 1.6 apply to all the above classes of schemes, without any restrictions, that is, to all toric varieties, all spherical varieties and all G*-quasi-projective varieties, irrespective of whether they are smooth or projective. See also 6.4 below.*

The example in 6.1.1 shows that for large classes of toric and spherical varieties, the equivariant algebraic K-theory with finite coefficients away from the characteristic is not isomorphic to the corresponding equivariant K-theory with the Bott element inverted, so that Thomason's theorem does not provide a completion theorem for equivariant K-theory unless the Bott element is inverted. Neither does [29, Theorem 1.2] unless these varieties are stratified by affine spaces.

6.4. Computation of the homotopy groups of the derived completion

Assume that X is a G-scheme as in 6.3, with G = T a split torus. Then the approximation $\mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{G}}{\times} \mathrm{X}$ is always a scheme, and therefore, for a fixed value of $-s-t$, the homotopy groups of its **G**-theory may be computed using the spectral sequence with m sufficiently large:

$$E_2^{s,t} = \mathbb{H}^{s-t}_{BM,M}(\mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{G}}{\times} \mathrm{X}, \mathbb{Z}(-t)) \Rightarrow \pi_{-s-t}(\mathbf{G}(\mathrm{EG}^{gm,m} \underset{\mathrm{G}}{\times} \mathrm{X})) \cong \pi_{-s-t}(\mathbf{G}(\mathrm{X},\mathrm{G})\widehat{\ }_{\rho_{\mathrm{G},m}})$$

The last isomorphism is from (5.0.13) and $\mathbb{H}_{BM,M}$ denotes Borel-Moore motivic cohomology, which identifies with the higher equivariant Chow groups. (This identification shows that the $E_2^{s,t} = 0$ for all but finitely many values of s and t, once $-s-t$ is fixed. Therefore, these E_2-terms, and hence the abutment, will be independent of m, if $m >> 0$.) In addition, rationally, the homotopy groups $\pi_{-s-t}(\mathbf{G}(\mathrm{X},\mathrm{G})\widehat{\ }_{\rho_{\mathrm{G,m}}})$ identify with the rational G-equivariant higher Chow groups. For example, if X has only finitely many G-orbits, then the homotopy groups of the derived completion may be computed from the higher Chow-groups of the classifying spaces of the stabilizer groups. Such computations will hold if X is *any* toric variety and G denotes the corresponding torus. For more general groups other than a split torus, there is still a similar spectral sequence, but only for the full $\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{G}}{\times} \mathrm{X}$. This spectral sequence converges only conditionally, and computes the homotopy groups of the full derived completion.

6.5. Actions on algebraic spaces

Recall that Theorem 1.2(iii) applies to actions by split tori on algebraic spaces. Here we will mention some well-known examples where one is forced to consider algebraic spaces. Consider the action of a finite group on a scheme. If each orbit is not contained in an affine subscheme, then the quotient for the group action may not exist as a scheme, but only as an algebraic space. A well-known example is the non-projective proper three-fold due to Hironaka (see [20] or [18, p. 443]) provided with the action of the cyclic group of order two. In this case, the quotient exists only as an algebraic space.

Thus taking geometric quotients of schemes by actions of finite and reductive groups on schemes often produce algebraic spaces that are not schemes. Therefore, derived completion theorems for algebraic spaces provided with actions by split tori, such as in Theorem 1.2(iv), greatly increases the breadth and utility of our theorems.

6.6. Higher equivariant Riemann-Roch and Lefschetz-Riemann-Roch theorems

An immediate consequence of our main theorems is a strong form of Riemann-Roch theorem relating higher equivariant **G**-theory with other forms of Borel-style equivariant homology theories for actions of linear algebraic groups on schemes (and also algebraic spaces, if the group actions are by split tori). We consider this briefly as follows. Let $f : \mathrm{X} \rightarrow \mathrm{Y}$ denote a proper H-equivariant map between two H-quasi-projective schemes, where H is a linear algebraic group. Let G denote either a GL_n or a finite product of groups of the form GL_n containing H as a closed subgroup-scheme. Then the squares

$$\begin{CD}
\mathbf{G}(\mathrm{X},\mathrm{H}) @>>> \mathbf{G}(\mathrm{X},\mathrm{H})\,\widehat{}_{\rho_{\mathrm{G},\alpha(\mathrm{m})}} @>>> \mathbf{G}(\mathrm{EG}^{gm,m}\times \mathrm{X},\mathrm{H}) @>{\simeq}>> \mathbf{G}(\mathrm{EG}^{gm,m}\underset{\mathrm{H}}{\times}\mathrm{X})\\
@V{Rf_*}VV @V{Rf_*}VV @V{Rf_*}VV @V{Rf_*}VV\\
\mathbf{G}(\mathrm{Y},\mathrm{H}) @>>> \mathbf{G}(\mathrm{Y},\mathrm{H})\,\widehat{}_{\rho_{\mathrm{G},\alpha(\mathrm{m})}} @>>> \mathbf{G}(\mathrm{EG}^{gm,m}\times \mathrm{Y},\mathrm{H}) @>{\simeq}>> \mathbf{G}(\mathrm{EG}^{gm,m}\underset{\mathrm{H}}{\times}\mathrm{Y})
\end{CD}$$

homotopy commute, for any fixed integer $m \geq 0$, with $\alpha(m)$ as in Definition 4.6. (The commutativity of the left-square follows from Proposition 4.5(ii), after one identifies $\mathbf{G}(\mathrm{X},\mathrm{H})$ with $\mathbf{G}(\mathrm{G}\times_{\mathrm{H}}\mathrm{X},\mathrm{G})$ and similarly for Y. The composition of the left two horizontal maps is given by sending a complex of H-equivariant coherent sheaves on X (Y) to its pull-back on $\mathrm{EG}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}$ ($\mathrm{EG}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{Y}$)), respectively. Therefore, the homotopy commutativity of the two left-most squares follows from Proposition 4.5.) One may compose with the usual Riemann-Roch transformation into any equivariant Borel-Moore homology theory such as equivariant higher Chow groups, to obtain higher equivariant Riemann-Roch theorems that hold in general. When the group G is a split torus, one may also assume in the above theorem that X and Y are separated algebraic spaces of finite type (over the given base field). Moreover, in such cases, one can also consider Lefschetz-Riemann-Roch theorems involving the fixed points of the torus actions. Such equivariant Riemann-Roch theorems will be pursued (in more detail) separately in forthcoming work: see [9]. Note that [1] is classical result in this flavor.

It may be important to point out that, in view of the difficulties with usual Atiyah-Segal type completion, (for example as in [29]), such Riemann-Roch theorems are currently known only for Grothendieck groups, unless one assumes that both X and Y are projective smooth schemes or in the non-equivariant case as in [41].

6.7. Actions on projective smooth schemes over a field

Finally we proceed to show that for projective smooth schemes over a field provided with an action by a connected split reductive group, the homotopy groups of the derived completions in fact reduce to the usual completions of the homotopy groups of equivariant K-theory at the augmentation ideal and thereby complete the proof of Theorem 1.7. This shows that the results of [29, Theorem 1.2] all follow as a corollary to our more general derived completion theorems.

Proposition 6.4. *Let* X *denote a scheme with a trivial action by a split torus* T*, all defined over a field k. Then* $\pi_*(\mathbf{K}(\mathrm{X},\mathrm{T}))\widehat{}_{\mathrm{I_T}} \cong \pi_*(\mathbf{K}(\mathrm{X},\mathrm{T})\widehat{}_{\rho_\mathrm{T}})$ *where the completion on the left denotes the completion at the augmentation ideal* $\mathrm{I_T}$ *in the usual sense and the completion on the right denotes the derived completion along* ρ_T*.*

Proof. Observe that $\pi_*(\mathbf{K}(\mathrm{X},\mathrm{T})) \cong \pi_*(\mathbf{K}(\mathrm{X}))\underset{\mathbb{Z}}{\otimes}\mathrm{R}(\mathrm{T})$ and that R(T) is Noetherian. Therefore, [7, Theorem 4.4] shows the derived completion of $\pi_*(\mathbf{K}(\mathrm{X},\mathrm{T}))$ at the ideal $\mathrm{I_T}$ identifies with the usual completion of $\pi_*(\mathbf{K}(\mathrm{X},\mathrm{T}))$ at $\mathrm{I_T}$. Finally, the first spectral sequence in (3.0.4) degenerates identifying the homotopy groups $\pi_*(\mathbf{K}(\mathrm{X},\mathrm{T})\widehat{}_{\rho_\mathrm{T}})$ with the derived completion of $\pi_*\mathbf{K}(\mathrm{X}) \otimes \mathrm{R}(\mathrm{T})$ at $\mathrm{I_T}$, which by the above arguments identify with $\pi_*\mathbf{K}(\mathrm{X}) \otimes \mathrm{R}(\mathrm{T})\widehat{}_{\mathrm{I_T}} \cong \pi_*(\mathbf{K}(\mathrm{X},\mathrm{T}))\widehat{}_{\mathrm{I_T}}$. □

An immediate consequence of these results is Theorem 1.7 stated in the introduction.

Proof of Theorem 1.7. First, we observe the commutative diagram:

$$\begin{tikzcd}
\pi_*(\mathbf{G}(\mathrm{X},\mathrm{G}))\widehat{}_{\mathrm{I_G}} \arrow[r] \arrow[d, "\pi^*"] & \pi_*(\mathbf{G}(\mathrm{X},\mathrm{G})\widehat{}_{\rho_\mathrm{G}})\widehat{}_{\mathrm{I_G}} \arrow[d, "\pi^*"] & \pi_*(\mathbf{G}(\mathrm{X},\mathrm{G})\widehat{}_{\rho_\mathrm{G}}) \arrow[l] \arrow[d, "\pi^*"] \\
\pi_*(\mathbf{G}(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X},\mathrm{G}))\widehat{}_{\mathrm{I_G}} \arrow[r] \arrow[d, "\pi_*"] & \pi_*(\mathbf{G}(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X},\mathrm{G})\widehat{}_{\rho_\mathrm{G}})\widehat{}_{\mathrm{I_G}} \arrow[d, "\pi_*"] & \pi_*(\mathbf{G}(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X},\mathrm{G})\widehat{}_{\rho_\mathrm{G}}) \arrow[l] \arrow[d, "\pi_*"] \\
\pi_*(\mathbf{G}(\mathrm{X},\mathrm{G}))\widehat{}_{\mathrm{I_G}} \arrow[r] & \pi_*(\mathbf{G}(\mathrm{X},\mathrm{G})\widehat{}_{\rho_\mathrm{G}})\widehat{}_{\mathrm{I_G}} & \pi_*(\mathbf{G}(\mathrm{X},\mathrm{G})\widehat{}_{\rho_\mathrm{G}}) \arrow[l]
\end{tikzcd} \tag{6.7.1}$$

Here B is a fixed Borel subgroup of G. The observation that there is a natural map $\mathbf{G}(\mathrm{Y},\mathrm{G}) \to \mathbf{G}(\mathrm{Y},\mathrm{G})\widehat{}_{\rho_\mathrm{G}}$ for any G-scheme Y readily provides the horizontal maps. Since the composition of the maps π^* and π_* are isomorphisms, it suffices to prove both maps in the middle row are isomorphisms. Let T denote split maximal torus contained in B. Then, one observes the isomorphisms: $\pi_*(\mathbf{G}(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X},\mathrm{G}))\widehat{}_{I_G} \cong \pi_*(\mathbf{G}(\mathrm{G}\underset{\mathrm{B}}{\times}\mathrm{X},\mathrm{G}))\widehat{}_{\mathrm{I_T}}$ and

$\pi_*(\mathbf{G}(G\underset{B}{\times}X, G)\widehat{\ }_{\rho_G})\widehat{\ }_{I_G} \cong \pi_*(\mathbf{G}(G\underset{B}{\times}X, G)\widehat{\ }_{\rho_T})\widehat{\ }_{I_T}$. These follow from the fact that the I_T-adic and the $I_G R(T)$-adic completions are the same on any R(T)-module (as proven in [4, p. 8]) and Theorem 3.6. These reduce the proof to the case where the group $G = T$ is a split torus.

Next, making use of Bialynicki-Birula decomposition, for the action of a split torus T on the smooth projective scheme X, one shows that the theorem holds for the action of T on X by reducing it to the trivial action of T on the fixed point scheme X^T, at which point Proposition 6.4 applies. (A key observation here is the following: there is an ordering of the connected components of the fixed point scheme $X^T = \sqcup_{i=0}^n Z_i$ so that there exists a corresponding filtration $\{\phi = X_{-1} \subsetneq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X\}$ so that each $X_i - X_{i-1} \rightarrow Z_i$ is a T-equivariant affine space bundle. Therefore, one may show $\mathbf{K}(X, T) \simeq \vee_i \mathbf{K}(Z_i, T)$.) This proves Theorem 1.7(i).

Theorem 1.2(i) with $G = GL_n$ and $H = G$ proves that $\pi_*(\mathbf{K}(X, G)\widehat{\ }_{\rho_{GL_n}}) \cong \pi_*(\mathbf{K}(EGL_n\underset{G}{\times}X))$
$\cong \pi_*(\mathbf{K}(EG\underset{G}{\times}X))$, since EGL_n also satisfies the properties required of EG: see (B.0.4). But by [29, Theorem 1.2], this identifies with $\pi_*(\mathbf{K}(X, G))\widehat{\ }_{I_G}$. This proves (ii). □

6.8. Various extensions of our current results

The key reason we have restricted to equivariant G-theory in this paper is because of the readily available localization sequences. However, already by the late 1980s (see [58]), Thomason had established localization sequences for the K-theory of perfect complexes and recent work, for example [44], considers an abstract framework in which localization sequences for K-theory holds: this framework applies to the equivariant situation by [44, section 5]. On considering the associated homotopy invariant version, one also recovers the homotopy property.

One of our immediate goals therefore is to extend the derived completion theorems proven here to equivariant (homotopy invariant) algebraic K-theory of perfect complexes and also to actions by families of linear algebraic groups. The first of this is already worked out in detail in [9].

Appendix A. Chain complexes vs. abelian group spectra

In this section we recall certain well-known relations between the above two categories, the basic references being [48] and [50]. Since this correspondence is discussed in detail in the above references, we will simply summarize the key points for the convenience of the reader. One may also consult [10], [17], [23] and [33] for related discussions.

A chain complex will mean one with differentials of degree −1. Let R denote a commutative ring with 1. Then $\mathbb{H}(R)$ will denote the Eilenberg-Maclane spectrum defined as follows. The space in degree n, $\mathbb{H}(R)_n$ is the underlying simplicial set of $R \otimes S^n$ which in degree k is the free R-module on the non-base point k-simplices of S^n (and with the base point identified with 0). The symmetric group Σ_n acts on $S^n \cong (S^1)^{\wedge n}$ and therefore on $\mathbb{H}(R)_n$. Clearly there is a map from the simplicial suspension $S^1 \wedge \mathbb{H}(R)_n \to \mathbb{H}(R)_{n+1}$ (compatible with the action of Σ_{n+1}), so that this defines a symmetric spectrum. This is the *Eilenberg-Maclane spectrum associated to* R. One may observe this is an Ω-spectrum and that it is also a symmetric ring spectrum.

Next one may define a functor $\mathbb{H} : Mod(R) \to Mod(\mathbb{H}(R))$, where $Mod(R)$ ($Mod(\mathbb{H}(R))$) denotes the category of modules over R (module spectra over $\mathbb{H}(R)$, respectively) by $\mathbb{H}(M) = M \otimes_R \mathbb{H}(R)$: the n-th space $\mathbb{H}(M)_n$ is now $M \otimes_R (R \otimes S^n)$.

Though the normalizing functor N sending a simplicial abelian group to its associated normalized chain complex is symmetric monoidal, its familiar inverse Γ (defined by the classical Dold-Kan equivalence) is *not* symmetric monoidal. This is the main difficulty in extending the functor $\mathbb{H}$ to a monoidal functor defined on all unbounded chain complexes of R-modules with the usual tensor product as the monoidal structure. Nevertheless, solutions to this problem have been worked out in detail in [48, Appendix B]. (See also [50] and [42].) The main result then is the following.

Let $C(Mod(R))$ denote the category of unbounded chain complexes (that is, with differentials of degree -1) of R-modules, with the tensor product of chain complexes as the monoidal product and provided with the projective model structure where the fibrations are degree-wise surjections, weak-equivalences are maps that induce isomorphism on homology and the cofibrations are defined by the lifting property. This is the model structure discussed in A.0.5 below. Let $Mod(\mathbb{H}(R))$ denotes the category of symmetric module spectra over $\mathbb{H}(R)$, provided with $\wedge_{\mathbb{H}(R)}$ as the monoidal product and with the model structure defined on it as in [47, section 4]. Then these two model categories are related by functors

$$\mathbb{H} : C(Mod(R)) \to Mod(\mathbb{H}(R)), \Theta : Mod(\mathbb{H}(R)) \to C(Mod(R)) \tag{A.0.1}$$

so that they are part of a Quillen equivalence between the corresponding derived categories. Since the two functors $\mathbb{H}$ and Θ are weak-monoidal functors, one also obtains a weak-equivalence

$$\mathbb{H}(M) \underset{\mathbb{H}(R)}{\overset{L}{\wedge}} \mathbb{H}(N) \simeq \mathbb{H}(M \underset{R}{\overset{L}{\otimes}} N), \ M, N \in C(Mod(R)). \tag{A.0.2}$$

that is natural in the arguments M and N. Here the left-derived functor of $\underset{\mathbb{H}(R)}{\wedge}$ $(\underset{R}{\otimes})$ is defined by replacing one of the arguments by a cofibrant replacement. The above weak-equivalence is not induced by a map from the left-hand-side to the right-hand-side: rather, the left-hand-side and the right-hand-side are related by maps which are weak-equivalences into intermediate spectra. Nevertheless, this suffices for us to reduce computations of the left-hand-side to computing $M \underset{R}{\overset{L}{\otimes}} N$.

One may make use of the discussion on model structures for $C(Mod(R))$ given below to find suitable cofibrant replacements of a given $K \in C(Mod(R))$. However, if $K = S$

is actually a commutative algebra over R, one may make use of the following device to produce a particularly nice cofibrant replacement of S in the category of commutative dg-algebras over R. This is discussed in [7, proof of Lemma 5.2].

A.0.3. Resolutions via the free commutative algebra functor

Let R denote a commutative ring with 1 and let S denote a commutative algebra over R. Given a set X, one first forms the free R-module on X: this will be denoted $F_R(X)$. Now one takes the R-symmetric algebra on $F_R(X)$: this will be denoted $\tilde{F}_R(X)$. Observe that $\tilde{F}_R(X)$ is a graded R-module which is free in each degree. Let $(R-algebras)$ denote the category of all commutative R-algebras and let (sets) denote the category of all (small) sets. Let $\tilde{U} : (R-algebras) \to (sets)$ be the forgetful functor. Then the functor $\tilde{F}_R$ defined above will be left-adjoint to $\tilde{U}$. Together, these two functors define a triple and we may apply them in the usual manner to S to produce a simplicial object in the category of R-algebras, $\tilde{S}_\bullet$ together with a quasi-isomorphism to S. Since the normalizing functor sending a simplicial abelian group to the associated normalized chain complex is strictly monoidal, one observes that $\tilde{S} = N(\tilde{S}_\bullet)$ is a dg-algebra, trivial in negative degrees, together with a quasi-isomorphism to S. (One may also observe that if each $\tilde{S}_n$ is a free R-module, then so is $N(\tilde{S}_\bullet)_n$.) In view of the explicit construction above, one may verify the following isomorphism:

$$\mathbb{H}(\tilde{S}) \underset{\mathbb{H}(R)}{\wedge} \mathbb{H}(M) \cong \mathbb{H}(\tilde{S} \underset{R}{\otimes} M),\ M \ \epsilon\ C(Mod(R)), \tag{A.0.4}$$

where we let $\mathbb{H}(\tilde{S}) = \mathbb{H}(\tilde{S}_\bullet) = \underset{\Delta}{\text{hocolim}}\{\mathbb{H}(\tilde{S}_n)|n\}$ and $\mathbb{H}(\tilde{S} \underset{R}{\otimes} M) = \mathbb{H}(\tilde{S}_\bullet \underset{R}{\otimes} M) = \underset{\Delta}{\text{hocolim}}\{\mathbb{H}(\tilde{S}_n \underset{R}{\otimes} M)|n\}$.

Finally it is important to observe that $\mathbb{H}(\tilde{S}_n)$ for each $n \geq 0$ and therefore $\underset{\Delta}{\text{hocolim}}\{\mathbb{H}(\tilde{S}_n)|n\}$ is commutative algebra spectrum over $\mathbb{H}(R)$. Moreover, each $\mathbb{H}(\tilde{S}_n)$ is a cofibrant object in $Mod(\mathbb{H}(R))$ and hence so is $\underset{\Delta}{\text{hocolim}}\{\mathbb{H}(\tilde{S}_n)|n\}$. Therefore we will in fact make use of this cofibrant replacement, along with Remark 3.2.

A.0.5. Model structure for the category of (unbounded) chain complexes

Let R denote a commutative ring and let $\mathcal{C}(R)$ denote the category of all (that is, possibly unbounded) chain-complexes of R-modules with a differential of degree -1. This category may be given the cofibrantly generated model category structure described in [21, Definition 2.3.3] and also [13]. Recall the generating cofibrations and generating trivial cofibrations are defined as follows. For each integer n, one lets S^n to be the complex concentrated in degree n where it is R and trivial elsewhere. D^n will be the complex concentrated in degrees n and $n-1$ where they are both R with the differential $D^n_n \to D^n_{n-1}$ being the identity. Then it is shown in [21, Proposition 4.2.13] that this model category is a cofibrantly generated symmetric monoidal model category with the generating cofibrations I, being maps of the form $\{S^{n-1} \to D^n|n\}$ and with the generating trivial cofibrations J, being maps of the form $\{0 \to D^n|n\}$. The monoidal structure is given by the tensor product of chain complexes, weak-equivalences are quasi-isomorphisms of chain complexes and fibrations are maps that are degree-wise surjective. It is straightforward to verify that this model category satisfies the monoidal axiom: see [13, Lemma 4.6] for more details.

A.0.6. Homotopy inverse limits of cosimplicial objects of chain complexes

Let R denote a commutative ring and let $\mathcal{C}(R)$ denote the category of chain complexes of possibly unbounded complexes of R-modules with differentials of degree -1. We provide $\mathcal{C}(R)$ with the model structure discussed above. In order to define homotopy inverse limits of diagrams in $\mathcal{C}(R)$, it suffices to first observe that the model category $\mathcal{C}(R)$ is *quasi-simplicial* (see [13, Definition 5.1]), though not simplicial. Then the usual definition of the homotopy inverse limit as an end carries over and therefore the homotopy inverse limit will have the usual properties.

Appendix B. Motivic slices and independence on the choice of the geometric classifying space

Throughout this section $\mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}})$ will denote the $\mathbb{A}^1$-localized category of S^1-spectra on the Nisnevich site of $\mathrm{S} = \operatorname{Spec} k$, with k a perfect field. We will let $E \,\epsilon\, \mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}})$ denote any spectrum that satisfies the following properties:

(i) it is -1-connected,
(ii) it is $\mathbb{A}^1$-homotopy invariant and has Nisnevich excision, and
(iii) is the 0-th spectrum of some $\mathbb{P}^1$-Ω spectrum $\mathcal{E} \,\epsilon\, \mathbf{Spt}(\mathrm{S}_{\mathrm{mot}})$.

One may observe that the above properties are satisfied by the S^1-spectrum representing algebraic K-theory. The work of [61], [31] and [38] show that now one can define a sequence of functors $f_n : \mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}}) \to \mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S})$ so that the following properties are true for any spectrum $E \,\epsilon\, \mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}})$:

B.0.1.

(i) One obtains a map $f_nE \to E$ that is natural in the spectrum E and universal for maps $F \to E$, with $F \,\epsilon\, \Sigma^n_{\mathbb{P}^1}\mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}})$.
(ii) One also obtains a tower of maps $\cdots \to f_{n+1}E \to f_nE \to \cdots \to f_0E = E$. Let s_pE denote the canonical homotopy cofiber of the map $f_{p+1}E \to f_pE$ and let $s_{\le q-1}E$ be defined as the canonical homotopy cofiber of the map $f_qE \to E$. Then one may show readily (see, for example, [38, Proposition 3.1.19]) that the canonical homotopy fiber of the induced map $s_{\le q}E \to s_{\le q-1}E$ identifies also with $s_q(E)$. One then also obtains a tower: $\cdots \to s_{\le q}E \to s_{\le q-1}E \to \cdots$.
(iii) Let Y be a smooth scheme of finite type over S and let $\mathrm{W} \subseteq \mathrm{Y}$ denote a closed not necessarily smooth subscheme so that $codim_{\mathrm{Y}}(\mathrm{W}) \ge \mathrm{q}$ for some $q \ge 0$. Then

the map $f_q E \to E$ induces a weak-equivalence (see [31, Lemma 2.3.2]) where $\mathcal{M}ap$ denotes the mapping spectrum (that is, the internal hom in the category $\mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S})$):

$$\mathcal{M}ap(\Sigma^{\infty}_{\mathrm{S}^1}(\mathrm{Y}/\mathrm{Y}-\mathrm{W})_+, f_{\mathrm{q}}\mathrm{E}) \to \mathcal{M}\mathrm{ap}(\Sigma^{\infty}_{\mathrm{S}^1}(\mathrm{Y}/\mathrm{Y}-\mathrm{W})_+, E), E \,\epsilon\, \mathbf{Spt}_{\mathrm{S}^1}(\mathrm{S}_{\mathrm{mot}}).$$

(iv) It follows that, then,

$$\mathcal{M}ap(\Sigma^{\infty}_{\mathrm{S}^1}(\mathrm{Y}/\mathrm{Y}-\mathrm{W})_+, s_{\leq \mathrm{q}-1}E) \simeq *$$

Next assume that G is a linear algebraic group and (W, U), $(\bar{\mathrm{W}}, \bar{\mathrm{U}})$ are both good pairs for G as in Definition 2.1. Let $\{(\mathrm{W}_{\mathrm{m}}, \mathrm{U}_{\mathrm{m}})|\mathrm{m} \geq 1\}$ and $\{(\bar{\mathrm{W}}_m, \bar{\mathrm{U}}_m)|m \geq 1\}$ denote the associated admissible gadgets. Then, since G acts freely on both U_{m} and $\bar{\mathrm{U}}_{\mathrm{m}}$, it is easy to see that $(\mathrm{W}\times\bar{\mathrm{W}}, \mathrm{U}\times\bar{\mathrm{W}}\cup\mathrm{W}\times\bar{\mathrm{U}})$ is also a good pair for G with respect to the diagonal action on $\mathrm{W}\times\bar{\mathrm{W}}$. Moreover, under the same hypotheses, one may readily check that $\{\mathrm{W}_{\mathrm{m}}\times\bar{\mathrm{W}}_{\mathrm{m}}, \mathrm{U}_{\mathrm{m}}\times\bar{\mathrm{W}}_{\mathrm{m}}\cup\mathrm{W}_{\mathrm{m}}\times\bar{\mathrm{U}}_{\mathrm{m}})|\mathrm{m}\geq 1\}$ is also an admissible gadget for G, for the diagonal action on $\mathrm{W}\times\bar{\mathrm{W}}$. Let X denote a smooth scheme of finite type over k on which G acts.

Since G acts freely on both U_{m} and $\bar{\mathrm{U}}_{\mathrm{m}}$, it follows that G has a free action on $\mathrm{U}_{\mathrm{m}}\times\bar{\mathrm{W}}_{\mathrm{m}}$ and also on $\mathrm{W}_{\mathrm{m}}\times\bar{\mathrm{U}}_{\mathrm{m}}$. We will let $\widetilde{\mathrm{U}}_{\mathrm{m}} = \mathrm{U}_{\mathrm{m}}\times\bar{\mathrm{W}}_{\mathrm{m}}\cup\mathrm{W}_{\mathrm{m}}\times\bar{\mathrm{U}}_{\mathrm{m}}$ for the following discussion. One may now compute the codimensions

$$codim_{\widetilde{\mathrm{U}}_m \underset{\mathrm{G}}{\times}\mathrm{X}}(\widetilde{\mathrm{U}}_m \underset{\mathrm{G}}{\times}\mathrm{X} - \mathrm{U}_{\mathrm{m}}\times\bar{\mathrm{W}}_{\mathrm{m}}\underset{\mathrm{G}}{\times}\mathrm{X}) = codim_{\mathrm{W}_{\mathrm{m}}}(\mathrm{W}_{\mathrm{m}}-\mathrm{U}_{\mathrm{m}}), \text{ and} \tag{B.0.2}$$

$$codim_{\widetilde{\mathrm{U}}_m \underset{\mathrm{G}}{\times}\mathrm{X}}(\widetilde{\mathrm{U}}_m \underset{\mathrm{G}}{\times}\mathrm{X} - \mathrm{W}_{\mathrm{m}}\times\bar{\mathrm{U}}_{\mathrm{m}}\underset{\mathrm{G}}{\times}\mathrm{X}) = codim_{\bar{\mathrm{W}}_m}(\bar{\mathrm{W}}_m-\bar{\mathrm{U}}_m) \tag{B.0.3}$$

In view of 8.0.1(iv), it follows that the induced maps

$$\begin{aligned}\mathcal{M}ap(\Sigma^{\infty}_{\mathrm{S}^1}(\mathrm{U}_{\mathrm{m}}\times_{\mathrm{G}}\mathrm{X})_+, s_{\leq\mathrm{q}-1}E) &\simeq \mathcal{M}\mathrm{ap}(\Sigma^{\infty}_{\mathrm{S}^1}((\mathrm{U}_{\mathrm{m}}\times\bar{\mathrm{W}}_{\mathrm{m}})\times_{\mathrm{G}}\mathrm{X})_+, s_{\leq\mathrm{q}-1}E)\\ &\leftarrow \mathcal{M}ap(\Sigma^{\infty}_{\mathrm{S}^1}(\widetilde{\mathrm{U}}_m\times_{\mathrm{G}}\mathrm{X})_+, s_{\leq\mathrm{q}-1}E) \text{ and}\\ \mathcal{M}ap(\Sigma^{\infty}_{\mathrm{S}^1}(\bar{\mathrm{U}}_m\times_{\mathrm{G}}\mathrm{X})_+, s_{\leq\mathrm{q}-1}E) &\simeq \mathcal{M}\mathrm{ap}(\Sigma^{\infty}_{\mathrm{S}^1}((\mathrm{W}_{\mathrm{m}}\times\bar{\mathrm{U}}_{\mathrm{m}})\times_{\mathrm{G}}\mathrm{X})_+, s_{\leq\mathrm{q}-1}E)\\ &\leftarrow \mathcal{M}ap(\Sigma^{\infty}_{\mathrm{S}^1}(\widetilde{\mathrm{U}}_m\times_{\mathrm{G}}\mathrm{X})_+, s_{\leq\mathrm{q}-1}E)\end{aligned} \tag{B.0.4}$$

are both weak-equivalences if $codim_{\mathrm{W}_{\mathrm{m}}}(\mathrm{W}_{\mathrm{m}}-\mathrm{U}_{\mathrm{m}})$ and $codim_{\bar{\mathrm{W}}_m}(\bar{\mathrm{W}}_m-\bar{\mathrm{U}}_m)$ are both greater than or equal to q. The first weak-equivalences in (B.0.4) are provided by the homotopy property for the spectrum E, which is inherited by the slices. Therefore, both maps in (B.0.4) induce a weak-equivalence on taking the homotopy inverse limit as $m, q \to \infty$. Finally, one may conclude that, since E is also assumed to be $\mathbb{A}^1$-homotopy invariant, $\underset{\infty\leftarrow q}{\mathrm{holim}}\mathcal{M}ap(\Sigma^{\infty}_{\mathrm{S}^1}(\mathrm{U}_{\mathrm{m}}\times_{\mathrm{G}}\mathrm{X})_+, \mathrm{s}_{\leq\mathrm{q}-1}E) \simeq \mathcal{M}\mathrm{ap}(\Sigma^{\infty}_{\mathrm{S}^1}(\mathrm{U}_{\mathrm{m}}\times_{\mathrm{G}}\mathrm{X})_+, E)$ and $\underset{\infty\leftarrow q}{\mathrm{holim}}\mathcal{M}ap(\Sigma^{\infty}_{\mathrm{S}^1}(\bar{\mathrm{U}}_m\times_{\mathrm{G}}\mathrm{X})_+, \mathrm{s}_{\leq\mathrm{q}-1}E) \cong \mathcal{M}\mathrm{ap}(\Sigma^{\infty}_{\mathrm{S}^1}(\bar{\mathrm{U}}_{\mathrm{m}}\times_{\mathrm{G}}\mathrm{X})_+, E)$.

Appendix C. Additional results on derived completion

In this section we collect together a few technical results that have been used in the body of the paper. Recall that $\mathbf{Spt}_{S^1}$ denotes the category of S^1-symmetric spectra: the spectra that we consider in this section will all be spectra in $\mathbf{Spt}_{S^1}$. A notion that we have found useful is that of *ideals* in ring spectra, which we use in Lemma 3.5, Proposition 4.5 and also appears in [7, Corollary 6.7]. (See also [22] for related results.)

C.1. Ideals in ring spectra

Definition C.1. Let E be a commutative ring spectrum, which we will assume is cofibrant in the category of algebra spectra over the sphere spectrum. An *ideal* in E is an E-module spectrum, I, which is *cofibrant as an E-module spectrum*, together with a specified map $i : I \to E$ of E-module-spectra so that the canonical homotopy cofiber of i is weakly-equivalent to a commutative ring spectrum *under* E (that is, with a map of commutative ring spectra from E). Moreover, we require that these can be chosen functorially in i.

Lemma C.2. *Let $f : E' \to E$ denote a map of commutative ring-spectra.*

(i) If $i : I \to E$ is an ideal in E, then the canonical homotopy pull-back of i by f is an ideal in E'.

(ii) If $j : J \to E'$ is an ideal in E', and E is cofibrant as an E'-module spectrum, then $i = j \underset{id_{E'}}{\wedge} id_E : J \underset{E'}{\wedge} E \to E$ is an ideal in E.

Proof. Observe first that, E', E and I are all module-spectra over E' and that the given maps between them are also maps of E'-module-spectra.

Next we consider (i). One may functorially replace i by a map which is a fibration in the category of E-module spectra. Let $j : J \to E'$ denote the pull-back of I over E' by f. Therefore, it follows that J is an E'-module spectrum. The homotopy fiber of the induced map $J \to E'$, which will be denoted $hofib(j)$ maps to the homotopy fiber of the map $i : I \to E$ (denoted $hofib(i)$) and the above map is a weak-equivalence. By our hypotheses, $hocofib(i)$ (that is, the homotopy cofiber of i, which is weakly equivalent to the suspension of $hofib(i)$, that is, $\Sigma hofib(i)$) maps by a weak-equivalence to a commutative ring-spectrum E/I under E. Since f is map of ring-spectra, E/I is a commutative a ring-spectrum under E' as well. Moreover the choice of E/I is functorial in i and hence in j.

(ii) Clearly $J \underset{E'}{\wedge} E$ is an E-module. The map i corresponds to the map $J \underset{E'}{\wedge} E \to E' \underset{E'}{\wedge} E = E$. Since E is assumed to be cofibrant as an E'-module,

$$hofib(j) \underset{E'}{\wedge} E \to J \underset{E'}{\wedge} E \to E' \underset{E'}{\wedge} E = E$$

is a fiber sequence. Let $\Sigma hofibf(j) \to hocofib(j) \simeq E'/J$ denote a weak-equivalence to a commutative ring spectrum under E' given by the ideal-structure of j. Since E is a commutative ring spectrum under E' (and cofibrant as an E'-module), it follows that, $hocofib(j)\underset{E'}{\wedge}E \simeq E'/J\underset{E'}{\wedge}E$. The latter is clearly a commutative ring spectrum under E. This proves (ii). □

Proposition C.3. *(i) Let R denote a commutative Noetherian ring with unit and let I, J denote two ideals in R with $J \subseteq I$ and where there exists a positive integer N so that $I^N \subseteq J$. If M is any chain complex (that is, with differentials of degree -1) of R-modules trivial in sufficiently small degrees, then $M\hat{}_I \simeq M\hat{}_J$, where the completion denotes the derived completion.*

(ii) Let E denote a -1-connected commutative ring spectrum so that $\pi_0(E)$ is a commutative Noetherian ring. Let I, J denote two ideals in E so that $\pi_0(J) \subseteq \pi_0(I)$ and there exists a positive integer N so that $\pi_0(I)^N \subseteq \pi_0(J)$. If M is any t-connected E-module spectrum (for some integer t), then one obtains a weak-equivalence $M\hat{}_I \simeq M\hat{}_J$.

Proof. We may reduce (ii) to (i) by making use of the arguments in 3.0.4. Now (i) follows by invoking Proposition 3.8. (See for example, the proof of (3.0.19).) □

Next we will prove that indeed the derived completion of modules over a commutative Noetherian ring at an ideal depends only the radical of the ideal.

First we show how to obtain functorial flat resolutions that are compatible with pairings. Let R denote a commutative Noetherian ring with 1. Let $Mod(R)$ denote the category of all R-modules and let $(sets)$ denote the category of all small sets. Then we obtain the following functors: $S : Mod(R) \to (sets)$ and $F : (sets) \to Mod(R)$ by letting $S(M) = Hom_R(R, M)$ and $F(T) =$ the free R-module on the set T. Here Hom_R denotes Hom in the category $Mod(R)$. Then F is left-adjoint to S and they provide a triple:

$F \circ S : Mod(R) \to Mod(R)$.

By iterating this triple, we obtain a simplicial resolution of any R-module M, which consists of flat-modules in each degree. One may then apply the normalization functor that sends a simplicial abelian group to a chain complex (that is, where the differentials are of degree -1) and then to the associated co-chain complex that is trivial in positive degrees and with differentials of degree $+1$. We will denote this functorial flat resolution of an R-module M by $\tilde{M}$ (or $\tilde{M}^\bullet$). This construction has the following properties.

(i) It is functorial in M, i.e. if $f : M \to N$ and $g : N \to P$ are maps in $Mod(R)$, then one obtains induced maps $\tilde{f} : \tilde{M} \to \tilde{N}$ and $\tilde{g} : \tilde{N} \to \tilde{P}$ so that $\widetilde{(g \circ f)} = \tilde{g} \circ \tilde{f}$.

(ii) It is compatible with pairings, that is, if $M \otimes_R N \to P$ is a map of R-modules, one obtains an induced map $\tilde{M} \otimes_R \tilde{N} \to \widetilde{(M \otimes_R N)} \to \tilde{P}$.

Theorem C.4. *Let R denote a commutative Noetherian ring with 1 and let $J \subseteq I$ denote two ideals in R so that $\sqrt{I} = \sqrt{J}$. Then if M is an R-module, not necessarily finitely generated, $M\hat{}_I \simeq M\hat{}_J$ where $M\hat{}_I$ ($M\hat{}_J$) denotes the derived completion of M with respect to I (J, respectively).*

Proof. First observe that since $\sqrt{J} = \sqrt{I} \supseteq I$ and I is finitely generated, it follows that $I^{n_0} \subseteq J$ for some $n_0 >> 0$, that is, now we obtain the chain of containments: $J^{n_0} \subseteq I^{n_0} \subseteq J \subseteq I$. Therefore, the conclusion follows from Proposition C.3. □

We conclude with following Lemma: we thank Pablo Pelaez for discussions on this.

Lemma C.5. *Let* E *denote an* S^1*-ring spectrum. Let* $\{p_n : \mathcal{X}_{n+1} \to \mathcal{X}_n | n \geq 0\}$ *denote a tower of maps of* E*-module spectra and let* $\{q_n : \mathcal{Y}_{n+1} \to \mathcal{Y}_n | n \geq 0\}$ *denote a tower of fibrations of* E*-module spectra.*

Assume that one is given, for each n*, a map* $f_n : \mathcal{X}_n \to \mathcal{Y}_n$*, so that each of the squares*

$$\begin{tikzcd} \mathcal{X}_{n+1} \arrow[r, "p_n"] \arrow[d, "f_{n+1}"] & \mathcal{X}_n \arrow[d, "f_n"] \\ \mathcal{Y}_{n+1} \arrow[r, "q_n"] & \mathcal{Y}_n \end{tikzcd}$$

homotopy commutes in the category of E*-module spectra. Then one may inductively replace each of the map* f_n *up to homotopy by a map* f'_n*,* $n \geq 0$*, so that the maps* $\{f'_n | n \geq 0\}$ *define a map of inverse systems of maps* $\{\mathcal{X}_n | n\}$ *and* $\{\mathcal{Y}_n | n\}$ *of* E*-module spectra.*

Proof. We will construct the replacement f'_n by ascending induction on n, starting with $f'_0 = f_0$. Assume we have found replacements of the maps f_i, by f'_i, for all $0 \leq i \leq n$, so that (i) each of the maps f'_i is homotopic to the given map f_i for all $0 \leq i \leq n$ and that the diagram

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, "p_n"] \arrow[d, "f'_n"] & \cdots \arrow[r, "p_1"] & \mathcal{X}_0 \arrow[d, "f'_0"] \\ \mathcal{Y}_n \arrow[r, "q_n"] & \cdots \arrow[r, "q_1"] & \mathcal{Y}_0 \end{tikzcd} \tag{C.0.1}$$

strictly commutes. Let $\mathrm{H} : \mathcal{X}_{\mathrm{n+1}} \times \Delta[1] \to \mathcal{Y}_{\mathrm{n}}$ denote a homotopy so that, $\mathrm{H}(\quad, 1) = \mathrm{q}_{\mathrm{n+1}} \circ \mathrm{f}_{\mathrm{n+1}}$ and $\mathrm{H}(\quad, 0) = \mathrm{f}'_{\mathrm{n}} \circ \mathrm{p}_{\mathrm{n+1}}$. Now we consider the diagram:

$$\begin{tikzcd} \mathcal{X}_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "i_1"'] & \mathcal{Y}_{n+1} \arrow[d, "q_{n+1}"] \\ \mathcal{X}_{n+1} \times \Delta[1] \arrow[r, "\mathrm{H}"] \arrow[ur, "\mathrm{H}'"] & \mathcal{Y}_n \\ \mathcal{X}_{n+1} \arrow[u, "i_0"] \arrow[r, "p_{n+1}"] & \mathcal{X}_n \arrow[u, "f'_n"'] \end{tikzcd}$$

Here the homotopy H' is a lifting of the given homotopy H: observe that such a lifting H' exists because the map i_1 is a trivial cofibration and q_{n+1} is a fibration.

Now we will replace the map f_{n+1} by the map $f'_{n+1} = \mathrm{H}' \circ \mathrm{i}_0$. Then one may observe that H' provides a homotopy between the two maps f_{n+1} and f'_{n+1}. Moreover, the diagram

$$\begin{array}{ccccccc} \mathcal{X}_{n+1} & \xrightarrow{p_{n+1}} & \mathcal{X}_n & \xrightarrow{p_n} & \cdots & \xrightarrow{p_1} & \mathcal{X}_0 \\ \downarrow{\scriptstyle f'_{n+1}} & & \downarrow{\scriptstyle f'_n} & & & & \downarrow{\scriptstyle f'_0} \\ \mathcal{Y}_{n+1} & \xrightarrow{q_{n+1}} & \mathcal{Y}_n & \xrightarrow{q_n} & \cdots & \xrightarrow{q_1} & \mathcal{Y}_0 \end{array}$$

strictly commutes. Clearly, one may repeat the above arguments to complete the proof. □

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