XPPAUT Tutorial III for Math 865L

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We already learn the basis functions of xppaut in the previous two tutorials. Here we will use xppaut to solve the models in virus dynamics and Epidemiology.

1 Virus Dynamics

The simple HIV model we will discuss here is proposed by Leentheer and Smith. The models considered have three state variables: T, the concentration of uninfected T cells; T^*, the concentration of productively infected T cells; and$V$, the concentration of free virus particles in the blood. In chemical reaction notation, the model can be written

$$T + V \to T^* \to NV$$

because mass action reaction terms are used and each infected T cell is assumed to produce N viral particles over its lifespan.The interaction between these cells and virus particles is then given by the following equations:

$$\begin{aligned}
\frac{dT}{dt} &= f(T) - kVT \\
\frac{dT^*}{dt} &= -\beta T^* + kVT \\
\frac{dV}{dt} &= -\gamma V + N\beta T^* - ikVT
\end{aligned}$$

where $f(T) = \delta - \alpha T$ and $i = 0$ if we choose to ignore the loss of a viral particle when it enters a target cell, or $i = 1$ when we do not. The main results: If R_0 is the reproduction number ($R_0 = \frac{k\bar{T}(N-i)}{\gamma}$)

1.If $R_0 < 1$, the virus is cleared.

2.If $R_0 > 1$,, then a chronic disease steady state $E_e = (T_e, T_e^*, V_e)$ given by

$$T_e = \frac{\gamma}{k(N-i)}, \quad T_e^* = \frac{\gamma V_e}{(N-i)\beta} \quad V_e = \frac{f(T_e)}{kT_e}$$

which is globally asymptotically stable.

Here the parameters are given as

$$\alpha = 0.2 day^{-1}, \quad \beta = 0.24 day^{-1}, \quad \gamma = 2 day^{-1}$$

$$\delta = 10 day^{-1} mm^{-3}, \quad k = 0.002 mm^3 day^{-1}, \quad i = 1$$

$$T_0 = 300, \quad T^*(0) = 100, \quad V(0) = 1000.$$

With these parameters, if $N = 21$, the reproduction number $R_0 = 1$. Try to demonstrate numerically with case (1) by choosing $N = 15$ and case (2) by choosing $N = 30$.

HW: In Perelson and Nelson's model, they proposed $f(T) = \delta - \alpha T + pT(1 - \frac{T}{T_{max}})$ instead. In this case, healthy T cells are assumed to proliferate logistically. Use XPP to solve this model for the following parameters:

$$\alpha = 0.02 day^{-1}, \quad \beta = 0.24 day^{-1}, \quad \gamma = 2.4 day^{-1}$$

$$\delta = 10 day^{-1} mm^{-3}, \quad k = 0.0027 mm^3 day^{-1}, \quad i = 1$$

$$N = 10, \quad T_{max} = 1500 mm^{-3}, \quad p = 3 day^{-1}$$

$$T_0 = 350, \quad T^*(0) = 1400, \quad V(0) = 900.$$

In this case, there exists an orbitally asymptotically stable periodic orbit.

2 SIR model

$$\begin{aligned} \frac{dS}{dt} &= -kSI \\ \frac{dI}{dt} &= kSI - \alpha I \\ \frac{dR}{dt} &= \alpha I \end{aligned}$$

where $S + I + R = const = N$. Thus we only need to solve the first two equations. Fixed points at $I = 0, S = S$. Stability at this fixed point (no Infectious) is determined by the eigenvalues of

$$J(S, 0) = \begin{bmatrix} 0 & -kS \\ 0 & kS - \alpha \end{bmatrix}.$$

The eigenvalues are 0 and $kS - \alpha$.

1. Fixed point is stable (not asymptotically) along the eigen-vector corresponding to the zero eigenvalue. ($I = 0$)

2. Fixed point is aymptotically stable or unstable (depending on the sign of $kS - \alpha$) along the eigen vector corresponding to the second eigen value.

Thus disease free state is not stable when ($R_0 = \frac{kS}{\alpha} > 1$) $kS - \alpha > 0$ so an epidemic can occur.

Ex1: Parameters: $S(0) = 499$, $I(0) = 1$, $R(0) = 0$, $k = 1.e-3$, $\alpha = 0.1$

3 SEIR model

$$\begin{aligned} \frac{dS}{dt} &= II - \mu S - \beta SI \\ \frac{dE}{dt} &= \beta SI - (\mu + k)E \\ \frac{dI}{dt} &= kE - (\mu + \alpha)I \\ \frac{dR}{dt} &= \alpha I - \mu R \end{aligned}$$

The reproduction number is $R_0 = \frac{\beta k}{(\mu+k)(\mu+\alpha)}$. It can be shown that for $R_0 > 1$ the model has a fixed point with $I = 0$ which is unstable, and a fixed point with $I > 0$ which is stable.

Ex1: Parameters: $S(0) = 499$, $E(0) = 200$, $I(0) = 1$, $R(0) = 0$, $\beta = 1.e - 3$, $\alpha = k = 0.1$, $\mu = 0.1$

Ex2: Parameters: $S(0) = 499$, $E(0) = 200$, $I(0) = 1$, $R(0) = 0$, $\beta = 0.5$, $\alpha = k = 0.1$, $\mu = 0.1$