

XPPAUT Tutorial III for Math 865L

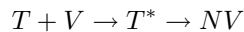
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We already learn the basis functions of xppaut in the previous two tutorials. Here we will use xppaut to solve the models in virus dynamics and Epidemiology.

1 Virus Dynamics

The simple HIV model we will discuss here is proposed by Leentheer and Smith. The models considered have three state variables: T , the concentration of uninfected T cells; T^* , the concentration of productively infected T cells; and V , the concentration of free virus particles in the blood. In chemical reaction notation, the model can be written



because mass action reaction terms are used and each infected T cell is assumed to produce N viral particles over its lifespan. The interaction between these cells and virus particles is then given by the following equations:

$$\begin{aligned}\frac{dT}{dt} &= f(T) - kVT \\ \frac{dT^*}{dt} &= -\beta T^* + kVT \\ \frac{dV}{dt} &= -\gamma V + N\beta T^* - ikVT\end{aligned}$$

where $f(T) = \delta - \alpha T$ and $i = 0$ if we choose to ignore the loss of a viral particle when it enters a target cell, or $i = 1$ when we do not. The main results: If R_0 is the reproduction number ($R_0 = \frac{kT(N-i)}{\gamma}$)

1. If $R_0 < 1$, the virus is cleared.

2. If $R_0 > 1$, then a chronic disease steady state $E_e = (T_e, T_e^*, V_e)$ given by

$$T_e = \frac{\gamma}{k(N-i)}, \quad T_e^* = \frac{\gamma V_e}{(N-i)\beta}, \quad V_e = \frac{f(T_e)}{kT_e}$$

which is globally asymptotically stable.

Here the parameters are given as

$$\begin{aligned}\alpha &= 0.2 \text{day}^{-1}, & \beta &= 0.24 \text{day}^{-1}, & \gamma &= 2 \text{day}^{-1} \\ \delta &= 10 \text{day}^{-1} \text{mm}^{-3}, & k &= 0.002 \text{mm}^3 \text{day}^{-1}, & i &= 1\end{aligned}$$

$$T_0 = 300, \quad T^*(0) = 100, \quad V(0) = 1000.$$

With these parameters, if $N = 21$, the reproduction number $R_0 = 1$. Try to demonstrate numerically with case (1) by choosing $N = 15$ and case (2) by choosing $N = 30$.

HW: In Perelson and Nelson's model, they proposed $f(T) = \delta - \alpha T + pT(1 - \frac{T}{T_{max}})$ instead. In this case, healthy T cells are assumed to proliferate logistically. Use XPP to solve this model for the following parameters:

$$\begin{aligned} \alpha &= 0.02 \text{day}^{-1}, \quad \beta = 0.24 \text{day}^{-1}, \quad \gamma = 2.4 \text{day}^{-1} \\ \delta &= 10 \text{day}^{-1} \text{mm}^{-3}, \quad k = 0.0027 \text{mm}^3 \text{day}^{-1}, \quad i = 1 \\ N &= 10, \quad T_{max} = 1500 \text{mm}^{-3}, \quad p = 3 \text{day}^{-1} \\ T_0 &= 350, \quad T^*(0) = 1400, \quad V(0) = 900. \end{aligned}$$

In this case, there exists an orbitally asymptotically stable periodic orbit.

2 SIR model

$$\begin{aligned} \frac{dS}{dt} &= -kSI \\ \frac{dI}{dt} &= kSI - \alpha I \\ \frac{dR}{dt} &= \alpha I \end{aligned}$$

where $S+I+R = \text{const} = N$. Thus we only need to solve the first two equations. Fixed points at $I = 0, S = S$. Stability at this fixed point (no Infectious) is determined by the eigenvalues of

$$J(S, 0) = \begin{bmatrix} 0 & -kS \\ 0 & kS - \alpha \end{bmatrix}.$$

The eigenvalues are 0 and $kS - \alpha$.

1. Fixed point is stable (not asymptotically) along the eigen-vector corresponding to the zero eigenvalue. ($I = 0$)
2. Fixed point is asymptotically stable or unstable (depending on the sign of $kS - \alpha$) along the eigen vector corresponding to the second eigen value.

Thus disease free state is not stable when ($R_0 = \frac{kS}{\alpha} > 1$) $kS - \alpha > 0$ so an epidemic can occur.

Ex1: Parameters: $S(0) = 499, I(0) = 1, R(0) = 0, k = 1.e - 3, \alpha = 0.1$

3 SEIR model

$$\begin{aligned} \frac{dS}{dt} &= II - \mu S - \beta SI \\ \frac{dE}{dt} &= \beta SI - (\mu + k)E \\ \frac{dI}{dt} &= kE - (\mu + \alpha)I \\ \frac{dR}{dt} &= \alpha I - \mu R \end{aligned}$$

The reproduction number is $R_0 = \frac{\beta k}{(\mu+k)(\mu+\alpha)}$. It can be shown that for $R_0 > 1$ the model has a fixed point with $I = 0$ which is unstable, and a fixed point with $I > 0$ which is stable.

Ex1: Parameters: $S(0) = 499, E(0) = 200, I(0) = 1, R(0) = 0, \beta = 1.e - 3,$
 $\alpha = k = 0.1, \mu = 0.1$

Ex2: Parameters: $S(0) = 499, E(0) = 200, I(0) = 1, R(0) = 0, \beta = 0.5,$
 $\alpha = k = 0.1, \mu = 0.1$